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Siegel modular forms of half integral weight and a lifting conjecture

By

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Abstract

A conjecture on lifting to Siegel cusp forms of half-integral weight $k-1/2$ of degree two from each pair of cusp forms of $SL_2(\mathbb{Z})$ of weight $2k-2$ and $2k-4$ is given with a conjectural relation of the L functions and numerical evidences. We also describe the space of Siegel modular forms of half-integral weight, its "plus subspace" and Jacobi forms of degree two by explicitly given theta functions.

This paper has two aims.
(1) We describe Siegel modular forms of half integral weight of $\Gamma_0(4)$ of degree two explicitly.
(2) We give a conjecture on lifting preserving L function from a pair of elliptic modular forms to Siegel modular forms of half integral weight of degree two with numerical evidences on coincidence of the Euler factors.

As for (1), we also describe the so-called "plus subspace" consisting of a kind of new forms which is isomorphic to the space of Jacobi forms of some sort. We state our results in section §1 (cf. Theorems 1.3, 1.5, 1.8, 1.9) and give the proof in §2. In the remaining sections we treat (2) (cf. Conjecture 3.1).

Now we explain more precise content of this paper. First of all, rough content of our conjecture mentioned above is as follows. We denote by $M_{k-1/2}(\Gamma_0(4))$ the space of Siegel modular forms of $\Gamma_0(4)$ of degree two of weight $k-1/2$ and by $S_{k-1/2}(\Gamma_0(4))$ the subspace of cusp forms. We denote by $S^+_{k-1/2}(\Gamma_0(4))$ the plus subspace of degree two. (This plus space was first introduced by W. Kohnen in case of one variable and later generalized by the present authors for general degree. As for the definition, see §2). Now our conjecture claims that for each pair of common eigen cusp forms f of weight

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$2k-2$ and g of weight $2k-4$ belonging to $SL_2(\mathbb{Z})$, there should exist a common eigen Siegel cusp form $F \in S^+_{k-1/2}(\Gamma_0(4))$ of weight $k-1/2$ of $\Gamma_0(4)$ such that $L(s,F) = L(s,f)L(s-1,g)$ (cf. §3, Conjecture 3.1). Here $L(s,f)$ and $L(s,g)$ are the usual Hecke L functions and $L(s,F)$ is a L function defined by Zhuravlev [22] (cf. also [6]). His Hecke theory on Siegel modular forms of half integral weight and the precise definition of L function will be reviewed in §3.

This conjecture is based on our numerical calculation of examples of L functions of explicitly given Siegel cusp forms of half integral weight. So we explain our explicit results on Siegel modular forms. Denote by $M_{k-1/2}(\Gamma_0(4),\chi)$ the space of Siegel modular forms of weight $k-1/2$ of $\Gamma_0(4)$ of degree two with character χ. Then the direct sum $\oplus_{k=1}^{\infty} M_{k-1/2}(\Gamma_0(4),\chi)$ is not a ring. But we can regard it as a module over a certain ring of Siegel modular forms of integral weight, and we can give explicit generators of modules of Siegel modular forms of half integral weight (with or without character) by theta constants (cf. Theorem 1.1, 1.2, 1.3). By this, we can give a dimension formula of Siegel modular forms of half integral weight of $\Gamma_0(4)$ as a corollary (cf. Corollary 1.2, 1.5) which was first obtained by Tsushima [17] by using holomorphic Lefschetz Theorem.

Then we need a description of the plus subspace. In degree one case, this space is isomorphic to holomorphic or skew holomorphic Jacobi forms of index one. (Eichler-Zagier [2], Skoruppa [15]). We can generalize the notion of the plus space for general degree so that the plus space of weight $k-1/2$ of degree n is isomorphic to the space of holomorphic or skew holomorphic Jacobi forms of index one of weight k of degree n of $Sp(n,\mathbb{Z})$, depending on parity of k or on character. (This is mostly known in Ibukiyama [9] and Hayashida [3]. The remaining case can be done in a similar way.) Now we have Tsushima's dimension formula for Jacobi forms of degree two. By using his result, we can also give each basis of the space of holomorphic or skew holomorphic Jacobi forms, or of the plus subspace explicitly(cf. Theorem 1.8, 1.9, 1.10). The result is very simple. Each space is a free module over the ring isomorphic to Siegel modular forms of even weight belonging to $Sp(2,\mathbb{Z})$. Extracting modular forms with small weights, the Euler factors at small primes in the plus space can be given by computer calculations, and we see that these examples support our conjecture (cf. §3).

The authors would like to express their thanks to Professor Tsushima for showing us his new results on dimensions.

1. Modules of Siegel modular forms

1.1. Graded rings of modular forms of integral weights

Let n or N be any natural number. For any commutative ring R, we denote by $Sp(n,R)$ the symplectic group of size $2n$ with components in R.

$$Sp(n,R) = \left\{g \in M_{2n}(R); gJ\,{}^t g = J\right\}$$

where $J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$ and 1_n is the unit matrix of size n. We put

$$\Gamma_0^{(n)}(N) = \left\{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{Z}); C \equiv 0 \bmod N \right\}.$$

Sometimes a conjugate of $\Gamma_0^{(n)}(4)$ is easier to treat, so we put

$$\rho_2 = \begin{pmatrix} 1_n & 0 \\ 0 & 2 \cdot 1_n \end{pmatrix},$$

and put $\Gamma^{(n)} = \rho_2^{-1}\Gamma_0^{(n)}(4)\rho_2$. Then, we get

$$\Gamma^{(n)} = \left\{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{Z}); B \equiv C \equiv 0 \bmod 2 \right\}.$$

If we define

$$\psi(g) = \left(\frac{-1}{\det(D)} \right) \text{ for } g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^{(n)}(4) \cup \Gamma^{(n)},$$

then this gives a character of the group $\Gamma_0^{(n)}(4)$ or $\Gamma^{(n)}$. For any integer k, any discrete subgroup Γ' of $Sp(n, \mathbb{R})$ with $\mathrm{vol}(\Gamma' \backslash Sp(n, \mathbb{R})) < \infty$ and a character χ of Γ', and any function $F(\tau)$ on the Siegel upper half space

$$H_n = \{\tau = X + iY = {}^t\tau \in M_n(\mathbb{C}); X, Y \in M_n(\mathbb{R}), Y > 0 \text{ (positive definite)}\},$$

we write

$$(F|_{k,\chi}\gamma)(\tau) = \chi(\gamma)^{-1} \det(C\tau + D)^{-k} F(\gamma\tau).$$

We say that a holomorphic function F on H_n is a modular form of weight k with character χ belonging to Γ' if it satisfies

$$F|_{k,\chi}\gamma = F$$

for all $\gamma \in \Gamma'$ and is bounded at each cusps of Γ'. The space of these modular forms is denoted by $M_k(\Gamma', \chi)$ and cusp forms by $S_k(\Gamma', \chi)$. When χ is the trivial character, we may sometimes omit χ in the above notation. For simplicity, we write

$$M(\Gamma', \chi) = \oplus_{k=0}^{\infty} M_k(\Gamma', \chi^k).$$

Then, this is obviously a graded ring.

In this paper, we mainly treat the case $n = 2$. So we write $\Gamma_0(4) = \Gamma_0^{(2)}(4)$ and $\Gamma = \Gamma^{(2)}$. The following formula for $n = 2$ was calculated by Tsushima, using [10] and [16].

Proposition 1.1 (Tsushima [17])**.**

$$\sum_{k=0}^{\infty} \dim M_k(\Gamma_0(4),\psi^k)t^k = \frac{1}{(1-t)(1-t^2)^2(1-t^3)},$$
$$\sum_{k=0}^{\infty} \dim M_k(\Gamma_0(4))t^k = \frac{(1+t^4)(1+t^{11})}{(1-t^2)^3(1-t^6)},$$
$$\sum_{k=0}^{\infty} \dim M_{2k}(\Gamma_0(4),\psi)t^{2k} = \frac{t^{12}+t^{14}}{(1-t^2)^3(1-t^6)}.$$

First, we shall obtain the graded ring $\oplus_{k=0}^{\infty} M_k(\Gamma_0(4),\psi^k)$. Instead of $\Gamma_0(4)$, we consider Γ, partly because $\Gamma \subset \Gamma_0(2)$ and $M(\Gamma_0(2))$ has been known in Ibukiyama [7]. Indeed the ring $M_{even}(\Gamma_0(2))$ of modular forms of even weights is generated by four algebraically independent modular forms X, Y, Z, K of degree two defined by

$$\begin{aligned}
X &= ((\theta_{0000})^4 + (\theta_{0001})^4 + (\theta_{0010})^4 + (\theta_{0011})^4)/4, \\
Y &= (\theta_{0000}\theta_{0001}\theta_{0010}\theta_{0011})^2, \\
Z &= ((\theta_{0100})^4 - (\theta_{0110})^4)^2/16384, \\
K &= (\theta_{0100}\theta_{0110}\theta_{1000}\theta_{1001}\theta_{1100}\theta_{1111})^2/4096,
\end{aligned}$$

(see [7]), where θ_m is the theta constant on H_2 defined by

$$\theta_m(\tau) \ = \ \sum_{p\in\mathbb{Z}^2} e\left(\frac{1}{2}\,{}^t\left(p+\frac{m'}{2}\right)\tau\left(p+\frac{m'}{2}\right) + {}^t\left(p+\frac{m'}{2}\right)\frac{m''}{2}\right),$$

for $m = {}^t({}^t m', {}^t m'') \in \mathbb{Z}^4$, $m', m'' \in \mathbb{Z}^2$, $\tau \in H_2$ and $e(x) = e^{2\pi i x}$. (cf. Igusa [11]). Now, we put

$$\begin{aligned}
f_1 &= (\theta_{0000})^2, \\
f_2 &= f_1^2, \\
g_2 &= (\theta_{0000})^4 + (\theta_{0100})^4 + (\theta_{1000})^4 + (\theta_{1100})^4 \\
f_3 &= (\theta_{0001}\theta_{0010}\theta_{0011})^2, \\
\chi_5 &= \theta_{0000}\theta_{0001}\theta_{0010}\theta_{0011}\theta_{0100}\theta_{0110}\theta_{1000}\theta_{1001}\theta_{1100}\theta_{1111}, \\
f_6 &= (\theta_{0001}^4 - \theta_{0010}^4)(\theta_{0001}^4 - \theta_{0011}^4)(\theta_{0010}^4 - \theta_{0011}^4), \\
f_{11} &= f_6\chi_5, \\
f_{21/2} &= f_{11}/\theta_{0000}.
\end{aligned}$$

By definition, we have $Y = f_1 f_3$ and it is not difficult to show that $Z = (g_2 + 2X - 3f_2)^2/36864$. (Since all the relations between θ_m^4 are known by Igusa [10], it is a routine calculation to show this anyway. We omit the details here.) Here the form $f_{21/2}$ is obviously holomorphic. The notation f_6 and χ_5 are introduced to make notation simpler.

Proposition 1.2. (1) *The function X, f_2, g_2, Z, Y, K, or f_{11} is a modular form of Γ of weight $2, 2, 2, 4, 4, 6$, or 11, respectively, and f_1 or f_3 is a modular form of Γ with character ψ of weight 1 or 3, respectively.*
(2) *The four forms X, f_1, g_2, K are algebraically independent.*
(3) *We have*

$$\begin{aligned} f_3^2 &= -4096K + \frac{1}{9} f_2(4g_2X - 6f_2g_2 + 24f_2X + g_2^2 - 32X^2) + Y(4X - 2f_2) \\ &= -4096K + f_2(4096Z - f_2^2 + 4f_2X - 4X^2) + Y(4X - 2f_2), \end{aligned}$$

and hence X, f_2, g_2, f_3 are also algebraically independent.

We denote by B the weighted polynomial ring generated by X, f_2, g_2, K.

$$B = \mathbb{C}[X, f_2, g_2, K].$$

Theorem 1.1. *The ring $M(\Gamma, \psi) = \oplus_{k=0}^{\infty} M_k(\Gamma, \psi^k)$ is given by a weighted polynomial ring*

$$M(\Gamma, \psi) = \mathbb{C}[f_1, g_2, X, f_3].$$

Also, the ring $M(\Gamma) = \oplus M_k(\Gamma)$ is given by

$$M(\Gamma) = B \oplus YB \oplus f_{11}(B \oplus YB).$$

The formula for f_{11}^2 is easily obtained but the result is complicated and not so interesting, so we omit it here.

Theorem 1.2. *The module of Siegel modular forms of even weight of Γ with character ψ is given by*

$$\oplus_{k=0}^{\infty} M_{2k}(\Gamma, \psi) = f_{11} f_1 B \oplus f_{11} f_3 B.$$

We note that the result for $M_k(\Gamma, \psi)$ for odd k is already contained in Theorem 1.1.

We can rewrite the above results for Γ to those for $\Gamma_0(4)$ very easily, since $M_k(\Gamma_0(4), \psi) = \{F(2\tau); F \in M_k(\Gamma, \psi)\}$ and $M_k(\Gamma_0(4)) = \{F(2\tau); F \in M_k(\Gamma)\}$. The latter spaces are also described by usual theta constants by using Riemann's theta relations (cf. Igusa [11, p. 233]). For example, we get

$$\begin{aligned} f_1(2\tau) &= (\theta_{0000}(\tau)^2 + \theta_{0001}(\tau)^2 + \theta_{0010}(\tau)^2 + \theta_{0011}(\tau)^2)/4, \\ X(2\tau) &= (2X(\tau) + 12(\theta_{0000}(\tau)\theta_{0001}(\tau)\theta_{0010}(\tau)\theta_{0011}(\tau)) \\ &\qquad + 3(\theta_{0000}(\tau)^2\theta_{0001}(\tau)^2 + \theta_{0000}(\tau)^2\theta_{0010}(\tau)^2 + \theta_{0000}(\tau)^2\theta_{0011}(\tau)^2 \\ &\qquad + \theta_{0001}(\tau)^2\theta_{0010}(\tau)^2 + \theta_{0001}(\tau)^2\theta_{0011}(\tau)^2 + \theta_{0010}(\tau)^2\theta_{0011}(\tau)^2))/32, \\ g_2(2\tau) &= X(\tau). \end{aligned}$$

1.2. Modular forms of half integral weights

We put $\theta(\tau) = \sum_{p\in\mathbb{Z}^n} \exp(2\pi i({}^t p\tau p))$. Let F be a holomorphic function on H_n. For any integer $k \geq 1$, we say that F is a Siegel modular form of weight $k-\frac{1}{2}$ belonging to $\Gamma_0(4)$ with character χ, if F satisfies the following condition

$$F(\gamma\tau) = \chi(\gamma)\left(\frac{\theta(\gamma\tau)}{\theta(\tau)}\right)^{2k-1} F(\tau) \quad \text{for every } \gamma \in \Gamma_0^{(n)}(4)\ .$$

We denote the space of above forms by $M_{k-\frac{1}{2}}(\Gamma_0^{(n)}(4),\chi)$. When χ is ψ or the trivial character, we also put

$$M_{k-1/2}(\Gamma^{(n)},\chi) = \left\{f(\tau/2);\ f\in M_{k-1/2}(\Gamma_0^{(n)}(4),\chi)\right\}.$$

The following Theorem 1.3 for $n = 2$ was first observed by Tsushima [17] by showing the dimension formulas in the Corollary 1.2 by Riemann Roch theorem and by comparing the dimensions of both sides in Theorem 1.3. We use a different argument, that is, without using the dimension formula, we first prove the following theorem directly by using ring theoretic argument, and next gives a dimension formula of modular forms of half integral weights as a corollary of this theorem.

Theorem 1.3. *We get*

$$\oplus_{k=0}^{\infty} M_{k+1/2}(\Gamma_0(4)) = \theta_{0000}(2\tau)(\oplus_{k=0}^{\infty} M_k(\Gamma_0(4),\psi^k)).$$

We put

$$M^{(1/2)}(\Gamma) = (\oplus_{k=0}^{\infty} M_k(\Gamma,\psi^k)) + (\oplus_{k=0}^{\infty} M_{k+1/2}(\Gamma)).$$

Since

$$(\theta_{0000}(\gamma\tau)/\theta_{0000}(\tau))^2 = \psi(\gamma)\det(c\tau+d)$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma$, the module $M^{(1/2)}(\Gamma)$ is a graded ring corresponding to the automorphic factors $(\theta_{0000}(\gamma\tau)/\theta_{0000}(\tau))^k$ ($k = 0, 1, \ldots, \gamma \in \Gamma$). By Theorem 1.1 and 1.3, we get

Corollary 1.1.

$$M^{(1/2)}(\Gamma) = \mathbb{C}[\theta_{0000}, g_2, X, f_3].$$

Corollary 1.2.

$$\sum_{k=0}^{\infty} \dim M_{k+1/2}(\Gamma_0(4))t^k = \sum_{k=0}^{\infty} \dim M_{k+1/2}(\Gamma)t^k = \frac{1}{(1-t)(1-t^2)^2(1-t^3)}.$$

We denote by $S(\Gamma)$ the space of cusp forms in $\oplus_{k=0}^{\infty} M_{k+1/2}(\Gamma)$. The description of cusp forms is given as follows.

Theorem 1.4. *The space $S(\Gamma)$ is generated as a $\oplus_{k=0}^{\infty} M_{2k}(\Gamma)$ module by four cusp forms*

$$\begin{array}{ll} \theta_{0000}(f_3(3f_2 - 2X - g_2)), & \theta_{0000}(g_2 - 4X)(-3f_3 + f_1(g_2 + 8X - 6f_2)), \\ \theta_{0000}K, & \theta_{0000}(8X + g_2 - 6f_2)(g_2 - 4X)(3f_2 - 2X - g_2) \end{array}$$

of weight $11/2, 11/2, 13/2, 13/2$.

