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Adams e-invariant, Toda bracket and $[X, U(n)]$

By

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Abstract

In the previous paper [1], the author investigated the group structure of the homotopy set $[X, U(n)]$ with the pointwise multiplication, under the assumption that X is a finite CW complex with its dimension $2n$ and $U(n)$ is the unitary group, and showed that $[X, U(n)]$ is an extension of $\widetilde{K}^1(X)$ by $N_n(X)$, where $N_n(X)$ is a group defined as the cokernel of a map $\Theta : \widetilde{K}^0(X) \to \mathrm{H}^{2n}(X; \mathbf{Z})$. In this paper, we offer another interpretation of $N_n(X)$ using Adams e-invariant and show that the extension $N_n(X) \to U_n(X) \to \widetilde{K}^1(X)$ is determined by some Toda brackets. Also we give some applications including the calculation of $[SO(4), U(3)]$.

1. Introduction

Let $U(n)$ be the unitary group and X be a finite CW-complex. We consider the homotopy set $U_n(X) = [X, U(n)]$ which forms a group by the pointwise multiplication. This association provide a functor from the category of finite CW-complexes to the category of (not always commutative) groups and, if $2n > \dim X$, $U_n(X)$ is merely equal to $\widetilde{K}^1(X)$.

As mentioned in [1], even if X is base-pointed, the group of the homotopy set and that of the base-point preserving homotopy set between X and $U(n)$ are naturally isomorphic. Thus in this paper we work in the base-pointed category and assume all spaces are pointed. Especially the base-point of $U(n)$ is the unit and we assume that X is connected and all attaching maps of cells of X are base-point preserving.

Now set $\dim X \leq 2n$. Considering the fibration involving $U(n)$, i.e., from the fibration $U(n) \xrightarrow{i} U(\infty) \xrightarrow{p} W_n$ (We set $W_n = U(\infty)/U(n)$.), one obtains the fibration sequence

$$\Omega U(\infty) \xrightarrow{\Omega p} \Omega W_n \xrightarrow{\delta} U(n) \xrightarrow{i} U(\infty) \xrightarrow{p} W_n,$$

which, applying $[X, \cdot\,]$, turns out to be the exact sequence:

$$(1.1) \qquad \widetilde{K}^0(X) \xrightarrow{\Theta(X)} \mathrm{H}^{2n}(X; \mathbf{Z}) \xrightarrow{\Phi(X)} U_n(X) \xrightarrow{\Pi(X)} \widetilde{K}^1(X) \to 0.$$

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(See Theorem 1.1 of [1].) Here, $\Theta(X)$, $\Phi(X)$ and $\Pi(X)$ are homomorphisms defined for each X ($\dim X \leq 2n$) and are obtained from Ωp_*, δ_* and i_* respectively. We may omit "(X)" when it makes no confusion. We denote the cokernel of $\Theta(X)$ by $N_n(X)$.

Dually, we consider a cofibration involving X. We denote the $(2n-1)$-skeleton of X by X' and set that $f_i : S^{2n-1} \to X'(i \in I)$ are the attaching maps of the $2n$-cells of X. Then we have the natural cofibration sequence:

$$\bigvee_{i \in I} S^{2n-1} \to X' \to X \to \bigvee_{i \in I} S^{2n} \to \Sigma X'$$

and this induces the exact sequence

$$U_n(\Sigma X') \to U_n\left(\bigvee_{i \in I} S^{2n}\right) \xrightarrow{\rho^*} U_n(X).$$

In this paper, we claim that $N_n(X) \cong \operatorname{Im}\rho$ and $\operatorname{Im}\rho$ can be determined by "e-invariant". (Theorem 2.1.)

After the investigation of $N_n(X)$ for $2n$-dimensional X, we consider the extension of $0 \to N_n(X) \to U_n(X) \to \widetilde{K}^1(X) \to 0$. (Despite of the non-commutativity of $U_n(X)$ in general, we use '0' as the unit group and denote the operation by '+'.) When $\widetilde{K}^1(X)$ is free $\mathbf{Z}$-module, all the relations in $\widetilde{K}^1(X)$ as a group are the commutativity and the above extension is determined by the commutators between the inverse images in $U_n(X)$ of the elements of $\widetilde{K}^1(X)$. This was done in the previous paper ([1]). When $\widetilde{K}^1(X)$ is not free, there are further relations which indicate the torsion part of $\widetilde{K}^1(X)$. In this paper we exhibit that the determination of the above extension in general case is essentially those of some secondary compositions. (Theorem 3.1.)

Finally, in Section 4, using the results of previous sections, we give some applications. We determine $[SO(4), U(3)]$. (Theorem 4.1.) Also we show examples, in which $U_n(X)$ is commutative but the above extension is not trivial. (Theorem 4.2.)

Throughout this paper, we use the following notation. We fix the orientation of spheres, i.e., select a generator of $\mathrm{H}^n(S^n;\mathbf{Z})$ and denote it b_n. Let η be the canonical line bundle over S^2 and denote the generator of $\widetilde{K}^0(S^{2n})$ by $\lambda_n = (\eta - 1)^n$. Since $U_n(S^{2k+1}) \cong \widetilde{K}^1(S^{2k+1}) \cong [S^{2k+1}, U(\infty)]$ ($k < n$), let $\epsilon_k \in U_n(S^{2k+1})$ ($k < n$) denote the generator which corresponds to $\lambda_k \in [S^{2k}, \Omega U(\infty)] \cong \widetilde{K}^0(S^{2k})$ by the adjoint isomorphism.

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2. e-invariant

The e-invariant is classically known and, among the several definitions of e-invariant, we adopt the following here. (See [2].)

