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On the pathwise uniqueness of solutions of stochastic differential equations driven by multi-dimensional symmetric α stable class

By

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Abstract

We will propose a sufficient condition which guarantees the pathwise uniqueness for jump type equations in multi-dimensional case. An example given in Section 3 shows that the condition is nearly best possible. Comparing our results with those known in the case of Brownian equations, we claim that essential difference between these two cases. It seems to be remarkable that we could explain these phenomena in the language of the Potential theory. Our principal method in the paper is based on the Fourier analysis, where effective tools such as Bessel functions, hypergeometric functions play essential roles.

1. Introduction

Stochastic differential equations driven by symmetric α stable processes are becoming gradually important. The principal aim of the present paper is to present a sufficient condition of the pathwise uniqueness. Related topics such as, time-change problems, some properties of exponential martingales are investigated.

We shall shortly refer to some results obtained in the case of d-dimensional Brownian motions. Consider the following stochastic differential equation (SDE), for each $1 \le i, j \le d$, $t \ge 0$:

$$dX^i(t) = \sigma^i_j(\boldsymbol{X}(t))dW^j(t), \qquad X^i(0) = x^i(0) \tag{1.1}$$

where $\boldsymbol{W} = \{(W^1(t), W^2(t), \ldots, W^d(t)); t \ge 0\}$ is a d-dimensional Brownian motion. In this case, the best possible sufficient condition is stated differently in each of following three cases: $d = 1$, $d = 2$, and $d \ge 3$. See [9] and also [8]. In one dimensional case, it is well known that the pathwise uniqueness holds when $\boldsymbol{\sigma}$ is *Hölder* continuous of order $\frac{1}{2}$. On the other hand, in multi-dimensional

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case, [8] also discusses that the best possible condition in the case of $d = 2$ differs from that in the case of $d \geq 3$.

Now consider the following SDE, for each $1 \leq i \leq d$, $t \geq 0$:

$$(1.2) \qquad dY^i(t) = \sigma^i_j(\boldsymbol{Y}(t-))dZ^j(t), \qquad Y^i(0) = y^i(0)$$

where $\boldsymbol{Z} = \{(Z^1(t), Z^2(t), \ldots, Z^d(t)); t \geq 0\}$ is a d-dimensional symmetric α stable process.

In one dimensional case, a corresponding result for a symmetric α stable process, was given by Komatsu [4] (See also [3]). According to his result, if coefficients are *Hölder* continuous of order $\frac{1}{\alpha}$, then the pathwise uniqueness holds.

In multi-dimensional case, we will state a sufficient condition which guarantees the pathwise uniqueness for the equation (1.2) under the assumption:

$$(1.3) \qquad \sigma^i_j(\boldsymbol{x}) = \delta_{ij}\sigma(\boldsymbol{x}).$$

Under the above assumption, we can use freely, as you will see later, powerful tools such as Bessel functions and hypergeometric functions in the calculus of Fourier transforms to obtain several concrete estimates. This is the reason why we should limit our consideration under the assumption (1.3) in the present stage.

In our main theorem, the common best possible condition is obtained for the case $d \geq 2$. This result contrasts considerably with conditions in the case of Brownian motion which we have mentioned in the above.

These phenomena could be understood, as you see in details in Section 3, by the difference between Newton potential coresponding to Brownian motions and Riesz potential to symmetric α stable processes. Indeed, in two dimensional case the Newton potential is logarithmic which differs essentially from that in the case $d \geq 3$ where the essential part of the potential is $|\boldsymbol{x}|^{2-d}$. However the essential part of Riesz potential is $|\boldsymbol{x}|^{\alpha-d}$ for $d \geq 2$.

The paper is organized as follows. In Section 2 we introduce notions and definitions, then present the main result. An example in Section 3 shows that the condition is best possible in some sense. At last Section, the key lemma will be proved in the use of Bessel functions and also of hypergeometric functions.

2. Pathwise uniqueness in d-dimensional case

2.1. Main theorem

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ be a filtered probability space and $\boldsymbol{Z} = \{\boldsymbol{Z}(t); t \geq 0\}$ be a d-dimensional $\mathcal{F}_t$-symmetric α stable process, for each $t \geq 0$:

$$\boldsymbol{E}[\exp\{i\langle\boldsymbol{\xi}, \boldsymbol{Z}(t) - \boldsymbol{Z}(s)\rangle\}|\mathcal{F}_s] = \exp\{-(t-s)|\boldsymbol{\xi}|^\alpha\} \quad \text{for every} \quad \boldsymbol{\xi} \in \mathbf{R}^d.$$

where Lévy measure ν is given by

$$(2.1) \qquad \nu(d\boldsymbol{y}) = \frac{c_\nu}{|\boldsymbol{y}|^{\alpha+d}}d\boldsymbol{y}.$$

In the present paper, let $\alpha \in (1,2)$ and ρ be a increasing continuous function defined on $[0,a)$ $(a > 0)$ with $\rho(0) = 0$ and $\boldsymbol{\sigma}$ is bounded continuous functions.

