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Special linearly unrelated sequences

By

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Abstract

The main result of this paper are a criteria giving conditions that the certain infinite sequence of rational numbers be linearly unrelated. The proof is direct and does not require any special theorems.

1. Introduction

In 1975 Erdős [1] defined irrational sequences.

Definition 1.1. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers. We say the sequence $\{a_n\}_{n=1}^{\infty}$ is irrational if for every sequence $\{c_n\}_{n=1}^{\infty}$ of positive integers the series

$$\sum_{n=1}^{\infty} \frac{1}{a_n c_n}$$

is an irrational number. If $\{a_n\}_{n=1}^{\infty}$ is not an irrational sequence, then we say it is a rational sequence.

Erdős also proved a theorem giving a criteria for an irrational sequences in the same paper. Other criteria for a sequences to be irrational can also be found in [2]. Hančl [3] gave an extension of the Erdős definition to linear independence in the following way.

Definition 1.2. Let $\{a_{i,n}\}_{n=1}^{\infty}$ for $i = 1, \ldots, K$ be sequences of positive real numbers. If for every sequence $\{c_n\}_{n=1}^{\infty}$ of positive integers the numbers $\sum_{n=1}^{\infty} \frac{1}{a_{1,n}c_n}$, $\sum_{n=1}^{\infty} \frac{1}{a_{2,n}c_n}$, $\ldots$, $\sum_{n=1}^{\infty} \frac{1}{a_{K,n}c_n}$, and 1 are linearly independent over rational numbers, then the sequences $\{a_{i,n}\}_{n=1}^{\infty}$ $i = 1, \ldots, K$ are said to be linearly unrelated.

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There are not many results in this field. Some criteria can be found in [3] and [4] for linear independence. Our main result is Theorem 2.1 below and it gives the criterion of linearly unrelated sequences.

2. Main result

Theorem 2.1. *Let K be a positive integer and ε, μ, ν be real numbers such that $0 < \varepsilon$, $0 \le \mu$, $0 \le \nu$ and $1-\mu-\nu > \frac{1}{1+\varepsilon}$. Suppose that $\{a_{i,n}\}_{n=1}^\infty$ and $\{b_{i,n}\}_{n=1}^\infty$ $i = 1,\dots,K$ are sequences of positive integers with $\{a_{1,n}\}_{n=1}^\infty$ non-decreasing, such that*

$$\limsup_{n\to\infty} a_{1,n}^{\frac{1}{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^n}} = \infty \tag{2.1}$$

$$a_{1,n} \ge n^{1+\varepsilon} \tag{2.2}$$

$$b_{i,n} \le a_{1,n}^{\mu}, \quad i = 1,\dots,K \tag{2.3}$$

$$\lim_{n\to\infty} \frac{a_{i,n}b_{j,n}}{b_{i,n}a_{j,n}} = 0, \quad i,j = 1,\dots,K, \quad i > j \tag{2.4}$$

and

$$a_{i,n}a_{1,n}^{-\nu} \le a_{1,n} \le a_{i,n}a_{1,n}^{\nu}, \quad i = 1,\dots,K \tag{2.5}$$

hold for every sufficiently large n. Then the sequences $\{\frac{a_{i,n}}{b_{i,n}}\}_{n=1}^\infty$ $i = 1,\dots,K$ are linearly unrelated.

Example 2.1. The sequences

$$\left\{\frac{n^{6\cdot 9^n}+7}{n^{9^n}+5}\right\}_{n=1}^{\infty}$$

and

$$\left\{\frac{n^{3\cdot 9^n}+11}{n^{9^n}+13}\right\}_{n=1}^{\infty}$$

are linearly unrelated. It is enough to put $K = 2$, $\mu = \frac{1}{6}$, $\nu = \frac{1}{2}$ and $\varepsilon = 4$ in Theorem 2.1.

Remark 1. Theorem 5 from [4] can not be used for Example 2.1 because condition (2.3) from Theorem 5 is not fulfilled.

Remark 2. Theorem 2.1 of this paper is not generalization of Theorem 5 in [4]. From Theorem 5 in [4] we obtain that the sequence $\{\frac{2^{n2^n}}{n}\}_{n=1}^\infty$ is irrational but Theorem 2.1 of this paper does not imply this fact.

Example 2.2. Let K be a positive integer with $K > 2$. Then the sequences

$$\left\{\frac{n^{j(K+5)^n}+j}{n^{(K+5)^n}+j}\right\}_{n=1}^{\infty}$$

$j = 1,2,\dots,K$ are linearly unrelated.

Remark 3. If we put $K = 1, \mu = 0, \nu = 0$ in Theorem 2.1 then we obtain Erdős's Theorem from [1].

Open problem 2.1. Are the sequences $\{2^{3^n}+1\}_{n=1}^{\infty}$ and $\{3^{2^n}+1\}_{n=1}^{\infty}$ linearly unrelated?

3. Proof

Lemma 3.1. *Let K, ε, μ, ν and the sequences $\{a_{i,n}\}_{n=1}^{\infty}$, $\{b_{i,n}\}_{n=1}^{\infty}$ $i = 1, \dots, K$ satisfy all conditions stated in Theorem 2.1. Then there is a positive real number $B = B(K, \varepsilon, \mu, \nu)$ which does not depend on n such that*

$$\sum_{i=1}^{K}\sum_{j=0}^{\infty}\frac{b_{i,n+j}}{a_{i,n+j}} < \frac{1}{a_{1,n}^{B}} \tag{3.1}$$

holds for all sufficiently large n.