Assume a CW-complex X' satisfies $\mathrm{H}^{2n}(X';\mathbf{Q}) = \mathrm{H}^{2n-1}(X';\mathbf{Q}) = 0$ and let $ch^n : BU \to K(\mathbf{Q}, 2n)$ be the n-th Chern character. For $f : S^{2n-1} \to X'$

and $\beta \in \widetilde{K}^0(X') = [X', BU]$, $\beta \circ f$ and $ch^n \circ \beta$ are null-homotopic.

$$S^{2n-1} \xrightarrow{f} X' \xrightarrow{\beta} BU \xrightarrow{ch^n} K(\mathbf{Q}, 2n)$$

Thus the secondary composition (Toda bracket) $\{ch^n, \beta, f\} \in [S^{2n}, K(\mathbf{Q}, 2n)]$ can be defined modulo $\mathrm{Im}(ch^n : \widetilde{K}^0(S^{2n}) \to \mathrm{H}^{2n}(S^{2n}; \mathbf{Q}))$. Remark that $\mathrm{Im}(\Sigma f^* : \mathrm{H}^{2n}(X'; \mathbf{Q}) \to \mathrm{H}^{2n}(S^{2n}; \mathbf{Q}))$ is 0. Thus $\{ch^n, \beta, f\}$ takes its value in $\mathrm{H}^{2n}(S^{2n}; \mathbf{Q}) / \mathrm{Im}\, ch^n \cong \mathbf{Q}/\mathbf{Z}$ and this value is denoted by $e(f)(\beta)$, i.e., $e(f)$ is defined as a map $\widetilde{K}^0(X') \to \mathrm{H}^{2n}(S^{2n}; \mathbf{Q}) / \mathrm{Im}\, ch^n$.

Now we offer yet another modified definition of the above one. Let X' be a CW-complex whose dimension is less than $2n$, $f : S^{2n-1} \to X'$ and $\beta \in \widetilde{K}^0(X') = [X', BU]$:

$$S^{2n-1} \xrightarrow{f} X' \xrightarrow{\beta} BU \xrightarrow{s_n} K(\mathbf{Z}, 2n).$$

Also we set $s_n : BU \to K(\mathbf{Z}, 2n)$ is the map which corresponds to the n-th primitive s-class of the universal bundle, in other words, $n!ch^n$. Then the secondary composition $\{s_n, \beta, f\} \in [S^{2n}, K(\mathbf{Z}, 2n)] = \mathrm{H}^{2n}(S^{2n}; \mathbf{Z})$ can be also defined modulo

$$\mathrm{Im}(s_n : \widetilde{K}^0(S^{2n}) \to \mathrm{H}^{2n}(S^{2n}; \mathbf{Z})) + \mathrm{Im}(\Sigma f^* : \mathrm{H}^{2n}(\Sigma X'; \mathbf{Z}) \to \mathrm{H}^{2n}(S^{2n}; \mathbf{Z})).$$

Here recall that $\Theta(X)$ in the exact sequence (1.1) is the map which associates $(-1)^n s_n(\alpha)$ to each $\alpha \in \widetilde{K}^0(X)$. (See Proposition 3.1 in [1].) Thus, applying this exact sequence to S^{2n}, $\Phi(S^{2n})(\{s_n, \beta, f\})$ takes its value in $U_n(S^{2n}) = \mathbf{Z}/n!\mathbf{Z}$ modulo $\mathrm{Im}(\Phi \circ \Sigma f^* : \mathrm{H}^{2n}(\Sigma X'; \mathbf{Z}) \to U_n(S^{2n}))$. We write this value as $e'(f)(\beta)$.

Lemma 2.1. *When* $\mathrm{H}^{2n-1}(X'; \boldsymbol{Q}) = 0$, *the ambiguity of* $e'(f)$ *disappears and* $e'(f)(\beta) \in U_n(S^{2n}) = \boldsymbol{Z}/n!\boldsymbol{Z}$. *Moreover,* $e'(f)$ *and* $e(f)$ *coincide by means of the injection which maps* $\langle k \rangle \in \boldsymbol{Z}/n!\boldsymbol{Z}$ *to* $k/n! \in \boldsymbol{Q}/\boldsymbol{Z}$.

Proof. Because $\mathrm{H}^*(S^{2n}; \mathbf{Z})$ is free, $\mathrm{H}^{2n}(\Sigma X'; \mathbf{Q}) = 0$ implies that $\mathrm{Im}(\Phi \circ \Sigma f^*) = 0$, i.e., the ambiguity of $e'(f)$ vanishes. The latter follows from the next commutative diagram:

$$\begin{array}{ccccccc} S^{2n-1} & \xrightarrow{f} & X' & \xrightarrow{\beta} & BU & \xrightarrow{s_n} & K(\mathbf{Z}, 2n) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \times \frac{1}{n!} \\ S^{2n-1} & \xrightarrow{f} & X' & \xrightarrow{\beta} & BU & \xrightarrow{ch^n} & K(\mathbf{Q}, 2n) \end{array}$$

□

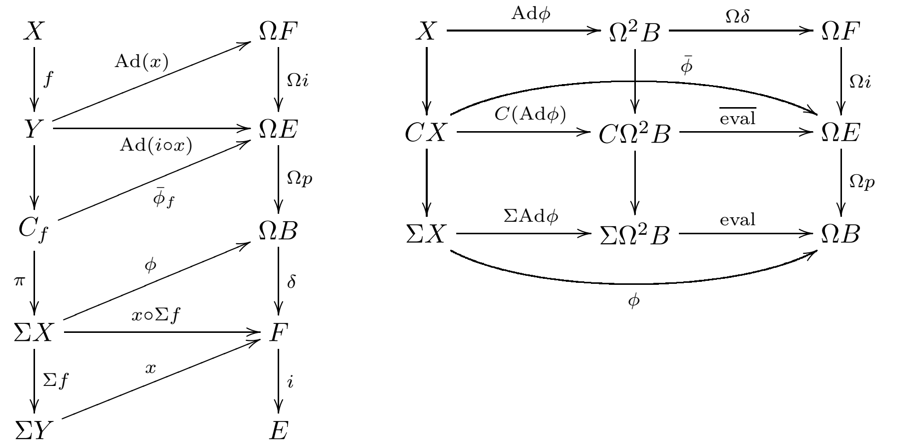
Remark. Especially, in case that $k < n$, $X' = S^{2k}$ and $f : S^{2n-1} \to S^{2k}$, since $\mathrm{H}^{2n}(\Sigma X'; \mathbf{Z}) = 0$, $e'(f) : \widetilde{K}^0(S^{2k}) \to U_n(S^{2n}) \cong \mathrm{H}^{2n}(S^{2n}; \mathbf{Z}) / \mathrm{Im}\, s_n$ has no ambiguity. Then let $a \in \mathbf{Z}/n!\mathbf{Z}$ satisfies $e'(f)(\lambda_k) = ab_n \in \mathrm{H}^{2n}(S^{2n}; \mathbf{Z}) / \mathrm{Im}\, s_n$ and we also denote this value a by $e'(f)$.