This module is not a free module. We can describe the module structure precisely (cf. the proof in §2) and get the following dimension formula of cusp forms, which was first obtained by Tsushima by Riemann Roch Theorem. Here we shall give a simple alternative proof based on the above theorem.

Corollary 1.3.

$$\sum_{k=0}^{\infty} \dim S_{k+1/2}(\Gamma) t^k = \frac{2t^5 + t^7 + t^9 - 2t^{11} + 4t^6 - t^8 + t^{10} - 3t^{12} + t^{14}}{(1-t^2)^3(1-t^6)}.$$

We also give Siegel modular forms of half integral weight with character ψ. The following Corollary 1.5 was also obtained by Tsushima first (cf. [17]). Our proof is independent of his argument.

Theorem 1.5. *We have*

$$\oplus_{k=0}^{\infty} M_{k+1/2}(\Gamma, \psi) = f_{21/2}(\oplus_{k=0}^{\infty} M_k(\Gamma, \psi^k)).$$

We denote by $S_{k+1/2}(\Gamma, \psi)$ the subspace of cusp forms of $M_{k+1/2}(\Gamma, \psi)$. Then we have

Corollary 1.4.

$$\oplus_{k=0}^{\infty} S_{k+1/2}(\Gamma, \psi) = \oplus_{k=0}^{\infty} M_{k+1/2}(\Gamma, \psi).$$

Corollary 1.5.

$$\begin{aligned} \sum_{k=0}^{\infty} \dim M_{k+1/2}(\Gamma_0(4), \psi) t^k &= \sum_{k=0}^{\infty} \dim S_{k+1/2}(\Gamma_0(4), \psi) t^k \\ &= \frac{t^{10}}{(1-t)(1-t^2)^2(1-t^3)}. \end{aligned}$$

1.3. The plus subspace of Siegel modular forms of half integral weight

In order to explain the relation between the plus space and Jacobi forms shortly, first we introduce holomorphic Jacobi forms of general degree following Ziegler [23]. Let k be a natural number and let $F(\tau, z)$ be a holomorphic function on $(\tau, z) \in H_n \times \mathbb{C}^n$. If F satisfies the next three conditions (1), (2), (3), we say that F is a holomorphic Jacobi form of weight k of index 1 of degree n.

(1) $F(M(\tau,z)) = e({}^tz(C\tau+D)^{-1}Cz)\det(C\tau+D)^kF(\tau,z)$ for any $M \in Sp(n,\mathbb{Z})$, where $M(\tau,z) = (M\tau, {}^t(C\tau+D)^{-1}z) \in H_n \times \mathbb{C}^n$.

(2) $F(\tau, z+\tau\lambda+\mu) = e(-{}^t\lambda\tau\lambda - 2{}^t\lambda z)\,F(\tau,z)$ for any $\lambda,\mu \in \mathbb{Z}^n$.

(3) $F(\tau,z)$ has the Fourier expansion of the following form,

$$F(\tau,z) = \sum_{N,r} A(N,r)e(\mathrm{tr}(N\tau) +^t rz).$$

where we denote by L_n^* the set of all half integral symmetric matrices, and N runs over all positive semi-definite elements in L_n^*, and r runs over all elements in $\mathbb{Z}^n$ satisfying $4N - r^tr \geq 0$ (i.e. positive semi-definite).

Moreover, if the Fourier coefficients $A(N,r)$ are zero unless $4N - r^tr > 0$ (i.e. positive definite), then we say that F is a holomorphic Jacobi cusp form.

Next we introduce skew holomorphic Jacobi forms following Skoruppa [15] and Arakawa [1]. Let k be a natural number. Let $F(\tau,z)$ be a function on $(\tau,z) \in H_n \times \mathbb{C}^n$ which is real analytic in the real and the imaginary part of τ and holomorphic in z. If F satisfies the next three conditions (1), (2) and (3), we say that F is a skew holomorphic Jacobi form of weight k of index 1 of degree n.

(1) $F(M(\tau,z)) = e({}^tz(C\tau+D)^{-1}Cz)\det(C\overline{\tau}+D)^{k-1}|\det(C\tau+D)|F(\tau,z)$ for any $M \in Sp(n,\mathbb{Z})$.

(2) $F(\tau, z+\tau\lambda+\mu) = e(-{}^t\lambda\tau\lambda - 2{}^t\lambda z)\,F(\tau,z)$ for any $\lambda,\mu \in \mathbb{Z}^n$.

(3) $F(\tau,z)$ has the Fourier expansion of the following form,

$$F(\tau,z) = \sum_{N,r} A(N,r)e\left(\mathrm{tr}\left(N\tau - \frac{1}{2}i(4N - r\,{}^tr)Y\right) +^t rz\right).$$

where Y is the imaginary part of τ, N runs over L_n^*, and r runs over all elements of $\mathbb{Z}^n$ satisfying $r\,{}^tr - 4N \geq 0$.

Moreover, if the Fourier coefficients $A(N,r)$ are zero unless $r\,{}^tr - 4N > 0$, then we say that F is a skew holomorphic Jacobi cusp form.

We denote by $J_{k,1}^{(n)}$ or $J_{k,1}^{(n),sk}$ the space of holomorphic Jacobi forms or skew holomorphic Jacobi forms of weight k of index 1 defined above. We denote the space of cusp forms of $J_{k,1}^{(n)}$ (resp. $J_{k,1}^{(n),sk}$) by $J_{k,1}^{(n),cusp}$ (resp. $J_{k,1}^{(n),sk,\,cusp}$).

Now we review shortly relations between Siegel modular forms of half integral weight and Jacobi forms of degree n. Let $l = 0$ or 1 and $F(\tau) \in M_{k-1/2}(\Gamma_0^{(n)}(4), \psi^l)$. We write the Fourier expansion of $F(\tau)$ as

$$F(\tau) = \sum_{T \geq 0} a(T)e(\mathrm{tr}(T\tau)),$$

where T runs over half-integral symmetric matrices. We say that F belongs to the plus space $M_{k-1/2}^+(\Gamma_0^{(n)}(4), \psi^l)$ if $a(T) = 0$ unless $T - (-1)^{k+l-1}\lambda^t\lambda \in 4L_n^*$

for some column vector $\lambda \in (\mathbb{Z}/2\mathbb{Z})^n$. We have a theorem for general degree n.

Theorem 1.6. *We have the following isomorphisms.*

$$J_{k,1}^{(n)} \cong M_{k-1/2}^{+}(\Gamma_0^{(n)}(4), \psi^k).$$
$$J_{k,1}^{(n),sk} \cong M_{k-1/2}^{+}(\Gamma_0^{(n)}(4), \psi^{k+1}).$$

When k is even, then $\psi^k = id$ and the above first isomorphism is the claim in Ibukiyama [9], and the second isomorphism for any k is the claim in Hayashida [3]. The remaining case is easy to prove and we omit the proof in this paper.

Now from now on, we consider the case $n = 2$ exclusively until the end of this paper. We put $J_{k,1} = J_{k,1}^{(2)}$, $J_{k,1}^{cusp} = J_{k,1}^{(2),cusp}$ and so on.

For any modular form $f(Z) \in M_k(Sp(2,\mathbb{Z}))$, if we take $g(Z) = f(4Z)$, then the Fourier coefficients of $g(Z)$ is non zero only at T with $T \in 4L_2^*$. Besides, we have $g(Z) \in M_k(\Gamma_0(4))$. Hence, if we put $A' = \{f(4Z); f \in \oplus_{k=0}^{\infty} M_{2k}(Sp(n,\mathbb{Z}))\}$, then $M_{k-1/2}^{+}(\Gamma_0(4), \psi^k)$ is A'-module. To make our calculation easier a little, in §1 and §2 we sometimes use the group $\Gamma = \rho_2^{-1}\Gamma_0(4)\rho_2$ instead of $\Gamma_0(4)$. So, for $l = 0$ or 1, we put

$$M_{k-1/2}^{+}(\Gamma, \psi^l) = \{f(\tau/2); f \in M_{k-1/2}^{+}(\Gamma_0(4), \psi^l)\},$$
$$A = \{f(\tau/2); f \in A'\}.$$

Of course every result on Γ can be easily interpreted to the one for $\Gamma_0(4)$ by taking the image of $f(\tau) \to f(2\tau)$. Also we put

$$M^{+}(\Gamma) = \oplus_{k=1}^{\infty} M_{k-1/2}^{+}(\Gamma),$$
$$M^{+}(\Gamma, \psi) = \oplus_{k=1}^{\infty} M_{k-1/2}^{+}(\Gamma, \psi).$$

Then $M^{+}(\Gamma)$ and $M^{+}(\Gamma, \psi)$ are A-modules. The following dimension formulae by Tsushima are very helpful to determine the A-module structures, and we can show they are free A-modules as the formulae may suggest.

Theorem 1.7 (Tsushima [18])**.**

$$\sum_{k=0}^{\infty} \dim(J_{k,1}) t^k = \frac{t^4 + t^6 + t^{10} + t^{12} + t^{21} + t^{27} + t^{29} + t^{35}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}.$$
$$\sum_{k=0}^{\infty} \dim(J_{k,1}^{sk}) t^k = \frac{t + t^7 + t^9 + t^{15} + t^{24} + t^{26} + t^{30} + t^{32}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}.$$

$$\sum_{t=0}^{\infty}\dim(J_{k,1}^{cusp})t^k$$
$$= \frac{t^{10}+t^{12}+t^{14}+2\,t^{16}+t^{18}-t^{26}-t^{28}+t^{21}+t^{27}+t^{29}+t^{35}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}.$$
$$\sum_{k=0}^{\infty}\dim(J_{k,1}^{sk\ cusp})t^k$$
$$= \frac{t^{11}+2\,t^{13}+t^{15}+t^{17}+2\,t^{19}+t^{21}-2\,t^{23}-t^{25}-t^{29}-t^{31}+t^{35}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}$$
$$+ \frac{t^{24}+t^{26}+t^{30}+t^{32}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}.$$

To make our expression slightly shorter, we replace the generators g_2 or f_3 by

$$R_2 = 6f_1^2 - 2g_2 - 4X,$$
$$V_3 = 2(f_1^3 - 2f_1X + f_3).$$

and put

$$P_{7/2} = \theta_{0000}\,(-48\,f_1{}^3 + 21\,V_3 + 112\,f_1\,X)/64,$$
$$P_{11/2} = \theta_{0000}\,(-1152\,f_1{}^5 - 11\,f_1\,R_2{}^2 + 792\,f_1{}^2V_3 + 792\,V_3X + 4224\,f_1\,X^2)/3072,$$
$$P_{19/2} = \theta_{0000}\,(f_1\,R_2{}^4 - 162\,V_3{}^3 + 36\,R_2{}^2V_3X)/1358954496,$$
$$P_{23/2} = \theta_{0000}\,(16\,f_1{}^3R_2{}^4 + 3\,R_2{}^4V_3 + 4\,R_2{}^4f_1\,X + 18\,f_1\,R_2{}^2V_3{}^2 - 864\,f_1{}^4R_2{}^2V_3$$
$$+1728\,f_1{}^2R_2{}^2V_3X + 3888\,f_1{}^2V_3{}^3 - 6480\,V_3{}^3X$$
$$-288\,R_2{}^2V_3X^2)/21743271936,$$
$$P_{1/2} = \theta_{0000},$$
$$P_{13/2} = \theta_{0000}(-192\,f_1^6 - 26\,f_1^2\,R_2^2 + 4992\,f_1^3\,f_3 + 7488\,f_3^2$$
$$+312\,f_1^3\,V_3 - 2808\,f_3\,V_3 + 117\,V_3^2)/12288,$$
$$P_{17/2} = \theta_{0000}(768\,f_1^8 + 13056\,f_1^5\,f_3 - 544\,f_1\,R_2^2\,f_3 + 104448\,f_1^2\,f_3^2 - 1632\,f_1^5\,V_3$$
$$+17\,f_1\,R_2^2\,V_3 - 1632\,f_1^2\,f_3\,V_3 + 408\,f_1^2\,V_3^2 + 78336\,f_3^2\,X - 29376\,f_3\,V_3\,X$$
$$+1224\,V_3^2\,X)/196608,$$
$$P_{29/2} = \theta_{0000}(144\,f_1^6\,R_2^4 + 5\,f_1^2\,R_2^6 + 3744\,f_1^3\,R_2^4\,f_3 + 720\,R_2^4\,f_3^2 - 684\,f_1^3\,R_2^4\,V_3$$
$$+180\,R_2^4\,f_3\,V_3 - 45\,R_2^4\,V_3^2 - 41472\,f_1\,R_2^2\,f_3\,V_3^2 + 9396\,f_1\,R_2^2\,V_3^3$$
$$-69984\,f_1^2\,V_3^4 + 116640\,V_3^4\,X)/100192997081088.$$

Then we have the following structure theorem.

Theorem 1.8. *The vector space $M^+(\Gamma)$ is a free A module of rank 8, and we have*

$$M^+(\Gamma) = AP_{7/2}\oplus AP_{11/2}\oplus AP_{19/2}\oplus AP_{23/2}\oplus AP_{1/2}\oplus AP_{13/2}\oplus AP_{17/2}\oplus AP_{29/2}.$$

We put $S^+(\Gamma) = S(\Gamma) \cap M^+(\Gamma)$ and we denote by A^{cusp} the space of cusp forms in A. We denote by $E_k(\tau)$ the Eisenstein series of $Sp(2,\mathbb{Z})$ of weight k such that the constant term of the Fourier expansion is one. We also put $E_k^*(\tau) = E_k(2\tau)$ and $B^{'} = \mathbb{C}[E_4^*, E_6^*]$. Then $S^+(\Gamma)$ is given as follows.

Theorem 1.9.

$$\begin{aligned} S^+(\Gamma) = {} & A^{cusp}P_{7/2} \oplus A^{cusp}P_{11/2} \oplus AP_{19/2} \oplus AP_{23/2} \\ & \oplus A^{cusp}P_{1/2} \oplus A^{cusp}P_{13/2} \oplus A^{cusp}P_{17/2} \oplus B^{'}P_{25/2} \oplus AP_{29/2} \ , \end{aligned}$$

where

$$P_{25/2} = \left(5\,(E_4^*)^3\,P_{1/2} - 5\,(E_6^*)^2\,P_{1/2} + 6\,E_6^*\,P_{13/2} - 6E_4^*\,P_{17/2}\right)/17280.$$

Let $S^+(\Gamma,\psi)$ be the space of cusp forms in $M^+(\Gamma,\psi)$. In order to describe the explicit structure $M^+(\Gamma,\psi)$ and $S^+(\Gamma,\psi)$, we put

$$\begin{aligned} P_{41/2} = {} & f_{21/2}R_2(2322432\,f_1{}^3V_3X + 1008\,f_1\,R_2{}^2V_3 + R_2{}^4 - 497664\,V_3f_1\,X^2 \\ & +9216\,f_1{}^2R_2{}^2X - 1824768\,f_1{}^5V_3 + 217728\,f_1{}^2V_3{}^2 - 10368\,V_3{}^2X \\ & -7962624\,f_1{}^6X + 7962624\,f_1{}^4X^2 \\ & -2654208\,f_1{}^2X^3 + 2654208\,f_1{}^8)/521838526464, \\ P_{53/2} = {} & f_{21/2}R_2(-4608\,f_1{}^6R_2{}^4 + 2\,f_1{}^2R_2{}^6 + 1296\,f_1{}^3R_2{}^4V_3 + 4608\,f_1{}^4R_2{}^4X \\ & +9\,R_2{}^4V_3{}^2 + 144\,R_2{}^4V_3f_1\,X + 41472\,f_1{}^4R_2{}^2V_3{}^2 \\ & +3888\,f_1\,R_2{}^2V_3{}^3 + 41472\,f_1{}^2R_2{}^2V_3{}^2X - 279936\,f_1{}^2V_3{}^4 \\ & -93312\,V_3{}^4X)/307792887033102336, \\ P_{57/2} = {} & f_{21/2}R_2(-16\,f_1{}^2R_2{}^6X - 72\,R_2{}^4V_3{}^2X - 3456\,f_1{}^5R_2{}^4V_3 \\ & +27648\,f_1{}^3R_2{}^4V_3X + 1658880\,f_1{}^4R_2{}^2V_3{}^2X + 62208\,V_3{}^3R_2{}^2f_1\,X \\ & +165888\,V_3{}^2R_2{}^2f_1{}^2X^2 - 1152\,R_2{}^4V_3X^2f_1\, - 48\,f_1{}^4R_2{}^6 \\ & +2239488\,f_1{}^4V_3{}^4 - 1119744\,f_1\,V_3{}^5 + 746496\,V_3{}^4X^2 \\ & -8957952\,f_1{}^2V_3{}^4X + 73728\,f_1{}^4R_2{}^4X^2 - 73728\,f_1{}^6R_2{}^4X \\ & +248832\,f_1{}^3R_2{}^2V_3{}^3 + 2376\,f_1{}^2R_2{}^4V_3{}^2 - 497664\,f_1{}^6R_2{}^2V_3{}^2 \\ & +3\,f_1\,R_2{}^6V_3)/4924686192529637376, \\ P_{69/2} = {} & f_{21/2}R_2(135\,f_1\,R_2{}^6V^3 - 870912\,f_1{}^5R_2{}^4V^3 + 71663616\,f_1{}^9R_2{}^2V^3 \\ & +5474304\,f_1{}^8R^4V^2 - 35831808\,f_1{}^6R_2{}^2V^4 + 9072\,f_1{}^4R_2{}^6V^2 \\ & +6718464\,V^6X^2 - 10077696\,f_1\,V^7 + 181398528\,f_1{}^4V^6 \\ & -322486272\,f_1{}^7V^5 + 110592\,f_1{}^{10}R_2{}^6 + 26542080\,f_1{}^9R_2{}^4VX \\ & +64512\,f_1{}^5R_2{}^6VX - 11114496\,f_1{}^6R_2{}^4V^2X - 31850496\,f_1{}^7R_2{}^4VX^2 \\ & +2448\,f_1{}^2R_2{}^6V^2X + 752467968\,f_1{}^5V^5X - 648\,R_2{}^4V^4X \\ & +68040\,f_1{}^2R_2{}^4V^4 + 18\,f_1{}^3R_2{}^8V - 76032\,f_1{}^7R_2{}^6V \\ & -7962624\,f_1{}^{11}R_2{}^4V - 36864\,f_1{}^4R_2{}^6X^3 + 107495424\,V^5X^3f_1 \\ & +128\,f_1{}^4R_2{}^8X - 537477120\,f_1{}^3V^5X^2 - 134369280\,f_1{}^2V^6X \end{aligned}$$

