J. Math. Kyoto Univ. (JMKYAZ)
46-1 (2006), 65–74

Stability analysis of difference systems via cone valued Liapunov's function method

By

A. A. Soliman

Abstract

Total stability for systems of ordinary differential equations, functional differential equations, and difference equations was introuced. In this paper, we will extend this notion to the so-called total ϕ_0-stability for systems of difference equations. given some new criteria and results. Our technique depends on cone-valued Liapunov's function method.

1. Introduction

Recently, difference equations problems has been considerable interest in studying and improving (see [2], [3], [7]–[11]). Furthermore it has been successfully in different approachs based on Liapunov's direct method, and was study with cone and cone-valued Liapunov function method (see [8]).

Our purpose in this paper is to extend total stability of [11] to new type of stability, namely total ϕ_0-stability of difference equations systems which lie somewhere between totally stability of [11] on one side and ϕ_0-stability of [8] on the othere side via cone-valued Liapunov function method that was studied in [5] and used in [1], [8].

Let $\Re^m$ be the m-dimensional Euclidean real space, $J = [t_0, \infty)$, and $\Re^+ = [0, \infty)$. The following definitions will be needed.

Definition 1.1 ([4]). A function $b(r)$ is said to be belong to the class $\mathcal{K}$ if $b(r) \in C[(0, \rho), \Re^+], b(0) = 0$ and $b(r)$ is strictly monotone increasing in r.

Definition 1.2 ([4]). A function $a(t)$ is said to be belong to the class $\mathcal{L}$ if $a(t) \in C[J, \Re^+], a(t) \to \infty$ as $t \to \infty$ and $a(t)$ is strictly monotone decreasing in t.

Definition 1.3 ([1]). A proper subset $K \subset \Re^m$ is called a cone if

(i) $\lambda K \subset K, \lambda \geq 0$, (ii) $K + K \subset K$, (iii) $\overline{K} = K$,
(iv) $K^\circ \neq \emptyset$, (v) $K \cap (-K) = 0$,

(MSC)-34D20
Received March 17, 2005

where $\overline{K}$ and K° denote the closure and interior of K respectively, and ∂K denotes the boundary of $K, x \in \partial K \Longleftrightarrow y-x=0$ for some $y \in K_0^*, K_0 = K-0$.

The order relation on $\Re^m$ induced by the cone K is defined as follows

Let $x, y \in K$, then

$$x \leq_K y \Longleftrightarrow y-x \in K, \text{ and } x \leq_{K^o} y \Longleftrightarrow y-x \in K^o.$$

The set K^* is called the adjoint cone if

$$K^* = \{\phi \in \Re^m : (\phi, x) \geq 0\}, \quad \text{for} \quad x \in K,$$

satisfies the properties $(i)-(v)$ of Definition 1.3.

Definition 1.4 ([1]). A function $g : D \to \Re^m, D \subset \Re^n$ is called quasi-monotone relative to the cone K, if $x, y \in D$ and $y - x \in \partial K$, then there exists $\phi_\circ \in K_\circ^\star$ such that $(\phi_\circ, y-x) = 0$ and $(\phi_\circ, g(y) - g(x)) \geq 0$.

Consider system of difference equations

$$x(n+1) = f(n, x(n)), \quad x(n_0) = \psi \tag{1.1}$$

and the perturbed system

$$x(n+1) = f(n, x(n)) + h(n, x(n)), \quad x(n_0) = \psi \tag{1.2}$$

where $f, h : Z^+ \times C \to \Re^m$ are continuous in x_n, y_n, Z^+ is the set of nonnegative integers $x, y \in \Re^m, f(n, 0) = h(n, 0) = 0$ for $n \in Z^+$, so that the equations (1.1), (1.2) always have the zero solution $x(n) = 0,\ y(n) = 0$. Let

$$\| \psi \| = max \mid \psi(s) \mid : s \in \{-r, , -r+1, \dots, 0\}$$

and $C = \{\psi : \{-r, -r+1, \dots, 0\} \to \Re^m\}$ for positive integer $r > 0,\ x_n(s) = x(n+s), y_n(s) = y(n+s) \text{for} s = -r, -r+1, \dots, 0$. Furthermore for any given $n_0 \in Z^+$ and given initial function $\psi \in C$, there is a unique solutions of $x(n_0, \psi)(n), y(n_0, \psi)(n)$ such that it satisfies (1.1), (1.2) and

$$x(n_0, \psi)(n_0 + s) = \psi(s), \quad y(n_0, \psi)(n_0 + s) = \psi(s), \text{ for } s = -r, -r+1, \dots, 0.$$

respectively for all integer $n \geq n_0$.