Next, we introduce a proposition about secondary composition.

Proposition 2.1. *Assume a fibration* $F \xrightarrow{i} E \xrightarrow{p} B$ *and a map* $X \xrightarrow{f} Y$. *For any map* $x : \Sigma Y \to F$ *such that* $i \circ x \circ \Sigma f \simeq *$,

$$\delta \circ \{\Omega p, \mathrm{Ad}(i \circ x), f\} \ni x \circ \Sigma f.$$

Proof. Consider the fibration sequence associated with $F \to E \to B$ shown in the left diagram below. Since $i \circ (x \circ \Sigma f) \simeq *$, there exists a lift $\phi : \Sigma X \to \Omega B$ such that $\delta \circ \phi \simeq x \circ \Sigma f$.



Now, recall that $\Omega\delta$ can be characterize as follows: Let $\mathrm{eval} : \Sigma\Omega^2 B \to \Omega B$ be the evaluation map. Using the CHP of $\Omega F \to \Omega E \to \Omega B$, there exists a map $\overline{\mathrm{eval}} : C\Omega^2 B \to \Omega E$ which makes the right above diagram commutative. Here CX means the cone of X, $[0,1] \times X/\{0\} \times X$. Therefore the restriction of $\overline{\mathrm{eval}}$ to the $\Omega^2 B = \{1\} \times \Omega^2 B$ induces a map $\Omega^2 B \to \Omega F$ and this is just $\Omega\delta$ up to homotopy.

We remark $\phi = \mathrm{eval} \circ \Sigma(\mathrm{Ad}\,\phi)$ and set $\bar{\phi} = \overline{\mathrm{eval}} \circ C(\mathrm{Ad}\,\phi)$ and $\bar{\phi}_X = \bar{\phi}|_X$. Then

$$\bar{\phi}_X \simeq \Omega i \circ \Omega\delta \circ \mathrm{Ad}\,\phi = \Omega i \circ \mathrm{Ad}(\delta \circ \phi) \simeq \Omega i \circ \mathrm{Ad}(x \circ \Sigma f) = \mathrm{Ad}(i \circ x) \circ f.$$

Using this homotopy, one can deform $\bar{\phi}$ into $\bar{\phi}'$ so that $\bar{\phi}'|_X = \mathrm{Ad}(i \circ x) \circ f$. This deformation enables us to combine this deformed $\bar{\phi}' : CX \to \Omega E$ and $\mathrm{Ad}(i \circ x) : Y \to \Omega E$ to make the new map $\bar{\phi}_f$ from C_f to ΩE. In other words, $\bar{\phi}_f$ is made from $Y \xrightarrow{\mathrm{Ad}(i\circ x)} \Omega E$ and the homotopy $\mathrm{Ad}(i \circ x) \circ f \simeq *$.

Here, one can easily verify $\Omega p \circ \bar{\phi}_f = \pi \circ \phi$ and this implies that ϕ belongs to the secondary composition $\{\Omega p, \mathrm{Ad}(i \circ x), f\}$ from its definition. Thus

$$\delta\{\Omega p, \mathrm{Ad}(i \circ x), f\} \ni \delta \circ \phi \simeq x \circ \Sigma f.$$

□

We turn to $U_n(X)$. For a connected CW-complex X of dim$2n$, let X' be its $(2n-1)$-skeleton and $f_i : S^{2n-1} \to X'$ $(i \in I)$ be the attaching maps of $2n$-cells. Applying the exact sequence (1.1) to the each member of the cofibration sequence:

$$\bigvee S^{2n-1} \xrightarrow{\vee f_i} X' \xrightarrow{j} X \xrightarrow{\rho} \bigvee S^{2n} \xrightarrow{\vee \Sigma f_i} \Sigma X',$$

we obtain the next diagram where all the rows and the columns are exact.

(2.1)

$$\begin{array}{ccccccccc}
0 & \xleftarrow{\vee {f_i}^*} & \widetilde{K}^0(X') & \xleftarrow{\text{epic}} & \widetilde{K}^0(X) & \longleftarrow & \bigoplus \widetilde{K}^0(S^{2n}) & & \\
 & & \downarrow & & \downarrow{\scriptstyle \Theta(X)} & & \downarrow & & \\
 & & 0 = \mathrm{H}^{2n}(X') & \longleftarrow & \mathrm{H}^{2n}(X) & \longleftarrow & \bigoplus \mathrm{H}^{2n}(S^{2n}) & \longleftarrow & \mathrm{H}^{2n}(\Sigma X') \\
 & & & & \downarrow{\scriptstyle \Phi(X)} & & \downarrow{\scriptstyle \Phi(S^{2n})} & & \downarrow{\scriptstyle \Phi(\Sigma X')} \\
 & & & & U_n(X) & \xleftarrow{\rho^*} & \bigoplus U_n(S^{2n}) & \xleftarrow{\vee \Sigma {f_i}^*} & U_n(\Sigma X') \\
 & & & & & & \downarrow{\scriptstyle \Pi(S^{2n})} & & \downarrow{\scriptstyle \Pi(\Sigma X')} \\
 & & & & & & \bigoplus \widetilde{K}^1(S^{2n}) = 0 & \longleftarrow & \widetilde{K}^1(\Sigma X') \\
 & & & & & & \downarrow & & \downarrow \\
 & & & & & & 0 & & 0
\end{array}$$

Theorem 2.1. *In the above diagram, let $M_i \subset U_n(S^{2n})$ be the submodule* $\mathrm{Im}(\Phi \circ \Sigma {f_i}^* : \mathrm{H}^{2n}(\Sigma X'; \mathbf{Z}) \to U_n(S^{2n}))$. *Then we have*

(1) $N_n(X) = \mathrm{Coker}\, \Theta(X) \cong \mathrm{Im}(\rho^* : \bigoplus U_n(S^{2n}) \to U_n(X))$.

