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The suspension order of the real even dimensional projective space

By

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Abstract

The purpose of this paper is to prove the truth of the conjecture in [12]: The suspension order of the real even dimensional projective space coincides with its stable order determined by Toda [21] (see Silberbush and Ucci [19]). We obtain the assertion by proving that the suspension order of the real 6-projective space is 8.

1. Introduction

In this paper all spaces, maps and homotopies are based. For a space X, we denote by ΣX a suspension of X and by ι_X the identity class of X. The order of $\iota_{\Sigma X} \in [\Sigma X, \Sigma X]$ is called the suspension order ([21]) or the characteristic [4] of X. The order of $\Sigma^\infty \iota_X \in \{X, X\}$ is called the stable order ([21]) of X. Let P^n be the real n-dimensional projective space. Adams [1, Theorem 7.4] showed that the KO-group $\widetilde{KO}(\mathrm{P}^n)$ is isomorphic to $\mathbb{Z}_{2^{\phi(n)}}$, where $\phi(n)$ is the number of integers i satisfying $1 \leq i \leq n$ and $i \equiv 0, 1, 2$ or $4 \bmod 8$. Toda [21, Corollary 3 to Theorem 4.3] determined the stable order of P^{2n} equal to $2^{\phi(2n)}$. The purpose of the present paper is to show the following.

Theorem 1.1. *The suspension order of* P^6 *is* 8.

As an application of the theorem, we conclude that the suspension order of P^{2n} coincides with its stable order (see [12, Appendix]). In other words, we obtain the following.

Corollary 1.2. *The suspension order of* P^{2n} *is* $2^{\phi(2n)}$.

For a space X and its subspace A, let us denote by $i_{A,X} : A \to X$ the inclusion map and by $p_{X,A} : X \to X/A$ the map pinching A to one point. We set $\mathrm{P}^n_k = \mathrm{P}^n/\mathrm{P}^{k-1}$, $i_{k,n} = i_{\mathrm{P}^k,\mathrm{P}^n}$ and $p_{n,k} = p_{\mathrm{P}^n,\mathrm{P}^k}$ for $k \leq n$. We set $\iota_n = \iota_{S^n}$. We denote by $\gamma_n : S^n \to \mathrm{P}^n$ the covering map.

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By abuse of notation, the same notation is often used for a mapping and its homotopy class.

Since $\pi_n(\mathrm{P}^{n+1}) = 0$, we have

$$i_{n,n+1} \circ \gamma_n = 0. \tag{1.1}$$

From the cell structure of P^n, we obtain

$$p_{n,n-1} \circ \gamma_n = (1 + (-1)^{n-1})\iota_n. \tag{1.2}$$

Let $\eta_n \in \pi_{n+1}(S^n)$ for $n \geq 2$ be the Hopf map and $\eta_n^2 = \eta_n \circ \eta_{n+1} \in \pi_{n+2}(S^n)$. We recall from [20] that

$$\pi_3(S^2) = \mathbb{Z}\{\eta_2\}, \quad \pi_{n+1}(S^n) = \mathbb{Z}_2\{\eta_n\} \qquad (n \geq 3)$$

and

$$\pi_{n+2}(S^n) = \mathbb{Z}_2\{\eta_n^2\} \qquad (n \geq 2).$$

Here, for example, the notation $\pi_{n+1}(S^n) = \mathbb{Z}_2\{\eta_n\}$ indicates that $\pi_{n+1}(S^n)$ is isomorphic to $\mathbb{Z}_2$ and generated by η_n.

We set $M^n = \Sigma^{n-2}\mathrm{P}^2$, $i_n = \Sigma^{n-2}i_{1,2}$ and $p_n = \Sigma^{n-2}p_{2,1}$. Let $\tilde{\eta}_2 \in \pi_4(M^3)$ be an element satisfying $p_3\tilde{\eta}_2 = \eta_3$ ([15, Lemma 4.1]) and set

$$\tilde{\eta}_n = \Sigma^{n-2}\tilde{\eta}_2 \qquad (n \geq 2).$$

Let $\bar{\eta}_3 \in [M^5, S^3]$ be an extension of η_3 and set

$$\bar{\eta}_n = \Sigma^{n-3}\bar{\eta}_3 \qquad (n \geq 3).$$

We obtain $\pi_3(M^3) = \mathbb{Z}_4\{i_3\eta_2\}$ and

$$\pi_n(M^n) = \mathbb{Z}_2\{i_n\eta_{n-1}\} \qquad (n \geq 4).$$

We recall an important relation ([3], [21])