Proof. (of Lemma 3.1)
From (2.3) and (2.5) we obtain

$$\sum_{j=0}^{\infty}\frac{b_{i,n+j}}{a_{i,n+j}} \le \sum_{j=0}^{\infty}\frac{a_{1,n+j}^{\mu}a_{1,n+j}^{\nu}}{a_{1,n+j}} = \sum_{j=0}^{\infty}\frac{1}{a_{1,n+j}^{1-\mu-\nu}} \tag{3.2}$$

for every n sufficiently large.
Now we have

$$\sum_{j=0}^{\infty}\frac{1}{a_{1,n+j}^{1-\mu-\nu}} = \sum_{n+j<a_{1,n}^{\frac{1}{1+\epsilon}}}\frac{1}{a_{1,n+j}^{1-\mu-\nu}} + \sum_{n+j\ge a_{1,n}^{\frac{1}{1+\epsilon}}}\frac{1}{a_{1,n+j}^{1-\mu-\nu}}. \tag{3.3}$$

We will estimate the first summand on the right hand side of (3.3) as

$$\sum_{n+j<a_{1,n}^{\frac{1}{1+\epsilon}}}\frac{1}{a_{1,n+j}^{1-\mu-\nu}} \le \frac{1}{a_{1,n}^{1-\mu-\nu}}a_{1,n}^{\frac{1}{1+\epsilon}} = \frac{1}{a_{1,n}^{1-\mu-\nu-\frac{1}{1+\varepsilon}}} = \frac{1}{a_{1,n}^{B_1}}. \tag{3.4}$$

Here $B_1 = 1 - \mu - \nu - \frac{1}{1+\varepsilon}$ is a positive real number which does not depend on n.
We now estimate the second summand on the right hand side of (3.3).
From (2.2) we obtain

$$\sum_{n+j\ge a_{1,n}^{\frac{1}{1+\epsilon}}}\frac{1}{a_{1,n+j}^{1-\mu-\nu}} \le \sum_{n+j\ge a_{1,n}^{\frac{1}{1+\varepsilon}}}\frac{1}{(n+j)^{(1+\varepsilon)(1-\mu-\nu)}} \le \int_{a_{1,n}^{\frac{1}{1+\varepsilon}}}^{\infty}\frac{dx}{x^{1+\frac{(1+\varepsilon)(1-\mu-\nu)-1}{2}}}$$

$$\le \frac{1}{\left(a_{1,n}^{\frac{1}{1+\varepsilon}}\right)^{\frac{(1+\varepsilon)(1-\mu-\nu)-1}{3}}} = \frac{1}{a_{1,n}^{B_2}}, \tag{3.5}$$

where $B_2 = \frac{(1+\varepsilon)(1-\mu-\nu)-1}{3(1+\varepsilon)}$ is a positive real constant which does not depend on n.

Hence (3.2), (3.3), (3.4) and (3.5) imply

$$\sum_{i=1}^{K}\sum_{j=0}^{\infty}\frac{b_{i,n+j}}{a_{i,n+j}} \le \sum_{i=1}^{K}\sum_{j=0}^{\infty}\frac{1}{a_{1,n+j}^{1-\mu-\nu}} \le K\left(\frac{1}{a_{1,n}^{B_1}}+\frac{1}{a_{1,n}^{B_2}}\right) \le \frac{1}{a_{1,n}^{B}},$$

where $B = \frac{1}{2}\min(B_1, B_2)$ is a positive real constant which does not depend on n and (3.1) follows. □

Lemma 3.2. *Let K, ε, μ, ν and the sequences $\{a_{i,n}\}_{n=1}^{\infty}$, $\{b_{i,n}\}_{n=1}^{\infty}$ $i = 1, \dots, K$ satisfy all conditions stated in Theorem* 2.1 *except that instead of* (2.2) *we have*

$$a_{1,n} > 2^n \tag{3.6}$$

for all sufficiently large n. Then

$$\sum_{i=1}^{K}\sum_{j=0}^{\infty}\frac{b_{i,n+j}}{a_{i,n+j}} \le \frac{2K\log_2 a_{1,n}}{a_{1,n}^{1-\mu-\nu}} \tag{3.7}$$

holds for every sufficiently large n.

Proof. (of Lemma 3.2)
As in the proof of Lemma 3.1 from (2.3) and (2.5) we obtain

$$\sum_{j=0}^{\infty}\frac{b_{i,n+j}}{a_{i,n+j}} \le \sum_{j=0}^{\infty}\frac{1}{a_{1,n+j}^{1-\mu-\nu}} = \sum_{n+j<\log_2 a_{1,n}}\frac{1}{a_{1,n+j}^{1-\mu-\nu}} + \sum_{n+j\ge\log_2 a_{1,n}}\frac{1}{a_{1,n+j}^{1-\mu-\nu}}. \tag{3.8}$$

We now estimate both sums on the right hand side of equation (3.8). For the first summand, we have

$$\sum_{n+j<\log_2 a_{1,n}}\frac{1}{a_{1,n+j}^{1-\mu-\nu}} \le \frac{\log_2 a_{1,n}}{a_{1,n}^{1-\mu-\nu}}. \tag{3.9}$$