Consider the stochastic differential equation driven by $\boldsymbol{Z}(t)$,

$$d\boldsymbol{Y}(t) = \boldsymbol{\sigma}(\boldsymbol{Y}(t-))d\boldsymbol{Z}(t) \qquad i = 1,2,\dots d.$$

It means that,

$$Y^i(t) - Y^i(0) = \sum_{j=1}^{d} \int_0^t \sigma^i_j(\boldsymbol{Y}(s-))dZ^j(s) \qquad i = 1,2,\dots d. \tag{2.2}$$

Assumption 2.1. *Assume that the coefficient matrix $\boldsymbol{\sigma} = [\sigma^i_j]$ satisfies*

$$\sigma^i_j(\boldsymbol{x}) = \delta_{ij}\sigma(\boldsymbol{x})$$

Under the assumption, we consider the following equation,

$$Y^i(t) - Y^i(0) = \int_0^t \sigma(\boldsymbol{Y}(s-))dZ^i(s) \qquad i = 1,2,\dots d. \tag{2.3}$$

The main theorem is here:

Theorem 2.2. *Let $\rho(u)$ satisfy*

$$\int_0^\epsilon \frac{1}{G(u)}du = \infty, \quad for\ any\ \epsilon > 0 \tag{2.4}$$

$$where \quad G(u) := \rho^\alpha(u^{\frac{1}{\alpha-1}})u^{\frac{-1}{\alpha-1}} \quad is\ concave. \tag{2.5}$$

Then, for every σ such that

$$|\sigma(\boldsymbol{x}) - \sigma(\boldsymbol{y})| \le \rho(|\boldsymbol{x}-\boldsymbol{y}|)\ for\ \boldsymbol{x},\boldsymbol{y} \in \mathbf{R}^d\ with\ |\boldsymbol{x}-\boldsymbol{y}| < a,$$

the pathwise uniqueness holds for the equation (2.3).

2.2. Remark

For examples,

$$\rho(\xi) = \xi, \rho(\xi) = \xi\left(\log\frac{1}{\xi}\right)^{1/\alpha}, \rho(\xi) = \xi\left(\log\frac{1}{\xi}\right)^{1/\alpha}\log\left((\alpha-1)\log\frac{1}{\xi}\right)^{1/\alpha}, \dots$$

satisfy (2.4) and (2.5).

Comparing our results with those known in Brownian motion case, the examples of modulus of continuity function ρ are shown in the following table.

<table>
<tr><th>Dimension</th><th>Brownian Motion</th><th>Symmetric α stable</th></tr>
<tr><td>$d=1$</td><td>$\frac{1}{2}$-Hölder [8]</td><td>$\frac{1}{\alpha}$-Hölder [4](See also [3])</td></tr>
<tr><td>$d=2$</td><td>[8]
$\xi,\ \xi(\log\frac{1}{\xi}),\ \xi(\log\frac{1}{\xi})(\log(\log\frac{1}{\xi}))^{\frac{1}{2}}.$</td><td rowspan="2">[T]
ξ
$\xi(\log\frac{1}{\xi})^{1/\alpha},$
$\xi(\log\frac{1}{\xi})^{1/\alpha}(\log((\alpha-1)\log\frac{1}{\xi}))^{1/\alpha}.$</td></tr>
<tr><td>$d\geq 3$</td><td>[8]
$\xi,\ \xi(\log\frac{1}{\xi})^{1/2},\ \xi(\log\frac{1}{\xi})^{1/2}(\log(\log\frac{1}{\xi}))^{1/2}.$</td></tr>
</table>

The situation in two dimensional case is crucial.

2.3. The proof of the theorem

A heuristic derivation is to use

$$\mathcal{L}|\boldsymbol{x}|^{\alpha-1} = const.\frac{1}{|\boldsymbol{x}|}$$

where $\mathcal{L}$ is the generator of $\boldsymbol{Z}$, we apply Ito's formula. The key in the proof is the next lemma, which will be proved in Section 4. Let $\boldsymbol{F}$ [resp. $\boldsymbol{F}^{-1}$] stands for the Fourier transform [resp. the inverse Fourier transform] and $\{\phi_n\}$ be a series of mollifier functions.

Key Lemma. *Let*

$$u(\boldsymbol{x}) = |\boldsymbol{x}|^{\alpha-1},\ u^{\epsilon}(\boldsymbol{x}) = |\boldsymbol{x}|^{\alpha-1}e^{-\epsilon|\boldsymbol{x}|} \text{ and } u_n = u^{\epsilon} * \phi_n \text{ where } \left(\epsilon = \frac{1}{n}\right).$$

Then, we can choose a series of mollifier functions ϕ_n such that

$$|\boldsymbol{F}^{-1}[|\boldsymbol{\xi}|^{\alpha}(\boldsymbol{F}u_n)[\boldsymbol{\xi}]][\boldsymbol{x}]| \leq C(\alpha,d)|\boldsymbol{x}|^{-1} \quad \boldsymbol{x}\in\mathbf{R}^d\setminus\{0\} \text{ holds,}$$

where $C(\alpha,d)$ is a constant which depends on (α,d) but is independent of n.

Using the lemma, we will prove the main theorem.

The proof of the main theorem. Let $(\boldsymbol{Y}^1(t),\boldsymbol{Z}(t))$ and $(\boldsymbol{Y}^2(t),\boldsymbol{Z}(t))$ be any two solution to (2.3) defined on the same probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\},P)$. Define

$$\Delta(t) := \boldsymbol{Y}^1(t) - \boldsymbol{Y}^2(t).$$

Using Ito formula [See Proposition 4.1], we have

$$u_n(\Delta(t)) - u_n(\Delta(0)) = c_{\nu}\int_0^t |\sigma(\boldsymbol{Y}^1(s-)) - \sigma(\boldsymbol{Y}^2(s-))|^{\alpha}\mathcal{L}u_n(\Delta(s-))ds \\ + M_n(t)$$

where c_ν is the constant given in (2.1), $\mathcal{L}$ is the generator of $\boldsymbol{Z}$, and M_n is a martingale. Then, by the key Lemma and also the condition (2.5), we obtain

