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Classification of regular and non-degenerate projectively Anosov flows on three-dimensional manifolds

By

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Abstract

We give a classification of C^2-regular and non-degenerate projectively Anosov flows on three-dimensional manifolds. More precisely, we prove that such a flow must be either an Anosov flow or represented as a finite union of $\mathbb{T}^2 \times I$-models.

1. Introduction

Mitsumatsu [8], and Eliashberg and Thurston [5] observed that any Anosov flow on a three-dimensional manifold induces a pair of mutually transverse positive and negative contact structures. They also showed that such pairs correspond to projectively Anosov flows, which form a wider class than that of Anosov flows. In [5], Eliashberg and Thurston studied projectively Anosov flows, which are called conformally Anosov flows in their book, from the viewpoint of confoliations.

The definition of a projectively Anosov flow is as follows: Let $\Phi = \{\Phi^t\}_{t\in\mathbb{R}}$ be a flow on a three-dimensional manifold M without stationary points. Let $T\Phi$ denote the one-dimensional subbundle of the tangent bundle TM that is tangent to the flow. The flow Φ induces a flow $\{N\Phi^t\}$ on $TM/T\Phi$. We call a decomposition $TM = E^s + E^u$ by two-dimensional subbundles E^u and E^s *a* $\mathbb{P}$A *splitting* associated with Φ if

1. $E^u(z) \cap E^s(z) = T\Phi(z)$ for any $z \in M$,
2. $D\Phi^t(E^\sigma(z)) = E^\sigma(\Phi^t(z))$ for any $\sigma \in \{u, s\}$, $z \in M$, and $t \in \mathbb{R}$, and
3. there exist constants $C > 0$ and $\lambda \in (0, 1)$ such that

$$\|(N\Phi^t|_{(E^u/T\Phi)(z)})^{-1}\| \cdot \|N\Phi^t|_{(E^s/T\Phi)(z)}\| \leq C\lambda^t$$

for all $z \in M$ and $t > 0$.

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Note that it can be shown that each subbundle is continuous and integrable, and that the splitting is unique by the same argument for hyperbolic splittings. A flow is called *a projectively Anosov flow* (or simply $\mathbb{P}$A flow) if it admits a $\mathbb{P}$A splitting. Remark that any Anosov flow is a $\mathbb{P}$A flow.

The subbundles E^s and E^u are not uniquely integrable, and hence, do not generate foliations in general. We say a $\mathbb{P}$A flow is (C^r-)*regular* when both subbundles generate (C^r-)smooth foliations. There are two known classes of regular $\mathbb{P}$A flows. One is the class of regular Anosov flows. Ghys [6] gave the complete classification of such flows. In fact, he showed that any regular Anosov flow must be equivalent to either a quasi-Fuchsian flow or the suspension of an Anosov automorphism on the torus. Another known class is that of flows represented by finite union of $\mathbb{T}^2 \times I$-models given by Noda [9]. Roughly speaking, a $\mathbb{T}^2 \times I$-model is a $\mathbb{P}$A flow on $\mathbb{T}^2 \times [0,1]$ which preserves the boundary tori, is equivalent to a linear flow on them, and is transverse to $\mathbb{T}^2 \times \{z\}$ for any $z \in (0,1)$. See [9] for the precise definition.

A natural question is whether there are other regular $\mathbb{P}$A flows or not. Noda and Tsuboi showed that no other $\mathbb{P}$A flows on certain manifolds. Their results are summarized as follows:

Theorem 1.1 ([9], [10], [11], and [13])**.** *Let M be a $\mathbb{T}^2$-bundle on S^1 or a Seifert fibered manifold. Then, any regular* $\mathbb{P}$A *flow on M must be either an Anosov flow or represented as a finite union of $\mathbb{T}^2 \times I$-models.*

We say a dynamical system is *non-degenerate* when all periodic orbits are hyperbolic. In this paper, we show that there are no new regular and non-degenerate $\mathbb{P}$A flows on *any* three-dimensional manifold.