Definition 1.5 ([8]). The zero solution of (1.1) is said to be ϕ_0-equistable if for $\epsilon > 0,\ n_0 \in Z^+$, there exist positive functions $\delta(n_0, \epsilon) > 0$ that is continuous in n_0, such that for $\phi_0 \in K_0^*$

$$(\phi_0, x^*(n_0, \psi)) < \epsilon, \quad \text{for} \quad n \geq n_0.$$

provided that $(\phi_0, \psi) < \delta$, where x^*, y^* here and in this paper denote the maximal solutions of (1.1) and (1.2) relative to the cone $K \subset \Re^n$ respectively. Other ϕ_0- stability can be similarly dedined.

The following definitions are somewhat new and related with that of [11].

Definition 1.6. The zero solution of (1.1) is said to be totally uniformly ϕ_0-stable if for $\epsilon > 0, n_0 \in Z^+$, there exist positive functions $\delta_1(n_0, \epsilon) > 0$ and $\delta_2(n_0, \epsilon) > 0$ that is continuous in n_0, such that for $\phi_0 \in K_0^*$

$$(\phi_0, x^*(n_0, \psi)) < \epsilon, \quad \text{for all} \quad n \geq n_0.$$

provided that $(\phi_0, \psi) < \delta_1$ and $(\psi, h(n, x(n))) < \delta_2$.

Definition 1.7. The zero solution of (1.1) is said to be totally ϕ_0-stable under permanent perturbations bounded in the mean if for $\epsilon > 0, n_0 \in Z^+$, there exist two positive functions $\delta_1(n_0, \epsilon) > 0$ and $\delta_2(n_0, \epsilon) > 0$ that is continuous in n_0,such that for $\phi_0 \in K_0^*$

$$(\phi_0, x^*(n_0, \psi)) < \epsilon, \quad \text{for all} \quad n \geq n_0.$$

provided that $(\phi_0, \psi) < \delta_1$, $(\psi, h(n, x(n))) < \delta_2$.

In the case of uniformly totally ϕ_0- stability δ_1 and δ_2 are independent of t_0.

Definition 1.8. The zero solution of (1.2) is said to be totally uniformly ϕ_0-stable under permanent perturbations bounded in the mean if for $\epsilon > 0, n_0 \in Z^+$, there exist two positive functions $\delta_1(n_0, \epsilon) > 0$ and $\delta_2(n_0, \epsilon) > 0$ that is continuous in n_0 such that for the maximal solution $y^*(n_0, \psi))$ of (1.2), and $\phi_0 \in K_0^*$

$$(\phi_0, y^*(n_0, \psi)) < \epsilon, \quad \text{for all} \quad n \geq n_0.$$

provided that $(\phi_0, \psi) < \delta_1$, $(\psi, h(n, y(n))) < \delta_2$, $n \geq n_0$, where

$$\mid h(n, y(n)) \mid = sup \mid h(n, \varphi) \mid : n \in Z^+, \parallel \varphi \parallel < \epsilon.$$

Definition 1.9. The zero solution of (1.2) is said to be totally uniformly asymptotically ϕ_0-stable if it is uniformly asymptotically ϕ_0-stable provided that for

$$\mid h(n, \varphi) \mid \leq \sigma(n), \qquad \text{uniformly for} \quad \parallel \varphi \parallel < \rho.$$

where $\sigma(n) \to 0$ as $n \to \infty$, and ρ is some constant.