$$2\iota_{M^n} = i_n\eta_{n-1}p_n \qquad (n \geq 3). \tag{1.3}$$

We also obtain $\pi_{n+2}(M^{n+1}) = \mathbb{Z}_4\{\tilde{\eta}_n\}$,

$$2\tilde{\eta}_n = i_{n+1}\eta_n^2 \qquad (n \geq 2), \tag{1.4}$$

$[M^{n+2}, S^n] = \mathbb{Z}_4\{\bar{\eta}_n\}$ and

$$2\bar{\eta}_n = \eta_n^2 p_{n+2} \qquad (n \geq 3). \tag{1.5}$$

Making use of the cofiber sequence

$$(*)_n \qquad S^n \xrightarrow{2\iota_n} S^n \xrightarrow{i_{n+1}} M^{n+1} \xrightarrow{p_{n+1}} S^{n+1} \longrightarrow \cdots$$

and by the groups $\pi_k(M^n)$ for $k = n, n+1$, we get that (see [16, Lemma 1.5 (i)])

$$[M^{n+1}, M^n] = \mathbb{Z}_2\{i_n\bar{\eta}_{n-1}\} \oplus \mathbb{Z}_2\{\tilde{\eta}_{n-1}p_{n+1}\} \qquad (n \geq 4). \tag{1.6}$$

To prove Theorem 1.1, we need the following.

Theorem 1.3. $2[M^7, \Sigma \mathrm{P}^n] = 0$ *for* $n \geq 3$.

We note that P^6_3 is identified with the mapping cone of $i_4\bar{\eta}_3$:

$$\mathrm{P}^6_3 = M^4 \cup_{i_4\bar{\eta}_3} CM^5.$$

We set $i' = i_{\mathrm{P}^4_3, \mathrm{P}^6_3}$ and $p' = p_{\mathrm{P}^6_3, \mathrm{P}^4_3}$. We consider an element $(\Sigma i_{1,4})\eta_2\bar{\eta}_3 \in [M^5, \Sigma \mathrm{P}^4]$. By [18, Lemma 5.2], we know a relation

$$(\Sigma i_{2,4})\tilde{\eta}_2\eta_4 = 0 \in \pi_5(\Sigma \mathrm{P}^4).$$

So, by (1.4) and (1.5),

$$\begin{aligned}(\Sigma i_{1,4})\eta_2\bar{\eta}_3 \circ i_5\bar{\eta}_4 &= (\Sigma i_{1,4})\eta_2^2\bar{\eta}_4 = (\Sigma i_{2,4})\tilde{\eta}_2 \circ 2\iota_4 \circ \bar{\eta}_4 \\ &= (\Sigma i_{2,4})\tilde{\eta}_2\eta_4^2 p_6 = 0 \in [M^6, \Sigma \mathrm{P}^4],\end{aligned}$$

and hence $(\Sigma i_{1,4})\eta_2\bar{\eta}_3$ is extendible to $\overline{(\Sigma i_{1,4})\eta_2\bar{\eta}_3} \in [\Sigma \mathrm{P}^6_3, \Sigma \mathrm{P}^4]$. By (1.3) and (1.6),

$$i_4\bar{\eta}_3 \circ i_5\eta_4 p_5 = 2(i_4\bar{\eta}_3) = 0 \in [M^5, M^4].$$

Therefore there exists a coextenstion $\widetilde{i_5\eta_4p_5} \in [M^6, \mathrm{P}^6_3]$ of $i_5\eta_4p_5$ satisfying $p'_*\widetilde{i_5\eta_4p_5} = i_6\eta_5p_6$. The Toda bracket

$$\{(\Sigma i_{1,4})\eta_2\bar{\eta}_3, i_5\bar{\eta}_4, i_6\eta_5p_6\}_1 \subset [M^7, \Sigma \mathrm{P}^4]$$

is represented by the composition $-\overline{(\Sigma i_{1,4})\eta_2\bar{\eta}_3} \circ \Sigma\widetilde{i_5\eta_4p_5}$ ([20, Proposition 1.7]) as shown in the commutative diagram ($\xi = (\Sigma i_{1,4})\eta_2\bar{\eta}_3$):

$$\begin{tikzcd}
 & M^5 \arrow[d, "\Sigma i'"] \arrow[dr, "\xi"] & \\
M^7 \arrow[r, "\Sigma\widetilde{i_5\eta_4p_5}"] \arrow[dr, "i_7\eta_6p_7"'] & \Sigma \mathrm{P}^6_3 \arrow[r, "\bar{\xi}"] \arrow[d, "\Sigma p'"] & \Sigma \mathrm{P}^4 \\
 & M^7. &
\end{tikzcd}$$

The key to proving Theorem 1.1 is to find out the following.