Theorem 1.2. *A C^2-regular and non-degenerate* $\mathbb{P}$A *flow on a connected and closed three-dimensional manifold must be either an Anosov flow or represented as a finite union of $\mathbb{T}^2 \times I$ models.*

The proof of Theorem 1.2 is divided into three parts. In Subsection 2.1, we review the stability of semi-proper annular leaves for a C^2 codimension-one foliation. Subsection 2.2 is the main step of the proof. We show that any regular and non-degenerate $\mathbb{P}$A flow with a periodic orbit is Anosov. In Subsection 2.3, we show that any regular and non-degenerate $\mathbb{P}$A flow without a periodic orbit is represented by a finite union of $\mathbb{T}^2 \times I$-models. It is an easy consequence of the results of Arroyo and Rodriguez Hertz [1] and the classification by Noda.

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2. Proof of Theorem 1.2

2.1. Semi-proper annular leaves of codimension-one foliations

In this subsection, we review the stability of semi-proper annular leaves for C^2 codimension-one foliation on three-dimensional manifolds.

For a foliation $\mathcal{F}$, let $\mathcal{F}(z)$ denote the leaf through a point z. Recall that a leaf L of transversely orientable codimension-one foliation $\mathcal{F}$ is called *semi-proper* if L does not accumulate to itself at lease from one side, and is called *proper* if L does not accumulate to itself from either side.

Proposition 2.1. *Let M be a closed three-manifold and $\mathcal{F}$ a C^2 codimension-one foliation on M. Suppose that a leaf L of $\mathcal{F}$ is semi-proper and is homeomorphic to $S^1 \times \mathbb{R}$. Then, L is a proper leaf with trivial holonomy.*

Proof. Suppose that L is a semi-proper but not proper leaf. By the level theory of Cantwell and Conlon, L is at finite level, and hence, it is contained in an exceptional local minimal set X. See Lemma 8.3.23 and Theorem 8.3.11 of [3], for instance. However, Duminy's theorem (see Theorem 1.1 of [4]) asserts that the end of any semi-proper leaf of an exceptional local minimal set must be a Cantor set. It contradicts that L is homeomorphic to $S^1 \times \mathbb{R}$. Therefore, L is a proper leaf of $\mathcal{F}$. A theorem of Cantwell and Conlon [2, Theorem 1] on the stability of ends of proper leaves implies that the leaf L has trivial holonomy. □

2.2. Regular and non-degenerate $\mathbb{P}$A flows

The main aim of this subsection is to show the following proposition.

Proposition 2.2. *Any C^2-regular and non-generate $\mathbb{P}$A flow with a periodic orbit is Anosov.*

Let $\Phi = \{\Phi^t\}$ be a $\mathbb{P}$A flow on a closed three-dimensional manifold M and $TM = E^s + E^u$ a $\mathbb{P}$A splitting associated with Φ. Suppose that Φ is C^2-regular and non-degenerate. Let $\mathrm{Per}(\Phi)$ be the set of all periodic points of Φ, in other words, the union of all periodic orbits. Let $\overline{\mathrm{Per}(\Phi)}$ denote the closure of $\mathrm{Per}(\Phi)$.

Let $\mathcal{F}^u$ and $\mathcal{F}^s$ be the C^2 foliations generated by E^u and E^s. Without loss of generality, we may assume that $\mathcal{F}^u$ and $\mathcal{F}^s$ are transversely orientable. For $z \in M$, let $\mathcal{O}(z)$ denote the orbit $\{\Phi^t(z) \mid t \in \mathbb{R}\}$ of z, and $\mathcal{F}^s(z)$ and $\mathcal{F}^u(z)$ denote the leaves of $\mathcal{F}^s$ and $\mathcal{F}^u$ through z. We define *the strong unstable set* $W^{uu}(z)$ and *the unstable set* $W^u(z)$ of $z \in M$ by

$$W^{uu}(z) = \{z' \in M \mid \lim_{t\to\infty} d(\Phi^{-t}(z), \Phi^{-t}(z')) = 0\},$$

and $W^u(z) = \bigcup_{z' \in \mathcal{O}(z)} W^{uu}(z')$. *The strong stable set* $W^{ss}(z)$ and *the stable set* $W^s(z)$ are defined by $W^{ss}(z) = W^{uu}(z; \{\Phi^{-t}\})$ and $W^s(z) = W^u(z; \{\Phi^{-t}\})$.