Estimating the second summand of equation (3.8) inequality (3.6) implies that

$$\begin{aligned}\sum_{n+j\ge\log_2 a_{1,n}}\frac{1}{a_{1,n+j}^{1-\mu-\nu}} &\le \sum_{n+j\ge\log_2 a_{1,n}}\frac{1}{(2^{(n+j)})^{(1-\mu-\nu)}}\\ &= \sum_{n+j\ge\log_2 a_{1,n}}\frac{1}{(2^{(1-\mu-\nu)})^{(n+j)}}\\ &\le \frac{1}{2^{(1-\mu-\nu)\log_2 a_{1,n}}}C\\ &= \frac{C}{a_{1,n}^{(1-\mu-\nu)}},\end{aligned} \tag{3.10}$$

where C is positive real constants which does not depend on n. Therefore (3.8), (3.9) and (3.10) together imply that

$$\sum_{i=1}^{K}\sum_{j=0}^{\infty}\frac{b_{i,n+j}}{a_{i,n+j}} \le \sum_{i=1}^{K}\sum_{j=0}^{\infty}\frac{1}{a_{1,n+j}^{1-\mu-\nu}} \le K\left(\frac{\log_2 a_{1,n}}{a_{1,n}^{1-\mu-\nu}}+\frac{C}{a_{1,n}^{1-\mu-\nu}}\right) \le \frac{2K\log_2 a_{1,n}}{a_{1,n}^{1-\mu-\nu}}$$

So (3.7) follows. □

Proof. (of Theorem 2.1)
Let $\{c_n\}_{n=1}^{\infty}$ be a sequence of positive integers. Then the sequences $\{a_{i,n}c_n\}_{n=1}^{\infty}$ and $\{b_{i,n}\}_{n=1}^{\infty}$ $i=1,\dots,K$ also satisfy conditions (2.1)–(2.5) and if in addition we reorder the sequence $\{a_{1,n}c_n\}_{n=1}^{\infty}$ and obtain the non-decreasing sequence $\{A_{1,n}\}_{n=1}^{\infty}$ then the new sequence together with the relevant sequences $\{A_{i,n}\}_{n=1}^{\infty}$ $i=2,\dots,K$ and $\{B_{i,n}\}_{n=1}^{\infty}$ $i=1,\dots,K$ will also immediatelly satisfy (2.1), (2.3), (2.4) and (2.5). From the fact that $A_{1,n}\ge a_{1,n}\ge n^{1+\varepsilon}$ we obtain that the sequence $\{A_{1,n}\}_{n=1}^{\infty}$ also satisfies condition (2.2). It follows that $\{A_{i,n}\}_{n=1}^{\infty}$ $i=1,\dots,K$ and $\{B_{i,n}\}_{n=1}^{\infty}$ $i=1,\dots,K$ will satisfy all the conditions stated in Theorem 2.1. Thus it suffices to prove that if K,μ,ν,ε and the sequences $\{a_{i,n}\}_{n=1}^{\infty}$, $\{b_{i,n}\}_{n=1}^{\infty}$ $i=1,\dots,K$ satisfy all conditions stated in Theorem 2.1 then the numbers $\sum_{n=1}^{\infty}\frac{b_{1,n}}{a_{1,n}},\ \dots,\ \sum_{n=1}^{\infty}\frac{b_{K,n}}{a_{K,n}}$ and the number 1 are linearly independent over the rational numbers. To establish this we will prove that for every K-tuple of integers $\alpha_1,\alpha_2,\dots,\alpha_K$ (not all equal to zero) the sum

$$I=\sum_{i=1}^{K}\alpha_i\sum_{n=1}^{\infty}\frac{b_{i,n}}{a_{i,n}}$$

is an irrational number. Suppose that I is a rational number. Let R be the maximal index such that $\alpha_R\ne 0$. Then we have

$$I=\sum_{i=1}^{K}\alpha_i\sum_{n=1}^{\infty}\frac{b_{i,n}}{a_{i,n}}=\sum_{n=1}^{\infty}\sum_{i=1}^{R}\alpha_i\frac{b_{i,n}}{a_{i,n}}=\sum_{n=1}^{\infty}\frac{b_{R,n}}{a_{R,n}}\left(\sum_{i=1}^{R-1}\alpha_i\frac{b_{i,n}a_{R,n}}{a_{i,n}b_{R,n}}+\alpha_R\right).$$

By (2.4) the number

$$\sum_{i=1}^{R-1}\alpha_i\frac{b_{i,n}a_{R,n}}{a_{i,n}b_{R,n}}+\alpha_R$$

and the number α_R have the same sign for all sufficiently large n. Without loss of generality assume that

$$\sum_{i=1}^{K}\alpha_i\frac{b_{i,n}}{a_{i,n}}>0 \tag{3.11}$$

for every sufficiently large n. Since I is a rational number there must be integers $p,q,(q>0)$ such that

$$I=\frac{p}{q}=\sum_{i=1}^{K}\alpha_i\sum_{n=1}^{\infty}\frac{b_{i,n}}{a_{i,n}}.$$