Theorem 1.4.
(i) $4\iota_{\Sigma \mathrm{P}^6} \equiv (\Sigma i_{4,6})\overline{(\Sigma i_{1,4})\eta_2\bar{\eta}_3} \circ \Sigma p_{6,2}$ mod $[M^7, \Sigma \mathrm{P}^6] \circ \Sigma p_{6,4}$.
(ii) $\{(\Sigma i_{1,4})\eta_2\bar{\eta}_3, i_5\bar{\eta}_4, i_6\eta_5p_6\}_1$ *consists of a single element and* $2\overline{(\Sigma i_{1,4})\eta_2\bar{\eta}_3}$ $= \{(\Sigma i_{1,4})\eta_2\bar{\eta}_3, i_5\bar{\eta}_4, i_6\eta_5p_6\}_1 \circ \Sigma p' \in [\Sigma \mathrm{P}^6_3, \Sigma \mathrm{P}^4]$.
(iii) $(\Sigma\gamma_5)\eta_6p_7 = (\Sigma i_{4,5}) \circ \{(\Sigma i_{1,4})\eta_2\bar{\eta}_3, i_5\bar{\eta}_4, i_6\eta_5p_6\}_1 \in [M^7, \Sigma \mathrm{P}^5]$.

Theorem 1.1 is a direct consequence of Theorems 1.3, 1.4 and (1.1).

We use the composition methods in [20] arranged for suspended real projective spaces. And we are based on the result in [18]. The exact sequence of James [10, Theorem 2.1] is used to determine the group structure of $\pi_6(\Sigma\mathrm{P}^2 \wedge \mathrm{P}^2)$ (Lemma 3.1 (ii)). To prove Theorem 1.3, it is essential to find out the triviality of some element of $[M^7, \Sigma\mathrm{P}^2 \wedge \mathrm{P}^2]$ (3.3) by considering the Whitehead product $[\iota_{M^4}, 2\iota_{M^4}]$.

2. Recollection of known results

Let ν' be a generator of the 2-primary component $\pi_6^3 \cong \mathbb{Z}_4$ of $\pi_6(S^3)$. We need the following facts [20]:

(2.1)
$$\begin{gathered}\pm\nu' = \bar{\eta}_3\tilde{\eta}_4,\\ 2\nu' = \eta_3^3,\ \eta_3\nu_4 = \nu'\eta_6,\\ \pi_7^4 = \mathbb{Z}\{\nu_4\} \oplus \mathbb{Z}_4\{\Sigma\nu'\}\end{gathered}$$

and

$$\pm[\iota_4, \iota_4] = 2\nu_4 - \Sigma\nu'.$$

We also recall from [13], [14], [15], [16] that

(2.2)
$$\begin{aligned}\pi_7(M^5) &= \mathbb{Z}_4\{i_5\nu_4\} \oplus \mathbb{Z}_2\{\tilde{\eta}_4\eta_6\},\\ \pi_6(M^4) &= \mathbb{Z}_4\{\delta\} \oplus \mathbb{Z}_2\{\tilde{\eta}_3\eta_5\},\\ 2\delta &= i_4\nu',\\ \Sigma\delta &= 2(i_5\nu_4) \in \pi_7(M^5),\\ \pi_7(M^4) &= \mathbb{Z}_2\{\delta\eta_6\} \oplus \mathbb{Z}_2\{\tilde{\eta}_3\eta_5^2\}\end{aligned}$$

and

(2.3)
$$[\iota_{M^4}, i_4] = \delta p_6.$$

Making use of the cofiber sequence $(*)_k$ for $k = 5, 6$, by the group structures $\pi_k(M^5)(k = 6, 7)$, $\pi_k^3(k = 5, 6)$, $\pi_k(M^4)(5 \le k \le 7)$ and by (1.4), (2.2), (1.3), we obtain the following (see [16, Lemma 1.5 (iii)]).