(2) *For $\beta \in U_n(\Sigma X')$, let $\beta' \in \widetilde{K}^0(X')$ be the element which corresponds to* $\Pi(\Sigma X')(\beta)$ *by the isomorphism* $\widetilde{K}^1(\Sigma X') = [\Sigma X', U(\infty)] \xrightarrow{Ad} [X', \Omega U(\infty)] = \widetilde{K}^0(X')$. *Then we have*

$$\Sigma {f_i}^*(\beta) = (-1)^n e'(f_i)(\beta') \text{ modulo } M_i.$$

(3) *Modulo $\bigoplus M_i$,* $\mathrm{Ker}\, \rho^*$ *coincides with* $\bigoplus \mathrm{Im}(e'(f_i))$. *Especially, when* $\mathrm{H}^{2n-1}(X'; \boldsymbol{Q}) = 0$,

$$N_n(X) \cong \bigoplus \pi_{2n}(U(n)) / \mathrm{Im}(e'(f_i)).$$

Proof. The first assertion can be easily deduced by a simple diagram chasing:

$$\mathrm{Coker}\, \Theta(X) \cong \mathrm{Im}\, \Phi(X) = \mathrm{Im}\, \Phi(X) \circ \rho^* = \mathrm{Im}\, \rho^* \circ \Phi(S^{2n}) = \mathrm{Im}\, \rho^*.$$

We apply the previous proposition to the case that $F \to E \to B$ is the fibration $U(n) \xrightarrow{i} U(\infty) \xrightarrow{p} W_n$, and f is the attaching map of a $2n$-cell of

X, i.e., $f_i : S^{2n-1} \to X'$. Then for $\beta \in U_n(\Sigma X')$, we obtain

$$\Sigma {f_i}^* \beta \in \delta\{\Omega p, \mathrm{Ad}(i \circ \beta), f_i\} \subset U_n(S^{2n}). \tag{2.2}$$

On the other hand, we recall that there is a map $a_{2n} : \Omega W_n \to K(\mathbf{Z}, 2n)$ which induces the isomorphism $[X, \Omega W_n] \cong \mathrm{H}^{2n}(X; \mathbf{Z})$ for CW complex X of the dimension no more than $2n$ and makes the next diagram commutative. (See Theorem 1.1 of [1].) Also remark that $\Phi(X) : \mathrm{H}^{2n}(X; \mathbf{Z}) \to U_n(X)$ was defined by $\delta_*(a_2 n_*)^{-1}$.

$$\begin{array}{ccccc} \Omega U(\infty) & \xrightarrow{\Omega p} & \Omega W_n & \xrightarrow{\delta} & U(n) \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle a_{2n}} & & \\ BU & \xrightarrow{(-1)^n s_n} & K(\mathbf{Z}, 2n) & & \end{array}$$

Therefore, under the isomorphism $[S^{2n}, \Omega W_n] \cong \mathrm{H}^{2n}(S^{2n}; \mathbf{Z})$ induced by a_{2n},

$$\{\Omega p, \mathrm{Ad}(i \circ \beta), f_i\} = (-1)^n \{s_n, \mathrm{Ad}(i \circ \beta), f_i\}$$

and

$$\begin{aligned} \delta\{\Omega p, \mathrm{Ad}(i \circ \beta), f_i\} &= \Phi(S^{2n})((-1)^n \{s_n, \mathrm{Ad}(i \circ \beta), f_i\}) \\ &= (-1)^n e'(f_i)(\mathrm{Ad}(i \circ \beta)). \end{aligned} \tag{2.3}$$

Then (2.2), (2.3) and the equation $\mathrm{Ad}(i \circ \beta) = \mathrm{Ad}(\Pi(\Sigma X')(\beta))$ imply the second assertion.

Since all rows and columns of diagram (2.1) are exact, $\mathrm{Ker}\, \rho^* = \bigoplus \mathrm{Im}\, \Sigma {f_i}^*$ and the third assertion follows. Especially, if $\mathrm{H}^{2n-1}(X'; \mathbf{Q}) \cong \mathrm{H}^{2n}(\Sigma X'; \mathbf{Q}) = 0$, $\mathrm{H}^{2n}(\Sigma X'; \mathbf{Z})$ has no free part, while $\mathrm{H}^{2n}(S^{2n}; \mathbf{Z})$ is free. Therefore $\mathrm{Im}(\Phi(S^{2n}) \circ \Sigma {f_i}^*) = 0$, i.e., $e'(f_i)$ has no ambiguity and $N_n(X) \cong \bigoplus U_n(S^{2n}) / \mathrm{Im}(e'(f_i))$ as asserted. □

3. Extension

As mentioned in Section 1, $U_n(X)$ is a extension of $\widetilde{K}^1(X)$. In this section, we formulate the group extension of $U_n(X)$ for a CW complex of dimension $2n$ by means of second composition.

Assume that X is a connected CW complex of dimension $2n$, X' is its $(2n-1)$-skeleton, $f_i : S^{2n-1} \to X' (1 \leq i \leq l)$ are attaching maps of $2n$-cells and $\rho : X \to \bigvee X/X' \cong \bigvee_i S^{2n}$ is the quotient map. Also we set the inclusion map $j : X' \to X$. Note that the following extension obtained from (1.1) is central. [1]

$$0 \to N_n(X) \xrightarrow{\Phi(X)} U_n(X) \xrightarrow{\Pi(X)} \widetilde{K}^1(X) \to 0. \tag{3.1}$$

Let $\alpha_1, \ldots, \alpha_k \in \widetilde{K}^1(X)$ be the generators as a $\mathbf{Z}$-module, m_i be the order of α_i in $\widetilde{K}^1(X)$ and $\widetilde{\alpha}_i \in U_n(X)$ be a inverse image of α_i in the above exact sequence.

We set $\mathrm{H}^*(U(n);\mathbf{Z}) = \bigwedge(x_1, x_3, x_5, \dots x_{2n-1})$ where x_{2k-1} is the cohomology suspension of the k-th universal Chern class. Finally, we set the map $\mu_m : U(n) \to U(n)$ which maps $A \in U(n)$ to A^m.

Theorem 3.1. *The extension of* (3.1) *can be determined by the followings*:

(1) *The commutator* $[\widetilde{\alpha}_i, \widetilde{\alpha}_j] = \Phi(X)\langle u\rangle$ *where*

$$u = \sum_{k+l+1=n} (\widetilde{\alpha}^*(x_{2k+1}) \cup \widetilde{\beta}^*(x_{2l+1})) \in \mathrm{H}^{2n}(X;\mathbf{Z})$$

and $\langle u\rangle$ *means the class of* $N_n(X) = \operatorname{Coker}\Theta(X)$ *represented by* u.

