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# Weinstein's theorem for Finsler manifolds

To the memory of Professor Makoto Matsumoto

By

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**Abstract**

Here we prove the generalization of Weinstein's theorem for Finsler manifolds: an isometry of a compact oriented Finsler manifold of positive flag curvature has a fixed point supposed that it preserves the orientation of the manifold if its dimension is even, or reverses it if odd.

## 1. Introduction

Alan Weinstein proved in 1968 [13] that a conformal diffeomorphism of a compact oriented Riemannian manifold of positive sectional curvature has a fixed point supposed that it preserves the orientation of the manifold if its dimension is even, or reverses it if odd. Especially, it is true for isometries. It is not known whether the theorem is still true for any diffeomorphism. If yes, this would imply that $S^2 \times S^2$ does not carry a metric of positive curvature, since the map which is the antipodal map on each factor preserves the orientation, and does not have a fixed point.

Weinstein's theorem implies Synge's earlier theorem [11] stating that a compact manifold $M$ with positive sectional curvature is simply connected if $M$ is orientable and its dimension is even. Synge's theorem has been generalized for Finsler manifold by Auslander [2], see also in [3, p. 221]. The odd dimensional case remained open there.

An isometry of a Finsler manifold is a diffeomorphism $f : M \to M$ which preserves the distance, or equivalently, preserves the Finsler norm of the tangent vectors. This equivalence, the generalization of Myers-Steenrod theorem of Riemannian geometry was proved in [7]. Few other results are known about isometries of Finsler manifolds. In [12] Szabó determines all the non-Riemannian Finsler spaces having a group of motions of the largest order, and further shows interesting and important results on the problems of Berwald spaces, scalar curvature and projective flatness.

Our aim is now to prove the following

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Weinstein's Theorem for Finsler manifolds: *Let $f$ be an isometry of a compact oriented positively homogeneous Finsler manifold $M$ of dimension $n$. If $M$ has positive flag curvature and $f$ preserves the orientation of $M$ for $n$ even and reverses the orientation of $M$ for $n$ odd, then $f$ has a fixed point.*

As immediate consequence we obtain Synge's theorem for both even and odd dimensions. The proof of the first part is different from that given in [2], [3], and the second assertion was not covered there. Further consequences of the generalized Weinstein theorem will be described in a forthcoming paper.

A *Finsler manifold* is a manifold equipped with a Banach norm $F(x, \cdot)$ at each tangent spaces $T_xM$, called a Finsler fundamental function if

1. $F(x, y) > 0 \ \forall x \in M, y \in TM, \ \ y \neq 0$
2. $F(x, \lambda y) = \lambda F(x, y) \ \forall \lambda \in \mathbb{R}^+, y \in TM$
3. $F$ is smooth except on the zero section
4. $g_{ij}(x, y) = \dfrac{\partial^2(\frac{1}{2}F^2)}{\partial y^i \partial y^j}(x, y)$ is positive definite for any $(x, y) \neq 0$.

We remark that in condition (3) the exclusion of the zero section ensures that the homogeneity does not imply linearity. The last condition implies that the indicatrix body is strongly convex, and conversely. Notice that in condition (2) the homogeneity is supposed for positive $\lambda$ only, therefore we deal with positively homogeneous, *non-reversible Finsler metrics.*

The arc length of a curve $\gamma : [a, b] \to M$ in a Finsler manifold $(M, F)$ is defined as

$$s = \int_a^b F(\gamma(t), \dot{\gamma}(t))\, dt.$$

From the recent flourishing literature of Finsler geometry, we refer the reader the books [1, 3, 6, 8, 9]. In the proof we shall utilize the Chern connection. See for details [3] or [9]. Specially, we need the Riemann curvature $R_y(u)$ and the flag curvature $K(P, y)$ for any $y \in T_xM, u \in T_xM, x \in M$ with $P = \mathrm{span}\{y, u\}$. Finsler manifolds with positive flag curvature have been extensively studied recently. Bryant in [4] and Shen in [10] constructed fine examples for such spaces.

## 2. The proof

We follow the line of Weinstein, carefully adapted for Finsler setting.

Suppose that the isometry $f$ has no fixed points:$f(x) \neq x$ for all $x \in M$. Since the manifold $M$ is compact, the function $h : M \to \mathbb{R}$ given by $h(x) = d(x, f(x))$ attains its minimum at a point $x \in M$, so $h(x) > 0$ for all $x \in M$.