From this and (3.11) we obtain that

$$C_N = \left(p - q\sum_{i=1}^{K}\alpha_i \sum_{n=1}^{N-1}\frac{b_{i,n}}{a_{i,n}}\right)\prod_{n=1}^{N-1}\prod_{i=1}^{K}a_{i,n} = q\left(\prod_{n=1}^{N-1}\prod_{i=1}^{K}a_{i,n}\right)\sum_{i=1}^{K}\alpha_i\sum_{n=N}^{\infty}\frac{b_{i,n}}{a_{i,n}} \tag{3.12}$$

is a positive integer for every sufficiently large N. So (3.12) implies

$$1 \le Q_1\left(\prod_{n=1}^{N-1}\prod_{i=1}^{K}a_{i,n}\right)\sum_{i=1}^{K}\sum_{n=N}^{\infty}\frac{b_{i,n}}{a_{i,n}} \tag{3.13}$$

for all sufficiently large N, where $Q_1 = q\max_{i=1,\dots,K} \mid \alpha_i \mid$ is a positive integer constant which does not depend on N. From (2.5) we obtain

$$\prod_{n=1}^{N-1}\prod_{i=1}^{K}a_{i,n} \le Q_2\left(\prod_{n=1}^{N-1}a_{1,n}\right)^K\left(\prod_{n=1}^{N-1}a_{1,n}^{\nu}\right)^{K-1} \tag{3.14}$$

for every sufficiently large N, where Q_2 is a positive real constant which does not depend on N. Then (3.13) and (3.14) imply

$$1 \le Q\left(\prod_{n=1}^{N-1}a_{1,n}\right)^K\left(\prod_{n=1}^{N-1}a_{1,n}^{\nu}\right)^{K-1}\sum_{i=1}^{K}\sum_{n=N}^{\infty}\frac{b_{i,n}}{a_{i,n}} = Q\left(\prod_{n=1}^{N-1}a_{1,n}\right)^{K+(K-1)\nu}\sum_{i=1}^{K}\sum_{n=N}^{\infty}\frac{b_{i,n}}{a_{i,n}} \tag{3.15}$$

for every sufficiently large N, there Q is a positive real constant which does not depend on N. Now the proof falls into several cases.

1. Let us assume that (3.6) holds for every sufficiently large n and there is a $\delta > 0$ such that

$$\limsup_{n\to\infty} a_{1,n}^{\frac{1}{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1+\delta\right)^n}} = \infty. \tag{3.16}$$

This implies that there exist infinitely many N such that

$$a_{1,N}^{\frac{1}{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1+\delta\right)^N}} > \max_{k=1,\dots,N-1} a_{1,k}^{\frac{1}{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1+\delta\right)^k}}.$$

It follows that

$$
\begin{aligned}
a_{1,N} &> \left(\max_{k=1,\dots,N-1} a_{1,k}^{\frac{1}{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1+\delta\right)^k}} \right)^{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1+\delta\right)^N} \\
&> \left(\max_{k=1,\dots,N-1} a_{1,k}^{\frac{1}{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1+\delta\right)^k}} \right)^{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+\delta\right)\left(\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1+\delta\right)^{N-1}+\cdots+1\right)} \\
&> \left(\prod_{n=1}^{N-1} a_{1,n} \right)^{\frac{K+(K-1)\nu}{1-\mu-\nu}+\delta}.
\end{aligned}
$$

From this we obtain

$$
a_{1,N}^{\frac{1}{\frac{K+(K-1)\nu}{1-\mu-\nu}+\delta}} > \prod_{n=1}^{N-1} a_{1,n}. \tag{3.17}
$$

Lemma 3.2, (3.15) and (3.17) imply that

$$
\begin{aligned}
1 &\le Q \left(\prod_{n=1}^{N-1} a_{1,n} \right)^{K+(K-1)\nu} \sum_{i=1}^{K} \sum_{n=N}^{\infty} \frac{b_{i,n}}{a_{i,n}} \\
&\le Q \left(\prod_{n=1}^{N-1} a_{1,n} \right)^{K+(K-1)\nu} \frac{2K \log_2 a_{1,N}}{a_{1,N}^{1-\mu-\nu}} \\
&< \frac{2KQ a_{1,N}^{\frac{K+(K-1)\nu}{\frac{K+(K-1)\nu}{1-\mu-\nu}+\delta}} \log_2 a_{1,N}}{a_{1,N}^{1-\mu-\nu}} \\
&= \frac{2KQ \log_2 a_{1,N}}{a_{1,N}^{1-\mu-\nu-\frac{K+(K-1)\nu}{\frac{K+(K-1)\nu}{1-\mu-\nu}+\delta}}} = \frac{2KQ \log_2 a_{1,N}}{a_{1,N}^{\frac{\delta(1-\mu-\nu)^2}{K+(K-1)\nu+\delta(1-\mu-\nu)}}} < 1
\end{aligned}
$$

for infinitely many sufficiently large N. This is a contradiction.
2. Let us assume that (3.6) holds for every sufficiently large n and there is no $\delta > 0$ such that (3.16) holds. Hence for every $\delta > 0$ we have

$$
\limsup_{n\to\infty} a_{1,n}^{\frac{1}{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1+\frac{\delta}{2}\right)^n}} < \infty.
$$