The key step of the proof of Proposition 2.2 is to show that our assumptions imply $W^u(z) = \mathcal{F}^u(z)$ for any $z \in \overline{\mathrm{Per}(\Phi)}$. We emphasize that a regular $\mathbb{P}$A flow may not satisfy $W^u(z) = \mathcal{F}^u(z)$ for some $z \in \overline{\mathrm{Per}(\Phi)}$ in general. For example, a regular $\mathbb{P}$A flow may admit a toral leaf T of $\mathcal{F}^u$ consisting of periodic orbits. In this case, it is easy to see that $W^u(z)$ coincides with $\mathcal{O}(z)$, and hence, is a proper subset of $T = \mathcal{F}^u(z)$ for any $z \in T$.

First, we investigate the topology of the unstable sets. We say a Φ-invariant embedded torus T is *irrational* if the restriction of the flow on T is topologically conjugate to an irrational linear flow on the torus. The following is an immediate corollary of Theorem B of [1].

Proposition 2.3. *Let $\Omega(\Phi)$ be the non-wandering set of Φ. Then, there exists a mutually disjoint decomposition $\Omega(\Phi) = \Omega_0 \sqcup \Omega_1 \sqcup \Omega_2$ such that*

1. *Ω_1 is a compact hyperbolic set of saddle type,*
2. *Ω_0 is the union of finitely many attracting periodic orbits and irrational toral leaves of $\mathcal{F}^u$.*
3. *Ω_2 is the union of finitely many repelling periodic orbits and irrational toral leaves of $\mathcal{F}^s$.*

Notice that an irrational toral leaf T of $\mathcal{F}^u$ is normally attracting, and hence, $W^u(z) = \mathcal{O}(z)$ for any $z \in T$ and $\bigcup_{z\in T} W^s(z)$ is a neighborhood of T which is homeomorphic to $\mathbb{T}^2 \times \mathbb{R}$. Using the theory of hyperbolic invariant sets (see Chapter 9 of [14] for example), we have $M = \{\bigcup_{z\in\Omega_1\cup\Omega_2} W^u(z)\} \cup \Omega_0$ and $M = \{\bigcup_{z\in\Omega_0\cup\Omega_1} W^s(z)\} \cup \Omega_2$.

Lemma 2.1. *Let σ be either u or s, p a repelling periodic point, and V an annular neighborhood of $\mathcal{O}(p)$ in $\mathcal{F}^\sigma(p)$. If a leaf L of $\mathcal{F}^\sigma$ contains a simple closed curve $\gamma \subset W^u(p)$ which is not null-homotopic in L, then $\Phi^{-t}(\gamma) \subset V$ for some $t > 0$. In particular, we have $L = \mathcal{F}^\sigma(p)$.*

Proof. Since p is repelling, there exists a neighborhood U of $\mathcal{O}(p)$ such that U is homeomorphic to $S^1 \times [-1,1]^2$, the restriction $\mathcal{F}^\sigma|_U$ of $\mathcal{F}^\sigma$ on U has a unique annular leaf L_0 with $\mathcal{O}(p) \subset L_0 \subset V$, and other leaves of $\mathcal{F}^\sigma|_U$ are homeomorphic to $\mathbb{R} \times [-1,1]$. See Figure 1.

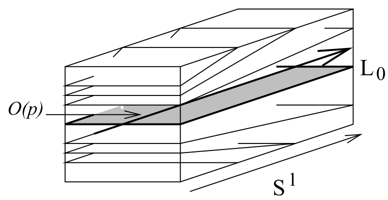

Figure 1. The restriction of $\mathcal{F}^\sigma$ on U

Suppose that $L \cap W^u(p)$ contains a simple closed curve γ which is not null-homotopic in L. Choose a sufficiently large $t_* > 0$ so that $\Phi^{-t_*}(\gamma)$ is contained in U. Since $\Phi^{-t_*}(\gamma)$ is not null-homotopic in L, we obtain $\Phi^{-t_*}(\gamma) \subset L_0 \subset V$. □

Lemma 2.2. *$W^u(z) \cap \mathcal{F}^u(z)$ is an open subset of $\mathcal{F}^u(z)$ for any $z \in M \backslash \Omega_0$.*