The completeness of the manifold $M$ implies that there exists a minimizing normalized forward geodesic $\sigma : [0, \ell] \to M$ joining $x$ and $f(x)$. We show that the curve formed by $\sigma$ and $f \circ \sigma$ gives a forward geodesic. Consider the forward geodesic $f \circ \sigma$ which joins $f(x)$ to $f^2(x)$, and a point $y = \sigma(t), \ \ t \in (0, \ell)$ on

$\sigma$ between $x$ and $f(x)$. Since $f$ is an isometry, $d(x,y) = d(f(x), f(y))$. By the triangle inequality it follows that:

$$\begin{aligned} d(y, f(y)) &\leq d(y, f(x)) + d(f(x), f(y)) \\ &= d(y, f(x)) + d(x, y) \\ &= d(x, f(x)). \end{aligned}$$

Since $x$ is a minimum for the function $h$, we have

$$d(y, f(y)) = d(y, f(x)) + d(f(x), f(y)),$$

so, the curve formed by $\sigma$ and $f \circ \sigma$ is a forward geodesic and this implies that it is smooth, that is

$$(f \circ \sigma)^{\cdot}(0) = \dot{\sigma}(\ell).$$

Clearly, if a map $f$ is an isometry of $(M, F)$: $F(x, u) = F(f(x), df_x(u))$ for $x \in M$ and $u \in T_x M$, i.e. $f$ is an isometry between the Minkowski spaces $(T_x M, F(x, \cdot))$ and $(T_f(x)M, F(f(x), df_x(\cdot))$ (cf. [3]), then, by the chain rule, we obtain that $g_{ij}(x, y)(v, w) = g_{ij}(f(x), df_x(y))(df_x(v), df_x(w))$, i.e. the isometry of a Finsler space gives rise to an isometry of the fibers over $(x, y)$ and $(f(x), df_x(y))$ in $\pi^* TM$.

Along a forward geodesic the Chern connection is metric compatible (see [3], p. 122). This implies that for any forward geodesic $\sigma(t), t \in [0, \ell]$, the linearly parallel transport $P_\sigma$ (see [6, p.73]) induced by the Chern connection along the forward geodesic $\sigma$, preserves the inner products $g_{\dot{\sigma}}$ along $\sigma$, that is

$$g_{\dot{\sigma}(t)}(P_{c(t)}(u), P_{c(t)}(v)) = g_{\dot{\sigma}(0)}(u, v), \text{ for } \ u, v \in T_{\sigma(0)}M.$$

See also [9], p. 89. This formula means that we have an isometry between the tangent spaces $T_{\sigma(t)}M$ along $\sigma$ with inner products $g_{\dot{\sigma}}$, induced by the linearly parallel transport.

Denote shortly by $P$ the linearly parallel transport along the forward geodesic $\sigma$ between the tangent spaces $T_x M$ and $T_{f(x)}M$. We can consider its inverse $G = P^{-1} : T_{f(x)}M \to T_x M$, which is an isometry, again. We consider now the map $B = G \circ df_x : T_x M \to T_x M$. By the above observations $B$ is an isometry.

We have the following relations

$$B(\dot{\sigma}(0)) = G \circ df_x(\dot{\sigma}(0)) = G((f \circ \sigma)^{\cdot}(0)) = G(\dot{\sigma}(\ell)) = \dot{\sigma}(0).$$

This means that $B$ leaves $\dot{\sigma}(0)$ fixed. Let $A$ be the restriction of $B$ to the $g_{\dot{\sigma}(0)}-$orthogonal complement of $\dot{\sigma}(0)$. $A$ is an isometry and since $P$ is an isometry which preserves the orientation it follows that

$$\det A = \det B = \det(G \circ df_x) = (-1)^n,$$

because of the hypothesis on $f$ and the fact that $G$ preserves the orientation.

By the Lemma from [5] p. 203, $A$ leaves a vector invariant. Let $E(t)$ be a unit linearly parallel vector field along $\sigma$ such that, $E(0)$ belongs to the $g_{\dot\sigma(0)}$– orthogonal complement of $\dot\sigma(0)$ and $E(0)$ is invariant by $A$: $A(E(0)) = E(0)$.

Next, take the forward geodesic $\alpha(s), s \in (-\epsilon, \epsilon)$, such that $\alpha(0) = x$, and $\dot\alpha(0) = E(0)$. We have $df_x(E(0)) = E(\ell)$ because $G \circ df_x(E(0)) = E(0)$, i.e., the forward geodesic $f \circ \alpha$ has the property that $(f \circ \alpha)(0) = f(x)$ and $(f \circ \alpha)^{\cdot}(0) = E(\ell)$.

Consider now the variation of $\sigma$ given by

$$h : (-\epsilon, \epsilon) \times [0, \ell] \to M$$
$$h(s, t) = \exp_{\sigma(t)}(sE(t)),\ s \in (-\epsilon, \epsilon),\ t \in [0, \ell].$$

Clearly $h(s, 0) = \alpha(s)$, moreover, we have

$$h(s, \ell) = \exp_{f(x)}(sE(\ell)) = (f \circ \alpha)(s),$$

for $(f \circ \alpha)^{\cdot}(0) = E(\ell)$. It follows then

$$\frac{\partial}{\partial s} \exp_{\sigma(t)}(sE(t))|_{s=0} = E(t),$$

so the transversal vector of the variation $h$ is linearly parallel transported along $\sigma$.