This and the fact that

$$
\lim_{n\to\infty} \frac{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1+\frac{\delta}{2}\right)^n}{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1+\delta\right)^n} = 0
$$

imply that

$$\limsup_{n\to\infty} a_{1,n}^{\frac{1}{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1+\delta\right)^n}} = \limsup_{n\to\infty} \left(a_{1,n}^{\frac{1}{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1+\frac{\delta}{2}\right)^n}}\right)^{\frac{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1+\frac{\delta}{2}\right)^n}{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1+\delta\right)^n}} = 1.$$

From this we see that

$$a_{1,n} < 2^{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+2\right)^n} \tag{3.18}$$

holds for every sufficiently large n. Equation (2.1) implies

$$a_{1,N}^{\frac{1}{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^N}} > \left(1+\frac{1}{N^2}\right) \max_{k=1,\dots,N-1} a_{1,k}^{\frac{1}{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^k}} \tag{3.19}$$

for infinitely many N. Otherwise there would exist n_0 such that for every $n \geq n_0$

$$\begin{aligned}
a_{1,n}^{\frac{1}{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^n}} &\leq \left(1+\frac{1}{n^2}\right) \max_{k=1,\dots,n-1} a_{1,k}^{\frac{1}{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^k}} \\
&\leq \left(1+\frac{1}{n^2}\right)\left(1+\frac{1}{(n-1)^2}\right) \max_{k=1,\dots,n-2} a_{1,k}^{\frac{1}{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^k}} \\
&\leq \dots \leq \prod_{j=n_0+1}^{n}\left(1+\frac{1}{j^2}\right) a_{1,n_0}^{\frac{1}{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^{n_0}}} \\
&\leq \dots \leq \prod_{j=n_0+1}^{\infty}\left(1+\frac{1}{j^2}\right) a_{1,n_0}^{\frac{1}{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^{n_0}}} < const.,
\end{aligned}$$

which contradicts (2.1). Hence for infinitely many N

(3.20)

$$\begin{aligned}
a_{1,N} &> \left(1+\frac{1}{N^2}\right)^{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^N} \left(\max_{k=1,\dots,N-1} a_{1,k}^{\frac{1}{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^k}}\right)^{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^N} \\
&> \left(1+\frac{1}{N^2}\right)^{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^N} \times \\
&\quad \times \left(\max_{k=1,\dots,N-1} a_{1,k}^{\frac{1}{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^k}}\right)^{\frac{K+(K-1)\nu}{1-\mu-\nu}\left(\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^{N-1}+\dots+1\right)} \\
&> \left(1+\frac{1}{N^2}\right)^{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^N} \left(\prod_{n=1}^{N-1} a_{1,n}\right)^{\frac{K+(K-1)\nu}{1-\mu-\nu}}.
\end{aligned}$$

Using Lemma 3.2, (3.15), (3.18) and (3.20) we obtain

$$
\begin{aligned}
1 &\le Q\left(\prod_{n=1}^{N-1} a_{1,n}\right)^{K+(K-1)\nu} \sum_{i=1}^{K}\sum_{n=N}^{\infty}\frac{b_{i,n}}{a_{i,n}}\\
&\le Q\left(\prod_{n=1}^{N-1} a_{1,n}\right)^{K+(K-1)\nu} \frac{2K\log_2 a_{1,N}}{a_{1,N}^{1-\mu-\nu}}\\
&< Q\frac{a_{1,N}^{1-\mu-\nu}}{\left(1+\frac{1}{N^2}\right)^{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^N(1-\mu-\nu)}}\frac{2K\log_2 a_{1,N}}{a_{1,N}^{1-\mu-\nu}}\\
&= \frac{2KQ\log_2 a_{1,N}}{\left(1+\frac{1}{N^2}\right)^{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^N(1-\mu-\nu)}}\\
&= \frac{2KQ\log_2 a_{1,N}}{2^{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^N(1-\mu-\nu)\log_2(1+\frac{1}{N^2})}}\\
&< \frac{2KQ\log_2 2^{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+2\right)^N}}{2^{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^N(1-\mu-\nu)\log_2(1+\frac{1}{N^2})}}\\
&= \frac{2KQ\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+2\right)^N}{2^{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^N(1-\mu-\nu)\log_2(1+\frac{1}{N^2})}} < 1
\end{aligned}
$$

for infinitely many N. This is a contradiction.
3. Now let us assume for infinitely many n that

$$a_{1,n} \le 2^n \tag{3.21}$$

and that there is a $\delta > 0$ such that (3.16) holds. Let A be a sufficiently large positive integer. From (3.16) we see that there exists n such that

$$a_{1,n}^{\frac{1}{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1+\delta\right)^n}} > A. \tag{3.22}$$

Let k be the least positive integer satisfying (3.22) and s be the greatest positive integer less than k such that (3.21) holds. So

$$a_{1,k} > A^{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1+\delta\right)^k} = 2^{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1+\delta\right)^k\log_2 A}. \tag{3.23}$$

Then there is a positive integer n such that

$$a_{1,n}^{\frac{1}{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1+\delta\right)^n}} > 2. \tag{3.24}$$

Let t be the least positive integer greater than s such that (3.24) holds. It follows that for every $r = s, s+1, \ldots, t-1$

$$a_{1,r} < 2^{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1+\delta\right)^r} \tag{3.25}$$

and

$$a_{1,t} > 2^{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1+\delta\right)^t}. \tag{3.26}$$