Lemma 2.1.
(i) $[M^7, M^5] = \mathbb{Z}_2\{i_5\eta_4\bar{\eta}_5\} \oplus \mathbb{Z}_2\{\tilde{\eta}_4\eta_6 p_7\} \oplus \mathbb{Z}_2\{i_5\nu_4 p_7\}$.
(ii) $[M^6, S^3] = \mathbb{Z}_2\{\eta_3\bar{\eta}_4\} \oplus \mathbb{Z}_2\{\nu' p_6\}$.
(iii) $[M^6, M^4] = \mathbb{Z}_2\{i_4\eta_3\bar{\eta}_4\} \oplus \mathbb{Z}_2\{\tilde{\eta}_3\eta_5 p_6\} \oplus \mathbb{Z}_2\{\delta p_6\}$.
(iv) $[M^7, M^4] = \mathbb{Z}_4\{\tilde{\eta}_3\bar{\eta}_5\} \oplus \mathbb{Z}_2\{\overline{i_4\nu'}\} \oplus \mathbb{Z}_2\{\delta\eta_6 p_7\}$, *where* $2\tilde{\eta}_3\bar{\eta}_5 = \tilde{\eta}_3\eta_5^2 p_7$ *and* $\overline{i_4\nu'}$ *is an extension of* $i_4\nu'$.

The smash product of P^2 with itself has the following cell structure:

$$\mathrm{P}^2 \wedge \mathrm{P}^2 = M^3 \cup_{2\iota_{M^3}} CM^3,$$

where we take $M^3 = \mathrm{P}^2 \wedge S^1$ and $2\iota_{M^3} = \iota_{\mathrm{P}^2} \wedge 2\iota_1$. So, by (1.3), it turns to the form:

$$\mathrm{P}^2 \wedge \mathrm{P}^2 = (M^3 \vee S^3) \cup_{f_1' i_3 \eta_2 + 2 f_2'} e^4,$$

where $f_1' : M^3 \to M^3 \vee S^3$ and $f_2' : S^3 \to M^3 \vee S^3$ be the embeddings to the first and second spaces, respectively. We set $i' = i_{M^3 \vee S^3, \mathrm{P}^2 \wedge \mathrm{P}^2}$, $p' = p_{\mathrm{P}^2 \wedge \mathrm{P}^2, M^3 \vee S^3}$ and $p'' = p_{\mathrm{P}^2 \wedge \mathrm{P}^2, M^3}$. Then we can take $i' f_1' = \iota_{\mathrm{P}^2} \wedge i_2$. Since $p'' \circ i' f_2' = i_4$, $i' f_2'$ is a coextension of i_3. We set $\tilde{i}_3 = i' f_2'$, $\tilde{i}_n = \Sigma^{n-3} \tilde{i}_3 (n \geq 3)$, $f_k = \Sigma f_k' (k = 1, 2)$, $i = \Sigma i'$ and $p = \Sigma p'$. By [17, Lemma 2.4],

$$\pi_3(\mathrm{P}^2 \wedge \mathrm{P}^2) = \mathbb{Z}_8\{\tilde{i}_3\}$$

and

$$\pi_n(\Sigma^{n-3}(\mathrm{P}^2 \wedge \mathrm{P}^2)) = \mathbb{Z}_4\{\tilde{i}_n\} \qquad (n \geq 4),$$

where

$$2\tilde{i}_n = (\Sigma^{n-3}(i_2 \wedge i_2))\eta_{n-1} \qquad (n \geq 3). \tag{2.4}$$

Let us recall that P^3 is homeomorphic to the 3-rd rotation group. Let $h : \Sigma \mathrm{P}^3 \wedge \mathrm{P}^3 \to \Sigma \mathrm{P}^3$ be the Hopf construction induced from the multiplication of the topological group P^3. We know the following ([5], [11]).

Lemma 2.2. *There exists a direct sum decomposition for a space X:*

$$[\Sigma^2 X, \Sigma \mathrm{P}^3] = h_*[\Sigma^2 X, \Sigma \mathrm{P}^3 \wedge \mathrm{P}^3] \oplus \Sigma[\Sigma X, \mathrm{P}^3].$$

The following is [18, Lemma 3.2].