2. Main results

In this section, we will discuss and obtain some results of total ϕ_0-stability of the system (1.1)

Theorem 2.1. *Let the zero solution of* (1.1) *be uniformly asymptotically ϕ_0- stable. Assume further that*

$$\parallel f(n, x) - f(n, y) \parallel \leq L(n) \parallel x - y \parallel,$$

for $(n, x), (n, y) \in Z^+ \times K, 0 \leq L(n) \leq \alpha T, \alpha$ is a positive constant.

Then there exists a cone-valued Liapunov function $V(n, x)$ with the following properties

I) $V(n,0) = 0, V(n,x(n)) : Z^+ \times C \to K$ *is continuous function and locally Lipschitzian in* $x(n)$ *relative to* K, *for a continuous* $\beta(n) > 0$,

II) $a(\phi_0, x^*(n)) \le (\phi_0, V(n, x^*(n))) \le b(\phi_0, x^*(n))$,
for $a, b \in \mathcal{K}, \phi_0 \in K_0^*$ *and* $(n, x(n)) \in Z^+ \times K$.

III) $(\phi_0, \triangle V(n, x^*(n)) \le -c(\phi_0, x^*(n)), c \in \mathcal{K}$, *where* $\triangle$ *is the difference operator define as*

$$\begin{aligned}\triangle(V(n,x(n)) \mid_{(1.1)} &= V(n, x(n+1)) - V(n,x)\\ &= V(n, f(n,x)) - V(n,x).\end{aligned}$$

Proof. From the hypotheses, solutions of (1.1) are exist and unique. Let $x(n_0.\psi)$ be a solution of (1.1), so that $x(n_0, \psi) = \psi$. Define the function c as

$$c(\phi_0, x^*(n)) = \frac{1}{A}[1 - exp(A(\phi_0, x^*(n)))],$$

where $A > 0$ is a constant. For

$$(\phi_0, x^*(t)) = 0, \quad \text{then} \quad \frac{1}{A}[1 - exp(A(\phi_0, x^*(n)))] = 0.$$

This implies that $c(0) = 0$. For

$$(\phi_0, x^*) > 0, \quad \text{then} \quad \frac{1}{A}[1 - exp(A(\phi_0, x^*(n)))]$$

is monotone increasing. It follows that $c \in K$.

This proves (I).

Now, we define a cone-valued Liapunov function by

$$V(n,0) = \sup_{\delta \ge 0} c[(\phi_0, x^*(n))]x(n+\delta, 0, \sigma_\omega(x(0, x(n)))) \left[\frac{1+B\delta}{1+\delta}\right] \tag{2.1}$$

where $\sigma_\omega : S_0(\rho) \longrightarrow K$, and $x^*(n)$ is the maximal solution of (1.1) relative to the cone $K \subset \Re^n$. For $x = 0$, thus from (2.1), V(n, 0)=0, and for $\delta = 0$, we have

$$c[(\phi_0, x^*(n))]x(0, \sigma_\omega(x(0, x(n))) \le_K V(n, x(n)).$$

Thus

$$c[(\phi_0, x^*)](\phi_0, x(0, \sigma_\omega(x(0, x(n))))) \le_K (\phi_0, V(n, x(n))).$$

and

$$c[(\phi_0, x^*)]\psi_0(\phi_0, e) = a[(\phi_0, x^*)] \le_K (\phi_0, V(n, x^*(n))). \tag{2.2}$$

where

$$\psi_0 = min \mid x(n_i) \mid, i = 1, 2, \ldots s; a(r) = u_0(\phi_0, e)c(r), \text{ and } e = (1, 1, \ldots 1)^T.$$

Since the zero solution of (1.1) is uniformly asymptotically ϕ_0-stable, given $\epsilon > 0$, there exist two numbers $\delta = \delta(\epsilon)$, and $T = T(\epsilon)$ are independent of n_0 such that

$$(\phi_0, \psi) < \delta(\phi_0, x^*) < \epsilon, \quad \text{for} \quad n \geq T(\epsilon).$$