$$
\begin{aligned}
&|\boldsymbol{E}u_n(\Delta(t)) - u_n(\Delta(0))| \\
&\leq c_\nu \boldsymbol{E}\int_0^t |\sigma(\boldsymbol{Y}^1(s-)) - \sigma(\boldsymbol{Y}^2(s-))|^\alpha |\mathcal{L}u_n(\Delta(s-))| ds \\
&\leq c_\nu C(\alpha,d)\boldsymbol{E}\left[\int_0^t \rho(|\Delta(s-)|)^\alpha \frac{1}{|\Delta(s-)|} ds\right] \\
&= c_\nu C(\alpha,d)\boldsymbol{E}\left[\int_0^t \frac{\rho(|\Delta(s)|)^\alpha}{|\Delta(s)|} ds\right] \\
&\leq c_\nu C(\alpha,d)\int_0^t G\boldsymbol{E}|\Delta(s)|^{\alpha-1} ds.
\end{aligned}
$$

As n goes to infinity, we have

$$\boldsymbol{E}|\Delta(t)|^{\alpha-1} \leq c_\nu C(\alpha,d)\int_0^t G\boldsymbol{E}|\Delta(s)|^{\alpha-1} ds.$$

By the condition (2.4), we can conclude

$$\boldsymbol{E}|\Delta(t)|^{\alpha-1} = 0$$

□

3. The result is nearly best possible

Here we will construct an example which shows the condition 2.4 is best possible in some sense.

The next proposition is a modification of Propositon 3.1 in [5].

Proposition 3.1. *Let H be an $\mathcal{F}_t$-adapted such that*

$$\boldsymbol{E}\exp\left\{\theta\int_0^t |H(s)|^\alpha ds\right\} < \infty \tag{3.1}$$

for every θ, $t > 0$. Then for every $\boldsymbol{\lambda} \in \mathbf{R}^d$

$$\varepsilon_H(t) := \exp\left\{i\langle\boldsymbol{\lambda}, \int_0^t H d\boldsymbol{Z}\rangle + |\boldsymbol{\lambda}|^\alpha \int_0^t |H|^\alpha ds\right\}$$

is a complex-valued martingale.

Proof. To begin with considering the case that H is simple:

$$H(t) = \sum_{k=0}^n H(t_k)1_{(t_k,t_{k+1}]}, \quad 0 = t_0 \leq t_1 \leq \cdots \leq t_n = t.$$

Then, we have

$$\varepsilon_H(t) = \exp\left\{ i\sum_{i=1}^{d}\lambda^i\sum_{k=0}^{n} H(t_k)(Z^i(t_{k+1}) - Z^i(t_k)) \right.$$
$$\left. + |\boldsymbol{\lambda}|^\alpha \left\{ \sum_{k=0}^{n} |H(t_k)|^\alpha (t_{k+1} - t_k) \right\} \right\}.$$

We define, for u, $v \in [0, \infty)$ with $u \le v$,

$$\varepsilon_H(u, v) = \exp\left\{ i\sum_{i=1}^{d}\lambda^i H(u)(Z^i(v) - Z^i(u)) + |\boldsymbol{\lambda}|^\alpha |H(u)|^\alpha (v - u) \right\},$$

then we obtain

$$\varepsilon_H(t) = \prod\nolimits_{k=0}^{n} \varepsilon_H(t_{k-1}, t_k),$$

and also we have

$$\boldsymbol{E}[\varepsilon_H(u, v)|\mathcal{F}_s] = \varepsilon_H(u, v)1_{\{v \le s\}} + \varepsilon_H(u, s)1_{\{u \le s \le v\}} + 1_{\{s \le u\}} \quad a.s.\ P.$$

Therefore, for $s \le t$, say $s \in [t_m, t_{m+1})(m \le n)$, we can conclude that

$$\boldsymbol{E}[\varepsilon_H(t)|\mathcal{F}_s] = \varepsilon_H(s) \quad a.s.\ P.$$

The case of general H follows by Lebesgue convergence theorem. □

The next proposition is also a modification of Theorem 3.1 in [5].

Proposition 3.2. *Let H be an $\mathcal{F}_t$-adapted satisfying*

$$\boldsymbol{E}\left[\int_0^t |H(s)|^\alpha ds\right] < \infty,$$

such that random time $\tau(u) := \int_0^t |H(s)|^\alpha ds$ satisfies $\tau(u) \to \infty$ as $u \to \infty$. Consider the inverse of τ and $\mathcal{G}_t = \mathcal{F}_{\tau^{-1}(t)}$, and the time changed stochastic integrals for $1 \le i \le d$,

$$\tilde{Z}^i(t) := \int_0^{\tau^{-1}(t)} H(s)dZ^i(s).$$

Then $\tilde{\boldsymbol{Z}} = \{(\tilde{Z}^1(t), \tilde{Z}^2(t), \ldots, \tilde{Z}^d(t)); t \ge 0\}$ is a $\mathcal{G}_t$-symmetric α stable process.