Lemma 2.3. $\Sigma \mathrm{P}^3 \wedge \mathrm{P}^3 = ((\Sigma \mathrm{P}^2 \wedge \mathrm{P}^2) \cup_{i\nu'} e^7) \vee M^6 \vee M^6$, *where* $i = i_3 \wedge i_2$.

Let $h_0 = h\mid_{(\Sigma \mathrm{P}^2 \wedge \mathrm{P}^2) \cup_{i\nu'} e^7}$, $h' = h\mid_{\Sigma \mathrm{P}^2 \wedge \mathrm{P}^2}$, $h'' = h\mid_{M^4}$ and $h''' = h\mid_{S^3}$ be the restrictions of h, respectively. By [18, Lemma 2.3 (i)],

$$h''' = (\Sigma i_{1,3})\eta_2 \in \pi_3(\Sigma \mathrm{P}^3) \tag{2.5}$$

and

$$h' \tilde{i}_4 = \pm(\Sigma i_{2,3})\tilde{\eta}_2 \in \pi_4(\Sigma \mathrm{P}^3).$$

By [18, Lemma 5.2, (6.3)],

$$\Sigma \gamma_4 = (\Sigma i_{3,4}) h'' \tilde{\eta}_3 \in \pi_5(\Sigma \mathrm{P}^4) \tag{2.6}$$

and

$$4\iota_{\Sigma \mathrm{P}^4} = (\Sigma i_{1,4})\eta_2 \bar{\eta}_3 \Sigma p_{4,2}. \tag{2.7}$$

3. Proof of the fact that $2[M^7, \Sigma \mathrm{P}^3] = 0$

First we show the following.

Lemma 3.1.
(i) $\pi_5(\Sigma \mathrm{P}^2 \wedge \mathrm{P}^2) = \mathbb{Z}_2\{(\iota_{M^3} \wedge i_2)\tilde{\eta}_3\} \oplus \mathbb{Z}_2\{\tilde{\imath}_4\eta_4\}$ *and* $\Sigma^n : \pi_5(\Sigma \mathrm{P}^2 \wedge \mathrm{P}^2) \to \pi_{n+5}(\Sigma^{n+1}\mathrm{P}^2 \wedge \mathrm{P}^2)$ *is an isomorphism for* $n \geq 1$.
(ii) $\pi_6(\Sigma \mathrm{P}^2 \wedge \mathrm{P}^2) = \mathbb{Z}_4\{(\iota_{M^3} \wedge i_2)\delta\} \oplus \mathbb{Z}_2\{(\iota_{M^3} \wedge i_2)\tilde{\eta}_3\eta_5\} \oplus \mathbb{Z}_2\{\tilde{\imath}_4\eta_4^2\} \oplus \mathbb{Z}_2\{[i_3 \wedge i_2, \tilde{\imath}_4]\}$, *where* $2((\iota_{M^3} \wedge i_2)\delta) = (i_3 \wedge i_2)\nu'$.

Proof. (i) is easily obtained (see [18, §4]).

The relation in (ii) is obtained from (2.2). We consider the homotopy exact sequence of a pair $(\Sigma \mathrm{P}^2 \wedge \mathrm{P}^2, M^4 \vee S^4)$:

$$\pi_7(\Sigma \mathrm{P}^2 \wedge \mathrm{P}^2, M^4 \vee S^4) \xrightarrow{\partial} \pi_6(M^4 \vee S^4) \xrightarrow{i_*} \pi_6(\Sigma \mathrm{P}^2 \wedge \mathrm{P}^2)$$
$$\xrightarrow{j_*} \pi_6(\Sigma \mathrm{P}^2 \wedge \mathrm{P}^2, M^4 \vee S^4) \xrightarrow{\partial} \cdots .$$

By Blakers-Massey [7], $\pi_6(\Sigma \mathrm{P}^2 \wedge \mathrm{P}^2, M^4 \vee S^4) \cong \pi_6(S^5)$. The generator of the relative homotopy group is denoted by $\hat{\eta}_4$, satisfying $p_*\hat{\eta}_4 = \eta_5$. We have $\pi_5(M^4 \vee S^4) \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2$. So, by (1.4),

$$\partial\hat{\eta}_4 = (f_1 i_4 \eta_3 + 2f_2) \circ \eta_4 = f_1 i_4 \eta_3^2 = 2f_1\tilde{\eta}_3 \neq 0.$$

Hence $i_* : \pi_6(M^4 \vee S^4) \to \pi_6(\Sigma \mathrm{P}^2 \wedge \mathrm{P}^2)$ is an epimorphism.