$$
\begin{aligned}
&+184320\, f_1{}^6 R_2{}^6 X^2 - 258048\, f_1{}^8 R_2{}^6 X + 2304\, f_1{}^3 R_2{}^6 V X^2 \\
&+1181952\, f_1{}^3 R_2{}^4 V^3 X + 6469632\, f_1{}^4 R_2{}^4 V^2 X^2 + 15925248\, f_1{}^5 R_2{}^4 V X^3 \\
&-167215104\, f_1{}^7 R_2{}^2 V^3 X + 23887872\, f_1{}^4 R_2{}^2 V^4 X \\
&+119439360\, f_1{}^5 R_2{}^2 V^3 X^2 - 62208\, R_2{}^4 V^3 X^2 f_1 - 829440\, R_2{}^4 V^2 X^3 f_1{}^2 \\
&-2654208\, R_2{}^4 V X^4 f_1{}^3 - 186624\, R_2{}^2 V^5 f_1\, X \\
&-23887872\, R_2{}^2 V^3 f_1{}^3 X^3)/29046981088226008356 61824,
\end{aligned}
$$

$$
\begin{aligned}
P_{47/2} = {}& f_{21/2}\, R_2\, (9216\, f_1^7\, R_2^2 - 32\, f_1^3\, R_2^4 - 18432\, f_1^4\, R_2^2\, f_3 + 9216\, f_1\, R_2^2\, f_3^2 \\
&-3456\, f_1^4\, R_2^2\, V_3 + R_2^4\, V_3 - 5760\, f_1\, R_2^2\, f_3\, V_3 - 41472\, f_1^5\, V_3^2 \\
&+1296\, f_1\, R_2^2\, V_3^2 + 124416\, f_1^2\, f_3\, V_3^2 \\
&-10368\, f_1^2\, V_3^3 - 82944\, f_3\, V_3^2\, X + 31104\, V_3^3\, X)/133590662774784,
\end{aligned}
$$

$$
\begin{aligned}
P_{51/2} = {}& f_{21/2}\, R_2\, (221184\, f_1^9\, R_2^2 + 384\, f_1^5\, R_2^4 - f_1\, R_2^6 - 221184\, f_1^6\, R_2^2\, f_3 \\
&-1920\, f_1^2\, R_2^4\, f_3 - 221184\, f_1^3\, R_2^2\, f_3^2 + 221184\, R_2^2\, f_3^3 \\
&-110592\, f_1^6\, R_2^2\, V_3 + 24\, f_1^2\, R_2^4\, V_3 - 165888\, f_1^3\, R_2^2\, f_3\, V_3 \\
&-165888\, R_2^2\, f_3^2\, V_3 - 2488320\, f_1^7\, V_3^2 - 10368\, f_1^3\, R_2^2\, V_3^2 + 4976640\, f_1^4\, f_3\, V_3^2 \\
&+17280\, R_2^2\, f_3\, V_3^2 - 4478976\, f_1\, f_3^2\, V_3^2 + 1741824\, f_1^4\, V_3^3 + 5184\, R_2^2\, V_3^3 \\
&+1990656\, f_1\, f_3\, V_3^3 - 248832\, f_1\, V_3^4 - 24\, R_2^4\, V_3\, X \\
&+1990656\, f_3\, V_3^2\, X^2 - 746496\, V_3^3\, X^2)/6412351813189632,
\end{aligned}
$$

$$
\begin{aligned}
P_{59/2} = {}& f_{21/2}\, R_2\, (48\, f_1^5\, R_2^6 + 16\, f_1^2\, R_2^6\, f_3 - 4032\, f_1^6\, R_2^4\, V_3 - 6\, f_1^2\, R_2^6\, V_3 \\
&+3456\, f_1^3\, R_2^4\, f_3\, V_3 + 576\, R_2^4\, f_3^2\, V_3 - 216\, f_1^3\, R_2^4\, V_3^2 - 360\, R_2^4\, f_3\, V_3^2 \\
&+72576\, f_1^4\, R_2^2\, V_3^3 + 45\, R_2^4\, V_3^3 + 10368\, f_1\, R_2^2\, f_3\, V_3^3 \\
&-3888\, f_1\, R_2^2\, V_3^4 - 279936\, f_1^2\, V_3^5 - 93312\, V_3^5\, X)/78794979080474198016,
\end{aligned}
$$

$$
\begin{aligned}
P_{63/2} = {}& f_{21/2}\, R_2\, (576\, f_1^7\, R_2^6 + 2\, f_1^3\, R_2^8 + 2688\, f_1^4\, R_2^6\, f_3 - 192\, f_1\, R_2^6\, f_3^2 \\
&-76032\, f_1^8\, R_2^4\, V_3 - 432\, f_1^4\, R_2^6\, V_3 - 76032\, f_1^5\, R_2^4\, f_3\, V_3 + 240\, f_1\, R_2^6\, f_3\, V_3 \\
&+165888\, f_1^2\, R_2^4\, f_3^2\, V_3 + 25056\, f_1^5\, R_2^4\, V_3^2 - 45\, f_1\, R_2^6\, V_3^2 \\
&-64800\, f_1^2\, R_2^4\, f_3\, V_3^2 + 1368576\, f_1^6\, R_2^2\, V_3^3 + 8424\, f_1^2\, R_2^4\, V_3^3 \\
&+2737152\, f_1^3\, R_2^2\, f_3\, V_3^3 - 124416\, R_2^2\, f_3^2\, V_3^3 - 590976\, f_1^3\, R_2^2\, V_3^4 \\
&+155520\, R_2^2\, f_3\, V_3^4 - 6718464\, f_1^4\, V_3^5 - 46656\, R_2^2\, V_3^5 \\
&-13436928\, f_1\, f_3\, V_3^5 + 3359232\, f_1\, V_3^6 - 13824\, R_2^4\, f_3^2\, V_3\, X \\
&+8640\, R_2^4\, f_3\, V_3^2\, X - 1080\, R_2^4\, V_3^3\, X \\
&+2239488\, V_3^5\, X^2)/37821589958627615047 68.
\end{aligned}
$$

Then we have the following structure theorem.

Theorem 1.10. *The vector space $M^+(\Gamma,\psi)$ is a free A module of rank 8, and we have*

$$
\begin{aligned}
M^+(\Gamma,\psi) &= S^+(\Gamma,\psi) \\
&= AP_{41/2} \oplus AP_{53/2} \oplus AP_{57/2} \oplus AP_{69/2} \oplus AP_{47/2} \oplus AP_{51/2} \\
&\quad \oplus AP_{59/2} \oplus AP_{63/2}.
\end{aligned}
$$

2. Proofs on explicit structures of modular forms

2.1. Generators

We quote the theta transformation formula from Igusa [11, p. 227]. For even theta characteristics $m = {}^t({}^tm', {}^tm'')$ with m', $m'' \in \mathbb{Z}^2$ (column vectors) and $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(2,\mathbb{Z})$, we write

$$M \cdot m = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} m + \begin{pmatrix} (c\,{}^td)_0 \\ (a\,{}^tb)_0 \end{pmatrix},$$

where for any symmetric matrix x, we denote x_0 the vector whose components consist of diagonal elements of x. Then we get

$$\theta_{M\cdot m}(M\tau) = \kappa(M)e(\phi_m(M))\det(c\tau+d)^{\frac{1}{2}}\theta_m(\tau),$$

where $\kappa(M)$ is a certain eighth root of unity, $e(x) = e^{2\pi ix}$ and

$$\phi_m(M) = -\frac{1}{8}({}^tm'\,{}^tbdm' + {}^tm''\,{}^tacm'' - 2\,{}^tm'\,{}^tbcm'' - 2\,{}^t(a\,{}^tb)_0(dm' - cm'')).$$

For any natural number N, we denote by $\Gamma(N)$ the principal congruence subgroup of $Sp(2,\mathbb{Z})$ of level N. Then $\Gamma \supset \Gamma(2)$, and any coset in $\Gamma/\Gamma(2)$ is represented by some element of the form $M = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$.

Proof of Proposition 1.2. The assertion for X, Y, K are in [7]. We shall show the rest. For the above M with $b = c = 0$, we get $\phi_m(M) = 0$ for all m, and

$$M \cdot m = \begin{pmatrix} dm' \\ am'' \end{pmatrix}.$$

So, we get $f_2|_2M = f_2$, and the action $f \to f|_2M$ gives a permutation on $\{(\theta_{0001})^4, (\theta_{0010})^4, (\theta_{0011})^4)\}$, or $\{(\theta_{1000})^4, (\theta_{0100})^4, (\theta_{1100})^4)\}$. Hence we get f_2, $g_2 \in A_2(\Gamma)$. The assertion for modular forms of odd weight can be obtained similarly. Now, we show the relation of modular forms given in the proposition. Using the notation of Igusa [10], we put

$$\begin{aligned} &y_0 = (\theta_{0110})^4, && y_1 = (\theta_{0100})^4, && y_2 = (\theta_{0000})^4, \\ &y_3 = -(\theta_{1000})^4 - (\theta_{0110})^4, && y_4 = -(\theta_{1100})^4 - (\theta_{0110})^4. && \end{aligned}$$

It is known that these forms generate the graded ring $A_{even}(\Gamma(2))$ of even weights modular forms with the fundamental relation

$$(y_1y_1 + y_0y_2 + y_1y_2 - y_3y_4)^2 - 4y_0y_1y_2(y_0+y_1+y_2+y_3+y_4) = 0.$$

Using Riemann's theta relation (cf. [10, Lemma 1]), and using the reduction process in Igusa [10, p. 393], we get

$$\begin{aligned}
f_2 &= y_2,\\
g_2 &= -2y_0 + y_1 + y_2 - y_3 - y_4,\\
X &= (y_0 + y_1 + 4y_2 + 2y_3 + 2y_4)/4,\\
Y &= (-y_0y_1 + y_0y_2 + y_1y_2 + y_3y_4 + 2y_2^2 + 2y_2y_3 + 2y_2y_4)/2,\\
K &= (-y_0^2y_1 + y_0^2y_2 - y_0y_1^2 - 2y_0y_1y_2 - 2y_0y_1y_3 - y_0y_3y_4\\
&\quad -2y_0y_1y_4 + y_1^2y_2 - y_1y_3y_4)/8192,\\
f_3^2 &= (y_2 + y_3)(y_2 + y_4)(y_0 + y_1 + y_2 + y_3 + y_4).
\end{aligned}$$

By direct calculation, we get

$$f_3^2 = -4096K + f_2(4g_2X - 6f_2g_2 + 24f_2X + g_2^2 - 8X^2)/9 + (4X - 2f_2)Y.$$

Hence we prove (1) and (3). Now we show that X, f_2, g_2, K are algebraically independent. We define the Witt operator W on any function $F(Z)$ on H_2 by

$$(WF)(\tau_1, \tau_2) = F\begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}.$$

For $i = 1$, 2, we put $x_i = \theta_{01}(\tau_i)$, $y_i = \theta_{10}(\tau_i)$, $z_i = \theta_{00}(\tau_i)$. It is well known that $z_i^4 = x_i^4 + y_i^4$. We get

$$\begin{aligned}
W(X) &= (x_1^4 + z_1^4)(x_2^4 + z_2^4)/4,\\
W(f_2) &= (z_1z_2)^4,\\
W(g_2) &= \prod_{i=1}^{2}(2z_i^4 - x_i^4),\\
W(K) &= 0.
\end{aligned}$$

Since the four forms x_1, x_2, z_1, z_2 are algebraically independent, three forms $W(X)$, $W(f_2)$, $W(g_2)$ are also algebraically independent. Now, assume that $P(X, f_2, g_2, K) = 0$ for a polynomial $P(X_1, X_2, X_3, X_4)$ of four variables. Writing $P = P_1(X_1, X_2, X_3) + X_4P_2(X_1, X_2, X_3, X_4)$ and applying W, we get $P_1(W(X), W(f_2), W(g_2)) = 0$. Hence we get $P_1 = 0$ as a polynomial. Hence $P_2(X, f_2, g_2, K) = 0$. Since the degree of P_2 is smaller than the one of P, we get $P = 0$ by induction. By using the relation between f_3^2 and K, we also get that f_2, g_2, X and K are algebraically independent. $\square$

Lemma 2.1. *If $F + YG = 0$ for any F, $G \in B$, then $F = G = 0$.*

Proof. Let P_i ($i = 1$, 2) be polynomials of four variables and assume that

$$F + YG = 0 \tag{2.1}$$

for $F = P_1(X, f_2, g_2, K)$ and $G = P_2(X, f_2, g_2, K)$. For each $i = 1$, 2, we take polynomials Q_{i1} of three variables and Q_{i2} of four variables such that

$$P_i(X_1, X_2, X_3, X_4) = Q_{i1}(X_1, X_2, X_3) + X_4Q_{i2}(X_1, X_2, X_3, X_4).$$

Taking the image of the Witt operator W of both sides of (2.1), we get

$$Q_{11}(W(f_2), W(g_2), W(X)) + W(Y)Q_{21}(W(f_2), W(g_2), W(X)) = 0.$$

Since we have $W(Y) = (z_1z_2x_1x_2)^4 = W(f_2(g_2 - 6f_2 + 8X))/3$ and three forms $W(f_2)$, $W(g_2)$, $W(X)$ are algebraically independent, we get

$$Q_{11}(X_1, X_2, X_3) = -\frac{1}{3}X_1(-6X_1 + X_2 + 8X_3)Q_{21}(X_1, X_2, X_3).$$

So, if we put $f = Y - f_2(g_2 - 6f_2 + 8X)/3$, then

$$fQ_{21}(f_2, g_2, X) + K(Q_{12}(f_2, g_2, X, K) + YQ_{22}(f_2, g_2, X, K)) = 0.$$

Now, we shall show $Q_{21} = 0$. Put

$$\gamma = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then, we get

$$\begin{aligned}
(\theta_{0000})^4|_2\gamma &= (\theta_{0000})^4, \\
(\theta_{1000})^4|_2\gamma &= (\theta_{1001})^4, \\
(\theta_{0100})^4|_2\gamma &= (\theta_{0110})^4, \\
(\theta_{1100})^4|_2\gamma &= (\theta_{1111})^4,
\end{aligned}$$

and $X|_2\gamma = X$, $Y|_4\gamma = Y$, $K|_6\gamma = K$. Hence $W(f|_4\gamma) = z_1^4z_2^4(x_1^4 - z_1^4)(x_2^4 - z_2^4) \neq 0$. We get $W(f_2|_2\gamma) = (z_1z_2)^4$, $W(g_2|_2\gamma) = (z_1^4 + x_1^4)(x_2^4 + z_2^4) - 3x_1^4x_2^4$, $W(X|_2\gamma) = (x_1^4 + z_1^4)(x_2^4 + z_2^4)/4$, so these three forms are also algebraically independent, and since $W(K|_6\gamma) = 0$, we get $Q_{21} = 0$ as a polynomial. Hence, we get

$$P_{12}(f_2, g_2, X, K) + YP_{22}(f_2, g_2, X, K) = 0.$$

Since the degree of P_{12}, or P_{22} is less than the degree of P_1, or P_2, respectively, we get $P_{12} = P_{22} = 0$ by induction. Hence $F = G = 0$. q.e.d.

It is also obvious that $f_1B + f_3B$ is a direct sum. By comparing the dimensions, we get Theorem 1.1 and 1.2. $\square$

Finally, we shall prove Theorem 1.3. We shall show

Proposition 2.1. *If $f \in M_{k+1/2}(\Gamma, \psi)$, then f/θ_{0000} is holomorphic on H_2.*

Theorem 1.3 and Corollary are easily obtained from this. For the proof of this Proposition, we use the explicit structure of $M(\Gamma, \psi)$. We need several Lemmas.