Proof. The mapping $t \mapsto \tau^{-1}(t)$ is right continuous and nondecreasing $\mathcal{F}_t$-stopping time. We first assume additionally that H satisfies condition (3.1). It follows from the above proposition 3.1 that the complex exponential martingale $\varepsilon_H(t)$ satisfies the equality

$$\boldsymbol{E}[\varepsilon_H(t)|\mathcal{F}_s] = \varepsilon_H(s) \quad a.s.\ P.$$

Applying Optional sampling theorem, we obtain

$$\boldsymbol{E}[\varepsilon_H(\tau^{-1}(t))|\mathcal{G}_s] = \varepsilon_H(\tau^{-1}(s)) \quad a.s.\ P.$$

Then

$$\boldsymbol{E}[\exp\{i\langle\boldsymbol{\lambda}, \tilde{\boldsymbol{Z}}(t) - \tilde{\boldsymbol{Z}}(s)\rangle\}] = \exp\{-(t-s)|\boldsymbol{\lambda}|^\alpha\} \quad a.s.\ P.$$

Now, the case of general H follows by the truncation argument, considering, instead of H satisfying the assumptions of the theorem, its restriction $H_n(t) := 1_{\{\tau_n \geq t\}} H(t)$ the same as H up to time τ_n, where

$$\tau_n := \inf\left\{t \in [0,\infty); \int_0^t |H(s)|^\alpha ds \leq n\right\} \wedge n.$$

□

Here we refer briefly to some notions in the Potential theory.

Definition 3.1. Let $\{\boldsymbol{Y}(t)\}$ be a Lévy process on $\mathbf{R}^d$ with the transition function $P(t, \boldsymbol{x}, B)$. For $\theta \geq 0$, $\boldsymbol{x} \in \mathbf{R}^d$, and $B \in \mathcal{B}(\mathbf{R}^d)$, we define

$$U^\theta(\boldsymbol{x}, B) := \int_0^\infty e^{-\theta t} P(t, \boldsymbol{x}, B) dt \tag{3.2}$$

When $\boldsymbol{x} = 0$, we write $U^\theta(0, B) = U^\theta(B)$ and it is said the θ-potential measure. If $U^\theta(B)$ is absolute continuous, then we call the density $u^\theta(\boldsymbol{x})$ the θ-potential density. As $U^0(B)$ denoted by $U(B)$ and called the potential measure, we write $u^\theta(\boldsymbol{x}) = u(\boldsymbol{x})$ and call it the potential density.

Proposition 3.3. *In the case of a symmetric α stable process $\boldsymbol{Z} = \{\boldsymbol{Z}(t)\}_{t\geq 0}$, for $d \geq 2$ and $1 < \alpha \leq 2$,*

$$u(\boldsymbol{x}) = K(d, \alpha)|\boldsymbol{x}|^{\alpha - d}, \tag{3.3}$$

where

$$K(d, \alpha) = \frac{\Gamma(d-\alpha)}{2^\alpha \pi^{\frac{d}{2}} \Gamma(\frac{\alpha}{2})}.$$

See [6, p.261].

Example. *The condition* (2.4) *in the main theorem* 2.1 *is best possible in some sense that, for $d \geq 2$, if ρ defined on $[0,\infty)$, bounded and subadditive such that $\int_{0+} \frac{1}{G(u)} du < \infty$. Then there exists $\boldsymbol{\sigma}$ for which the pathwise uniqueness for the equation* (2.2) *does not hold.*

The proof of the example. Let $\sigma^i_j(\boldsymbol{x}) = \delta_{ij}\rho(|\boldsymbol{x}|)$ $(\boldsymbol{x} \in \mathbf{R}^d)$, Then,

$$|\sigma(\boldsymbol{x}) - \sigma(\boldsymbol{y})| \leq |\rho(|\boldsymbol{x}|) - \rho(|\boldsymbol{y}|)| \leq \rho(|\boldsymbol{x} - \boldsymbol{y}|).$$

Consider the equation

$$\begin{cases} d\boldsymbol{y}(t) = \boldsymbol{\sigma}(\boldsymbol{y}(t-))d\boldsymbol{Z}(t) \\ \boldsymbol{y}(0) = 0 \end{cases} \tag{3.4}$$

Let $\boldsymbol{Z} = \{\boldsymbol{Z}(t); t \geq 0\}$ be a d-dimensional $\mathcal{F}_t$-symmetric α stable process on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ such that $\boldsymbol{Z}(0) = 0$. Let $H(t) := \rho(|\boldsymbol{Z}(t-)|)^{-1}$ and $\tau(t) := \displaystyle\int_0^t \frac{ds}{\rho(|\boldsymbol{Z}(s-)|)^{\alpha}}$. We show that $\tau(t)$ is integrable for $t \geq 0$. Note that

$$\begin{aligned} \boldsymbol{E}[\tau(t)] &= \boldsymbol{E}\left[\int_0^t \frac{ds}{\rho(|\boldsymbol{Z}(s-)|)^{\alpha}}\right] = \int_{\mathbf{R}^d}\int_0^t \frac{p(s,\boldsymbol{x})}{\rho(|\boldsymbol{x}|)^{\alpha}} ds d\boldsymbol{x} \\ &\leq \int_{|\boldsymbol{x}|\leq\delta} d\boldsymbol{x} \int_0^t \frac{p(s,\boldsymbol{x})}{\rho(|\boldsymbol{x}|)^{\alpha}} ds + \int_{|\boldsymbol{x}|\geq\delta} d\boldsymbol{x} \int_0^t \frac{p(s,\boldsymbol{x})}{\rho(|\boldsymbol{x}|)^{\alpha}} ds = I + II \end{aligned}$$

where $p(s, \boldsymbol{x})$ is the probability density of the symmetric α stable process. First, applying the property of (3.3), we have