Since $\pi_7(M^4 \times S^4, M^4 \vee S^4) \cong \pi_7(M^8)$, $M^4 \times S^4 = (M^4 \vee S^4) \cup_{[f_1,f_2]} CM^7$ and $[f_1, f_2] \circ i_7 = [f_1 i_4, f_2]$, we obtain

$$\pi_6(M^4 \vee S^4) = \mathbb{Z}_4\{f_1\delta\} \oplus \mathbb{Z}_2\{f_1\tilde{\eta}_3\eta_5\} \oplus \mathbb{Z}_2\{f_2\eta_4^2\} \oplus \mathbb{Z}_2\{[f_1 i_4, f_2]\}.$$

By [10, Theorem 2.1], $\pi_7(\Sigma \mathrm{P}^2 \wedge \mathrm{P}^2, M^4 \vee S^4) = \mathbb{Z}_2\{\widehat{\eta_4^2}\} \oplus \mathbb{Z}_2\{[\omega, f_1 i_4]\}$, where ω is the characteristic map of the 5-cell of $\Sigma \mathrm{P}^2 \wedge \mathrm{P}^2$, $[\ ,\ ]$ stands for the relative Whitehead product ([8]) and $\widehat{\eta_4^2}$ is an element satisfying $p_*\widehat{\eta_4^2} = \eta_5^2$. We have $\partial(\widehat{\eta_4^2}) = f_1 i_4 \eta_3^3 = 0$. By [8] and the fact that $[\eta_3, \iota_3] = 0$, we see that

$$\partial[\omega, f_1 i_4] = -[f_1 i_4 \eta_3 + 2f_2, f_1 i_4] = (f_1 i_4)[\eta_3, \iota_3] + 2[f_2, f_1 i_4] = 0.$$

This leads to (ii), completing the proof. □

By use of [2, Theorem 2.4] and [6, Proposition II. 3.2], we obtain the following (see [15, Remark, p. 273]).

Lemma 3.2. *Let* $\alpha \in [\Sigma A, X], \beta \in [\Sigma B, X], \delta \in [D, A]$ *and* $\varepsilon \in [E, B]$, *where* A, B, D, E *are polyhedra and* X *is a space. Then*

$$[\alpha \circ \Sigma\delta, \beta \circ \Sigma\varepsilon] = [\alpha, \beta] \circ \Sigma(\delta \wedge \varepsilon).$$

Next we show the following.

Lemma 3.3. $2[M^7, \Sigma \mathrm{P}^2 \wedge \mathrm{P}^2] = 0.$

Proof. Since $[\iota_{M^3} \wedge i_2, \tilde{i}_4] \circ \Sigma((i_2 \wedge \iota_1) \wedge \iota_3) = [i_3 \wedge i_2, \tilde{i}_4]$ by Lemma 3.2, $[\iota_{M^3} \wedge i_2, \tilde{i}_4]$ is an extension of $[i_3 \wedge i_2, \tilde{i}_4]$. By use of $(*)_6$ and by Lemma 3.1 (ii), we obtain

$$[M^7, \Sigma\mathrm{P}^2 \wedge \mathrm{P}^2] = \{(\iota_{M^3} \wedge i_2)\overline{\tilde{\iota}_4\nu'}, (\iota_{M^3} \wedge i_2)\tilde{\eta}_3\bar{\eta}_5, \tilde{i}_4\eta_4\bar{\eta}_5,$$
$$[\iota_{M^3} \wedge i_2, \tilde{i}_4]\} + \pi_7(\Sigma\mathrm{P}^2 \wedge \mathrm{P}^2) \circ p_7.$$