Lemma 2.2. *For any $F \in C = \mathbb{C}[g_2, X, K]$, assume that F/f_1 is holomorphic. Then $F = 0$.*

Proof. By Theorem 1.1, it is easy to see that $\sum_{k=1}^{\infty} M_{2k-1}(\Gamma, \psi) = f_1 B \oplus f_1 Y B \oplus f_3 C$. Since $F/f_1 \in M_{2k-1}(\Gamma, \psi)$ for some integer k, we write $F = f_1(f_1\alpha_1 + f_1\alpha_2 Y + f_3\alpha_3)$ for some α_1, $\alpha_2 \in B$ and $\alpha_3 \in C$. Hence $F - f_2\alpha_1 = f_2 Y \alpha_2 + Y\alpha_3 \in B \cap YB = \{0\}$ and we get $F = f_2\alpha_1$. Since $F \in C$ and f_2, g_2, X, K are algebraically independent, we get $F = 0$. $\square$

Lemma 2.3. *For any $F \in M(\Gamma, \psi) = \oplus_{k=0}^{\infty} M_k(\Gamma, \psi^k)$, assume that F/θ_{0000} is holomorphic. Then, F/f_1 is also holomorphic.*

Proof. First, we assume that F is of odd weight. Then $F = f_1\alpha_1 + f_3\alpha_2$ for $\alpha_2 \in C$ and $\alpha_1 \in B + YB$. Since F/θ_{0000} is holomorphic, and $f_1 = \theta_{0000}^2$, we see $(f_3\alpha_2/\theta_{0000})$ is holomorphic and hence $(f_3\alpha_2/\theta_{0000})^2$ also. Since $f_3^2 = -4096K + f_1 h$ for some holomorphic function h, we see $K\alpha_2^2/f_1$ is also holomorphic. Since the numerator belongs to C, we get $\alpha_2 = 0$ by virtue of the previous lemma. Hence $F/f_1 = \alpha_1$ is holomorphic. Secondly, we assume that F is of even weight. We write $F = \alpha_1 + f_2\alpha_2 + Y\alpha_3$, where $\alpha_1 \in C$, α_2, $\alpha_3 \in B$. Since $f_2 = \theta_{0000}^4$, $Y = \theta_{0000}^2 f_3$, and F/θ_{0000} is holomorphic, we see α_1/θ_{0000} is holomorphic, hence α_1^2/f_1 also. Since $\alpha_1^2 \in C$, we get $\alpha_1 = 0$ by the previous lemma. Hence $F/f_1 = f_1\alpha_2 + f_3\alpha_3$ is holomorphic. $\square$

Proof of Proposition 2.1. Since $f \in M_{k+1/2}(\Gamma)$, we see that $F := \theta_{0000} f \in M_{k+1}(\Gamma, \psi^{k+1})$. Since $f = F/\theta_{0000}$ is holomorphic, $F/f_1 = f/\theta_{0000}$ is again holomorphic by Lemma 2.3. $\square$

2.2. Cusp forms

We define a maximal parabolic subgroup $P_1(\mathbb{Q})$ of $Sp(2, \mathbb{Q})$ corresponding to the one dimensional cusps by

$$P_1(\mathbb{Q}) = \left\{ \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \in Sp(2, \mathbb{Q}) \right\}.$$

We can take a complete set of representatives of $\Gamma \backslash Sp(2, \mathbb{Q})/P_1(\mathbb{Q})$ as follows.

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad M_4 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

A modular form F of Γ is a cusp form, if and only if $\Phi(F|_k M_i) = 0$ for all i with $1 \leq i \leq 4$, where Φ is the usual Siegel Φ-operator. For the characteristic $m \in \mathbb{Z}^4$ whose second component is odd, we get $\Phi(\theta_m) = 0$. Otherwise, we get $\Phi(\theta_{m_1,0,m_3,m_4}) = \theta_{m_1 m_3}$. Now we put $x = \theta_{01}$ and $z = \theta_{00}$. Then $\theta_{10}^4 = z^4 - x^4$. By the theta transformation formula, the forms $\Phi(F|M) = \Phi(F|_k M_i)$ is obtained for generators F as follows.

1. $\Phi(K) = \Phi(f_3 + f_1 f_2 - 2X f_1) = \Phi(2X - 3f_2 + g_2) = 0$. $\Phi(f_1) = z^2$ and $\Phi(X) = (x^4 + z^4)/2$ are algebraically independent.

2. $\Phi(K|M_2) = \Phi(f_3|M_2) = \Phi((g_2 + 8X - 6f_2)|M_2) = 0$. $\Phi(f_1|M_2) = z^2$ and $\Phi(X|M_2) = (x^4 + z^4)/4$ are algebraically independent.

3. $\Phi(K|M_3) = \Phi(f_1|M_3) = \Phi(f_3|M_3) = 0$. $\Phi(X|M_3) = (x^4 + z^4)/4$ and $\Phi(g_2|M_3) = 2z^4 - x^4$ are algebraically independent.

4. $\Phi(K|M_4) = \Phi(f_3|M_4) = \Phi((g_2 - 4X)|M_4) = 0$. $\Phi(f_1|M_4) = z^2$ and $\Phi(X|M_4) = (x^4 + z^4)/4$ are algebraically independent.

For any $f = \theta_{0000} F \in M_{k+1/2}(\Gamma)$ with odd k, we see that f is a cusp form if and only if F is a cusp form. Indeed, $\Phi(\theta_{0000}|M_i) = 0$ only for $i = 3$ in the above four cases. In this case, we also get $\Phi(f_1|M_3) = \Phi(f_3|M_3) = 0$, so $\Phi(F|M_3) = 0$ for any $F = f_1 G + f_3 H \in M_k(\Gamma, \psi)$. When k is even, the above observation is false. For example, $F = (2X - 3f_2 + g_2)(g_2 + 8X - 6f_2)(g_2 - 4X)$ is not a cusp form, while $\theta_{0000} F$ is a cusp form. Anyway, in order to show that $f = \theta_{0000} F$ is a cusp form, we have to check only the conditions (1), (2), (4) for F of integral weight. We first treat the case of odd weight for Γ. Since K is a cusp form, we can assume that

$$F = f_1(P_1(X, f_2, g_2) + Y P_2(X, f_2, g_2)) + f_3 P_3(X, g_2),$$

up to the ideal generated by K. From (4) above, we get $P_1(X_1, X_2, 4X_1) = 0$ and hence $P_1 = (X_3 - 4X_1) Q_1(X_1, X_2, X_3)$ for some polynomial Q_1. Also from (2), we get $P_1 = (X_3 + 8X_1 - 6X_2)(X_3 - 4X_1) Q_2(X_1, X_2, X_3)$ for some polynomial Q_2. Now, we put $X_2^* = X_2 - (2X_1 + X_3)/3$. Then from the condition (1), we get

$$\begin{aligned}
&-9(4X_1 - X_3)^2 Q_2(X_1, X_2^* + (2X_1 + X_3)/3, X_3) \\
&+ (2X_1 + X_3)(4X_1 - X_3) P_2(X_1, X_2^* + (2X_1 + X_3)/3, X_3) \\
&+ 3(4X_1 - X_3) P_3(X_1, X_2^* + (2X_1 + X_3)/3, X_3) = 0.
\end{aligned}$$

This means that there are polynomials R_i ($i \leq i \leq 4$) such that

$$\begin{aligned}
P_1 &= (4X_1 - X_3 - 6X_2^*)(X_3 - 4X_1)(R_1(X_1, X_3) + X_2^* R_2(X_1, X_2^*, X_3), \\
P_2 &= R_3(X_1, X_3) + X_2^* R_4(X_1, X_2^*, X_3), \\
P_3 &= 3(4X_1 - X_3) R_1(X_1, X_3) + \frac{1}{3}(2X_1 + X_3) R_3(X_1, X_3).
\end{aligned}$$

Hence, $\theta_{0000} F$ is a cusp form for odd weight F, if and only if

$$
\begin{aligned}
F = {} & (3f_1Y - f_3(2X+g_2))R_3(X,g_2) \\
& + (g_2-4X)(-3f_3+f_1(g_2+8X-6f_2))R_1(X,g_2) \\
& + (f_1Y(3f_2-2X-g_2))R_4(X,f_2^*,g_2) \\
& + f_1(8X+g_2-6f_2)(g_2-4X)(3f_2-2X-g_2)R_2(X,f_2^*,g_2) \\
& + f_1K(R_5(X,f_2,g_2,K)+YR_6(X,f_2,g_2,K)) + f_3KR_7(X,g_2,K)
\end{aligned}
$$

for some polynomials R_i $(1 \le i \le 7)$, where we put $f_2^* = f_2 - (2X+g_2)/3$. By the structure Theorem 1.1, we can see that the above polynomials R_i depends only on F. Hence, the generating function of the dimension of cusp forms is given by

$$
\frac{2t^5}{(1-t^2)^2} + \frac{2t^7}{(1-t^2)^3} + \frac{t^7(1+t^4)}{(1-t^2)^3(1-t^6)} + \frac{t^9}{(1-t^2)^2(1-t^6)}.
$$

Now, we assume that k is even. Then, we can assume $F = P_1(X,f_2,g_2) + YP_2(X,f_2,g_2)$ as before. By the condition (2) and (4), we get $P_1(X_1,X_2,X_3) = (X_3-4X_1)(X_3+8X_1-6X_2)Q_1(X_1,X_2,X_3)$ for some polynomial Q_1. Now we put $X_3^* = X_3+2X_1-3X_2$. We write

$$
\begin{aligned}
Q_1(X_1,X_2,X_3) &= R_1(X_1,X_2) + X_3^*R_2(X_1,X_2,X_3^*), \\
P_2(X_1,X_2,X_3) &= R_3(X_1,X_2) + X_3^*R_4(X_1,X_2,X_3^*).
\end{aligned}
$$

Then by the condition (1), we get

$$
9(X_2-2X_1)R_1(X_1,X_2) + X_2R_3(X_1,X_2) = 0.
$$

So, we get $R_1(X_1,X_2) = X_2R_5(X_1,X_2)$ and $R_3(X_1,X_2) = 9(2X_1-X_2)R_5(X_1, X_2)$ for some polynomial R_5. Hence, any modular form $\theta_{0000}F$ with F with even weight is a cusp form if and only if

$$
\begin{aligned}
F = {} & Y(g_2+2X-f_2)R_4(X,f_2,g_2^*) \\
& + (g_2-4X)(g_2+8X-6f_2)(g_2+2X-3f_2)R_2(X,f_2,g_2^*) \\
& (f_2(g_2-4X)(g_2+8X-6f_2) + 9Y(2X-f_2))R_5(X,f_2) \\
& + K(R_6(X,f_2,g_2,K)+YR_7(X,f_2,g_2,K))
\end{aligned}
$$

for some polynomials R_i $(i = 2, 4, 5, 6, 7)$, where we put $g_2^* = g_2+2X-3f_2$. We can show by the structure Theorem 1.1 that these polynomials R_i are uniquely determined by F. Hence the generating function of the dimension of cusp forms is given by

$$
\frac{2t^6}{(1-t^2)^3} + \frac{t^6}{(1-t^2)^2} + \frac{t^6(1+t^4)}{(1-t^2)^3(1-t^6)}.
$$

Thus we complete the proof of Theorem 1.4 and its Corollary 1.3. $\square$

Let $f \in M_{k-1/2}(\Gamma, \psi)$, then $\theta_{0000} f \in M_k(\Gamma, \psi^{k+1})$. By Theorem 1.1, 1.2, we have

$$\begin{aligned}\oplus_{k=0}^{\infty} M_k(\Gamma, \psi^{k+1}) &= f_{11}(B \oplus YB \oplus f_1 B \oplus f_3 B) \\ &= f_{11}\mathbb{C}[f_1, X, g_2, f_3] \\ &= f_{11}\, M(\Gamma, \psi)\ .\end{aligned}$$

Moreover, $f_{11} = \theta_{0000} f_{21/2}$. Hence we have Theorem 1.5. □

Let M_i (i=1,2,3,4) be representatives of $\Gamma\backslash Sp(2,\mathbb{Q})/P_1(\mathbb{Q})$ which were defined before. For each i, we see $\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in M_i^{-1}\,\Gamma\, M_i$. So, for any $f \in M_{k-1/2}(\Gamma, \psi)$, we have $\Phi(f|M_i) = \Phi(f|M_i \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}) = -\Phi(f|M_i)$. Hence f belongs to $S_{k-1/2}(\Gamma, \psi)$. Thus we prove Corollary 1.4.

Corollary 1.5 is obvious by Theorem 1.5 and Corollary 1.4. □

2.3. Plus space

Proof of Theorem 1.6. This is mostly known by [9] and [3]. The remaining case uncovered by these papers can be easily proved in the same way and the proof is omitted here. □

Proof of Theorem 1.8. Although the plus space is originally defined for $\Gamma_0(4)$, we are taking its conjugate Γ for some convenience of calculation. As we explained, the original plus space is obtained by taking $f(2\tau)$ for our $f(\tau)$ for Γ. Now if the basis of $M_{k-1/2}(\Gamma)$ is concretely given, the basis of the plus space $M^+_{k-1/2}(\Gamma)$ is obtained in principle as follows. The condition of the plus space is the linear conditions on Fourier coefficients. By demanding this condition for Fourier coefficients at several T, we get a linear subspace M of $M_{k-1/2}(\Gamma)$ which contains $M^+_{k-1/2}(\Gamma)$. If we impose the condition at more T, the space M becomes smaller or is unchanged. Since we know $\dim M^+_{k-1/2}(\Gamma)$ by Theorem 1.7, we can continue the process until we get $\dim M = \dim M^+_{k-1/2}(\Gamma)$ and we get the plus space.

For example, in the case of weight 7/2, a basis of $M_{\frac{7}{2}}(\Gamma)$ is given by $\theta_{0000} f_1^3$, $\theta_{0000} f_1 g_2$, $\theta_{0000} f_1 X$, and $\theta_{0000} f_3$. For any $f \in M_{k-1/2}(\Gamma)$, we have the following Fourier expansion

$$F(\tau) = \sum_T c(T) e\left(\frac{1}{2} tr(T\tau)\right)$$

where T runs over half integral symmetric matrices. We give Fourier coefficients of the above four forms in the following table, where (a, c, b) means the Fourier coefficient $c(T)$ at $T = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$.

	$\theta_{0000} f_1^3$	$\theta_{0000} f_1 g_2$	$\theta_{0000} f_1 X$	$\theta_{0000} f_3$
$(1,0,0)$	14	30	6	-2
$(1,1,0)$	168	456	24	8
$(1,1,1)$	0	192	0	0

Writing down the condition that a linear combination vanishes at $(1,0,0)$, $(1,1,0)$ and $(1,1,1)$, the following modular form is the unique candidate of the element of the plus space up to constant:

$$-3\theta_{0000} f_1^3 + 14\theta_{0000} f_1 X + 21\theta_{0000} f_3.$$

Since $\dim M_{7/2}^+(\Gamma) = 1$, this actually belongs to the plus space. By similar method, we can give basis of $M_{k-1/2}(\Gamma)$ for $k - 1/2 = 7/2$, $11/2$, $19/2$, $23/2$, $1/2$, $13/2$, $17/2$, $29/2$, using the fact that the dimension of the plus space of each of these weights is 1, 1, 3, 3, 1, 2, 2, 5, respectively. We also see that $P_{k-1/2} \in M_{k-1/2}^+(\Gamma)$. Now, we would like to show that these elements are linearly independent over A. A set of generators of $\oplus_{k=0}^{\infty} M_{2k}(Sp(2,\mathbb{Z}))$ is given by

$$\begin{aligned}
E_4(\tau) &= 4X^2 - 3Y + 12288Z, \\
E_6(\tau) &= -8X^3 + 9XY + 73728XZ - 27648K, \\
\chi_{10}(\tau) &= YK, \\
\chi_{12}(\tau) &= 3Y^2 Z - 2XYK + 2072K^2,
\end{aligned}$$

where $E_4(\tau)$ and $E_6(\tau)$ are the Eisenstein series of weight 4 or 6 with constant term one as before and χ_{10} or χ_{12} is a cusp form of weight 10 or 12 , respectively, which is normalized so that the coefficient at $\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$ is one. (cf. Igusa [10]). We write $E_k^*(\tau) = E_k(2\tau)$ for $k = 4$, 6 and $\chi_k^*(\tau) = \chi_k(2\tau)$ for $k = 10$, 12. Then we have $A = \mathbb{C}[E_4^*, E_6^*, \chi_{10}^*, \chi_{12}^*]$. We also get

$$\begin{aligned}
E_4^* &= (-768\,Z + X^2 + 3\,Y)/4, \\
E_6^* &= (-3456\,K - X^3 + 9\,XY + 1152\,XZ)/8, \\
\chi_{10}^* &= KZ, \\
\chi_{12}^* &= (3\,K^2 - 2\,KXZ + 3\,YZ^2)/4.
\end{aligned}$$

Let W be the Witt operator defined before. For $i = 1$, 2, we put $x_i =$

$\theta_{01}(\tau_i)$, $y_i = \theta_{10}(\tau_i)$, $z_i = \theta_{00}(\tau_i)$. It is well known that $z_i^4 = x_i^4 + y_i^4$. We get

$$
\begin{aligned}
W(E_4^*) &= \prod_{i=1}^{2} \left(16\,{x_i}^8 + 16\,{x_i}^4\,{y_i}^4 + {y_i}^8\right)/2^8, \\
W(E_6^*) &= \prod_{i=1}^{2} \left(2\,{x_i}^4 + {y_i}^4\right)\left(32\,{x_i}^8 + 32\,{x_i}^4\,{y_i}^4 - {y_i}^8\right)/2^{12}, \\
W(\chi_{10}^*) &= 0, \\
W(\chi_{12}^*) &= 3\prod_{i=1}^{2} {x_i}^4\,{y_i}^{16}\,\left({x_i}^4 + {y_i}^4\right)/2^{30}, \\
W(\theta_{0000}P_{7/2}) &= 2^{-6}\prod_{i=1}^{2}\left({x_i}^4 + {y_i}^4\right)\left(8\,{x_i}^4 + {y_i}^4\right), \\
W(\theta_{0000}P_{11/2}) &= 2^{-10}\prod_{i=1}^{2}\left({x_i}^4 + {y_i}^4\right)\left(32\,{x_i}^8 + 20\,{x_i}^4\,{y_i}^4 - {y_i}^8\right), \\
W(\theta_{0000}P_{19/2}) &= 2^{-23}\prod_{i=1}^{2}{x_i}^4\,{y_i}^{12}\,\left({x_i}^4 + {y_i}^4\right), \\
W(\theta_{0000}P_{23/2}) &= 2^{-27}\prod_{i=1}^{2}{x_i}^4\,{y_i}^{12}\,\left({x_i}^4 + {y_i}^4\right)\left(4\,{x_i}^4 + 5\,{y_i}^4\right).
\end{aligned}
$$

It is clear that the A-module spanned by $P_{7/2}$, $P_{11/2}$, $P_{19/2}$, $P_{23/2}$ and the A-module spanned by $P_{1/2}$, $P_{13/2}$, $P_{17/2}$, $P_{29/2}$ have no intersection except for 0, since the weights of the forms are $k-1/2$ with even k for the former but odd k for the latter. We shall show linear independence of four forms $P_{7/2}$, $P_{11/2}$, $P_{19/2}$, $P_{23/2}$ over A. Linear independence of four forms $P_{1/2}$, $P_{13/2}$, $P_{17/2}$, $P_{29/2}$ are shown almost in the same way and the proof will be omitted here.