$$\begin{aligned} I &\leq \int_{|\boldsymbol{x}|\leq\delta} d\boldsymbol{x} \int_0^{\infty} \frac{p(s,\boldsymbol{x})}{\rho(|\boldsymbol{x}|)^{\alpha}} ds \leq K(d,\alpha) \int_{|\boldsymbol{x}|\leq\delta} \frac{|\boldsymbol{x}|^{\alpha-d}}{\rho(|\boldsymbol{x}|)^{\alpha}} d\boldsymbol{x} \\ &\leq K(d,\alpha)\omega_{d-1} \int_0^{\delta} \frac{\xi^{\alpha-1}}{\rho(\xi)^{\alpha}} d\xi = \frac{\omega_{d-1}K(d,\alpha)}{\alpha-1} \int_0^{\delta^{\alpha-1}} \frac{du}{G(u)} < \infty. \end{aligned}$$

where ω_{d-1} is surface area of S^{d-1}. Second,

$$II \leq \int_{|\boldsymbol{x}|\geq\delta} d\boldsymbol{x} \int_0^t \frac{p(s,\boldsymbol{x})}{\rho(\delta)^{\alpha}} ds \leq \rho(\delta)^{-\alpha} \int_0^t \int_{\mathbf{R}^d} p(s,\boldsymbol{x}) d\boldsymbol{x} ds = \rho(\delta)^{-\alpha} t.$$

Thus we see $\boldsymbol{E}[\tau(t)] < \infty$. Since ρ is bounded function, τ satisfies $\tau(u) \to \infty$, a.s. $u \to \infty$.

Define $\bar{\boldsymbol{Z}} = \{(\bar{Z}^1(t), \bar{Z}^2(t), \ldots, \bar{Z}^d(t)); t \geq 0\}$ such that

$$\bar{Z}^i(t) := \int_0^t \rho(|\boldsymbol{Z}(s-)|)^{-1} dZ^i(s) \qquad i = 1, 2, \ldots d.$$

Then we have

$$Z^i(t) = \int_0^t \rho(|\boldsymbol{Z}(s-)|) d\bar{Z}^i(s) \qquad i = 1, 2, \ldots d. \tag{3.5}$$

On the other hand, in the use of Proposition 3.2, $\tilde{\boldsymbol{Z}} = \{\tilde{\boldsymbol{Z}}(t); t \geq 0\}$ is a $\mathcal{G}_t$-symmetric α stable process, where

$$\tilde{Z}^i(t) := \bar{Z}^i(\tau^{-1}(t)) = \int_0^{\tau^{-1}(t)} \rho(|\boldsymbol{Z}(s-)|)^{-1} dZ^i(s) \qquad i = 1, 2, \ldots d.$$

Here, for $1 \le i \le d$, we define $Y^i(t) := Z^i(\tau^{-1}(t))$. Then by (3.5) , $\boldsymbol{Y} = \{(Y^1(t), Y^2(t), \dots, Y^d(t)); t \ge 0\}$ satisfies

$$Y^i(t) = \int_0^t \rho(|\boldsymbol{Y}(s-)|)d\tilde{Z}^i(s) \qquad i = 1, 2, \dots d.$$

It means that $(\boldsymbol{y}(t) = \boldsymbol{Y}(t), \tilde{\boldsymbol{Z}}(t))$ solves the equation (3.4). However, $(\boldsymbol{y}(t) = 0, \tilde{\boldsymbol{Z}}(t))$ is also a solution to the equation (3.4) and thus, the pathewise uniqueness fails. □

Remark 1. In symmetric α stable case, the condition given in the above example shows that best possible condition for $d \ge 3$ can not be be improved for $d = 2$. But, in Brownian motions case, [8] (see the table in Section 2) points out that the best possible condition for $d \ge 3$ can be relaxed for $d = 2$. These phenomena are due to the difference between the logarithmie potential corresponding to two dimensional Brownian motion and the Riesz potential to symmetric α stable processes $d \ge 2$.

4. Key Lemma

At the begining of the Section, we will modify an Ito formula given in [B, Proposition 2.1] to the multi-dimensional case. The proof can be shown by similar method as in [B, Proposition 2.1]. Let N be the Poisson random measure associated to the symmetric α stable process $\boldsymbol{Z}$ and $\tilde{N}$ be its compensated Poisson random measure.

Proposition 4.1. *Suppose $1 \le \alpha \le 2$, $f \in S(\mathbf{R}^d)$. Consider the following SDE, for each $1 \le i \le d$ and $t \ge 0$,*

$$Y^i(t) = Y^i(0) + \int_0^t\int_{|x|<1} x^i H(s)\tilde{N}(ds, dx) + \int_0^t\int_{|x|\ge 1} x^i H(s) N(ds, dx)$$

where H is an $\mathcal{F}_t$-adapted process such that

$$E\int_0^T |H(s)|^\alpha ds < \infty.$$

Then

$$f(\boldsymbol{Y}(t)) = f(\boldsymbol{Y}(0)) + M(t) + c_\nu \int_0^t |H(s)|^\alpha \mathcal{L} f(\boldsymbol{Y}(s-))ds$$

holds where $\mathcal{L}$ is the generator of $\boldsymbol{Z}$ and $M = \{M(t)\}$ is a martingale.

In the second part of this Section, we concentrate to the proof of the key lemma. To prove the key lemma we prepare following lemmas. In the next lemma, we will discuss the convergence of a improper integral in the use of the second mean-valued theorem.

Lemma 4.1. *Let $\phi(s,\delta)$ and $\tau(s,\delta)$ be monotone continuous functions for fixed δ and bounded uniformly with respect to (s,δ). If the improper integral of $\theta(s)$ converges and $\theta(s)$ is bounded, then the integral*

$$\int_0^\infty \phi(s,\delta)\tau(s,\delta)\theta(s)ds$$

converges uniformly with respect to δ.