Proof. The lemma is trivial if z is contained in the unstable set of a repelling periodic point. Hence, it is sufficient to show that $W^u(z)$ is tangent to E^u for any $z \in \Omega_1 \cup \Omega_2$ which is not a repelling periodic point. By the local unstable manifold theorem or the local strong unstable manifold theorem, there exists an injectively immersed open two-dimensional manifold V such that $\bigcap_{t>0} \Phi^{-t}(V) = \mathcal{O}(z)$, $V \subset \Phi^t(V)$ for any $t > 0$, and $T_{z'}V = E^u(z')$ for any $z' \in \mathcal{O}(z)$. Then, the domination property and the invariance of the splitting $TM = E^s + E^u$ implies that V, and hence, $W^u(z)$ must be tangent to E^u. □

For $z \in M \backslash \Omega_0$, let $V^u(z)$ denote the connected component of $W^u(z) \cap \mathcal{F}^u(z)$ that contains z. By the above lemma, $V^u(z)$ is an open subset of $\mathcal{F}^u(z)$.

For a topological space X and its subspace Y, we say a point $x \in X \backslash Y$ is *accessible* from Y if there exists a continuous map $l : [0,1] \to X$ such that $l(1) = x$ and $l([0,1)) \subset Y$.

Lemma 2.3. *For any $z \in M \backslash \Omega_0$, if $q \in \mathcal{F}^u(z) \backslash V^u(z)$ is accessible from $V^u(z)$ then it is an attracting periodic point.*

Proof. By Proposition 2.3, there are three possibilities: $q \in W^u(\Omega_1 \cup \Omega_2)$, q is contained in an irrational toral leaf T of $\mathcal{F}^u$, or q is an attracting periodic point. If the first occurs, then $V^u(q) \cap V^u(z) \neq \emptyset$ since $V^u(q)$ is an open subset of $\mathcal{F}^u(z)$. It implies that $W^u(z) = W^u(q)$. However, it contradicts $q \notin W^u(z)$. The second implies $T = \mathcal{F}^u(q) = \mathcal{F}^u(z)$ is a subset of Ω_0. It contradicts that the assumption $z \in M \backslash \Omega_0$. □

Lemma 2.4. *For a repelling periodic point p of Φ,*

1. *$V^u(z) = \mathcal{F}^u(z)$ for any $z \in W^u(p) \backslash V^u(p)$, and*
2. *if $V^u(p) \neq \mathcal{F}^u(p)$, then there exists an attracting periodic point $q \in \mathcal{F}^u(p)$ and an embedded closed annulus $A \subset \mathcal{F}^u(p)$ such that $\partial A = \mathcal{O}(p) \cup \mathcal{O}(q)$ and* Int $A \subset V^u(p) \cap W^s(q)$.

Proof. Suppose $V^u(z) \neq \mathcal{F}^u(z)$ for $z \in W^u(p)$. Take a point $q \in \mathcal{F}^u(z) \backslash V^u(z)$ which is accessible from $V^u(z)$. By Lemma 2.3, q is an attracting periodic point. Hence, we can take a compact annulus $A_0 \subset \mathcal{F}^u(z) = \mathcal{F}^u(q)$ so that $A_0 \backslash \mathcal{O}(q) \subset V^u(z)$, where $\mathcal{O}(q)$ is a boundary component of A_0 and the other boundary component γ is transverse to the flow. By the Poincaré-Bendixon theorem, γ is not null-homotopic in $\mathcal{F}^u(z)$. Take an annular neighborhood V of $\mathcal{O}(p)$ in $V^u(p)$ and apply Lemma 2.1 to $\sigma = u$, $L = \mathcal{F}^u(z)$, and V. Then, we obtain $t > 0$ satisfying $\Phi^{-t}(\gamma) \subset V$. In particular, we have $\gamma \subset V^u(p)$, and hence, $V^u(z) = V^u(p)$. It is easy to find the required closed annulus in $\Phi^{-t}(A_0) \cup V$. □

The next proposition is the key step of the proof.

Proposition 2.4. *The flow Φ has neither repelling nor attracting periodic points. In particular, $\mathcal{F}^u(z) = W^u(z)$ and $\mathcal{F}^s(z) = W^s(z)$ for any $z \in \Omega_1$.*

Proof. We show that Φ has no repelling periodic points. It also implies the non-existence of attracting periodic points once we replace $\Phi = \{\Phi^t\}$ by $\{\Phi^{-t}\}$. Then, the latter assertion follows from Lemma 2.3 and the fact that $W^u(z)$ is an immersed manifold tangent to E^u for any $z \in \Omega_1$.