By using the fact that $(1 + B\delta)/(1 + \delta) < B$, from (2.1) we get

$$\begin{aligned}(\phi_0, x^*) &= \sup_{\delta \geq 0} c[(\phi_0, x^*)][(\phi_0, x(n + \delta, 0, \sigma_\omega(n, x(n)))))] \left[\frac{1 + B\delta}{1 + \delta}\right] \\ &\leq \sup_{\delta \geq 0} c[(\phi_0, x^*)][(\phi_0, x^*) \left[\frac{1 + B\delta}{1 + \delta}\right] \\ &\leq \sup_{\delta \geq 0} c[(\phi_0, x^*)][(\phi_0, x^*) \left[\frac{1 + B\delta}{1 + \delta}\right] \\ &\leq \epsilon Bc[(\phi_0, x^*)] = b[(\phi_0, x^*)]\end{aligned}$$

that is

$$(\phi_0, V(n, x^*(n))) \leq b[(\phi_0, x^*)], \quad b \in \mathcal{K}. \tag{2.3}$$

Comparing this with (2.2), we have

$$a[(\phi_0, x^*)] \leq (\phi_0, V(n, x^*(n))) \leq b[(\phi_0, x^*)], \quad a, b \in \mathcal{L}. \tag{2.4}$$

This proves(II).

Now, for $\delta \leq T(\epsilon)$, where $T(\epsilon)$ is monotonic decreasing function, we have from uniform asymptotic ϕ_0-stability that

$$(\phi_0, x^*) < \epsilon.$$

Hence, if $\delta \geq T(\gamma(\phi_0, x^*))$, for $\gamma > 0$, then

$$(\phi_0, x^*) < (\gamma(\phi_0, x^*)) \Longrightarrow c(\phi_0, x^*) < c(\gamma(\phi_0, x^*))$$

and

$$\begin{aligned}&c[(\phi_0, x^*)]x(n + \delta, 0, \sigma_\omega(x(0, n, x))) \left[\frac{1 + B\delta}{1 + \delta}\right] \\ &\quad \leq Bc[(\phi_0, x^*)](\phi_0, x^*) \\ &\quad \leq \epsilon bc[(\gamma(\phi_0, x^*))] \\ &\quad \leq (\phi_0, V(n, x^*(n)))\end{aligned}$$

Thus

$$c[(\phi_0, x^*)]x(n + \delta, 0, \sigma_\omega(x(0, x(n))) \left[\frac{1 + B\delta}{1 + \delta}\right] \leq V(n, x).$$

This implies that $V(n, x(n))$ is defined only for $0 \leq \delta \leq T(\gamma(\phi_0, x^*))$. As

$$V(n, x) = \sup_{0 \leq \delta \leq T} c[(\phi_0, x^*)]x(n + \delta, 0, \sigma_\omega(x(0, x(n))) \left[\frac{1 + B\delta}{1 + \delta}\right]$$

where $T = T(\gamma(\phi_0, x_n^*))$.

By Corollary 2.7.1 of [4], and for $x(n_1),\ x(n_2) \in S(\rho)$, we have

$$\begin{aligned}
&\| V(n, x(n_1)) - V(n, x(n_2)) \| \\
&= \left\| \sup_{0\le\delta\le T} c[(\phi_0, x^*)]x(n+\delta, 0, \sigma_\omega(x(n_1)(0, x(n_1)))) \left(\frac{1+B\delta}{1+\delta}\right) \right. \\
&\quad \left. - \sup_{0\le\delta\le T} c[(\phi_0, x^*)]x(n+\delta, 0, \sigma_\omega(x(n_2)(0, x(n_2)))) \left(\frac{1+B\delta}{1+\delta}\right) \right\| \\
&\le \left| \sup_{0\le\delta\le T} c[(\phi_0, x^*)] \left(\frac{1+B\delta}{1+\delta}\right) \right| \| \sigma_\omega(x*(n_1)(0, x(n_1))) \\
&\quad - \sigma_\omega(x(n_2)(0, n, x(n_2))) \| \\
&\le k(n,w) \left| \sup_{0\le\delta\le T} c[(\phi_0, x^*)] \left(\frac{1+B\delta}{1+\delta}\right) \right| exp \left(\sum_{s=0}^{s=n} L(s) \right) \| x(n_1) - x(n_2) \| \\
&\le \beta(n) \| x(n_1) - x(n_2) \|.
\end{aligned}$$

where

$$\beta(n) = k(n,w) \left| \sup_{0\le\delta\le T} c(\phi_0, x^*) \left(\frac{1+B\delta}{1+\delta}\right) \right| exp \left(\sum_{s=0}^{s=n} L(s) \right)$$

is locally Lipschitzian in $x(n_1)$ and $x(n_2)$. Therefore $V(n, x(n))$ is locally Lipschitzian in x.