The fact that the number A is sufficiently large such that $A > 2$ and the definitions of the numbers t and k imply $t \le k$. From (3.25) and (3.26) we obtain

$$\begin{aligned} a_{1,t} &> 2^{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1+\delta\right)^t} \\ &> 2^{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+\delta\right)\left(\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1+\delta\right)^{t-1}+\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1+\delta\right)^{t-2}+\cdots+1\right)} \\ &> \left(\prod_{n=1}^{t-1} 2^{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1+\delta\right)^n}\right)^{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+\delta\right)} \\ &> \left(\prod_{n=1}^{t-1} a_{1,n}\right)^{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+\delta\right)} \left(\prod_{n=1}^{s} a_{1,n}\right)^{-\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+\delta\right)}. \end{aligned} \tag{3.27}$$

The sequence $\{a_{1,n}\}_{n=1}^{\infty}$ is non-decreasing and $a_{1,s} \le 2^s$. It follows that

$$\prod_{n=1}^{s} a_{1,n} < 2^{s^2}. \tag{3.28}$$

Together with (3.27) this implies that

$$\begin{aligned} a_{1,t} &> \left(\prod_{n=1}^{t-1} a_{1,n}\right)^{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+\delta\right)} \left(\prod_{n=1}^{s} a_{1,n}\right)^{-\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+\delta\right)} \\ &> \left(\prod_{n=1}^{t-1} a_{1,n}\right)^{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+\delta\right)} \cdot 2^{-\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+\delta\right)s^2}. \end{aligned} \tag{3.29}$$

Inequalities (3.25) and (3.28) yield

$$\begin{aligned} \prod_{n=1}^{t-1} a_{1,n} = \prod_{n=1}^{s-1} a_{1,n} \cdot \prod_{n=s}^{t-1} a_{1,n} &< \prod_{n=1}^{s-1} a_{1,n} \cdot \prod_{n=1}^{t-1} 2^{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1+\delta\right)^n} \\ &< 2^{s^2} \cdot 2^{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1+\delta\right)^t \frac{1}{\frac{K+(K-1)\nu}{1-\mu-\nu}+\delta}}. \end{aligned} \tag{3.30}$$

The definitions of the numbers s, t and k imply that $a_{1,n} > 2^n$ for all $n = t, t+1, \ldots, k$. From this fact, Lemma 3.1 and Lemma 3.2 we obtain

$$\sum_{i=1}^{K}\sum_{n=t}^{\infty} \frac{b_{i,n}}{a_{i,n}} = \sum_{i=1}^{K}\sum_{n=t}^{k-1} \frac{b_{i,n}}{a_{i,n}} + \sum_{i=1}^{K}\sum_{n=k}^{\infty} \frac{b_{i,n}}{a_{i,n}} < \frac{2K\log_2 a_{1,t}}{a_{1,t}^{1-\mu-\nu}} + \frac{1}{a_{1,k}^{B}}. \tag{3.31}$$

Now (3.15), (3.23), (3.29), (3.30) and (3.31) imply

$$
\begin{aligned}
1 &\le Q\left(\prod_{n=1}^{t-1} a_{1,n}\right)^{K+(K-1)\nu} \sum_{i=1}^{K}\sum_{n=t}^{\infty}\frac{b_{i,n}}{a_{i,n}} \\
&< Q\left(\prod_{n=1}^{t-1} a_{1,n}\right)^{K+(K-1)\nu}\left(\frac{2K\log_2 a_{1,t}}{a_{1,t}^{1-\mu-\nu}}+\frac{1}{a_{1,k}^{B}}\right) \\
&= \frac{\left(\prod_{n=1}^{t-1} a_{1,n}\right)^{K+(K-1)\nu} 2KQ\log_2 a_{1,t}}{a_{1,t}^{1-\mu-\nu}} + \frac{Q\left(\prod_{n=1}^{t-1} a_{1,n}\right)^{K+(K-1)\nu}}{a_{1,k}^{B}} \\
&= \frac{\left(\prod_{n=1}^{t-1} a_{1,n}\right)^{K+(K-1)\nu}}{a_{1,t}^{(1-\mu-\nu)\frac{K+(K-1)\nu}{K+(K-1)\nu+\delta(1-\mu-\nu)}}}\cdot\frac{2KQ\log_2 a_{1,t}}{a_{1,t}^{(1-\mu-\nu)\frac{\delta(1-\mu-\nu)}{K+(K-1)\nu+\delta(1-\mu-\nu)}}} \\
&\quad + \frac{Q\left(\prod_{n=1}^{t-1} a_{1,n}\right)^{K+(K-1)\nu}}{a_{1,k}^{B}} \\
&< \frac{a_{1,t}^{\frac{K+(K-1)\nu}{\frac{K+(K-1)\nu}{1-\mu-\nu}+\delta}} 2^{s^2(K+(K-1)\nu)}}{a_{1,t}^{(1-\mu-\nu)\frac{K+(K-1)\nu}{K+(K-1)\nu+\delta(1-\mu-\nu)}}}\cdot\frac{2KQ\log_2 a_{1,t}}{a_{1,t}^{\frac{\delta(1-\mu-\nu)^2}{K+(K-1)\nu+\delta(1-\mu-\nu)}}} \\
&\quad + \frac{Q\left(\prod_{n=1}^{t-1} a_{1,n}\right)^{K+(K-1)\nu}}{a_{1,k}^{B}} \\
&= 2^{s^2(K+(K-1)\nu)}\cdot\frac{2KQ\log_2 a_{1,t}}{a_{1,t}^{\frac{\delta(1-\mu-\nu)^2}{K+(K-1)\nu+\delta(1-\mu-\nu)}}} + \frac{Q\left(\prod_{n=1}^{t-1} a_{1,n}\right)^{K+(K-1)\nu}}{a_{1,k}^{B}} \\
&< 2^{s^2(K+(K-1)\nu)}\cdot\frac{2KQ\log_2 a_{1,t}}{a_{1,t}^{\frac{\delta(1-\mu-\nu)^2}{K+(K-1)\nu+\delta(1-\mu-\nu)}}} \\
&\quad + \frac{Q2^{s^2(K+(K-1)\nu)}\cdot 2^{(K+(K-1)\nu)\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1+\delta\right)^{t}\frac{1}{\frac{K+(K-1)\nu}{1-\mu-\nu}+\delta}}}{a_{1,k}^{B}} \\
&< 2^{s^2(K+(K-1)\nu)}\cdot\frac{2KQ\log_2 a_{1,t}}{a_{1,t}^{\frac{\delta(1-\mu-\nu)^2}{K+(K-1)\nu+\delta(1-\mu-\nu)}}} \\
&\quad + \frac{Q2^{s^2(K+(K-1)\nu)}\cdot 2^{(K+(K-1)\nu)\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1+\delta\right)^{t}\frac{1}{\frac{K+(K-1)\nu}{1-\mu-\nu}+\delta}}}{2^{B\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1+\delta\right)^{k}\log_2 A}} < 1
\end{aligned}
$$