Assume that there exists a repelling periodic point p. Take an annular neighborhood V_0 of p in $\mathcal{F}^u(p)$ such that $\Phi^{-t}(V_0) \subset V_0$ for any $t > 0$ and $\bigcap_{t>0} \Phi^{-t}(V_0) = \mathcal{O}(p)$. Note that $V^u(p) = \bigcup_{t>0} \Phi^t(V_0)$. We also take a neighborhood U of $\mathcal{O}(p)$ in M such that $U \cap \partial V_0 = \emptyset$ and $\Phi^{-t}(U) \subset U$ for any $t > 0$. We claim $U \cap V^u(p) = U \cap V_0$. In fact, suppose $\Phi^{t_0}(z) \in U$ for $z \in V_0$ and $t_0 > 0$. Then, we have $\Phi^t(z) \in U$ for any $t \in [0, t_0]$. If $\Phi^{t_0}(z) \notin V_0$, then there exists $t_1 \in (0, t_0)$ such that $\Phi^{t_1}(z) \in \partial V_0$ since $\Phi^t(V_0) \supset V_0$ for any $t > 0$. It contradicts that $U \cap \partial V_0 = \emptyset$. Hence, $\Phi^{t_0}(z)$ is a point of V_0.

If $\mathcal{F}^u(p) = V^u(p)$, the leaf $\mathcal{F}^u(p)$ is proper since $U \cap V^u(p) = U \cap V_0$. However, it contradicts Proposition 2.1 since $\mathcal{F}^u(p) = V^u(p) = \bigcup_{t>0} \Phi^t(V_0)$ is homeomorphic to $S^1 \times \mathbb{R}$ but the linear holonomy along $\mathcal{O}(p)$ is not trivial.

Second, we assume $\mathcal{F}^u(p) \neq V^u(p)$. By Lemma 2.4, there exist an attracting periodic point q and an embedded compact annulus A such that $\partial A = \mathcal{O}(p) \cup \mathcal{O}(q)$ and Int $A \subset V^u(p) \cap W^s(q)$. It is important to remark that the orientations of the orbits of p and q are opposite in A. It is because the holonomy of $\mathcal{F}^u$ along $\mathcal{O}(p)$ is expanding and that along $\mathcal{O}(q)$ is contracting.

Since q is attracting, there exists an annular neighborhood A^s of $\mathcal{O}(q)$ in $\mathcal{F}^s(q)$ such that $\Phi^t(A^s) \subset A^s$ for any $t > 0$, $\bigcap_{t\geq 0} \Phi^t(A^s) = \mathcal{O}(q)$, and $\mathcal{F}^u(z) \cap (U \backslash V_0) \neq \emptyset$ for any $z \in A^s \backslash \mathcal{O}(q)$. Since $U \cap V^u(p) = U \cap V_0$, Lemma 2.4 implies that $\mathcal{F}^u(z') \subset W^u(p)$ for any $z' \in U \backslash V_0$. Hence, we have $A^s \backslash \mathcal{O}(q) \subset W^u(p)$. See Figure 2. The Poincaré-Bendixon theorem implies that each boundary

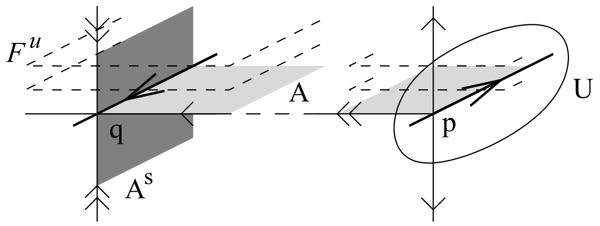

Figure 2. Proof of Lemma 2.4.

component of A^s is not null-homotopic in $\mathcal{F}^s(q)$. Applying Lemma 2.1 to $\sigma = s$ and $L = \mathcal{F}^s(q)$, we obtain $\mathcal{F}^s(p) = \mathcal{F}^s(q)$.