Proof. Let $L := \sup_{(s,\delta)}\{|\tau(s,\delta)| \vee |\phi(s,\delta)|\}$. Since the improper integral of $\theta(s)$ converges, for every $\eta > 0$, there exists $M_\eta > 0$ such that $a, b \geq M_\eta$,

$$\left|\int_a^b \theta(s)ds\right| < \frac{\eta}{2L}$$

Note that $\tau(\cdot,\delta)$ is monotone continuous for fixed δ. By the second mean-valued theorem, we can find some ξ in $[a,b]$ such that

$$\int_a^b \tau(s,\delta)\theta(s)ds = \tau(a,\delta)\int_a^\xi \theta(s)ds + \tau(b,\delta)\int_\xi^b \theta(s)ds.$$

Therefore

$$\left|\int_a^b \tau(s,\delta)\theta(s)ds\right| < \frac{\eta}{2} + \frac{\eta}{2} = \eta.$$

Thus, the improper integral of $\tau(s,\delta)\theta(s)$ converges uniformily with respect to δ. Indeed, $\tau(s,\delta)\theta(s)$ is bounded uniformly with respect to (s,δ) and $\phi(\cdot,\delta)$ is monotone continuous for fixed δ. Apply the second mean-valued theorem again, the conclusion follows immediately. □

Lemma 4.2. *For $d = 2, 3$, the integral on $[0,\infty)$ of the function $s^{1-\frac{d}{2}}J_{\frac{d-2}{2}}(s)$ is convergent in the sense of improper integral. In the case of $d \geq 4$, the function $s^{1-\frac{d}{2}}J_{\frac{d-2}{2}}(s)$ is integrable on $[0,\infty)$.*

Proof. Note that the asymptotic behaviour of Bessel function for $\nu \geq -\frac{1}{2}$:

$$J_\nu(x) = O(x^\nu) \ \ as\ x \to 0+,$$
$$J_\nu(x) = O(x^{-\frac{1}{2}}) \ \ as\ x \to \infty.$$

There exists c_1 such that

$$\left|\int_0^1 s^{1-\frac{d}{2}}J_{\frac{d-2}{2}}(s)ds\right| \leq \int_0^1 s^{1-\frac{d}{2}}|J_{\frac{d-2}{2}}(s)|ds \leq c_1\int_0^1 s^{1-\frac{d}{2}}s^{\frac{d-2}{2}}ds = c_1 < \infty.$$

For $d \geq 4$, we can find c_2 such that

$$\left|\int_1^\infty s^{1-\frac{d}{2}}J_{\frac{d-2}{2}}(s)ds\right| \leq \int_1^\infty |s^{1-\frac{d}{2}}J_{\frac{d-2}{2}}(s)|ds < c_2\int_1^\infty s^{1-\frac{d}{2}}s^{-\frac{1}{2}}ds < \infty.$$

In the case of $d = 2, 3$,

$$\int_1^N s^{1-\frac{d}{2}} J_{\frac{d-2}{2}}(s)ds = \int_1^N s^{1-d}(s^{\frac{d}{2}} J_{\frac{d}{2}}(s))' ds$$

$$= [s^{1-\frac{d}{2}} J_{\frac{d}{2}}(s)]_1^N + (d-1)\int_1^N s^{-\frac{d}{2}} J_{\frac{d}{2}}(s)ds = a_N + (d-1)b_N, \ \textit{say}.$$

For the sequense $\{a_N\}_{N=1}^{\infty}$, we can choose c_3 as follows,

$$a_N = N^{1-\frac{d}{2}} J_{\frac{d}{2}}(N) - J_{\frac{d}{2}}(1), \qquad N^{1-\frac{d}{2}} |J_{\frac{d}{2}}(N)| \le c_3 N^{\frac{1-d}{2}}.$$

Then we have

$$\lim_{N\to\infty} a_N = -J_{\frac{d}{2}}(1).$$

On the other hand, we can find c_4 such that

$$|b_N| \le \int_1^N s^{-\frac{d}{2}} |J_{\frac{d}{2}}(s)|ds \le \int_1^N c_4 s^{-\frac{d+1}{2}} ds < c_4 \int_1^{\infty} s^{-\frac{d+1}{2}} ds < \infty$$

Then we see

$$|\lim_{N\to\infty} b_N| < \infty.$$

Finally, we can conclude

$$\left| \lim_{N\to\infty} \int_1^N s^{1-\frac{d}{2}} J_{\frac{d-2}{2}}(s)ds \right| < \infty.$$

$\square$

Lemma 4.3. *For a.e. $\boldsymbol{x} \in \mathbf{R}^d$,*

$$\begin{aligned}
&\boldsymbol{F}^{-1}|\boldsymbol{\xi}|^{\alpha}\boldsymbol{F}u^{\epsilon}[\boldsymbol{x}] \\
&\quad = |\boldsymbol{x}|^{-\frac{d-2}{2}} C_1(\alpha, d) \\
&\quad\quad \lim_{N\to\infty} \int_0^N \frac{\rho^{\alpha+\frac{d}{2}}}{(\epsilon^2+\rho^2)^{\frac{d+\alpha-1}{2}}} F\left(\frac{d+\alpha-1}{2}, -\frac{\alpha}{2}; \frac{d}{2}; \frac{\rho^2}{\epsilon^2+\rho^2}\right) J_{\frac{d-2}{2}}(|\boldsymbol{x}|\rho)d\rho \\
&\quad = \frac{1}{|\boldsymbol{x}|} C_1(\alpha, d) \\
&\quad\quad \lim_{N\to\infty} \int_0^N \frac{s^{\alpha+\frac{d}{2}}}{(|\boldsymbol{x}|\epsilon^2+s^2)^{\frac{d+\alpha-1}{2}}} F\left(\frac{d+\alpha-1}{2}, -\frac{\alpha}{2}; \frac{d}{2}; \frac{s^2}{|\boldsymbol{x}|\epsilon^2+s^2}\right) J_{\frac{d-2}{2}}(s)ds
\end{aligned}$$

where F is a hypergeometric function and we define

$$C_1(\alpha, d) = \frac{\left(\frac{1}{2}\right)^{\frac{d-2}{2}} \Gamma(d+\alpha-1)}{\Gamma(\frac{d}{2})}$$

Moreover this convergence is independent of ϵ and $|\boldsymbol{x}|$.