Now,

$$\begin{aligned}
\| V(n+\delta, x) - V(n, y) \| &\le \| V(n+\delta, x) - V(n+\delta, y) \| \\
&\quad + \| V(n+\delta, y) - V(n+\delta, y)(n+\delta, y)) \| \\
&\quad + \| V(n+\delta, y) - V(n, y) \|
\end{aligned} \tag{2.5}$$

Since $V(n,x)$ is locally Lipschitzian in y, y is continuous in δ, the first two terms in the right hand side of the inequality (2.5) are small whenever $\| y - x \|$ and δ are small. By using (2.1), the term tends to zero. Therefor $V(n,x)$ is continuous in all its arguments.

Let $x = x(n_0, \psi), x(n_\rho) = x(n_\rho)(n+\rho, n, x),\ \rho > 0$, then we have

$$V(n+\rho, x(n_\rho)) = \sup_{0\le\delta\le T} c[(\phi_0, x^*)]x(n+\rho+\delta_\rho, 0, \sigma_\omega(x(0, x(n+\rho))) \left[\frac{1+B\delta}{1+\delta}\right]$$

The continuity of V and the uniqeness of a solution od (1.1) imply that there exists a point δ_ρ in which the upper bound is reached so that we have

$$V(n+\rho, x(n_\rho)) = c[(\phi_0, x^*)]x(n+\rho+\delta_\rho, 0, \sigma_\omega(x(0, x(n+\rho))) \left[\frac{1+B\delta}{1+\delta}\right].$$

By putting $\delta_\rho = \delta_1 - \rho$, and using the fact

$$\frac{1+B\delta_\rho}{1+\delta_\rho} = \left[\frac{1+B\delta_1}{1+\delta_1}\right] \left[1 - \frac{(B-1)\rho}{(1+B\delta_1)(1+\delta_\rho)}\right],$$

we get

$$\begin{aligned}
&V(n+\rho, x(n_\rho)) \\
&\quad = c[(\phi_0, x^*)]x(n+\rho+\delta_\rho, 0, \sigma_\omega(x(0, x(n+\rho))) \left(\frac{1+B\delta_1}{1+\delta_1}\right) \\
&\qquad \times \left[1 - \frac{(B-1)\rho}{(1+B\delta_1)(1+\delta_\rho)}\right] \\
&\quad \leq_K V(n,x) - \left[\frac{(B-1)\rho V(n,x)}{(1+B\delta_1)(1+\delta_\rho)}\right].
\end{aligned}$$

Since $0 \leq \delta_\rho < T,\ 0 < \rho < \delta_1 \leq \rho + T,\ T$ is monotonic decreasing and using (2.4), we have

$$\frac{V(n+\rho, x(n_\rho)) - V(n,x)}{\rho} \leq_K - \left[\frac{(B-1)V(n,x)}{(1+B\delta_1)(1+\delta_\rho)}\right].$$

$$\left(\phi_0, \frac{V(n+\rho, x(n_\rho)) - V(n,x)}{\rho}\right) \leq_K - \left[\frac{(B-1)(\phi_0, V(n,x))}{1+B\delta_1)(1+\delta_\rho)}\right].$$

So that

$$\begin{aligned}
\triangle(\phi_0, V(n,x))) &\leq - \left[\frac{(B-1)(\phi_0, V(n,x))}{(1+BT(\gamma(\phi_0,x^*)))(1+T\gamma(\phi_0,x^*))) + B\rho}\right] \\
&\leq -\beta(\phi_0, V(n,x))), \beta \in \kappa \\
&\leq -\beta a[(\phi_0, x^*)] \leq c(\phi_0, x^*), c \in \mathcal{K}.
\end{aligned}$$