We assume that there exist polynomials $Q_i(X_1, X_2, X_3, X_4) (i = 1, 2, 3, 4)$ which satisfy the following relation

$$\sum_{i=1}^{4} Q_i(E_4^*, E_6^*, \chi_{10}^*, \chi_{12}^*)G_i = 0,$$

where we put $G_1 = \theta_{0000}P_{7/2}$, $G_2 = \theta_{0000}P_{11/2}$, $G_3 = \theta_{0000}P_{19/2}$, $G_4 = \theta_{0000}P_{23/2}$. If we define polynomials R_i , $R_i^{'}$ by

$$Q_i(X_1, X_2, X_3, X_4) = R_i(X_1, X_2, X_4) + X_3R_i^{'}(X_1, X_2, X_3, X_4)\ ,\ (i = 1, 2, 3, 4)$$

we get

$$\sum_{i=1}^{4} R_i(E_4^*, E_6^*, \chi_{12}^*)\,G_i + \chi_{10}^*\sum_{i=1}^{4} R_i^{'}(E_4^*, E_6^*, \chi_{10}^*, \chi_{12}^*)\,G_i = 0.$$

By taking the image of both sides under Witt operator, we get

$$\sum_{i=1}^{4} R_i(W(E_4^*), W(E_6^*), W(\chi_{12}^*))\, W(G_i) \;=\; 0.$$

As we wrote before, the forms $W(E_4^*)$, $W(E_6^*)$, $W(\chi_{12}^*)$, $W(G_1)$, $W(G_2)$, $W(G_3)$, $W(G_4)$ are polynomials of four algebraically independent variables x_1, x_2, y_1, y_2. For each $W(f)$ for E_4^* etc. above, we denote by $W(f)_0$ the polynomial of x_1, x_2, y_2 obtained by substituting y_1^4 by $-2x_1^4$. Then we get

$$\begin{aligned}
W(E_4^*)_0 &= -3\; {x_1}^8 \left(16\, {x_2}^8 + 16\, {x_2}^4\, {y_2}^4 + {y_2}^8\right)/2^6,\\
W(E_6^*)_0 &= 0,\\
W(\chi_{12}^*)_0 &= -3\; {x_1}^{24}\, {x_2}^4\, {y_2}^{16} \left({x_2}^4 + {y_2}^4\right)/2^{26},\\
W(G_1)_0 &= -3\, {x_1}^8 \left({x_2}^4 + {y_2}^4\right), \left(8\, {x_2}^4 + {y_2}^4\right)/2^5,\\
W(G_2)_0 &= 3\, {x_1}^{12} \left({x_2}^4 + {y_2}^4\right)\left(32\, {x_2}^8 + 20\, {x_2}^4\, {y_2}^4 - {y_2}^8\right)/2^8,\\
W(G_3)_0 &= \; {x_1}^{20}\, {x_2}^4\, {y_2}^{12} \left({x_2}^4 + {y_2}^4\right)/2^{20},\\
W(G_4)_0 &= -3\, {x_1}^{24}\, {x_2}^4\, {y_2}^{12} \left({x_2}^4 + {y_2}^4\right)\left(4\, {x_2}^4 + 5\, {y_2}^4\right)/2^{23}.
\end{aligned}$$

Now we write R_i as

$$R_i(X_1, X_2, X_3) = R_{i,1}(X_1, X_3) + X_2 R_{i,2}(X_1, X_2, X_3).$$

Dividing the relation into the part where the total degree is $0 \bmod 16$ and $8 \bmod 16$, we get

$$R_{1,1}(W(E_4^*)_0, W(\chi_{12}^*)_0)W(G_1)_0 + R_{4,1}(W(E_4^*)_0, W(\chi_{12}^*)_0)W(G_4)_0 \;=\; 0.$$

and

$$R_{2,1}(W(E_4^*)_0, W(\chi_{12}^*)_0)W(G_2)_0 + R_{3,1}(W(E_4^*)_0, W(\chi_{12}^*)_0)W(G_3)_0 \;=\; 0.$$

Since four forms x_1, x_2, y_1, y_2 are algebraic independent, $W(G_4)_0$ is divisible by y_2, so $R_{1,1}(W(E_4^*)_0, W(\chi_{12}^*)_0)W(G_1)_0$ must be divisible by y_2. But $W(G_1)_0$ is not divisible by y_2, so the polynomial $R_{1,1}(X_1, X_3)$ is also divisible by X_3. In the same argument we see that $R_{4,1}(X_1, X_3)$ is divisible by X_3. Repeating the process, we see $R_{1,1}(X_1, X_3) = R_{4,1}(X_1, X_3) = 0$. We get $R_{2,1}(X_1, X_3) = R_{3,1}(X_1, X_3) = 0$ in the same way. So we get $R_i(X_1, X_2, X_3) = 0$, and $Q_i(X_1, X_2, X_3, X_4) = 0$. Thus we have proved Theorem 1.8. □

The proof of Theorem 1.10 is almost same as the proof of Theorem 1.8. But the computation is more complicate. To determine $P_{69/2}$, we need Fourier coefficients of basis of $M_{69/2}(\Gamma, \psi)$ at $\begin{pmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{pmatrix}$ with $0 \le a \le 20$, $0 \le c \le 20$, and $0 \le b \le 40$. We omit details here. □

Next we shall prove Theorem 1.9. By applying the Siegel Φ operator at each cusp, we can show that $P_{19/2}$, $P_{23/2}$, $P_{29/2}$ are all cusp forms. So in order to prove the theorem, it is sufficient to determine cusp forms in $AP_{7/2} \oplus AP_{11/2}$ and in $AP_{1/2} \oplus AP_{13/2} \oplus AP_{17/2}$. We see

$$\begin{aligned}\Phi(E_4^*) &= 2^{-4}\left(16\,{x_1}^8 + 16\,{x_1}^4\,{y_1}^4 + {y_1}^8\right),\\ \Phi(E_6^*) &= 2^{-6}\left(2\,{x_1}^4 + {y_1}^4\right)\,\left(32\,{x_1}^8 + 32\,{x_1}^4\,{y_1}^4 - {y_1}^8\right),\\ \Phi(\theta_{0000}P_{7/2}) &= 2^{-3}\,\left({x_1}^4 + {y_1}^4\right)\,\left(8\,{x_1}^4 + {y_1}^4\right),\\ \Phi(\theta_{0000}P_{11/2}) &= 2^{-5}\,\left({x_1}^4 + {y_1}^4\right)\,\left(32\,{x_1}^8 + 20\,{x_1}^4\,{y_1}^4 - {y_1}^8\right).\end{aligned}$$

We assume that $R_1(E_4^*, E_6^*)P_{7/2} + R_2(E_4^*, E_6^*)P_{11/2}$ is a cusp form for some polynomials $R_i(X_1, X_2)$. By definition we get

$$R_1(\Phi(E_4^*), \Phi(E_6^*))\Phi(\theta_{0000}P_{7/2}) + R_2(\Phi(E_4^*), \Phi(E_6^*))\Phi(\theta_{0000}P_{11/2}) \;=\; 0.$$

For $f = E_4^*$ etc., we denote by $(\Phi(f))_0$ the polynomial of x_1 obtained from $\Phi(f)$ by substituting y_1^4 by $-2x_1^4$. Then,

$$\begin{aligned}(\Phi(E_4^*))_0 &= -3\;{x_1}^8/4,\\ (\Phi(E_6^*))_0 &= 0,\\ (\Phi(\theta_{0000}P_{7/2}))_0 &= -3\,{x_1}^8/4,\\ (\Phi(\theta_{0000}P_{11/2}))_0 &= 3\,{x_1}^{12}/8.\end{aligned}$$

Let $R_i(X_1, X_2) = R_{i,1}(X_1) + X_2R_{i,2}(X_1, X_2)$. We get

$$R_{1,1}((\Phi(E_4^*))_0,\; (\Phi(\theta_{0000}P_{7/2}))_0) + R_{2,1}((\Phi(E_4^*))_0,\; (\Phi(\theta_{0000}P_{11/2}))_0) \;=\; 0.$$

Regarding this as an equality between polynomials of x_1, we get $R_i(X_1, X_2) = 0$. So, there are no cusp forms in $\mathbb{C}[E_4^*, E_6^*]P_{7/2} + \mathbb{C}[E_4^*, E_6^*]P_{11/2}$ except for 0. By similar calculation, we can show that there are no cusp forms in $\mathbb{C}[E_4^*, E_6^*]P_{1/2} + \mathbb{C}[E_4^*, E_6^*]P_{13/2} + \mathbb{C}[E_4^*, E_6^*]P_{17/2}$ except for the ideal generated by $P_{25/2}$. Thus we complete the proof of Theorem 1.9. $\square$

3. A lifting conjecture

3.1. Statement of Conjecture

For Siegel cusp forms of half integral weight of degree two, we propose the following conjecture.

Conjecture 3.1. For any $f \in S_{2k-2}(SL(2,\mathbb{Z}))$ and $g \in S_{2k-4}(SL(2,\mathbb{Z}))$ which are common eigen forms of Hecke operators, there exists a common eigen form $F \in S^+_{k-1/2}(\Gamma_0(4))$ of Hecke operators such that

$$L(s, F) = L(s, f)\,L(s-1, g).$$

Here $L(s,f)$ and $L(s,g)$ are the usual Hecke L function and $L(s,F)$ is the L function of F defined by Zhuravlev which will be reviewed in §3.2. This conjecture is based on numerical examples of Euler factors of cusp forms given in §3.5. Conceptually, it can also be regarded as half-integral analogue of vector valued version of Yoshida lifting in [19]. We have also some similar experimental results for Siegel cusp forms outside the plus space but our knowledge would be too vague to state any conjecture in that case.

As for the L function of common eigen non-cusp forms of half integral weight, we can prove a theorem given below which is similar to the theorem by Zharkovskaya [20] for integral weight. The theorem below may be regarded as a non-cusp form version of the above conjecture, though this seems very different from usual liftings. Let k be a positive integer and let F be an element of $M^+_{k-1/2}(\Gamma_0(4))$. If F is not a cusp form, then we get $\Phi(F)\neq 0$ and $\Phi(F)$ belongs to the plus space of modular forms of one variable.

Theorem 3.1. *Let $F\in M^+_{k-1/2}(\Gamma_0(4))$ be a Hecke eigen form with $\Phi(F)\neq 0$. Then $\Phi(F)$ is an eigen form of $T_1(p^2)$ of weight $k-1/2$, where $T_1(p^2)$ is the usual Hecke operator on modular forms of half integral weight of degree one at p. Besides we have*

$$L(s,F)=L(s-1,E_{2k-4})\ L(s,\Phi(F)),$$

where E_{2k-4} is the Eisenstein series in $M_{2k-4}(SL(2,\mathbb{Z}))$.

The proof of this theorem will be given in §3.4. Almost the same theorem was given in [4] for $M_{k-1/2}(\Gamma_0^{(n)}(4))$, but here we assumed that f is in the plus space, hence our theorem includes the claim for Euler 2 factors too.

3.2. Hecke theory for Siegel modular forms of half integral weigh of degree 2 at odd prime

The Hecke theory at odd primes on Siegel modular forms of half integral weight is developed in Zhuravlev [21], [22]. We review his result in case of degree two. The definition of L function is not very clearly written there in terms of Hecke operators, so we review some argument also. (See also the definition in [6]). As for modular forms of $\Gamma_0(4)$, two is a bad prime, but if we restrict ourselves to the plus space, we have a good theory also at two. We shall explain this in the next section.

Now we put

$$GSp^+(2,\mathbb{R})=\left\{M=\begin{pmatrix}A&B\\C&D\end{pmatrix}\in M(4,\mathbb{R})\ ;\ MJ^tM=n(M)J\ ,\ n(M)\in\mathbb{R}^+\right\}.$$

We denote by $\widetilde{GSp^+(2,\mathbb{R})}$ the covering group of $GSp^+(2,\mathbb{R})$ defined as follows. The underlying set of $\widetilde{GSp^+(2,\mathbb{R})}$ consists of pairs $(M,\phi(\tau))$, where $M\in GSp^+(2,\mathbb{R})$ and $\phi(\tau)$ is any holomorphic function on H_2 such that $|\phi(\tau)|=|\det M|^{-1/4}|C\tau+D|^{1/2}$. The group operation on $\widetilde{GSp(2,\mathbb{R})}$ is given by $(M,\phi(\tau))\,(M^{'},\phi^{'}(\tau)) = \left(MM^{'},\phi(M^{'}\tau)\phi^{'}(\tau)\right)$. Then, we can embed $\Gamma_0(4)$ into the group $\widetilde{GSp^{+}(2,\mathbb{R})}$ by

$$\Gamma_0(4) \ni M \mapsto (M,\theta(M\tau)\theta(\tau)^{-1}) \in \widetilde{GSp^{+}(2,\mathbb{R})},$$

where $\theta(\tau)=\theta_{0000}(2\tau)$.

For any odd prime p, we put

$$K_1 = \begin{pmatrix} 1 & & & \\ & p & & \\ & & p^2 & \\ & & & p \end{pmatrix}, \; K_2 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & p^2 & \\ & & & p^2 \end{pmatrix}.$$

Then $(K_s, p^{\frac{1}{2}s}) \in \widetilde{GSp^{+}(2,\mathbb{R})}$ $(s=1,2)$. We put $\widetilde{K_s}=(K_s,p^{\frac{1}{2}s})$. The left $\widetilde{\Gamma_0(4)}$-coset decomposition of $\widetilde{\Gamma_0(4)}\widetilde{K_s}\widetilde{\Gamma_0(4)}$ is explicitly given by Zhuravlev [21]. For the sake of simplicity, we put

$$X_0(p)=\widetilde{\Gamma_0(4)}(p1_4,1)\widetilde{\Gamma_0(4)}\ ,\ X_1(p)=\widetilde{\Gamma_0(4)}\widetilde{K_1}\widetilde{\Gamma_0(4)}\ ,\ X_2(p)=\widetilde{\Gamma_0(4)}\widetilde{K_2}\widetilde{\Gamma_0(4)}.$$

We take a left $\widetilde{\Gamma_0(4)}$-coset decomposition

$$X_s(p)=\bigcup_v \widetilde{\Gamma_0(4)}\widetilde{M_v}.$$

where $\widetilde{M_v} = (M_v,\phi_v(\tau)) \in \widetilde{GSp^{+}(2,\mathbb{R})}$, We define an action of $\widetilde{M_v} = (M_v,\phi_v(\tau)) \in \widetilde{GSp^{+}(2,\mathbb{R})}$ on $F \in M_{k-1/2}(\Gamma_0(4),\psi^l)$ by

$$F|_{k-\frac{1}{2},\psi^l}\widetilde{M_v} = n(M_v)^{k-\frac{7}{2}}\psi(M_v)^l\phi_v(\tau)^{-2k+1}F(M_v\tau),$$

and an action of $X_s(p)$ by $F\,|\,X_s(p)=\sum_v F|_{k-\frac{1}{2},\psi^l}\widetilde{M_v}$. By abuse of language, we denote these operators also by $X_s(p)$ which are double cosets originally.

Let L_p be a Hecke ring generated by operators $X_0(p)^{\pm 1}$, $X_1(p)$, $X_2(p)$. According to [22], this L_p is isomorphic to a certain ring of W_2-invariant polynomials $\mathbb{C}^{W_2}[x_0^{\pm1},x_1^{\pm1},x_2^{\pm1}]$, where W_2 is the automorphism group generated by the following elements :

$$\begin{aligned}
&\sigma : x_0 \to x_0,\ x_1 \to x_2,\ x_2 \to x_1,\\
&\sigma_i : x_0 \to x_0x_i,\ x_i \to x_i^{-1},\ x_j \to x_j \quad (i=1,2, \quad i\neq j),\\
&\sigma' : x_0 \to -x_0,\ x_1 \to x_1,\ x_2 \to x_2.
\end{aligned}$$

By using Proposition 4.1 and Lemma 3.2 in [22], we see there is an isomorphism $\phi:\ L_p \to \mathbb{C}^{W_2}[x_0^{\pm1},x_1^{\pm1},x_2^{\pm1}]$ such that

$$\begin{aligned}
&\phi(X_0(p)) = p^{-3}x_0^2x_1x_2,\\
&\phi(X_1(p)) = p^{-1}x_0^2(x_1+x_2)(1+x_1x_2),\\
&\phi(X_2(p)) = x_0^2\left(1+x_1^2+x_2^2+(1-p^{-2})x_1x_2+x_1^2x_2^2\right).
\end{aligned}$$

Define a polynomial $\gamma(z)$ of z by $\gamma(z) = \prod_{i=1}^{2}(1 - x_i z)(1 - x_i^{-1} z)$, and write its expansion as $\gamma(z) = \sum_{j=0}^{4}(-1)^j R_j z^j$. By using the above three relations, the inverse image $\phi^{-1}(R_j)$ of R_j is given:

$$\begin{aligned}
\phi^{-1}(R_0) = \phi^{-1}(R_4) &= 1, \\
\phi^{-1}(R_1) = \phi^{-1}(R_3) &= p^{-2} X_0(p)^{-1} X_1(p), \\
\phi^{-1}(R_2) = p^{-3} X_0(p)^{-1} X_2(p) &+ (1 + p^{-2}).
\end{aligned}$$

We say that $F \in M_{k-1/2}(\Gamma_0(4), \psi^l)$ is a Hecke eigen form, if F is a common eigen function for the action of $X_1(p)$, $X_2(p)$ for all odd prime p. For a Hecke eigen form $F \in M_{k-1/2}(\Gamma_0(4), \psi^l)$, we denote by $\beta_{j,p}$ $(j = 0, \ldots, 4)$ the Hecke eigen value of F of $\phi^{-1}(R_j)$. Then, there exists $\alpha_{1,p}^{\pm}$, $\alpha_{2,p}^{\pm}$ which satisfy $\sum_{j=0}^{4}\beta_{j,p} z^j = \prod_{i=1}^{2}(1 - \alpha_{i,p} z)(1 - \alpha_{i,p}^{-1} z)$. The L-function of F is defined in Zhuravlev [22] by

$$L(s, F) = \prod_p \prod_{i=1}^{2}(1 - \psi(p)^l \, \alpha_{i,p} \, p^{-s+k-3/2})^{-1}(1 - \psi(p)^l \, \alpha_{i,p}^{-1} \, p^{-s+k-3/2})^{-1}.$$

We rewrite this by eigen values. We denote by $\lambda(p)$ or $\omega(p)$ the Hecke eigen values of F of $X_1(p)$ or $X_2(p)$, respectively. Then, we have

$$\begin{aligned}
L(s, F) = \prod_p \big(1 - \lambda^*(p) p^{-s} + (p\,\omega(p) + p^{2k-5}(1 + p^2)) p^{-2s} \\
- \lambda^*(p) p^{2k-3-3s} + p^{4k-6-4s}\big)^{-1},
\end{aligned}$$

where we put $\lambda^*(p) = \lambda(p) \left(\frac{-1}{p}\right)^l p^{-k+7/2}$. In the above product, at moment we defined Euler p factors only for odd primes, but an Euler 2 factor will be defined for elements of plus subspace later.