The second variation formula of the arc-length has the following form ([9, p. 161])

$$\begin{aligned} L''(0) = &\int_0^\ell \{g_{\dot\sigma}(\nabla_{\dot\sigma} E, \nabla_{\dot\sigma} E) - g_{\dot\sigma}(R_{\dot\sigma}(E), E)\}dt \\ &+ g_{\dot\sigma(\ell)}(\kappa_\ell(0), \dot\sigma(\ell)) - g_{\dot\sigma(0)}(\kappa_0(0), \dot\sigma(0)) \\ &+ \mathbf{T}_{\dot\sigma(0)}(E(0)) - \mathbf{T}_{\dot\sigma(\ell)}(E(\ell)) \end{aligned}$$

Here the quantities $\kappa_\ell(0)$ and $\kappa_0(0)$ are the geodesic curvatures of the transversal curves $\alpha(s)$ for $s = 0$. Being the transversal curves forward geodesics, the geodesic curvatures are zero. Furthermore, $\mathbf{T}$ represents the T-curvature (see [9, p. 153]), which depends on the Finsler metric, only. The points $\sigma(0)$ and $\sigma(\ell)$ are coupled by the isometry $f$, moreover, $df_x(E(0)) = E(\ell)$ holds, therefore $\mathbf{T}_{\dot\sigma(0)}(E(0)) = \mathbf{T}_{\dot\sigma(\ell)}(E(\ell))$. Finally in the first term $\nabla_{\dot\sigma} E$ is zero along the forward geodesic $\sigma$, since $E$ is linearly parallel transported along $\sigma$. By the above observations the second variation formula reduces to

$$L''(0) = -\int_0^\ell g_{\dot\sigma}(R_{\dot\sigma}(E), E)dt = -\int_0^\ell K(P, \dot\sigma) g_{\dot\sigma}(E, E)dt = -\int_0^\ell K(P, \dot\sigma)dt$$

so the second variation is negative because the flag curvature is positive. But this contradicts the minimality of $\sigma$, the curve which joins $x$ and $f(x)$. Therefore $d(x, f(x)) > 0$ is impossible. □

## 3. Synge's theorem for Finsler manifolds

In this section we prove the Synge theorem in the Finslerian context, using our main result.

**Theorem 3.1.** *Let $(M, F)$ be a compact Finsler manifold of positive flag curvature of dimension $n$.*

1. *If $M$ is orientable and $n$ is even, then $M$ is simply connected.*
2. *If $n$ is odd, then $M$ is orientable.*

*Proof.* (1) Consider the universal covering $\pi : \widetilde{M} \to M$ and the covering metric on $\widetilde{M}$. We can choose an orientation on $\widetilde{M}$ such that the covering map $\pi$ preserves the orientation. Because $M$ is compact, the flag curvature is strictly positive, that is, there exists $\delta > 0$ such that the flag curvature is greater or equal to $\delta$. The same bound on the curvature holds on $\widetilde{M}$ because $\pi$ is a local isometry (see [3], p. 197). Consider now a covering transformation $\tau : \widetilde{M} \to \widetilde{M}$: $\pi \circ \tau = \pi$. $\tau$ is an isometry of $\widetilde{M}$ which preserves the orientation. Because $n$ is even, due to our main theorem $\tau$ has a fixed point, so $\tau$ is the identity (because a covering transformation which has a fixed point is the identity). This implies that the group of covering transformations reduces to the identity, and therefore $M$ is simply connected.

(2) Suppose that $M$ is not orientable and consider the orientable double cover $\widetilde{M}$ of $M$ (see [5], p. 34), with the covering metric and denote the covering map by $p : \widetilde{M} \to M$. $\widetilde{M}$ is compact because it is the double cover of a compact manifold. Consider a deck transformation $\tau$ of $\widetilde{M}$, $\tau \neq \mathrm{id}$ with the covering metric, that is $p \circ \tau = p$. From the unique lifting property of the covering space and the fact that $M$ is connected, so $\widetilde{M}$ is, it follows that the deck transformation is completely determined by the image of a point, particularly only the identity transformation of $\widetilde{M}$ has fixed points.

Denote by $\widetilde{F} = p^*F$ the covering metric, that is $\widetilde{F}(\widetilde{x}, \widetilde{v}) = F(x, v)$ where $p(\widetilde{x}) = x,\ dp(\widetilde{v}) = v$. Because $p \circ \tau = p$ and the Finsler metric on $\widetilde{M}$ is the pull back of the metric of $M$, $\tau$ is an isometry of $\widetilde{M}$ which reverses the orientation of $\widetilde{M}$ because $M$ is not orientable. Consequently, $\tau$ has a fixed point, for $n$ is odd, which gives a contradiction. □

**Remark.** It is clear that the assumptions (1) of this theorem are really necessary. Namely, the case of real projective space of dimension 2 shows the necessity of orientability, and the real projective space of dimension 3 gives a counterexample for odd dimension. See also [3], page 224.

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