This proves (III), and the proof is completed. □

Theorem 2.2. *Let the hypotheses og Theorem* 2.1 *be satisfied. Then the zero solution of* (1.1) *is totally* ϕ_0*-stable.*

Proof. From Theorem 2.1, property (I) holds. Let $\epsilon > 0$ be given, choose $\delta_1 = \delta_1(\epsilon)$ such that

$$a(\epsilon) > b(\delta_1(\epsilon)), \qquad a, b \in \mathcal{K}.$$

Let $x(n) = x(n, \psi)$ be a solution of (1.1) such that

$$(\phi_0, \psi) < \delta_1 \quad \text{and} \quad (\phi_0, h(n,x)) < \delta_2, \quad \text{for} \quad \delta_2 = \delta_2(\epsilon) > 0.$$

By (II) of Theorem 2.1, we have $V(n_0, \psi) = b(\delta_1(\epsilon))$.

Now, we claim that

$$(\phi_0, V(n,x)) < a(\epsilon), n \geq 0.$$

This claim leads to

$$a(\phi_0, x^*) < (\phi_0, V(n,x)) < a(\epsilon).$$

Then $(\phi_0, x^*) < \epsilon$, and this show that the trivial solution of (1.1) is totally ϕ_0-stable.

Now, we justify this cliam. Define

$$T(n) = (\phi_0, V(n,x)).$$

Suppose that this claim is false, then there exist two numbers n_1 and n_2 with $n_0 < n_1 < n_2$ such that

$$T(n_1) = b(\delta_1(\epsilon)), \quad T(n_2) = a(\epsilon), \text{ and } T(n) \geq b(\delta_1(\epsilon)) \quad \text{for} \quad n_1 \leq n \leq n_2.$$

This show that $T(n)$ is nondecreasing in $[n_1, n_2]$ and, we have

$$\triangle T(n_1) \geq 0 \tag{2.6}$$

From (II)and (III) of Theorem 2.1 and for any $c^* \in \kappa$, we have

$$\triangle(\phi_0, V(n,x)) \leq -c^*(\phi_0, V(n,x)).$$

This implies that

$$\begin{aligned}\triangle T &\leq -c^*(T) + M[(\phi_0, h(n,x))], M > 0\\ &\leq -c^*(T) + M\delta_2\\ &\leq -c^*(b(\delta_1(\epsilon))) + M\delta_2\\ &= -b^*(\delta_1(\epsilon)) + M\delta_2,\end{aligned}$$

where $c^*(b(r)) = b^*(r) \in \mathcal{K}$. Now, we choose

$$\delta_2 = b^*\left(\frac{\delta_1}{M}\right). \qquad \Longrightarrow \qquad \triangle T < 0.$$

This contradicts (2.6) and our claim is justfied. Therefore the zero solution of (1.1) is totally ϕ_0-stable, and the proof is completed. □

Theorem 2.3. *Let the hypotheses og Theorem* 2.1 *be satisfied. Then the zero solution of* (1.1) *is totally ϕ_0-stable under permanent perturbation bounded in the mean.*

Proof. From Theorem 2.1, the property (I) holds. Let $x(n) = x(n_0, \psi)$ be a solution of (1.1) such that

$$(\phi_0, \psi) < \delta_1 \quad \text{and} \quad (\phi_0, h(n,x)) \leq \gamma(n), \text{ where } \sum_{s=n_0}^{s=n_0+T} \gamma(s)ds < \delta_2.$$

Now, we are continuous as in the proof of Theorem 2.2, we arrive the inequality (2.6). From (I), (II), and(III), we have

$$\begin{aligned}\triangle T \leq -c^*(T) &\leq -c^*(T) + M|(\phi_0, h(n,x))|, M < 0\\ &\leq -c^*(T) + M\gamma(n).\end{aligned}$$

Integrating on $[n_0, T^*]$, we get

$$T(n) \leq -\sum_{n}^{T^*} c^*(T(s))ds + M \sum_{s=n}^{s=T^*} \gamma(s)ds$$
$$\leq -\sum_{s=n}^{s=T^*} c^*(T(s))ds + M\delta_2.$$