Put $\lambda_p^\sigma = \|N\Phi^{T_p}|_{(E^\sigma/T\Phi)}(p)\|$ and $\lambda_q^\sigma = \|N\Phi^{T_q}|_{(E^\sigma/T\Phi)}(q)\|$ for $\sigma = u, s$, where T_p and T_q are the period of p and q respectively. Since $\mathcal{F}^u(p) = \mathcal{F}^u(q)$, $\mathcal{F}^s(p) = \mathcal{F}^s(q)$, and the orientations of the orbits of p and q are opposite, we have $\lambda_p^u \lambda_q^u = \lambda_p^s \lambda_q^s = 1$. On the other hand, we also have $\lambda_p^s < \lambda_p^u$ and $\lambda_q^s < \lambda_q^u$ since Φ is a $\mathbb{P}\mathbb{A}$ flow. It is a contradiction. □

Lemma 2.5. *The set* Ω_1 *coincides with* $\overline{\mathrm{Per}(\Phi)}$. *For* $z \in \Omega_1$, *the leaf* $\mathcal{F}^s(z)$ *contains a periodic orbit if it is semi-proper.*

Proof. Since Ω_0 consists of normally attracting invariant tori and Ω_2 consists of normally repelling invariant tori, we can take a neighborhood U of Ω_1 so that $\Omega_1 = \bigcap_{t \in \mathbb{R}} \Phi^t(U)$, in other words, Ω_1 is a locally maximal hyperbolic set. By the theory of hyperbolic invariant sets, we obtain $\overline{\mathrm{Per}(\Phi)} = \Omega_1$. See [14].

The second assertion follows from Proposition 2.4 and a variant of Proposition 1 of [12] for three-dimensional flows. □

Proof of Proposition 2.2. We show $M = \Omega_1$. It implies that Φ is an Anosov flow since Ω_1 is a hyperbolic set by Proposition 2.3.

By the Birkhoff-Smale theorem (see *e.g.* Corollary 6.5.6 of [7]), we have $W^u(p) \cap W^s(p) \subset \overline{\mathrm{Per}(\Phi)}$ for any $p \in \mathrm{Per}(\Phi)$. Proposition 2.4 and Lemma 2.5 imply $\overline{\mathrm{Per}(\Phi)} = \Omega_1$ and $\mathcal{F}^\sigma(z) = W^\sigma(z)$ for any $z \in \Omega_1$ and any $\sigma = u, s$. Since $\mathcal{F}^u$ and $\mathcal{F}^s$ are mutually transverse, we obtain that $\mathcal{F}^u(p) \cap \mathcal{F}^s(p) \subset \overline{\mathrm{Per}(\Phi)} = \Omega_1$ for any $z \in \Omega_1$.

Suppose $\mathcal{F}^u(z) \not\subset \Omega_1$ for some $z \in \Omega_1$. Then, there exists $z_0 \in \mathcal{F}^u(z) \cap \Omega_1$ which is accessible from $\mathcal{F}^u(z) \backslash \Omega_1$. It implies that $\mathcal{F}^s(z_0)$ is a semi-proper leaf. By Lemma 2.5, $\mathcal{F}^s(z_0) = W^s(z_0)$ contains a periodic point. In particular, it is diffeomorphic to $\mathbb{R} \times S^1$. However, it contradicts Proposition 2.1 since $\mathcal{F}^u(z_0)$ has the non-trivial linear holonomy along the periodic orbit. Therefore, we obtain $\mathcal{F}^u(z) \subset \Omega_1$ for any $z \in \Omega_1$. Similarly, we also have $\mathcal{F}^s(z) \subset \Omega_1$ for any $z \in \Omega_1$. Since $\mathcal{F}^u$ and $\mathcal{F}^s$ are mutually transverse, it implies that Ω_1 is an open subset of M. Therefore, $\Omega_1 = \overline{\mathrm{Per}(\Phi)}$ coincides with M if Φ has a periodic point. □

2.3. Regular $\mathbb{P}$A flows without periodic points

To complete the proof of Theroem 1.2, we show that any regular $\mathbb{P}$A flow Φ without periodic points is represented by a finite union of $\mathbb{T}^2 \times I$-models. In [9], Noda showed that any C^2 regular $\mathbb{P}$A flow on $\mathbb{T}^2$-bundle over the circle is either an Anosov flow or is represented by a finite union of $\mathbb{T}^2 \times I$-models. Hence, it is sufficient to show that M is such a manifold.