Proof. Since u^ϵ are rotation invariant, the Fourier transform of u^ϵ can be represented in the use of Bessel functions. Thus we have

$$\begin{aligned}
&|\boldsymbol{\xi}|^\alpha \boldsymbol{F} u^\epsilon[\boldsymbol{\xi}] \\
&\quad = |\boldsymbol{\xi}|^\alpha (2\pi)^{\frac{d}{2}} |\boldsymbol{\xi}|^{-\frac{d-2}{2}} \int_0^\infty t^{\frac{d}{2}+(\alpha-1)} e^{-\epsilon t} J_{\frac{d-2}{2}}(|\boldsymbol{\xi}|t)dt \\
&\quad = \frac{2\pi^{\frac{d}{2}}\Gamma(d+\alpha-1)}{\Gamma(\frac{d}{2})} \frac{|\boldsymbol{\xi}|^\alpha}{(\epsilon^2+|\boldsymbol{\xi}|^2)^{\frac{d+\alpha-1}{2}}} F\left(\frac{d+\alpha-1}{2}, -\frac{\alpha}{2}; \frac{d}{2}; \frac{|\boldsymbol{\xi}|^2}{\epsilon^2+|\boldsymbol{\xi}|^2}\right).
\end{aligned}$$

The second equality in the above is due to the formula concerning hypergeometric functions [7, p.385].

Note that the function $|\boldsymbol{\xi}|^\alpha \boldsymbol{F} u^\epsilon[\boldsymbol{\xi}]$ is again rotation invariant. Thus we have

$$\begin{aligned}
&\boldsymbol{F}^{-1} |\boldsymbol{\xi}|^\alpha \boldsymbol{F} u^\epsilon[\boldsymbol{x}] \\
&\quad = |\boldsymbol{x}|^{-\frac{d-2}{2}} C_1(\alpha, d) \\
&\quad \lim_{N\to\infty} \int_0^N \frac{\rho^{\alpha+\frac{d}{2}}}{(\epsilon^2+\rho^2)^{\frac{d+\alpha-1}{2}}} F\left(\frac{d+\alpha-1}{2}, -\frac{\alpha}{2}; \frac{d}{2}; \frac{\rho^2}{\epsilon^2+\rho^2}\right) J_{\frac{d-2}{2}}(|\boldsymbol{x}|\rho)d\rho.
\end{aligned}$$

Therefore we obtain the first equality in the lemma. The second equality follows making the change of variables. Now, let us concern the problem of the convergence. Define

$$\tau(s,\delta) := \frac{s^{\alpha+d-1}}{(\delta^2+s^2)^{\frac{d+\alpha-1}{2}}}, \quad \phi(s,\delta) := F\left(\frac{d+\alpha-1}{2}, -\frac{\alpha}{2}; \frac{d}{2}; \frac{s^2}{\delta^2+s^2}\right)$$

and

$$\theta(s) := s^{1-\frac{d}{2}} J_{\frac{d-2}{2}}(s).$$

Note that $\tau(s,\delta)$ is uniformly bounded with respect to (s,δ) and monotone increasing function of s for fixed δ. By the definition of the hypergeometric functions, $\phi(s,\delta)$ is also uniformly bounded with respect to (s,δ) and monotone decreasing function of s for fixed δ.

In the case of $d \geq 4$, the uniform convergence with respect to δ of the integral follows immediately by Lemma 4.2.

Applying lemma 4.1 and Lemma 4.2 to the case $d = 2, 3$, we can conclude that the improper integrals of the function, $\tau(s,\delta)\phi(s,\delta)J_{\frac{d-2}{2}}(s)$, is convergent, which is independent of δ. Therefore, putting $\delta = |\boldsymbol{x}|\epsilon$, we obtain the result. □

Lemma 4.4. *There exists a series of mollifier functions $\{\phi_n\}$ satisfying*

$$\left(\frac{1}{|\cdot|} * \phi_n\right)(\boldsymbol{x}) \leq K\frac{1}{|\boldsymbol{x}|} \quad \boldsymbol{x} \in \mathbf{R}^d. \tag{4.1}$$

where constance K is independent of n.

Proof. It is easy to choose a series of functions ϕ_n satisfying

1. $\phi_n \in C^\infty(\mathbf{R}^d)$
2. $\int_{\mathbf{R}^d} \phi_n(\boldsymbol{x})d\boldsymbol{x} = 1$
3. supp $\phi_n \subset \{x \in \mathbf{R}^d; |\boldsymbol{x}| \leq \frac{1}{n}\}$
4. $0 \leq \phi_n \leq \frac{2}{\Omega_d} n^d$

where Ω_d is the volume of a unit ball in $\mathbf{R}^d$ and ω_{d-1} is surface area of S^{d-1}.