Now, if we choose

$$\delta_2 = M^{-1} \sum_{s=n}^{s=T^*} c^*(T(s))ds,$$

then $T < 0$, that is $(\phi_0, V(n,x)) < 0$. But this is impossible since the properties (II),

$$(\phi_0, V(n,x)) \geq a(\phi_0, x^*), a \in \mathcal{K}.$$

Therefore the result is immediated. □

Theorem 2.4. *Let the hypotheses of Theorem* 2.1 *be satisfied,and assume further that* $h(n,x)$ *is locally Lipschitzian in* x *relative to the cone* $K \subset \Re^n$*, for each* $t \in \Re^+$*. Then the zero solution of* (1.1) *is uniformly totally* ϕ_0*-stable.*

Proof. From Theorem 2.1, it follows that

$$\triangle(\phi_0, V(n,x)) \leq -c^*[(\phi_0, V(n,x))] + M[(\phi_0, h(n,x))]$$
$$\leq -c^*[(\phi_0, V(n,x))] + M\sigma(n), M > 0.$$

since $\sigma \in \text{Ł}$, then there exists $T = T(\epsilon)$ sufficiently large such that for $t \geq T(\epsilon)$, we have that $\sigma \longrightarrow 0$. Therefore

$$\triangle(\phi_0, V(n,x)) \leq -c^*[(\phi_0, V(n,x))].$$

From (II), we have

$$\triangle(\phi_0, V(n,x)) \leq -c^*[(\phi_0, V(n,x))] + M\sigma(n) = c^*[(\phi_0, x^*)],$$

where $c^*, a \in \mathcal{K}$and$c^*[a(r)] = c(r)$ so that $c \in \mathcal{K}$. Now, by using the condition (I), (II) and (III), we see that the conditions of Theorem 3.1 of [1] are satisfied. Since

$$\| h(n,x) - h(n,y) \| \leq L(n) \| x - y \|, \quad \text{for} \quad x, y \in K,$$

then putting $y = 0$, we get

$$\| h(n,x) \| \leq L(n) \| x \|,$$

when $x = 0$, we have $\| h(n,x) \| = 0$. Therefore from Theorem 3.4 of [1], and definition of uniformly totally ϕ_0-stable, the result is immediated. □

Department of Mathematics, Faculty of Sciences
Benha University
Benha 13518, Kalubia, Egypt

present address: Department of Mathematics
Faculty of Teachers
Al-Jouf, Skaka, P.O. Box. 269, Saudia Arabia
e-mail: a_a_Soliman@hotmail.com

References

[1] E. P. Akpan and O. Akinyele, *On the ϕ_0-stability of comparison differential systems*, J. Math. Anal. Appl. **164**-2 (1992), 307–324.

[2] W. H. Chen, *Uniformly asymptotic stability for class of Difference equations with infinit delay*, J. Math. Res. Exposition **22** (2002), 299–306.

[3] S. Elaydi and S. Zhang, *Stability and periodicity of Difference equations with infinit delay*, Funkcial. Ekavac. **37**-3 (1994), 404–413.

[4] V. Lakshmikantham and S. Leela, *Differential and integral inequalities*, vol. I, Academic Press, New York, 1969.

[5] ______, *Cone-valued Liapunov functions*, Nonlinear Anal. **1** (1977), 215–222.

[6] V. Lakshmikantham and D. Trigiante, *Theory of difference equations; numerical method and applications*, Academic Press, Boston, 1988.

[7] H. Wu and X. Lu, *Uniformly asymptotic stability for class of Difference equations with infinit delay*, J. Math. Res. Exposition **22** (2002), 299–306.

[8] P. Wang and F. Geng, *On ϕ_0-stability of difference equations*, Appl. Math. Lett. **18** (2005), 139–147.

[9] S. Zhang, *Stability analysis of delay difference systems*, Comput. Math. Appl. **33**-10 (1987), 41–52.

[10] ______, *Stability of finite delay difference systems*, Nonlinear Analysis, TMA, **22**-9 (1994), 1121–1129.

[11] ______, *Estimate of total stability of delay difference systems*, Comput. Math. Appl. **37** (1999), 31–38.