(2) *The element* $m_j\widetilde{\alpha}_j$ *comes from* $N_n(X) \cong \bigoplus_i \pi_{2n}(U(n))/\operatorname{Ker}\rho^*$ *and*

$$m_j\widetilde{\alpha}_j \in \bigoplus_i \{\mu_{m_j}, \widetilde{\alpha}_j \circ j, f_i\} \subset \bigoplus_i \pi_{2n}(U(n))/\operatorname{Ker}\rho^*.$$

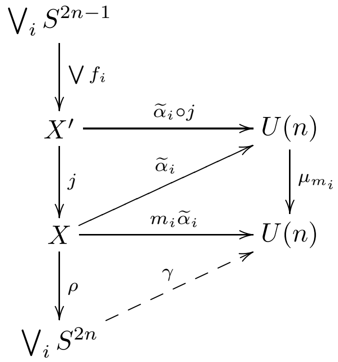
Remark. The secondary composition $\{\mu_{m_j}, \widetilde{\alpha}_j \circ j, f_i\}$ is a coset and, in $N_n(X)$, it has an ambiguity of $m_jN_n(X) = \{m_jx | x \in N_n(X)\}$. This ambiguity comes from the selection of the inverse images $\widetilde{\alpha}_j$. In spite of this ambiguity, the above equations uniquely determines the group extension of (3.1).

Proof. First assertion is just the statement of Theorem 1.4 of [1]. We remark that, since (3.1) is a central extension, $[\widetilde{\alpha}_i, \widetilde{\alpha}_j]$ does not depend on the selection of inverse images $\widetilde{\alpha}_i$ of α_i.

Now, we check the second assertion. See the next diagram which demonstrate the exact sequence (1.1) applied to each member of cofibration $X' \xrightarrow{j} X \xrightarrow{\rho} \bigvee S^{2n}$:

(3.2)
$$\begin{array}{ccccc}
0 & \longleftarrow & \mathrm{H}^{2n}(X;\mathbf{Z}) & \longleftarrow & \bigoplus_i \mathrm{H}^{2n}(S^{2n};\mathbf{Z}) \\
\downarrow & & \downarrow & & \downarrow \\
U_n(X') & \xleftarrow{j^*} & U_n(X) & \xleftarrow{\rho^*} & \bigoplus_i U_n(S^{2n}) \\
\downarrow{\scriptstyle \Pi(X')} & & \downarrow{\scriptstyle \Pi(X)} & & \downarrow \\
\widetilde{K}^1(X') & \xleftarrow{j^*} & \widetilde{K}^1(X) & \longleftarrow & 0
\end{array}$$

Since $j^*(m_i\alpha_i) = 0$ in $\widetilde{K}^1(X')$ and $\Pi(X')$ is injective, we have $j^*(m_i\widetilde{\alpha}_i) = 0$ in $U_n(X')$. Therefore there exists $\gamma \in [\bigvee S^{2n}, U(n)] \cong \bigoplus U_n(S^{2n})$ such that $\rho^*\gamma = m_i\widetilde{\alpha}_i$. We consider the relation of $\widetilde{\alpha}_i$, μ_m, f_i and j in the following diagram.



It can be seen that γ is obtained from $\mu_{m_i} \circ \widetilde{\alpha}_i$ and the homotopy of $\mu_{m_i} \circ \widetilde{\alpha}_i \circ j \simeq *$, i.e.,

$$\gamma \in \left\{\mu_{m_i}, \widetilde{\alpha}_i \circ j, \bigvee f_i\right\} = \bigoplus_i \{\mu_{m_i}, \widetilde{\alpha}_i \circ j, f_i\}.$$

□

In [3], A. T. Lundell introduced the unstable Bott map β_n: The restriction of Bott map $\beta|_{U(n)} : U(n) \hookrightarrow U(\infty) \xrightarrow{\beta} \Omega^2 U(\infty)$ can be factorized as $U(n) \xrightarrow{\beta_n} \Omega^2 U(n+1) \hookrightarrow \Omega^2 U(\infty)$. This map induces a map $U_n(X) \xrightarrow{\beta_{n*}} U_{n+1}(\Sigma^2 X)$ but we should remark that this is NOT a group homomorphism in general, unless X is a co-H space. Concerning this map, we have the following theorem.

Theorem 3.2. *Let X be a CW complex and $\dim X = 2n$. Then the next diagram commutes.*

(3.3)

$$\begin{array}{ccccccc}
\widetilde{K}^0(X) & \xrightarrow{\Theta(X)} & \mathrm{H}^{2n}(X;\mathbf{Z}) & \xrightarrow{\Phi(X)} & U_n(X) & \xrightarrow{\Pi(X)} & \widetilde{K}^1(X) \\
\downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \mu'_{(n+1)}} & & \downarrow{\scriptstyle \beta_{n*}} & & \downarrow{\scriptstyle \beta} \\
\widetilde{K}^0(\Sigma^2 X) & \xrightarrow{\Theta(\Sigma^2 X)} & \mathrm{H}^{2n+2}(\Sigma^2 X;\mathbf{Z}) & \xrightarrow{\Phi(\Sigma^2 X)} & U_{n+1}(\Sigma^2 X) & \xrightarrow{\Pi(\Sigma^2 X)} & \widetilde{K}^1(\Sigma^2 X)
\end{array}$$

In the above diagram, β means the Bott periodicity map and $\mu'_{(n+1)}$ is the map which maps $a \in \mathrm{H}^{2n}(X;\mathbf{Z})$ to $(n+1)\Sigma^2 a \in \mathrm{H}^{2n+2}(\Sigma^2 X;\mathbf{Z})$.

Proof. From the property of β_n, the commutativity of the right square in (3.3) follows.

For the commutativity of the left square, we recall that for $\alpha \in \widetilde{K}^0(X) = [X, BU]$, $\beta(\alpha) = \zeta \circ \Sigma^2 \alpha \in \widetilde{K}^0(\Sigma^2 X) = [\Sigma^2 X, BU]$ where $\zeta \in [\Sigma^2 BU, BU]$ is the classifying map of $\lim_{N\to\infty}(\eta - 1) \hat{\otimes} (\xi_N - N)$, η is the canonical line bundle over CP^1 and ξ_N is the universal bundle over $BU(N)$.