and this is a contradiction for A large enough because $s \le t \le k$ tend to infinity with A.
4. Finally let us assume that (3.21) holds for infinitely many n and that there is no $\delta > 0$ such that (3.16) holds. This implies that (3.18) holds for every sufficiently large n. Let A be also sufficiently large. From (2.1) we obtain

$$a_{1,n}^{\frac{1}{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^n}} > A \tag{3.32}$$

for infinitely many n. Let k be the least positive integer satisfying (3.32). Then

$$a_{1,k} > A^{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^k} = 2^{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^k \log_2 A}. \tag{3.33}$$

Let s be the greatest positive integer less than k such that (3.21) holds. As in case 2, (3.19) holds for infinitely many N. Let t be the least positive integer greater than s satisfying

$$a_{1,t}^{\frac{1}{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^t}} > \left(1+\frac{1}{t^2}\right) \max_{j=s,\dots,t-1} a_{1,j}^{\frac{1}{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^j}} \tag{3.34}$$

and

$$a_{1,r}^{\frac{1}{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^r}} \le \left(1+\frac{1}{r^2}\right) \max_{j=s,\dots,r-1} a_{1,j}^{\frac{1}{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^j}} \tag{3.35}$$

for every $r = s+1, \dots, t-1$. From (3.35) we obtain

$$\begin{aligned} a_{1,r}^{\frac{1}{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^r}} &\le \left(1+\frac{1}{r^2}\right) \max_{j=s,\dots,r-1} a_{1,j}^{\frac{1}{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^j}} \\ &\le \left(1+\frac{1}{r^2}\right)\left(1+\frac{1}{(r-1)^2}\right) \max_{j=s,\dots,r-2} a_{1,j}^{\frac{1}{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^j}} \\ &\le \dots \le \prod_{j=s+1}^{r} \left(1+\frac{1}{j^2}\right) a_{1,s}^{\frac{1}{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^s}} \le D, \end{aligned}$$

where $D < \prod_{j=1}^{\infty}\left(1+\frac{1}{j^2}\right)$ is a positive real constant which does not depend on A and k. It follows that

$$a_{1,r} \le D^{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^r} = 2^{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^r \log_2 D} \tag{3.36}$$

for every $r = s+1, \dots, t-1$. From this together with $a_{1,s} < 2^s$ and the fact that the sequence $\{a_{1,n}\}_{n=1}^{\infty}$ is non-decreasing, we obtain that

$$
\begin{aligned}
\left(\prod_{r=1}^{t-1} a_{1,r}\right) &= \left(\prod_{r=1}^{s} a_{1,r}\right)\left(\prod_{r=s+1}^{t-1} a_{1,r}\right)\\
&\le \left(\prod_{r=1}^{s} 2^{s}\right)\left(\prod_{r=s+1}^{t-1} 2^{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^{r}\log_2 D}\right)\\
&= 2^{s^2}\cdot 2^{\frac{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^{t}-\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^{s+1}}{\frac{K+(K-1)\nu}{1-\mu-\nu}}\log_2 D}\\
&\le 2^{\frac{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^{t}}{\frac{K+(K-1)\nu}{1-\mu-\nu}}\log_2 D}.
\end{aligned}
\tag{3.37}
$$

Notice that (3.33) and (3.36) also imply that $t \le k$. Now from (3.34) with $a_{1,s} \le 2^s$ and the fact that the sequence $\{a_{1,n}\}_{n=1}^{\infty}$ is non-decreasing, we obtain that