Next, we explain how to get eigen values by using Fourier coefficients of Hecke eigen forms. First, we prepare some notations. For i, $j \in \{0, 1, 2\}$, $i + j \le 2$, we put $d_{i,j} = \begin{pmatrix} 1_{2-i-j} & & \\ & p1_i & \\ & & p^2 1_j \end{pmatrix} \in M_2(\mathbb{Z})$. We denote by $M_{l,m}(p^\delta)$

a complete set of representatives of matrices of $M_{l,m}(\mathbb{Z})$ modulo p^{δ}, and put

$$R_{s,i,j} = \left\{ B = \begin{pmatrix} 0_{2-i-j} & 0 & 0 \\ 0 & A & pB_1 \\ 0 & {}^tB_1 & B_2 \end{pmatrix} ; \right.$$
$$\left. \begin{array}{l} A \in M_{i,i}(p),\ B_1 \in M_{i,j}(p),\ B_2 \in M_{j,j}(p^2), \\ A = {}^t A,\ B_2 = {}^t B_2, \text{ and } rank_p(A) = i-2+s \end{array} \right\}.$$

For a matrix $B = \begin{pmatrix} 0_{2-i-j} & 0 & 0 \\ 0 & A & pB_1 \\ 0 & {}^tB_1 & B_2 \end{pmatrix} \in R_{s,i,j}$ and for a fixed $\gamma = i-2+s$, we define a function $\kappa(B)$ by $\kappa(B) = 1$ or $\varepsilon_p^{\gamma}\left(\frac{(-1)^{\gamma}\det A_1}{p}\right)$ for $\gamma = 0$ or $\gamma > 0$, respectively, where $\varepsilon_p = 1$ or $\sqrt{-1}$ if $p \equiv 1$ or $3 \bmod 4$, A_1 is any matrix of size γ such that $A \equiv {}^tM \left(\begin{smallmatrix} A_1 & 0 \\ 0 & 0 \end{smallmatrix}\right) M \bmod p$ for some $M \in M_{i,i}(p) \cap SL(i,\mathbb{Z})$, and $\left(\frac{*}{p}\right)$ means the Legendre symbol. We write the Fourier expansion of $F(\tau) \in M_{k-1/2}(\Gamma_0(4), \psi^l)$ as $F(\tau) = \sum_{N \geq 0} C(N)e(\mathrm{tr}(N\tau))$. We define $\alpha_{s,i,j}(T)$ by

$$\alpha_{s,i,j}(T) = p^{(s+2-i-2j)(k-\frac{1}{2})-3s}\,\psi(p^{i+2j})^l \times \sum_{U,B} C\left(\frac{1}{p^2}d_{i,j}UT{}^tUd_{i,j}\right) e\left(\frac{1}{p^2}\mathrm{tr}\left(UT{}^tUd_{i,j}B\right)\right)\kappa(B)^{-2k+1}.$$

where the matrix B runs over all elements of $R_{s,i,j}$, U runs over a complete set of representatives of $\left(d_{i,j}^{-1}SL(2,\mathbb{Z})d_{i,j} \cap SL(2,\mathbb{Z})\right)\backslash SL(2,\mathbb{Z})$, and we regard $C(M) = 0$ if M is not a half integral matrix.

Now let F be a Hecke eigen form and we assume that $C(T) \neq 0$ for some semi-positive definite $T = \left(\begin{smallmatrix} a & b/2 \\ b/2 & c \end{smallmatrix}\right)$. By using [21, Proposition 7.1], we have

$$\lambda^*(p) = C(T)^{-1} \sum_{\substack{i,j \\ 1 \leq i \leq i+j \leq 2}} \alpha_{1,i,j}(T), \omega(p) = C(T)^{-1} \sum_{\substack{i,j \\ 0 \leq i \leq i+j \leq 2}} \alpha_{2,i,j}(T).$$

Now for any prime p, we put

$$R(p) = \left\{\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \ ;\ x \mod p\right\},$$
$$R(p^2) = \left\{\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} py & 1 \\ -1 & 0 \end{smallmatrix}\right) \ ;\ x \mod p^2 \ ,\ y \mod p\right\}.$$

Then we can calculate each $\alpha_{s,i,j}(T)$ for odd p explicitly as follows.

$$\alpha_{1,1,0}(T) = p^{2k-4} \sum_{U \in R(p)} C\left(\begin{pmatrix} p^{-1} & \\ & 1 \end{pmatrix} UT{}^tU \begin{pmatrix} p^{-1} & \\ & 1 \end{pmatrix}\right),$$

$$\alpha_{1,1,1}(T) = \sum_{U \in R(p)} C\left(\begin{pmatrix} 1 & \\ & p \end{pmatrix} UT{}^tU \begin{pmatrix} 1 & \\ & p \end{pmatrix}\right),$$

$$\alpha_{1,2,0}(T) = \begin{cases} \left(\frac{(-1)^{k+l-1}a}{p}\right) p^{k-2} C(T) & \text{if } p \nmid a \text{ and } p \mid \det 2T, \\ \left(\frac{(-1)^{k+l-1}c}{p}\right) p^{k-2} C(T) & \text{if } p \mid a \text{ and } p \mid \det 2T, \\ 0 & \text{otherwise,} \end{cases}$$

$$\alpha_{2,0,0}(T) = p^{4k-8}\, C\left(\frac{1}{p^2}T\right),$$

$$\alpha_{2,0,1}(T) = p^{2k-5} \sum_{U \in R(p^2)} C\left(\begin{pmatrix} p^{-1} & \\ & p \end{pmatrix} UT{}^tU \begin{pmatrix} p^{-1} & \\ & p \end{pmatrix}\right),$$

$$\alpha_{2,0,2}(T) = C(p^2T),$$

$$\alpha_{2,1,0}(T) = p^{3k-7} \sum_{U \in R(p)} \left(\frac{(-1)^{k+l-1}c_U}{p}\right) C\left(\begin{pmatrix} p^{-1} & \\ & 1 \end{pmatrix} UT{}^tU \begin{pmatrix} p^{-1} & \\ & 1 \end{pmatrix}\right),$$

$$\text{where } \begin{pmatrix} * & * \\ * & c_U \end{pmatrix} = UT{}^tU,$$

$$\alpha_{2,1,1}(T) = p^{k-3} \sum_{U \in R(p)} \left(\frac{(-1)^{k+l-1}a_U}{p}\right) C\left(\begin{pmatrix} 1 & \\ & p \end{pmatrix} UT{}^tU \begin{pmatrix} 1 & \\ & p \end{pmatrix}\right),$$

$$\text{where } \begin{pmatrix} a_U & * \\ * & * \end{pmatrix} = UT{}^tU,$$

$$\alpha_{2,2,0}(T) = \begin{cases} -\,p^{2k-6}\, C(T) & \text{if } p \nmid \det 2T, \\ (p-1)p^{2k-6}\, C(T) & \text{if } p \mid \det 2T. \end{cases}$$

3.3. Hecke theory on plus space at two

Although 2 is a bad prime for $\Gamma_0(4)$, it is a good prime for the plus subspace, since it is isomorphic to the space of Jacobi forms of "level" one. Namely we know that for odd primes the Hecke theory of Jacobi forms and Siegel modular forms of half integral weight corresponds well (cf. [9], [3], also see the correction at the end of this paper.), and for $F \in M^+_{k-1/2}(\Gamma_0(4), \psi^l)$ and for every odd prime p, we can interpret $X_1^*(p) = \psi(p)^l p^{-k+7/2} X_1(p)$ or $X_2(p)$ as a pull back of a Hecke operator on Jacobi forms. Now we can define operators $X_1^*(2)$ and $X_2(2)$ on $M^+_{k-1/2}(\Gamma_0(4), \psi^l)$ as the pull backs of the same Hecke operators at two on Jacobi forms. Hence we say that F is a common eigen form if the image of F in Jacobi forms of index one is a common eigen form of all the Hecke operators on Jacobi forms, including those at two. For F we can define $\lambda^*(2)$ and $\omega(2)$ in the same way by using $X_1^*(2)$ and $X_2(2)$, and also the Euler 2 factor is defined in the same formula as in the case of odd primes. Hence $L(s,F)$ is defined as the product of Euler factors at all primes p by the formula in the previous section.

The formula for $\lambda^*(2)$ and $\omega(2)$ using the Fourier coefficients is almost the same as in the odd case. Here we explain the necessary modification of the formula in the previous section. Let $C(T)$ be the Fourier coefficient of F as before. If $C(T) \neq 0$ for $T = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$, then since F belongs to the plus space, we get

$$\begin{aligned} a &= (-1)^{k+l-1}\lambda_1^2 + 4\alpha, \\ b &= (-1)^{k+l-1}2\lambda_1\lambda_2 + 4\beta, \\ c &= (-1)^{k+l-1}\lambda_2^2 + 4\gamma, \end{aligned}$$

where $(\lambda_1,\lambda_2) = (1,1)$, $(1,0)$, $(0,1)$ or $(0,0)$ and α, β, γ are integers. So, we get

$$\det(T) = 4(4\alpha\gamma - \beta^2) + 4(-1)^{k+l-1}(\alpha\lambda_2^2 + \gamma\lambda_1^2 - \beta\lambda_1\lambda_2),$$

and hence $\det(T) \equiv 0 \bmod 4$. The condition that $p|\det(2T)$ or $p \nmid \det(2T)$ in the previous formula should be replaced by the condition that $8|\det(T)$ or $8 \nmid \det(T)$, respectively. The Legendre symbol $\left(\frac{x}{p}\right)$ for odd p in the formula before is now interpreted as follows. First of all, we can easily show that each x which appears in the Legendre symbol, namely each of $x = (-1)^{k+l-1}a$, $(-1)^{k+l-1}c$, $(-1)^{k+l-1}a_U$ or $(-1)^{k+l-1}c_U$ for $p=2$ satisfies the condition $x \equiv 0$ or $1 \bmod 4$. So, we define

$$\left(\frac{x}{p}\right) = \begin{cases} 0 & \text{if } x \equiv 0 \bmod 4, \\ 1 & \text{if } x \equiv 1 \bmod 8, \\ -1 & \text{if } x \equiv 5 \bmod 8. \end{cases}$$

3.4. Proof of Theorem 3.1

For any prime p including $p=2$ and any eigen form $F \in M_{k-1/2}^{+}(\Gamma_0(4))$, we take the operators $X_1(p)$, $X_2(p)$ such that $F|X_1(p) = p^{k-7/2}\lambda^*(p)F$ and $F|X_2(p) = \omega(p)F$ as before. First, we calculate Fourier coefficients of $\Phi(F|X_1(p))$ and $\Phi(F|X_2(p))$.

If a Siegel modular form F has the Fourier expansion

$$F(\tau) = \sum_{T \geq 0} C(T)\, e(\mathrm{tr}(T\tau)),$$

then we have $\Phi(F)(z) = \sum_{m\geq 0} c(m)e(mz)$, where we put $c(m) = C\begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix}$. For $s = 1$, 2, we write $\Phi(F|X_s(p))(z) = \sum_{m\geq 0} a_s(m)e(mz)$. Then we get $a_s(m)$

$= \sum_{2-s\leq i\leq i+j\leq 2} \alpha_{s,i,j}\begin{pmatrix} m & 0 \\ 0 & 0\end{pmatrix}$, where $\alpha_{s,i,j}(T)$ is the same notation as in the previous section. By straightforward calculation, we have:

$$
\begin{aligned}
a_1(m) &= c(m/p^2)p^{2k-3} + c(p^2m) + c(m)\left(p^{2k-4} + p + \left(\frac{(-1)^{k-1}m}{p}\right)p^{k-2}\right), \\
a_2(m) &= c(m/p^2)(p^{4k-8} + p^{2k-3}) + c(p^2m)(p^{2k-5}+1) \\
&\quad + c(m)\left(p^{3k-7}\left(\frac{(-1)^{k-1}m}{p}\right) + p^{k-2}\left(\frac{(-1)^{k-1}m}{p}\right) + p^{2k-6}(p^2-1)\right),
\end{aligned}
$$

where the symbol $\left(\frac{x}{p}\right)$ for $p = 2$ is defined as before. Let f be a modular form of weight $k - 1/2$ of degree 1 in the *plus-subspace* and take its Fourier expansion $f(z) = \sum_{m\geq 0} b(m)e(mz)$. Then for any prime p including $p = 2$, the Hecke operator $T_1(p^2)$ is defined by

$$
\begin{aligned}
(f|T_1(p^2))(z) = \sum_{m\geq 0} \Bigg(& b(p^2m) + p^{2k-3}b(m/p^2) \\
& + \left(\frac{(-1)^{k-1}m}{p}\right)p^{k-2}b(m)\Bigg) e(mz).
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
\Phi(F|p^{-k+7/2}X_1(p)) &= \Phi(F)|T_1(p^2) + (p + p^{2k-4})\Phi(F), \\
\Phi(F|X_2(p)) &= (p^{2k-5}+1)(\Phi(F)|T_1(p^2)) + (p^{2k-4} - p^{2k-6})\Phi(F).
\end{aligned}
$$

If $\Phi(F) \neq 0$, then by the above relation, it is obvious that $\Phi(F)$ is also an eigenform of $T_1(p^2)$. If we put

$$\Phi(F)|T_1(p^2) = \mu(p^2)\Phi(F),$$

then we get

$$
\begin{aligned}
\lambda^*(p) &= \mu(p^2) + p + p^{2k-4}, \\
\omega(p) &= (p^{2k-5}+1)\mu(p^2) + p^{2k-4} - p^{2k-6}.
\end{aligned}
$$

so we have,

$$
\begin{aligned}
L(s,F) &= \prod_p \left((1 - \mu(p^2)p^{-s} + p^{2k-3-2s})(1-p^{1-s})(1-p^{2k-4-s})\right)^{-1} \\
&= L(s,\Phi(F))\zeta(s-1)\zeta(s-2k+4) \\
&= L(s,\Phi(F))L(s-1,E_{2k-4}).
\end{aligned}
$$

This completes the proof of Theorem 3.1. □

3.5. Numerical examples of Euler factors

In this section, we give some examples of Euler factors of forms in the plus space. The dimensions of the plus space and the space of elliptic modular forms are given in the following table for small weights.

k	$0 \sim 6$	7	8	9	10	11	12	13	14	15
dim $S^+_{k-1/2}(\Gamma_0(4))$	0	0	0	0	1	1	1	2	2	2
dim $S_{2k-2}(SL(2,\mathbb{Z}))$	0	1	0	1	1	1	1	2	1	2
dim $S_{2k-4}(SL(2,\mathbb{Z}))$	0	0	1	0	1	1	1	1	2	1

k	16	17	18	19	20	21	22	23	24	25
dim $S^+_{k-1/2}(\Gamma_0(4))$	4	4	4	6	6	7	9	9	10	13
dim $S_{2k-2}(SL(2,\mathbb{Z}))$	2	2	2	3	2	3	3	3	3	4
dim $S_{2k-4}(SL(2,\mathbb{Z}))$	2	2	2	2	3	2	3	3	3	3

For any Hecke eigen form $F \in S^+_{k-1/2}(\Gamma_0(4), \psi^l)$, we define the Hecke polynomial $H_p(T,F)$ at p by

$$\begin{aligned} H_p(T,F) = 1 - \lambda^*(p)T + (p\,\omega(p) + p^{2k-5}(1+p^2)))T^2 \\ - \lambda^*(p)p^{2k-3}T^3 + p^{4k-6}T^4. \end{aligned}$$

The dimension of cusp forms of plus space of weight 19/2 is 1. So $P_{19/2}(2\tau)$ is a Hecke eigen form, since the plus space is closed under the action of Hecke operators. Some of Fourier coefficients of $P_{19/2}(2\tau)$ are given as follows:

$wt\,19/2$	$(3,3,2)$	$(24,3,0)$	$(11,8,8)$	$(19,4,4)$	$(27,27,18)$
$P_{19/2}(2\tau)$	1	-5022	-861	-3423	23088645

where (a,c,b) means the Fourier coefficient at $\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$.