Therefore, referring to Proposition 3.1 of [1], we have

$$\Theta(\Sigma^2 X)(\beta(\alpha)) = (\beta(\alpha))^*((-1)^n s_{n+1}) = (\Sigma^2\alpha)^*\zeta^*((-1)^n s_{n+1}). \tag{3.4}$$

Also, it is known that $\zeta^*(c_i) = (-1)^{i-1}\Sigma^2 s_{i-1}$ where c_i is the i-th universal Chern class and s_i is the primitive s-class in $\mathrm{H}^i(BU;\mathbf{Z})$. (See Proposition 3.1 of [1].) Using Newton's formula and (3.4),

$$\zeta^*((-1)^n s_{n+1}) = \zeta^*((-1)^n(n+1)c_{n+1}+\text{decomposable elements}) = (n+1)\Sigma^2 s_n$$

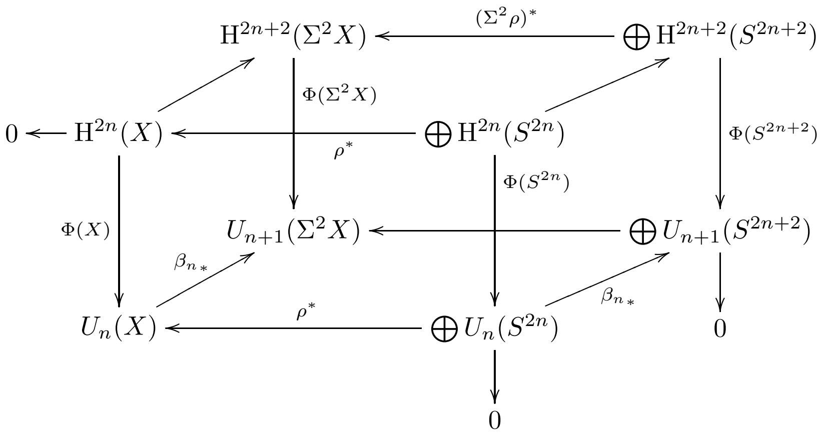
and

$$\Theta(\Sigma^2 X)(\beta(\alpha)) = (n+1)\Sigma^2\alpha^*(s_n) = (n+1)\Sigma^2(\Theta(X)(\alpha)).$$

This is just the commutativity of the left square of (3.3).

Now we prove the commutativity of the middle square of (3.3). In [3], it is shown that β_{n*} maps $1 \in \mathbf{Z}/n!\mathbf{Z} \cong U_n(S^{2n})$ to $(n+1) \in \mathbf{Z}/(n+1)!\mathbf{Z} \cong U_{n+1}(S^{2n+2})$.

Now we consider the diagram below.



In the above diagram, ρ means the quotient map $X \to X/X^{(n-1)} \cong \bigvee S^{2n}$ as before. We see that for any $x \in \mathrm{H}^{2n}(X)$, taking $\bar{x} \in \bigoplus \mathrm{H}^{2n}(S^{2n})$ so that $\rho^*(\bar{x}) = x$,

$$\begin{aligned}
\beta_{n*}\Phi(X)x &= \beta_{n*}\Phi(X)\rho^*(\bar{x}) \\
&= \beta_{n*}\rho^*\Phi(S^{2n})\bar{x} \\
&= (\Sigma^2\rho)^*\beta_{n*}\Phi(S^{2n})\bar{x} \\
&= (\Sigma^2\rho)^*\Phi(S^{2n+2})(n+1)\Sigma^2\bar{x} \\
&= \Phi(\Sigma^2 X)(\Sigma^2\rho)^*(n+1)\Sigma^2\bar{x} \\
&= \Phi(\Sigma^2 X)((n+1)\Sigma^2(\rho^*\bar{x})) \\
&= \Phi(\Sigma^2 X)((n+1)\Sigma^2 x).
\end{aligned}$$

And the statement is proved. □

Corollary 3.1. *Let* $\dim X = 2n$ *and assume that* X *is a co-H space. If* $0 \to N_n(X) \to U_n(X) \to \widetilde{K}^1(X) \to 0$ *is a trivial extension, so is* $0 \to N_{n+1}(\Sigma^2 X) \to U_{n+1}(\Sigma^2 X) \to \widetilde{K}^1(\Sigma^2 X) \to 0$.

Proof. When X is a co-H space, β_{n*} is a group homomorphism. Thus, by Theorem 3.2, we obtain the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_n(X) & \longrightarrow & U_n(X) & \longrightarrow & \widetilde{K}^1(X) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow\cong & & \\ 0 & \longrightarrow & N_{n+1}(\Sigma^2 X) & \longrightarrow & U_{n+1}(\Sigma^2 X) & \longrightarrow & \widetilde{K}^1(\Sigma^2 X) & \longrightarrow & 0 \end{array}$$

which says that the splitting of the upper extension induces the splitting of the lower. Also, $U_{n+1}(\Sigma^2 X)$ is abelian and the statement follows. □

4. Application

In this section we see some examples of the computation of $U_n(X)$. One of them is $U_3(SO(4))$.

In [1], examples are offered, in which $U_n(X)$ is not commutative, i.e., $U_n(X)$ is a non-trivial extension of $\widetilde{K}^1(X)$ in the sense of Theorem 3.1 (1). Here we also see a example in which the extension is not trivial in the sense of Theorem 3.1 (2).

We start with the next lemma.

Lemma 4.1. *For $n > 2$, we have $N_n(\Sigma^{2n-3}RP^3) \cong U_n(S^{2n}) = \boldsymbol{Z}/n!\boldsymbol{Z}$.*

Proof. We apply Theorem 2.1 (3).

The $(2n-1)$-skeleton of $\Sigma^{2n-3}RP^3$ is $\Sigma^{2n-3}RP^2$ and let f be the attaching map of the $2n$-cell. Since $\mathrm{H}^{2n}(\Sigma^{2n-3}RP^2; \mathbf{Z})$ has no free part, while $\mathrm{H}^{2n}(S^{2n}; \mathbf{Z})$ is free, thus $\mathrm{Im}(\Phi \circ \Sigma f^*)$ vanishes and $N_n(\Sigma^{2n-3}RP^3) \cong U_n(S^{2n})/\operatorname{Im} e'(f)$. But $\widetilde{K}^0(\Sigma^{2n-3}RP^2) = 0$ and $\operatorname{Im} e'(f)$ vanishes as well. □

Lemma 4.2. *The next extension is trivial*:

$$0 \to N_2(\Sigma RP^3) \to U_2(\Sigma RP^3) \to \widetilde{K}^1(\Sigma RP^3) \to 0.$$

Proof. We set $X = \Sigma RP^3$, $X' = \Sigma RP^2$, f is the attaching map of the 4-cell of X and apply Theorem 3.1 (2). Also we use the same notations as Theorem 3.1.