(3.38)

$$
\begin{aligned}
a_{1,t} &> \left(1+\frac{1}{t^2}\right)^{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^{t}}\left(\max_{j=s,\dots,t-1} a_{1,j}^{\frac{1}{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^{j}}}\right)^{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^{t}}\\
&> \left(1+\frac{1}{t^2}\right)^{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^{t}}\\
&\quad \left(\max_{j=s,\dots,t-1} a_{1,j}^{\frac{1}{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^{j}}}\right)^{\frac{K+(K-1)\nu}{1-\mu-\nu}\left(\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^{t-1}+\dots+\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^{s}\right)}\\
&> \left(1+\frac{1}{t^2}\right)^{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^{t}}\left(\prod_{j=1}^{t-1} a_{1,j}\right)^{\frac{K+(K-1)\nu}{1-\mu-\nu}}\left(\prod_{j=1}^{s-1} a_{1,j}\right)^{-\frac{K+(K-1)\nu}{1-\mu-\nu}}\\
&> \left(1+\frac{1}{t^2}\right)^{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^{t}}\left(\prod_{j=1}^{t-1} a_{1,j}\right)^{\frac{K+(K-1)\nu}{1-\mu-\nu}} 2^{-\frac{K+(K-1)\nu}{1-\mu-\nu}t^2}.
\end{aligned}
$$

As in the third case Lemma 3.1 and Lemma 3.2 imply (3.31) for our definition of the number t.

Finally from (3.15), (3.18), (3.31), (3.33), (3.37), (3.38) we obtain

$$1 \leq Q\left(\prod_{n=1}^{t-1} a_{1,n}\right)^{K+(K-1)\nu} \sum_{i=1}^{K}\sum_{n=N}^{\infty} \frac{b_{i,n}}{a_{i,n}}$$

$$< Q\left(\prod_{n=1}^{t-1} a_{1,n}\right)^{K+(K-1)\nu} \left(\frac{2K\log_2 a_{1,t}}{a_{1,t}^{1-\mu-\nu}} + \frac{1}{a_{1,k}^{B}}\right)$$

$$= Q\left(\prod_{n=1}^{t-1} a_{1,n}\right)^{K+(K-1)\nu} \frac{2K\log_2 a_{1,t}}{a_{1,t}^{1-\mu-\nu}} + \frac{Q\left(\prod_{n=1}^{t-1} a_{1,n}\right)^{K+(K-1)\nu}}{a_{1,k}^{B}}$$

$$< \frac{Qa_{1,t}^{1-\mu-\nu} 2^{(K+(K-1)\nu)t^2}}{\left(1+\frac{1}{t^2}\right)^{(1-\mu-\nu)\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^t}} \frac{2K\log_2 a_{1,t}}{a_{1,t}^{1-\mu-\nu}} + \frac{Q\left(\prod_{n=1}^{t-1} a_{1,n}\right)^{K+(K-1)\nu}}{a_{1,k}^{B}}$$

$$= \frac{2KQ2^{(K+(K-1)\nu)t^2}\log_2 a_{1,t}}{\left(1+\frac{1}{t^2}\right)^{(1-\mu-\nu)\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^t}} + \frac{Q\left(\prod_{n=1}^{t-1} a_{1,n}\right)^{K+(K-1)\nu}}{a_{1,k}^{B}}$$

$$< \frac{2KQ2^{(K+(K-1)\nu)t^2}\log_2 2^{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+2\right)^t}}{2^{(1-\mu-\nu)\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^t \log_2\left(1+\frac{1}{t^2}\right)}} + \frac{Q\left(\prod_{n=1}^{t-1} a_{1,n}\right)^{K+(K-1)\nu}}{a_{1,k}^{B}}$$

$$= \frac{2KQ2^{(K+(K-1)\nu)t^2}\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+2\right)^t}{2^{(1-\mu-\nu)\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^t \log_2\left(1+\frac{1}{t^2}\right)}} + \frac{Q\left(\prod_{n=1}^{t-1} a_{1,n}\right)^{K+(K-1)\nu}}{a_{1,k}^{B}}$$

$$\leq \frac{2KQ2^{(K+(K-1)\nu)t^2}\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+2\right)^t}{2^{(1-\mu-\nu)\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^t \log_2\left(1+\frac{1}{t^2}\right)}} + \frac{Q\left(2^{\frac{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^t}{\frac{K+(K-1)\nu}{1-\mu-\nu}}\log_2 D}\right)^{K+(K-1)\nu}}{a_{1,k}^{B}}$$

$$= \frac{2KQ2^{(K+(K-1)\nu)t^2}\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+2\right)^t}{2^{(1-\mu-\nu)\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^t \log_2\left(1+\frac{1}{t^2}\right)}} + \frac{Q2^{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^t(1-\mu-\nu)\log_2 D}}{a_{1,k}^{B}}$$

$$< \frac{2KQ2^{(K+(K-1)\nu)t^2}\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+2\right)^t}{2^{(1-\mu-\nu)\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^t \log_2\left(1+\frac{1}{t^2}\right)}} + \frac{Q2^{\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^t(1-\mu-\nu)\log_2 D}}{2^{B\left(\frac{K+(K-1)\nu}{1-\mu-\nu}+1\right)^k \log_2 A}} < 1$$

This is a contradiction. Now the proof of Theorem 2.1 is complete. □

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