Next we calculate eigen values $\lambda^*(3)$ and $\omega(3)$ of $P_{19/2}(2\tau)$. We put $C(a,c,b) = C\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$, then we have:

$$\begin{aligned} \lambda^*(3) &= C(24,3,0) \times 2 + C(11,8,8) + C(19,4,4) \\ &= -14328, \\ \omega(3) &= C(3,3,2) \times 2 \times 3^{15} + C(27,27,18) \\ &\qquad + (C(11,8,8) - C(19,4,4)) \times 3^7 + C(3,3,2) \times (-3^{14}) \\ &= 52606584. \end{aligned}$$

Then the Euler factor of $P_{19/2}(2\tau)$ at $p = 3$ is given by $H_3(3^{-s}, P_{19/2}(2\tau))$, where

$$\begin{aligned} H_3(T, P_{19/2}(2\tau)) &= 1 + 14328\,T + 301308822\,T^2 + 1850320255464\,T^3 + 3^{34}\,T^4 \\ &= (1 + 10044\,T + 3^{17}\,T^2)(1 + 4284\,T + 3^{17}\,T^2)\ . \end{aligned}$$

We denote by Δ_{16} or Δ_{18} the normalized Hecke eigen form belonging to $SL(2,\mathbb{Z})$ of weight 16 or 18, respectively. For any common eigen form $f \in S_k(SL(2,\mathbb{Z}))$, we denote by $\lambda(p,f)$ the eigen value of the Hecke operator $T(p)$ on f at prime p. We denote by $L_p(s,f)$ the Euler p - factor of the L function of f. We see

$$\lambda(3,\Delta_{16}) = -3348\ ,\lambda(3,\Delta_{18}) = -4284.$$

So, we get

$$H_3(3^{-s}, P_{19/2}(2\tau)) = L_3(s,\Delta_{18})L_3(s-1,\Delta_{16}).$$

By similar calculation, we get $\lambda^*(2) = -96$ and $\omega(2) = -64896$ for $P_{19/2}(2\tau)$. On the other hand, we see $\lambda(2,\Delta_{16}) = 216$, $\lambda(2,\Delta_{18}) = -528$, so we get

$$H_2(2^{-s}, P_{19/2}(2\tau)) = L_2(s,\Delta_{18})L_2(s-1,\Delta_{16}).$$

Let Δ_k $(k = 20, 22, 26)$ be the normalized cusp forms of weight k in the one dimensional space $S_k(SL(2,\mathbb{Z}))$, and Δ_{24}^+, Δ_{24}^- be a Hecke eigen basis of $S_{24}(SL(2,\mathbb{Z}))$. The Euler factors of Hecke eigen forms in the plus space of weight 21/2, 23/2, 25/2, 27/2, 29/2, 31/2, 33/2, and 35/2 are given as follows.

weight 21/2.

We have $\dim S_{21/2}^+(\Gamma_0(4)) = 1$ and this space $S_{21/2}^+(\Gamma_0(4))$ is generated by $\chi_{10}(4\tau)\,P_{1/2}(2\tau)$, which is of course a Hecke eigen form. We have

$$H_2(T, \chi_{10}(4\tau)\,P_{1/2}(2\tau)) = (1 + 1056\,T + 2^{19}\,T^2)\,(1 - 456\,T + 2^{19}\,T^2),$$
$$H_3(T, \chi_{10}(4\tau)\,P_{1/2}(2\tau)) = (1 + 12852\,T + 3^{19}\,T^2)\,(1 - 50652\,T + 3^{19}\,T^2),$$

and we also have

$$\lambda(2,\Delta_{18}) = -528, \qquad \lambda(2,\Delta_{20}) = 456,$$
$$\lambda(3,\Delta_{18}) = -4284, \qquad \lambda(3,\Delta_{20}) = 50652.$$

So we get

$$H_2(2^{-s}, \chi_{10}(4\tau)\,P_{1/2}(2\tau)) = L_2(s-1,\Delta_{18})L_2(s,\Delta_{20}),$$
$$H_3(3^{-s}, \chi_{10}(4\tau)\,P_{1/2}(2\tau)) = L_3(s-1,\Delta_{18})L_3(s,\Delta_{20}).$$

weight 23/2.

We have $\dim S_{23/2}^+(\Gamma_0(4)) = 1$ and $P_{23/2}(2\tau) \in S_{23/2}^+(\Gamma_0(4))$ is a Hecke eigen form. We have

$$H_2(T, P_{23/2}(2\tau)) = (1 - 912\,T + 2^{21}\,T^2)\,(1 + 288\,T + 2^{21}\,T^2),$$
$$H_3(T, P_{23/2}(2\tau)) = (1 - 151956\,T + 3^{21}\,T^2)(1 + 128844\,T + 3^{21}\,T^2),$$

and we also have

$$\lambda(2,\Delta_{20}) = 456, \qquad \lambda(2,\Delta_{22}) = -288,$$
$$\lambda(3,\Delta_{20}) = 50652, \qquad \lambda(3,\Delta_{22}) = -128844\ .$$

So we get

$$\begin{aligned}
H_2(2^{-s}, P_{23/2}(2\tau)) &= L_2(s-1, \Delta_{20}) L_2(s, \Delta_{22}), \\
H_3(3^{-s}, P_{23/2}(2\tau)) &= L_3(s-1, \Delta_{20}) L_3(s, \Delta_{22}).
\end{aligned}$$

weight 25/2.
We have $\dim S^+_{25/2}(\Gamma_0(4)) = 2$. We put

$$\begin{aligned}
\chi^+_{25/2} &= \left(119 - \sqrt{144169}\right) \chi_{12}(4\tau)\, P_{1/2}(2\tau) + P_{25/2}(2\tau), \\
\chi^-_{25/2} &= \left(119 + \sqrt{144169}\right) \chi_{12}(4\tau)\, P_{1/2}(2\tau) + P_{25/2}(2\tau).
\end{aligned}$$

Then $\chi^{(\pm)}_{25/2} \in S^+_{25/2}(\Gamma_0(4))$ and these are Hecke eigen forms. The Euler 2-factor and 3-factor of these forms are given by

$$\begin{aligned}
H_2(T, \chi^{(\pm)}_{25/2}) &= \left(1 + 576\, T + 2^{23}\, T^2\right) \\
&\quad \times \left(1 - \left(540 \mp 12\sqrt{144169}\right) T + 2^{23}\, T^2\right), \\
H_3(T, \chi^{(\pm)}_{25/2}) &= \left(1 + 386532\, T + 3^{23}\, T^2\right) \\
&\quad \times \left(1 - \left(169740 \pm 576\sqrt{144169}\right) T + 3^{23}\, T^2\right).
\end{aligned}$$

The eigen values of Δ_{22} and $\Delta^{(\pm)}_{24}$ at 2 and 3 are given by

$$\begin{aligned}
\lambda(2, \Delta_{22}) &= -288, & \lambda(2, \Delta^{(\pm)}_{24}) &= 540 \mp 12\sqrt{144169}\ , \\
\lambda(3, \Delta_{22}) &= -128844, & \lambda(3, \Delta^{(\pm)}_{24}) &= 169740 \pm 576\sqrt{144169}.
\end{aligned}$$

So we get

$$\begin{aligned}
H_2(2^{-s}, \chi^{(\pm)}_{25/2}) &= L_2(s-1, \Delta_{22}) L_2(s, \Delta^{(\pm)}_{24}), \\
H_3(3^{-s}, \chi^{(\pm)}_{25/2}) &= L_3(s-1, \Delta_{22}) L_3(s, \Delta^{(\pm)}_{24}).
\end{aligned}$$

weight 27/2.
We have $\dim S^+_{27/2}(\Gamma_0(4)) = 2$. We put

$$\chi^{(\pm)}_{27/2} = \left(-427 \pm \sqrt{144169}\right) \chi_{10}(4\tau) P_{7/2}(2\tau) + 9\, E_4(4\tau) P_{19/2}(2\tau).$$

Then $\chi^{(\pm)}_{27/2} \in S^+_{27/2}(\Gamma_0(4))$ and these are Hecke eigen forms. We have

$$\begin{aligned}
H_2(T, \chi^{(\pm)}_{27/2}) &= \left(1 - \left(1080 \mp 24\sqrt{144169}\right) T + 2^{25}\, T^2\right) \left(1 + 48\, T + 2^{25}\, T^2\right), \\
H_3(T, \chi^{(\pm)}_{27/2}) &= \left(1 - \left(509220 \pm 1728\sqrt{144169}\right) T + 3^{25}\, T^2\right) \\
&\quad \left(1 + 195804\, T + 3^{25}\, T^2\right).
\end{aligned}$$

We also have

$$\lambda(2,\Delta_{24}^{\pm}) = 540 \mp 12\sqrt{144169}, \qquad \lambda(2,\Delta_{26}) = -48,$$
$$\lambda(3,\Delta_{24}^{(\pm)}) = 169740 \pm 576\sqrt{144169}, \qquad \lambda(3,\Delta_{26}) = -195804.$$

So we get

$$H_2(2^{-s},\chi_{27/2}^{\pm}) = L_2(s-1,\Delta_{24}^{\pm})L_2(s,\Delta_{26}),$$
$$H_3(3^{-s},\chi_{27/2}^{\pm}) = L_3(s-1,\Delta_{24}^{\pm})L_3(s,\Delta_{26}).$$

weight 29/2.

We have $\dim S_{29/2}^{+}(\Gamma_0(4)) = 2$. We put

$$\chi_{29/2}^{(\pm)} = \left(47 \pm \sqrt{18409}\right) E_4(4\tau)\,\chi_{10}(4\tau)\,P_{1/2}(2\tau) + 81\,P_{29/2}(2\tau).$$

Then $\chi_{29/2}^{(\pm)} \in S_{29/2}^{+}(\Gamma_0(4))$ and these are Hecke eigen forms. We have

$$H_2(T,\chi_{29/2}^{(\pm)}) = (1+96\,T+2^{27}\,T^2)\,(1+(4140 \mp 108\sqrt{18209})\,T+2^{27}\,T^2),$$
$$H_3(T,\chi_{29/2}^{(\pm)}) = (1+587412\,T+3^{27}\,T^2)$$
$$(1+(643140 \mp 20736\sqrt{18209})\,T+3^{27}\,T^2).$$

We also have

$$\lambda(2,\Delta_{26}) = -48, \qquad \lambda(2,\Delta_{28}^{\pm}) = -4140 \pm 108\sqrt{18209},$$
$$\lambda(3,\Delta_{26}) = -195804, \qquad \lambda(3,\Delta_{28}^{\pm}) = -643140 \pm 20736\sqrt{18209}.$$

So we get

$$H_2(2^{-s},\chi_{29/2}^{\pm}) = L_2(s-1,\Delta_{26})L_2(s,\Delta_{28}^{\pm}),$$
$$H_3(3^{-s},\chi_{29/2}^{\pm}) = L_3(s-1,\Delta_{26})L_3(s,\Delta_{28}^{\pm}).$$

weight 31/2.

We have $\dim S_{31/2}^{+}(\Gamma_0(4)) = 4$. For $\epsilon_1 = \pm 1$ and $\epsilon_2 = \pm 1$, we put

$$\chi_{31/2}^{\epsilon_1,\,\epsilon_2} =$$
$$2(1087273 + 19401\,\epsilon_1\sqrt{18209} - \epsilon_2\sqrt{51349}(6889 + 33\,\epsilon_1\sqrt{18209}))$$
$$\times\,\chi_{12}(4\tau)P_{7/2}(2\tau) + 10(-661583 - 3855\,\epsilon_1\sqrt{18209}$$
$$+\,\epsilon_2\sqrt{51349}(2327 - 27\,\epsilon_1\sqrt{18209}))\chi_{10}(4\tau)P_{11/2}(2\tau)$$
$$+\,(-11759 + 517\,\epsilon_2\sqrt{51349} - \epsilon_1\sqrt{18209}(283 + \epsilon_2\sqrt{51349}))E_6(4\tau)P_{19/2}(2\tau)$$
$$+\,100590E_4(4\tau)P_{23/2}(2\tau).$$

These are in $S^+_{31/2}(\Gamma_0(4))$ and Hecke eigen forms. Then we have

$$\begin{aligned}
H_2(T,\chi_{31/2}^{\epsilon_1,\,\epsilon_2}) &= (1+(8280-216\,\epsilon_1\sqrt{18209})\,T+2^{29}\,T^2)\\
&\quad\times(1-(4320\;-96\,\epsilon_2\sqrt{51349})\,T+2^{29}\,T^2),\\
H_3(T,\chi_{31/2}^{\epsilon_1,\,\epsilon_2}) &= (1+(1929420\;-62208\,\epsilon_1\sqrt{18209})\,T+3^{29}\,T^2)\\
&\quad\times(1+(2483820\;-52992\,\epsilon_2\sqrt{51349})\,T+3^{29}\,T^2).
\end{aligned}$$

We also have

$$\begin{aligned}
\lambda(2,\Delta_{28}^{\pm}) &= -4140\pm 108\sqrt{18209},\\
\lambda(2,\Delta_{30}^{\pm}) &= 4320\mp 96\sqrt{51349},\\
\lambda(3,\Delta_{28}^{\pm}) &= -\,643140\pm 20736\sqrt{18209},\\
\lambda(3,\Delta_{30}^{\pm}) &= -\,2483820\pm 52992\sqrt{51349}.
\end{aligned}$$

So we get

$$\begin{aligned}
H_2(2^{-s},\chi_{31/2}^{\pm,\,\pm}) &= L_2(s-1,\Delta_{28}^{\pm})L_2(s,\Delta_{30}^{\pm}),\\
H_3(3^{-s},\chi_{31/2}^{\pm,\,\pm}) &= L_3(s-1,\Delta_{28}^{\pm})L_3(s,\Delta_{30}^{\pm}).
\end{aligned}$$

weight 33/2.

We have $\dim S^+_{31/2}(\Gamma_0(4))=4$. For $\epsilon_1=\pm1$ and $\epsilon_2=\pm1$, we put

$$\begin{aligned}
&\chi_{33/2}^{\epsilon_1,\,\epsilon_2} =\\
&(-198304+10027\epsilon_1\sqrt{51349}-128\epsilon_2\sqrt{18295489}\\
&-\epsilon_1\epsilon_2\sqrt{51349}\sqrt{18295489})E_6(4\tau)\chi_{10}(4\tau)P_{1/2}(2\tau)\\
&+189(1131+4\epsilon_1\sqrt{51349}+\epsilon_2\sqrt{18295489})E_4(4\tau)\chi_{12}(4\tau)P_{1/2}(2\tau)\\
&-13608\left(8+\epsilon_1\sqrt{51349}\right)\chi_{10}(4\tau)P_{13/2}(2\tau)+189E_4(4\tau)P_{25/2}(2\tau).
\end{aligned}$$

Then $\chi_{33/2}^{\pm,\,\pm}\in S^+_{33/2}(\Gamma_0(4))$ and these are Hecke eigen forms. We have

$$\begin{aligned}
H_2(T,\chi_{33/2}^{\epsilon_1,\,\epsilon_2}) &= \left(1-\left(8640\;+192\,\epsilon_1\sqrt{51349}\right)\,T+2^{31}\,T^2\right)\\
&\quad\times\left(1-\left(19980\;+12\,\epsilon_2\sqrt{18295489}\right)\,T+2^{31}\,T^2\right),\\
H_3(T,\chi_{33/2}^{\epsilon_1,\,\epsilon_2}) &= \left(1+108(68995+1472\,\epsilon_1\sqrt{51349})T+3^{31}\,T^2\right)\\
&\quad\times\left(1-324(26795+16\,\epsilon_2\sqrt{18295489})\,T+3^{31}\,T^2\right).
\end{aligned}$$

We also have

$$\begin{aligned}
\lambda(2,\Delta_{30}^{\pm}) &= 4320\pm 96\sqrt{51349},\\
\lambda(2,\Delta_{32}^{\pm}) &= 19980\pm 12\sqrt{18295489},\\
\lambda(3,\Delta_{30}^{\pm}) &= -36(68995\pm 1472\sqrt{51349}),\\
\lambda(3,\Delta_{32}^{\pm}) &= 324(26795\pm 16\sqrt{18295489}).
\end{aligned}$$

So we get

$$H_2(2^{-s},\chi_{33/2}^{\pm,\,\pm}) = L_2(s-1,\Delta_{30}^{\pm})L_2(s,\Delta_{32}^{\pm}),$$
$$H_3(3^{-s},\chi_{33/2}^{\pm,\,\pm}) = L_3(s-1,\Delta_{30}^{\pm})L_3(s,\Delta_{32}^{\pm}).$$

weight 35/2.