In this case, since $X' = S^2 \cup_{2\iota} e_3$, using the cofibration sequence $S^2 \xrightarrow{2\iota} S^2 \to X' \xrightarrow{\pi} S^3 \xrightarrow{2\iota} S^3$,

$$0 = U_2(S^2) \longleftarrow U_2(X') \xleftarrow{\pi^*} U_2(S^3) \xleftarrow{\times 2} U_2(S^3),$$

is exact and $U_2(X') \cong \mathbf{Z}/2\mathbf{Z}$, which has a generator $\alpha' = \pi^*\epsilon_1$. Also $\widetilde{K}^1(X) \cong \widetilde{K}^1(X') \cong U_2(X')$, since

$$\widetilde{K}^1(S^3) \xleftarrow{0} \widetilde{K}^1(X') \longleftarrow \widetilde{K}^1(X) \longleftarrow \widetilde{K}^1(S^4) = 0.$$

Thus we can take $\alpha \in \widetilde{K}^1(X)$ and $\widetilde{\alpha} \in U_2(X)$ so that $\widetilde{\alpha} \circ j \simeq \alpha'$ and $\Pi(X)(\widetilde{\alpha}) = \alpha$. (See diagram (3.2).)

By Theorem 3.1, we should consider $\{\mu_2, \alpha', f\} \subset \pi_{2n}(U(n))$. Recall that f is the suspension of the natural projection $S^2 \to RP^2$, π is the suspension of $RP^2 \to RP^2/RP^1$ and it can be seen that $\pi \circ f$ is null homotopic.

Thus $\{\mu_2, \alpha', f\} = \{\mu_2, \epsilon \circ \pi, f\} \supset \{\mu_2, \epsilon, \pi \circ f\} = 0$-coset and the extension is trivial. (In fact, in this case $\{\mu_2, \alpha', f\}$ has no ambiguity and exactly $\{0\}$.) □

Proposition 4.1. *For* $n > 2$*, the extension*

$$0 \to N_n(\Sigma^{2n-3}RP^3) \to U_n(\Sigma^{2n-3}RP^3) \to \widetilde{K}^1(\Sigma^{2n-3}RP^3) \to 0$$

is trivial and $U_n(\Sigma^{2n-3}RP^3) = \boldsymbol{Z}/n!\boldsymbol{Z} \oplus \boldsymbol{Z}/2\boldsymbol{Z}$.

Proof. This statement follows directly from Lemmas 4.1, 4.2 and Corollary 3.1. □

Theorem 4.1. *We can take generators* α, β, β', $\gamma \in U_3(SO(4))$ *and all their relations are*

$$6\gamma = 0, \quad 2\alpha = 0, \quad [\beta, \gamma] = [\beta', \gamma] = [\beta, \alpha] = [\beta', \alpha] = [\alpha, \gamma] = 0, \quad [\beta, \beta'] = \gamma.$$

Proof. It is well known that the fibre bundle $SO(3) \to SO(4) \to S^3$ is the trivial bundle and $SO(3) \cong RP^3$. Thus $SO(4) \cong S^3 \times RP^3$ and we consider the cofibration

$$S^3 \vee RP^3 \xrightarrow{i'} S^3 \times RP^3 \xrightarrow{p'} \Sigma^3 RP^3$$

which induces the short exact sequence

$$0 \leftarrow U_3(S^3) \oplus U_3(RP^3) \xleftarrow{i'^*} U_3(S^3 \times RP^3) \xleftarrow{p'^*} U_3(\Sigma^3 RP^3) \leftarrow 0. \tag{4.1}$$

Here $U_3(RP^3) \cong U_3(S^3) \cong \mathbf{Z}$ and we can take their generators as $\beta \in U_3(S^3)$ and $\beta' = \beta \circ \pi' \in U_3(RP^3)$ where $\pi' : RP^3 \to RP^3/RP^2$ is the quotient map. Then we take the inverse image of β and β' in $U_3(SO(4))$ as $\widetilde{\beta} = \beta \circ p_1$ and $\widetilde{\beta'} = \beta' \circ p_2$ where p_i is the i-th projection map of $SO(4) = S^3 \times RP^3$.

Also by Lemma 4.1 and Proposition 4.1, we know

$$\begin{aligned} U_3(\Sigma^3 RP^3) &\cong \widetilde{K}^1(\Sigma^3 RP^3) \oplus N_3(\Sigma^3 RP^3) \\ &\cong \widetilde{K}^1(\Sigma^3 RP^3) \oplus U_3(S^6) \cong \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/6\mathbf{Z} \end{aligned}$$

and set generators as $\alpha \in \widetilde{K}^1(\Sigma^3 RP^3)$ and $\gamma \in U_3(S^6)$.

All we have to do is to examine the extension of (4.1). Since the subgroup $\langle\gamma\rangle$ of $U_3(SO(4))$ generated by γ coincide with $N_3(SO(4)) = \operatorname{Im} \rho^*$ where ρ is the quotient map smashing the $(2n-1)$-skeleton of $SO(4) = S^3 \times RP^2$, recalling the central extension (3.1), $U_3(SO(4))/\langle\gamma\rangle \cong \widetilde{K}^1(SO(4))$ is commutative, all commutators in $U_3(SO(4))$ are in $\langle\gamma\rangle$ and γ belongs to the center of $U_3(SO(4))$. Also by Corollary 1.5 of [1], α belongs to the center, because its order is finite.