In the case of $|\boldsymbol{x}| \geq \frac{2}{n}$,

$$\frac{1}{|\cdot|} * \phi_n(\boldsymbol{x}) = \int_{\mathbf{R}^d} \frac{1}{|\boldsymbol{y}|}\phi_n(\boldsymbol{x}-\boldsymbol{y})d\boldsymbol{y} = \int_{|\boldsymbol{x}-\boldsymbol{y}|\leq\frac{1}{n}} \frac{1}{|\boldsymbol{y}|}\phi_n(\boldsymbol{x}-\boldsymbol{y})d\boldsymbol{y}$$

holds. Then we have

$$\frac{1}{|\cdot|} * \phi_n(\boldsymbol{x}) \leq \frac{1}{|\boldsymbol{x}| - \frac{1}{n}} \int_{|\boldsymbol{x}-\boldsymbol{y}|\leq\frac{1}{n}} \phi_n(\boldsymbol{x}-\boldsymbol{y})d\boldsymbol{y} \leq \frac{1}{|\boldsymbol{x}| - \frac{1}{n}} \leq \frac{2}{|\boldsymbol{x}|}.$$

On the other hand, in the case of $0 < |x| \leq \frac{2}{n}$, we see

$$\begin{aligned}\frac{1}{|\cdot|} * \phi_n(\boldsymbol{x}) &\leq \frac{2}{\Omega_d} n^d \int_{|\boldsymbol{x}-\boldsymbol{y}|\leq\frac{1}{n}} \frac{1}{|\boldsymbol{y}|}d\boldsymbol{y} \\ &\leq \frac{2}{\Omega_d} n^d \int_{|\boldsymbol{y}|\leq\frac{3}{n}} \frac{1}{|\boldsymbol{y}|}d\boldsymbol{y} \leq \frac{3^{d-1}4\omega_{d-1}}{(d-1)\Omega_d}\frac{n}{2}.\end{aligned}$$

Therefore, setting

$$K := \max\left\{2, \frac{3^{d-1}4\omega_{d-1}}{(d-1)\Omega_d}\right\}$$

we obtain the inequality (4.1). □

Finally we are now in the position to prove the key lemma.

The proof of Key Lemma. Choose the series of mollifier functions as in Lemma 4.4. The notation is the same as in Lemma 4.3. Let $\eta = 1$, using lemmma 4.3 we can choose $M_1 > 1$ such that

$$\left|\int_{M_1}^\infty \phi(s,\delta)\tau(s,\delta)s^{1-\frac{d}{2}}J_{\frac{d-2}{2}}(s)ds\right| \leq 1.$$

Using asymptoie behaviours of Bessel functions such that $J_\nu(s) = O(x^\nu)$ $(x \to 0+)$, we have

$$\left|\int_0^{M_1} \phi(s,\delta)\tau(s,\delta)s^{1-\frac{d}{2}}J_{\frac{d-2}{2}}(s)ds\right| \leq c_1 L^2 \int_0^{M_1} s^{1-\frac{d}{2}}s^{\frac{d-2}{2}}ds = c_1 L^2 M_1 < \infty$$

where c_1 is given in Lemma 4.2 and L in Lemma 4.1. Then we obtain

$$\begin{aligned}
&\boldsymbol{F}^{-1}|\boldsymbol{\xi}|^{\alpha}\boldsymbol{F}u^{\epsilon}[\boldsymbol{x}] \\
&\quad = \frac{1}{|\boldsymbol{x}|}C_1(\alpha, d)\lim_{N\to\infty}\int_0^N \tau(s, |\boldsymbol{x}|\epsilon)\phi(s, |\boldsymbol{x}|\epsilon)s^{1-\frac{d}{2}}J_{\frac{d-2}{2}}(s)ds \\
&\quad \le \frac{1}{|\boldsymbol{x}|}C_1(\alpha, d)\lim_{N\to\infty}\bigg(\Big|\int_0^{M_1} \tau(s, |\boldsymbol{x}|\epsilon)\phi(s, |\boldsymbol{x}|\epsilon)s^{1-\frac{d}{2}}J_{\frac{d-2}{2}}(s)ds\Big| \\
&\quad +\Big|\int_{M_1}^N \tau(s, |\boldsymbol{x}|\epsilon)\phi(s, |\boldsymbol{x}|\epsilon)s^{1-\frac{d}{2}}J_{\frac{d-2}{2}}(s)ds\Big|\bigg) \\
&\quad \le \frac{1}{|\boldsymbol{x}|}C_1(\alpha, d)(c_1L^2M_1+1).
\end{aligned}$$

Setting $C(\alpha, d) = KC_1(\alpha, d)(c_1L^2M_1+1)$, we can conclude

$$\begin{aligned}
&\boldsymbol{F}^{-1}|\boldsymbol{\xi}|^{\alpha}\boldsymbol{F}u_n[\boldsymbol{x}] = \boldsymbol{F}^{-1}|\boldsymbol{\xi}|^{\alpha}(\boldsymbol{F}u^{\epsilon}\boldsymbol{F}\phi_n)[\boldsymbol{x}] = (\boldsymbol{F}^{-1}|\boldsymbol{\xi}|^{\alpha}\boldsymbol{F}u^{\epsilon}) * \phi_n[\boldsymbol{x}] \\
&\quad \le C_1(\alpha, d)(c_1L^2M_1+1)\Big(\frac{1}{|\cdot|} * \phi\Big)(\boldsymbol{x}) \le C(\alpha, d)\frac{1}{|\boldsymbol{x}|}.
\end{aligned}$$

where $\epsilon = \dfrac{1}{n}$. $\square$

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