We have $\dim S_{35/2}^{+}(\Gamma_0(4)) = 4$. For $\epsilon_1 = \pm 1$ and $\epsilon_2 = \pm 1$, we put

$$\begin{aligned}
&\chi_{35/2}^{\epsilon_1,\,\epsilon_2} = \\
&\quad 80(-447232006969 + 489062419\,\epsilon_2\sqrt{2356201} \\
&\quad + \epsilon_1\sqrt{18295489}(-34677047 + 89597\,\epsilon_2\sqrt{2356201}))E_4(4\tau)\chi_{10}(4\tau)P_{7/2}(2\tau) \\
&\quad + 800(34455469783 - 39825301\,\epsilon_2\sqrt{2356201} \\
&\quad + \epsilon_1\sqrt{18295489}(-588847 + 1453\,\epsilon_2\sqrt{2356201}))\chi_{12}(4\tau)P_{11/2}(2\tau) \\
&\quad + 3(-121215233603 - 79606447\,\epsilon_2\sqrt{2356201} + 137801891\,\epsilon_1\sqrt{18295489} \\
&\quad - 83441(\epsilon_1\sqrt{18295489})(\epsilon_2\sqrt{2356201}))E_4(4\tau)^2P_{19/2}(2\tau) \\
&\quad + 9492701472E_6(4\tau)P_{23/2}(2\tau).
\end{aligned}$$

Then $\chi_{35/2}^{\epsilon_1,\,\epsilon_2} \in S_{35/2}^{+}(\Gamma_0(4))$ and these are Hecke eigen forms. We have

$$\begin{aligned}
H_2(T,\chi_{35/2}^{\epsilon_1,\,\epsilon_2}) &= \left(1 - \left(39960 + 24\,\epsilon_1\sqrt{18295489}\right)T + 2^{33}T^2\right) \\
&\quad \times \left(1 + \left(60840 + 72\,\epsilon_2\sqrt{2356201}\right)T + 2^{33}T^2\right), \\
H_3(T,\chi_{35/2}^{\epsilon_1,\,\epsilon_2}) &= \left(1 - \left(26044740 + 15552\,\epsilon_1\sqrt{18295489}\right)T + 3^{33}T^2\right) \\
&\quad \times \left(1 - \left(18959940 + 22464\,\epsilon_2\sqrt{2356201}\right)T + 3^{33}T^2\right).
\end{aligned}$$

We also have

$$\begin{aligned}
\lambda(2,\Delta_{32}^{\pm}) &= 19980 \pm 12\sqrt{18295489}, \\
\lambda(2,\Delta_{34}^{\pm}) &= -60840 \mp 72\sqrt{2356201}, \\
\lambda(3,\Delta_{32}^{\pm}) &= 8681580 \pm 5184\sqrt{18295489}, \\
\lambda(3,\Delta_{34}^{\pm}) &= 18959940 \pm 22464\sqrt{2356201}.
\end{aligned}$$

So we get

$$H_2(2^{-s},\chi_{35/2}^{\pm,\,\pm}) = L_2(s-1,\Delta_{32}^{\pm})L_2(s,\Delta_{34}^{\pm}),$$
$$H_3(3^{-s},\chi_{35/2}^{\pm,\,\pm}) = L_3(s-1,\Delta_{32}^{\pm})L_3(s,\Delta_{34}^{\pm}).$$

Finally we give examples of Siegel modular forms of weight 41/2 and 47/2 which cannot be obtained by this kind of lifting. We put

$$
\begin{aligned}
&\chi_{41/2} = \\
&(738592\, E_4^2(4\tau)\, \chi_{12}(4\tau) + 545630\, E_4(4\tau)\, E_6(4\tau)\, \chi_{10}(4\tau) \\
&+ 65820297600\, \chi_{10}(\tau)^2\,) P_{1/2}(2\tau) - 395994\, E_4(4\tau)\, \chi_{10}(4\tau)\, P_{13/2}(2\tau) \\
&- 838926\, \chi_{12}(4\tau)\, P_{17/2}(2\tau) + 3\, E_4(4\tau)^2\, P_{25/2}(2\tau) \\
&- 191004\, E_6(4\tau)\, P_{29/2}(2\tau).
\end{aligned}
$$

Then $\chi_{41/2}$ is a Hecke eigen form in $S^+_{41/2}(\Gamma_0(4))$. The Hecke polynomial of $\chi_{41/2}$ at two is given by

$$
\begin{aligned}
H_2(T, \chi_{41/2}) = 1 &- 105600\, T - 723412582400\, T^2 \\
&- 58054213946572800\, T^3 + 2^{78}\, T^4,
\end{aligned}
$$

which is irreducible over $\mathbb{Q}$. We also put

$$
\begin{aligned}
&\chi_{47/2} = \\
&(-946246\, E_4(4\tau)^2\, \chi_{12}(4\tau) + 2194570\, E_4(4\tau)\, E_6(4\tau)\, \chi_{10}(4\tau) \\
&- 1553434778880\, \chi_{10}(4\tau)^2) \times P_{7/2}(2\tau) + (-1747462\, E_4(4\tau)^2\, \chi_{10}(4\tau) \\
&- 580106\, E_6(4\tau)\, \chi_{12}(4\tau))\, P_{11/2}(2\tau) + (-27675\, E_4(4\tau)^2\, E_6(4\tau) \\
&+ 323725375872\, E_4(4\tau)\, \chi_{10}(4\tau))\, P_{19/2}(2\tau) + (38788\, E_4(4\tau)^3 - 8377\, E_6(4\tau)^2 \\
&+ 26731596672\, \chi_{12}(4\tau))\, P_{23/2}(2\tau).
\end{aligned}
$$

Then $\chi_{47/2}$ is a Hecke eigen form in $S^+_{47/2}(\Gamma_0(4))$. The Hecke polynomial of $\chi_{47/2}$ at 2 is given by

$$
\begin{aligned}
H_2(T, \chi_{47/2}) = 1 &- 3048960\, T - 21597142384640\, T^2 \\
&-107275743123965214720\, T^3 + 2^{90}\, T^4,
\end{aligned}
$$

which is irreducible over $\mathbb{Q}$.

4. Appendix

In this appendix, we give data of Fourier coefficients of forms in $S^+_{k-1/2}(\Gamma_0(4))$ used in the numerical examples of liftings in the previous section. In the table below, (a, c, b) means the Fourier coefficient at $\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$.

$wt\, 19/2\,,\ 23/2$	$P_{19/2}(2\tau)$	$P_{23/2}(2\tau)$
$(3,3,2)$	1	1
$(8,4,0)$	144	−1008
$(11,3,2)$	8	1328
$(11,8,8)$	−861	64827
$(12,12,8)$	−79232	1752832
$(19,4,4)$	−3423	188649
$(24,3,0)$	−5022	−115182
$(27,27,18)$	23088645	−2926756395

$wt\, 21/2$	$\chi_{10}(4\tau)\, P_{1/2}(2\tau)$
$(4,4,4)$	1
$(12,4,0)$	−200
$(12,12,12)$	13959
$(16,16,16)$	21376
$(28,4,4)$	1386
$(36,36,36)$	14006520

$wt\,25/2, 29/2$	$\chi_{12}(4\tau)\,P_{1/2}(2\tau)$	$P_{25/2}(2\tau)$	$E_4(4\tau)\,\chi_{10}(4\tau)\,P_{1/2}(2\tau)$	$P_{29/2}(2\tau)$
$(4,1,0)$	0	1	0	0
$(4,4,0)$	10	28	−2	0
$(4,4,4)$	1	2	1	1
$(5,4,4)$	2	−20	2	0
$(8,8,0)$	17072	−508128	−18064	−43200
$(12,4,0)$	472	519056	3160	4096
$(12,12,12)$	63	28377342	1372599	2025783
$(16,4,0)$	−1408	3839744	8192	0
$(16,16,0)$	6818304	1830706176	−201236736	2073600
$(16,16,16)$	3620608	−2025854464	66649216	66514432
$(20,8,8)$	−838986	−238148508	1142610	−1689600
$(28,4,4)$	−154566	−34637580	−521334	−559350
$(36,4,0)$	−595758	−138350004	−886650	−2457600
$(36,36,0)$	156778877538	84717349291692	−2797068474522	2440370073600
$(36,36,36)$	35411540472	15764814302832	2136037038840	2091081836664

$wt\,27/2$	$\chi_{10}(4\tau)P_{7/2}(2\tau)$	$E_4(4\tau)P_{19/2}(2\tau)$
$(3,3,2)$	0	1
$(4,4,0)$	−2	−20
$(4,4,4)$	1	14
$(8,4,0)$	−216	−4656
$(8,8,0)$	2048	−161920
$(11,3,2)$	0	−232
$(11,8,8)$	−7872	−993261
$(12,12,8)$	5184	7658368
$(12,12,12)$	462195	−1345014
$(16,4,0)$	−10144	71360
$(16,16,0)$	−16337152	−167288320
$(19,4,4)$	23424	525297
$(24,3,0)$	0	759618
$(27,27,18)$	4528944576	281757016485
$(28,4,4)$	−45822	5392092
$(36,36,36)$	164646611496	4840269943536

$wt\,31/2$	$\chi_{12}(4\tau)P_{7/2}(2\tau)$	$\chi_{10}(4\tau)P_{11/2}(2\tau)$	$E_6(4\tau)P_{19/2}(2\tau)$	$E_4(4\tau)P_{23/2}(2\tau)$
$(3,3,2)$	0	0	1	1
$(4,3,0)$	0	0	−6	18
$(4,4,0)$	10	−2	−20	164
$(4,4,4)$	1	1	14	18
$(7,7,2)$	16688	−4048	220288	−5184
$(8,4,0)$	1128	696	10224	38352
$(8,8,0)$	68000	−116128	−2232256	5083072
$(11,3,2)$	0	0	−11584	−21952
$(11,8,8)$	129408	198528	−4073685	−3187413
$(12,4,0)$	−8336	2896	259616	116064
$(12,12,8)$	−11090304	−5284224	24024064	−162364928
$(12,12,12)$	7042059	3786507	181538010	358763526
$(16,3,0)$	0	0	−485376	−223488
$(16,4,0)$	10784	−86944	292544	−187328
$(16,7,4)$	2973936	1380720	101693652	−92752956
$(16,12,0)$	−214834944	−145991424	−4721147904	−4407211008
$(16,16,0)$	1684279808	−1956802048	−62392864768	1386465280
$(16,16,16)$	141109504	337852672	1405675520	−10388302848
$(19,4,4)$	155136	−38400	−279111	2516409
$(20,8,8)$	2779566	29706666	436331172	377782668
$(24,3,0)$	0	0	3614706	6144498
$(27,4,0)$	2505504	51744	−9970860	82405764
$(27,27,18)$	1202110035840	172113144192	6472654435701	11781463880565
$(28,4,4)$	4249362	1795602	24906300	82405764
$(36,3,0)$	0	0	108990846	82853766
$(36,4,0)$	18430266	12121230	434457996	900033156
$(36,27,0)$	7987676467968	9536425747200	−235559672754480	637455917157264
$(36,36,0)$	242566342473618	94742289886230	6077098262322396	7656574058280180
$(36,36,36)$	51224741006760	16520031298728	826849235895408	1546543276410384

$wt\, 33/2$	$E_6(4\tau)\,\chi_{10}(4\tau)\,P_{1/2}(2\tau)$	$E_4(4\tau)\,\chi_{12}(4\tau)\,P_{1/2}(2\tau)$	$\chi_{10}(4\tau)\,P_{13/2}(2\tau)$	$E_4(4\tau)\,P_{25/2}(2\tau)$
(4, 1, 0)	0	0	0	1
(4, 4, 0)	−2	10	−2	28
(4, 4, 4)	1	1	1	2
(5, 4, 0)	−4	20	0	70
(5, 4, 4)	2	2	0	−20
(8, 8, 0)	−276016	292592	−184048	8302752
(9, 9, 2)	−374384	796240	−592800	−55430560
(12, 4, 0)	16936	−4808	20008	4534736
(13, 5, 2)	12760	−54056	0	−23593328
(16, 1, 0)	0	0	0	13888
(16, 4, 0)	−184960	−62848	−192384	164198144
(16, 5, 0)	−463188	−159060	0	530612040
(16, 16, 0)	−11449555968	5034906624	−10981181440	−114429609984
(16, 16, 16)	1296672256	−178237952	1174703104	167786771456
(20, 1, 0)	0	0	0	52110
(20, 4, 0)	198884	111500	80100	2164310920
(20, 8, 8)	68436450	92066454	51856866	96214121892
(20, 16, 0)	−18886241024	−5455811840	−8077728000	4168864371200
(20, 16, 16)	1354989568	−1115420672	266576896	2413351972864
(36, 4, 0)	62755254	−1611918	58306998	232075659276
(36, 36, 0)	1619966262888774	2512774604319138	1902755390009670	−4910744231383634388

$wt\, 35/2$	$E_4(4\tau)\chi_{10}(4\tau)P_{7/2}(2\tau)$	$\chi_{12}(4\tau)P_{11/2}(2\tau)$	$E_4(4\tau)^2 P_{19/2}(2\tau)$	$E_6(4\tau)P_{23/2}(2\tau)$
(3, 3, 2)	0	0	1	1
(4, 3, 0)	0	0	−6	18
(4, 4, 0)	−2	10	−20	164
(4, 4, 4)	1	1	14	18
(7, 7, 2)	−1456	43472	192496	319536
(8, 4, 0)	−696	−3432	−9456	−83664
(8, 8, 0)	−135232	820160	−2625280	30012928
(11, 3, 2)	0	0	57128	38120
(11, 8, 8)	1027008	145728	3140739	−12035709
(12, 4, 0)	−53408	−30944	−1169728	−2215872
(12, 12, 8)	44800704	−34981056	6847216768	3814468480
(12, 12, 12)	54130275	73539171	2708246826	2513884086
(16, 3, 0)	0	0	−986352	−820080
(16, 4, 0)	170336	281888	−12301120	−5300672
(16, 7, 4)	−2835504	−6920496	−276347628	577071108
(16, 12, 0)	2103486336	2581070976	−23181552384	2609240832
(16, 16, 0)	−21290651392	19967907584	−1185467722240	49881418240
(16, 16, 16)	3339448960	396575104	66876720896	232574683392
(19, 4, 4)	−809856	−2116224	−38862783	−87666111
(24, 3, 0)	0	0	12929058	25930530
(27, 4, 0)	−27288864	−20860704	−1142226828	−749346012
(27, 27, 18)	−32384503422144	−27722365577280	−114447493042875	−355161384812475
(28, 4, 4)	15561378	21989538	−285221988	−749346012
(28, 8, 8)	34009722	23805342	−2892760380	−4687859988
(36, 3, 0)	0	0	373316526	1464979446
(36, 4, 0)	26318718	226282122	−6270772500	−1534580316
(36, 27, 0)	−1039178537612928	−1327808930988672	−41328190976162544	27915068810095056
(36, 36, 0)	−1637127809425674	24532811136719154	20113692992657820	434756523224983860
(36, 36, 36)	2036597796248616	2709582596303400	9374314429341936	−46503207364243824

$wt\, 39/2$	$E_4(4\tau)\chi_{12}(4\tau)P_{7/2}(2\tau)$	$E_6(4\tau)\chi_{10}(4\tau)P_{7/2}(2\tau)$	$E_4(4\tau)\chi_{10}(4\tau)P_{11/2}(2\tau)$
(3, 3, 2)	0	0	0
(4, 3, 0)	0	0	0
(4, 4, 4)	1	1	1
(4, 4, 0)	10	−2	−2
(7, 4, 0)	560	−112	176
(7, 7, 2)	16688	−1456	−4048
(8, 4, 0)	3528	792	216
(8, 4, 4)	278	−394	−106
(8, 8, 0)	938720	−24160	184352
(11, 3, 2)	0	0	0
(11, 11, 6)	247495808	192665984	87688832
(12, 4, 0)	283984	144880	165616
(12, 12, 8)	735837696	824117760	1057556736
(16, 3, 0)	0	0	0
(16, 12, 0)	6965769216	5218139136	9307900416
(16, 4, 0)	513824	2441312	2098016
(16, 7, 0)	−55173216	25406304	1145760
(16, 16, 0)	853418006528	271104100352	417542423552
(16, 16, 16)	116461149184	77180695552	69248553472
(28, 4, 0)	18287584	193226656	293768608
(28, 7, 4)	−8121890560	1192126208	−1149009664
(28, 16, 0)	102085925255168	27995367827456	13757140745216
(28, 28, 8)	−7189200652320768	−5864058898563072	−5809348678508544

$wt\ 39/2$	$E_4(4\tau)E_6(4\tau)P_{19/2}(2\tau)$	$\chi_{10}(4\tau)P_{19/2}(2\tau)$	$E_4(4\tau)^2 P_{23/2}(2\tau)$
(3, 3, 2)	1	0	1
(4, 3, 0)	−6	0	18
(4, 4, 0)	−20	0	164
(4, 4, 4)	14	0	18
(7, 4, 0)	1820	0	7660
(7, 7, 2)	17248	−1	102816
(8, 4, 0)	5424	0	77712
(8, 4, 4)	−3820	0	7660
(8, 8, 0)	2446784	68	28935232
(11, 3, 2)	−132784	0	12368
(11, 11, 6)	4730629568	−2216	1443375040
(12, 4, 0)	2670176	0	9674784
(12, 12, 8)	47171325184	−5656	49424250112
(16, 3, 0)	19134624	0	15099552
(16, 4, 0)	84549824	0	111610432
(16, 7, 0)	−2710680000	6482	5438462400
(16, 12, 0)	−140332967424	−19120	1175375596032
(16, 16, 0)	−3997818293248	−2364512	7610050155520
(16, 16, 16)	5058824092160	651984	6534551112192
(28, 4, 0)	135962176	0	2851862720
(28, 7, 4)	317666936320	114320	−1124037987840
(28, 16, 0)	3457494645251072	−1003899936	3570785990425600
(28, 28, 8)	− 754120391288610816	−54289682176	−670354007327760384

Correction

[9] p. 114 Theorem 2 line 5, $p^{-2kn-j/2}$ in LHS should read $p^{-(2k+1)n-j/2}$.
[9] p. 123 line 1, $(\mathbb{Z}/p\mathbb{Z})^n$ should read $(\mathbb{Z}/p^2\mathbb{Z})^n$.
[9] p. 123 line 10, the LHS should be multiplied by p^{-n}.
[3] p. 208 Theorem 2 line 3, $p^{-2kn-s/2}$ in LHS should read $p^{-(2k+1)n-s/2}$.
[3] p. 216 Lemma 7 line 4, p^{2kn} in RHS should read $p^{(2k+1)n}$.

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