Thus (4.1) is also a central extension and, since $U_3(S^3) \oplus U_3(RP^3)$ is free, the extension is determined by $[\widetilde{\beta}, \widetilde{\beta'}] \in \langle \gamma \rangle$. By Theorem 3.1 (1), we have

$$[\widetilde{\beta}, \widetilde{\beta'}] = \Phi\langle u \rangle, u = \sum_{i+j=2} \widetilde{\beta}^*(x_{2i+1}) \cup \widetilde{\beta'}^*(x_{2j+1}).$$

One can easily check that u is a generator of $\mathrm{H}^6(SO(4); \mathbf{Z})$ and obtain $[\widetilde{\beta}, \widetilde{\beta'}] = \pm\gamma$. Retake γ and the statement follows. □

Now we turn to another examples. Let $M = S^{2k} \cup_{m\iota} e_{2k+1}$ be the Moore space of type $(\mathbf{Z}/m\mathbf{Z}, 2k)$ and $X = M \cup_f e_{2n}$, where $f : S^{2n-1} \to S^{2k} \subset M$ and $k < n - 1$. We can easily see (for example by AHSS) $\widetilde{K}^1(X) = \mathbf{Z}/m\mathbf{Z}$. Also, since $\mathrm{H}^{2n-1}(M; \mathbf{Z}) = 0$ and $\widetilde{K}^0(M) = 0$, by Theorem 2.1 (3), we have $N_n(X) \cong U_n(S^{2n}) \cong \mathbf{Z}/n!\mathbf{Z}$. Thus we have an extension

$$0 \to \mathbf{Z}/n!\mathbf{Z} \to U_n(X) \to \mathbf{Z}/m\mathbf{Z} \to 0.$$

We shall take the generators $\alpha \in \widetilde{K}^1(X)$, $\gamma \in U_n(S^{2n})$ and the inverse image $\widetilde{\alpha}$ of α in $U_n(X)$.

Theorem 4.2. *Let X be as above. Then we can take α, γ and $\widetilde{\alpha}$ so that $U_n(X)$ has two generators γ and $\widetilde{\alpha}$ and their all relations are*

$$n!\gamma = 0, \quad m\widetilde{\alpha} = e'(f)\gamma.$$

Proof. See the commutative diagram (3.2) in the proof of Theorem 3.1. In this situation, we see $\widetilde{K}^1(X) \cong \widetilde{K}^1(M) \cong U_n(M)$ and $N_n(X) \cong U_n(S^{2n})$. Thus we consider the extension

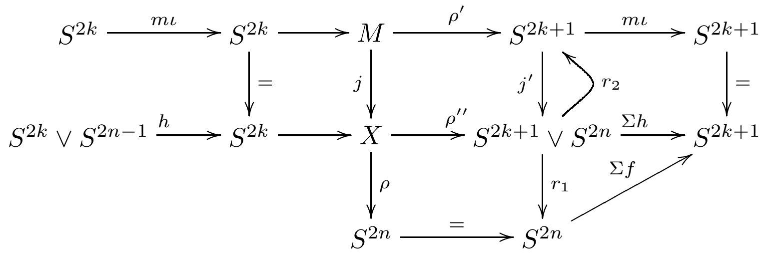
$$0 \to U_n(S^{2n}) \to U_n(X) \to U_n(M) \to 0 \tag{4.2}$$

instead of $0 \to N_n(X) \to U_n(X) \to \widetilde{K}^1(X) \to 0$.

Next we fix some maps as

$$M \xrightarrow{\rho'} S^{2k+1}, \qquad X \xrightarrow{\rho''} S^{2k+1} \vee S^{2n},$$
$$S^{2k+1} \vee S^{2n} \xrightarrow{r_1} S^{2k+1}, \qquad S^{2k+1} \vee S^{2n} \xrightarrow{r_2} S^{2n},$$
$$S^{2k} \vee S^{2n-1} \xrightarrow{h} S^{2k}$$

where ρ' and ρ'' are quotient maps smashing the $2k$-skeleton, r_1 and r_2 are retraction maps smashing the first or second component, $h = \nabla \circ (m\iota \vee f)$ and $\nabla : S^{2k} \vee S^{2k} \to S^{2k}$ is the folding map. Then we have the next commutative diagram in which all columns and rows are cofibration sequences:



where j and j' are natural inclusion.

Applying $U_n(\cdot)$ to the diagram, we know that $0 \leftarrow U_n(M) \leftarrow U_n(S^{2k+1}) \overset{\times m}{\longleftarrow} U_n(S^{2k+1})$ is exact and can take the generator of $U_n(M)$ as $\alpha = \epsilon_k \circ \rho'$ where $\epsilon_k \in U_n(S^{2k+1}) \cong \mathbf{Z}$ is the generator.

Now we set $\widetilde{\alpha} = \epsilon_k \circ r_2 \circ \rho''$. Then

$$j^* \widetilde{\alpha} = \epsilon_k \circ r_2 \circ \rho'' \circ j = \epsilon_k \circ r_2 \circ j' \circ \rho' = \epsilon_k \circ \rho' = \alpha,$$

i.e., $\widetilde{\alpha}$ is a inverse image of α in (4.2). Next we set $\beta = -\epsilon_k \circ \Sigma f \in U_n(S^{2n})$ and show that $\rho^* \beta = m\widetilde{\alpha}$. In fact, since

$$\rho^* \beta = \beta \circ \rho = -\epsilon_k \circ \Sigma f \circ \rho = -\epsilon_k \circ \Sigma f \circ r_1 \circ \rho'' = \rho''^*(-\epsilon_k \circ \Sigma f \circ r_1),$$

$$m\widetilde{\alpha} = m(\epsilon_k \circ r_2 \circ \rho'') = \rho''^*((m\epsilon_k) \circ r_2),$$

therefore we can proceed as

$$m\widetilde{\alpha} - \rho^* \beta = \rho''^*(\epsilon_k \circ \Sigma f \circ r_1 + \epsilon_k \circ (m\iota) \circ r_2) = \rho''^*(\Sigma h)^* \epsilon_k = 0.$$

Now all we have to do is to prove that $\beta = e'(f)\gamma$. But, applying Theorem 2.1 (2) to the CW-complex $Y = S^{2k} \cup_f e_{2n}$, it can be see that

$$\Sigma f^*(-\epsilon_k) = (-1)^{n+1} e'(f) \cdot b_{2n} \in \mathrm{H}^{2n}(S^{2n}; \mathbf{Z}) / \operatorname{Im} s_n \cong U_n(S^{2n}).$$

Thus we can take the generator γ satisfying $\beta = e'(f)\gamma$. □

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