We have $2(\eta_4\bar{\eta}_5) = 0$ and $2\overline{\tilde{\iota}_4\nu'} = 0$ by Lemma 2.1 (iv). By the relation $2i_2 = 0$,

$$2[\iota_{M^3} \wedge i_2, \tilde{i}_4] = [\iota_{M^3} \wedge i_2, \tilde{i}_4] \circ \Sigma((i_2 \wedge 2\iota_1) \wedge \iota_3) = 0$$

and

$$2((\iota_{M^3} \wedge i_2)\tilde{\eta}_3\bar{\eta}_5) = 0.$$

Hence, by the relation $2p_7 = 0$, the assertion follows. □

We examine the Whitehead product $[\iota_{M^4}, 2\iota_{M^4}]$. By (1.3), Lemma 3.2, (2.3) and by the fact that

$$\iota_{\mathrm{P}^2} \wedge \eta_2 = i_4\bar{\eta}_3 + \tilde{\eta}_3 p_5 \in [M^5, M^4],$$

we get the following in $[\Sigma M^3 \wedge M^3, M^4]$:

$$\begin{aligned}[\iota_{M^4}, 2\iota_{M^4}] = [\iota_{M^4}, i_4\eta_3 p_4] &= [\iota_{M^4}, i_4] \circ \Sigma(\iota_{M^3} \wedge (\eta_2 p_3)) \\ &= [\iota_{M^4}, i_4] \circ \Sigma(\iota_{M^3} \wedge \eta_2) \circ \Sigma(\iota_{M^3} \wedge p_3) \\ &= \delta\eta_6 \circ p_7 \circ \Sigma(\iota_{M^3} \wedge p_3).\end{aligned}$$

Since

$$p_7 \circ \Sigma(\iota_{M^3} \wedge p_3) = \Sigma(p_3 \wedge p_3) \in [\Sigma M^3 \wedge M^3, S^7], \tag{3.1}$$

we obtain

$$[\iota_{M^4}, 2\iota_{M^4}] = \delta\eta_6 \circ \Sigma(p_3 \wedge p_3) \in [\Sigma M^3 \wedge M^3, M^4]. \tag{3.2}$$

Now we show the following.

Lemma 3.4. $2[M^7, (\Sigma\mathrm{P}^2 \wedge \mathrm{P}^2) \cup_{i\nu'} e^7] = 0.$

Proof. By (3.2) and the fact that $2(\iota_{M^3} \wedge i_2) = 0$,

$$\begin{aligned}(\iota_{M^3} \wedge i_2)\delta\eta_6 \circ \Sigma(p_3 \wedge p_3) &= [\iota_{M^3} \wedge i_2, 2(\iota_{M^3} \wedge i_2)] \\ &= 0 \in [\Sigma M^3 \wedge M^3, \Sigma\mathrm{P}^2 \wedge \mathrm{P}^2].\end{aligned}$$

So, by making use of the cofiber sequence

$$M^6 \xrightarrow{\iota_{M^4} \wedge i_3} \Sigma M^3 \wedge M^3 \xrightarrow{\iota_{M^4} \wedge p_3} M^7 \xrightarrow{2\iota_{M^7}} M^7 \longrightarrow \cdots$$

and by (3.1), we obtain $(\iota_{M^3} \wedge i_2)\delta\eta_6 p_7 \in 2[M^7, \Sigma\mathrm{P}^2 \wedge \mathrm{P}^2]$. Hence, by Lemma 3.3,

$$(\iota_{M^3} \wedge i_2)\delta\eta_6 p_7 = 0 \in [M^7, \Sigma\mathrm{P}^2 \wedge \mathrm{P}^2]. \tag{3.3}$$

We set $B = (\Sigma\mathrm{P}^2 \wedge \mathrm{P}^2) \cup_{i\nu'} e^7$ and $r = i_{\Sigma\mathrm{P}^2 \wedge \mathrm{P}^2, B}$. From the homotopy exact sequence of a pair $(B, \Sigma\mathrm{P}^2 \wedge \mathrm{P}^2)$ and by Lemma 3.1 (ii), we obtain

$$\pi_6(B) = \{r(\iota_{M^3} \wedge i_2)\delta, r(\iota_{M^3} \wedge i_2)\tilde{\eta}_3\eta_5, r\tilde{i}_4\eta_4^2, r[i_3 \wedge i_2, \tilde{i}_4]\} \cong (\mathbb{Z}_2)^4.$$