By Proposition 2.3 and Lemma 2.5, the non-wandering set of Φ is the union of irrational toral leaves of $\mathcal{F}^u$ and $\mathcal{F}^s$. Put $W^u(T) = \bigcup_{z \in T} W^u(z)$ and $W^s(T) = \bigcup_{z \in T} W^s(z)$ for such a toral leaf T. Recall that if T is an irrational toral leaf of $\mathcal{F}^u$, then $W^s(T)$ is a neighborhood of T which is homeomorphic to $\mathbb{T}^2 \times (-1, 1)$. Since $M = \bigcup_{z \in \Omega_0} W^s(z) \cup \Omega_2$, each boundary component of $W^s(T)$ is a toral leaf of $\mathcal{F}^s$. The similar holds for normally repelling tori. Therefore, there exists a covering map $h : \mathbb{T}^2 \times \mathbb{R} \to M$ such that

1. $T_{2k} = h(\mathbb{T}^2 \times \{2k\})$ is an irrational toral leaf of $\mathcal{F}^s$ with $W^u(T_{2k}) = h(\mathbb{T}^2 \times (2k-1, 2k+1))$ and
2. $T_{2k+1} = h(\mathbb{T}^2 \times \{2k+1\})$ is an irrational toral leaf of $\mathcal{F}^u$ with $W^s(T_{2k+1}) = h(\mathbb{T}^2 \times (2k, 2k+2))$

for any integer k. It implies that M is a $\mathbb{T}^2$-bundle over the circle.

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References

[1] A. Arroyo and F. Rodriguez Hertz, *Homoclinic bifurcations and uniform hyperbolicity for three-dimensional flows*, Ann. Inst. H. Poincaré Anal. Non Linéaire **20** (2003), 805–841.

[2] J. Cantwell and L. Conlon, *Reeb stability for noncompact leaves in foliated 3-manifolds*, Proc. Amer. Math. Soc. **33**-2 (1981), 408–410.

[3] A. Candel and L. Conlon, Foliations, I, Grad. Stud. Math. **23**, American Mathematical Society, Providence, RI, 2000.

[4] J. Cantwell and L. Conlon, Endsets of exceptional leaves; a theorem of G. Duminy, Foliations: geometry and dynamics (Warsaw, 2000), 225–261, World Sci. Publishing, River Edge, NJ, 2002.

[5] Y. Eliashberg and W. Thurston, Confoliations, University Lecture Series **13**, Amer. Math. Soc., Providence, RI, 1998.

[6] E. Ghys, *Rigidité différentiable des groupes fuchiens*, Inst. Hautes Études Sci. Publ. Math. **78** (1993), 163–185.

[7] A. Katok and B. Hasselblatt, Introduction to the modern theory of dynamical systems, Encyclopedia Math. Appl. **54**, Cambridge University Press, Cambridge, 1995.

[8] Y. Mitsumatsu, *Anosov flows and non-stein symplectic manifolds*, Ann. Inst. Fourier **45** (1995), 1407–1421.

[9] T. Noda, *Projectively Anosov flows with differentiable (un)stable foliations*, Ann. Inst. Fourier **50** (2000), 1617–1647.

[10] ———, Regular projectively Anosov flows with compact leaves. thesis, Graduate School of Mathematical Sciences, University of Tokyo, 2001.

[11] T. Noda and T. Tsuboi, Regular projectively Anosov flows without compact leaves, Foliations: geometry and dynamics (Warsaw, 2000), 403–419, World Sci. Publishing, River Edge, NJ, 2002.

[12] S. Newhouse and J. Palis, Hyperbolic nonwandering sets on two-dimensional manifolds, Dynamical systems (Proc. Sympos., Univ. Bahia, Salvador, 1971), 293–301, Academic Press, New York, 1973.

[13] T. Tsuboi, *Regular projectively Anosov flows on the Seifert fibered 3-manifolds*, J. Math. Soc. Japan. **56**-4 (2004), 1233–1253.

[14] M. Shub, Global stability of dynamical systems, Springer-Verlag, Berlin-New York, 1986.