So, by use of $(*)_6$, we obtain

$$[M^7, B] = \{\overline{r(\iota_{M^3} \wedge i_2)\delta}, r(\iota_{M^3} \wedge i_2)\tilde{\eta}_3\bar{\eta}_5, r\tilde{i}_4\eta_4\bar{\eta}_5, r[\iota_{M^3} \wedge i_2, \tilde{i}_4]\} + \pi_7(B) \circ p_7,$$

and hence, $2[M^7, B] = \{2\overline{r(\iota_{M^3} \wedge i_2)\delta}\}$. By (1.3) and (3.3),

$$2\overline{r(\iota_{M^3} \wedge i_2)\delta} = r(\iota_{M^3} \wedge i_2)\delta\eta_6 p_7 = 0.$$

This leads to the assertion, completing the proof. □

Since $r \circ (i_3 \wedge i_2)\nu' = 0$ and $h'''\nu' = h_0 \circ r(i_3 \wedge i_2)\nu'$, we obtain

$$(\Sigma i_{1,3})\eta_2\nu' = h'''\nu' = 0 \in \pi_6(\Sigma\mathrm{P}^3). \tag{3.4}$$

By Lemma 2.3, we have

$$\pi_6(\Sigma\mathrm{P}^3 \wedge \mathrm{P}^3) \cong \pi_6(B) \oplus \pi_6(M^6) \oplus \pi_6(M^6).$$

Since $2\pi_6(B) = 2\pi_6(M^6) = 0$, we get that $2\pi_6(\Sigma\mathrm{P}^3 \wedge \mathrm{P}^3) = 0$. We have $2\Sigma\pi_5(\mathrm{P}^3) = 0$. So, by Lemma 2.2, we obtain the following.

Lemma 3.5. $2\pi_6(\Sigma\mathrm{P}^3) = 0$.

We show the following.

Lemma 3.6. $(\Sigma i_{3,n})_* : \pi_6(\Sigma\mathrm{P}^3) \to \pi_6(\Sigma\mathrm{P}^n)$ *is an epimorphism for* $n = 4$, $n \geq 6$ *and* $\pi_6(\Sigma\mathrm{P}^5) = \mathbb{Z}\{\Sigma\gamma_5\} \oplus (\Sigma i_{3,5})_*\pi_6(\Sigma\mathrm{P}^3)$.

Proof. In the homotopy exact sequence of a pair $(\Sigma\mathrm{P}^4, \Sigma\mathrm{P}^3)$, the connecting homomorphism $\partial : \pi_6(\Sigma\mathrm{P}^4, \Sigma\mathrm{P}^3) \to \pi_5(\Sigma\mathrm{P}^3)$ is a monomorphism by [18, Theorem 5.3] and its proof. Hence $(\Sigma i_{3,4})_* : \pi_6(\Sigma\mathrm{P}^3) \to \pi_6(\Sigma\mathrm{P}^4)$ is an epimorphism.

By making use of the homotopy exact sequence of a pair $(\Sigma\mathrm{P}^5, \Sigma\mathrm{P}^4)$ and by (1.2), we conclude that

$$\pi_6(\Sigma\mathrm{P}^5) = \mathbb{Z}\{\Sigma\gamma_5\} \oplus (\Sigma i_{4,5})_*\pi_6(\Sigma\mathrm{P}^4).$$

Obviously $(\Sigma i_{5,n})_* : \pi_6(\Sigma\mathrm{P}^5) \to \pi_6(\Sigma\mathrm{P}^n)$ for $n \geq 6$ is an epimorphism. This completes the proof. □

Now we prove Theorem 1.3.

Proof of Theorem 1.3.
By Lemma 3.5, any element $\alpha \in \pi_6(\Sigma \mathrm{P}^3)$ is extendible to $\bar{\alpha} \in [M^7, \Sigma \mathrm{P}^3]$. So, by use of $(*)_6$ and by Lemma 3.6, we obatin

$$[M^7, \Sigma \mathrm{P}^n] = \{(\Sigma i_{3,n})\bar{\alpha}\} + \pi_7(\Sigma \mathrm{P}^n) \circ p_7 \ (\alpha \in \pi_6(\Sigma \mathrm{P}^3)).$$

Therefore it suffices to prove that $2[M^7, \Sigma \mathrm{P}^3] = 0$. By [7], $[CM^7, M^7; B \times M^6, B \vee M^6] \cong [M^8, B \wedge M^6]$ $(B = (\Sigma \mathrm{P}^2 \wedge \mathrm{P}^2) \cup_{i\nu'} e^7)$. So, by Lemma 2.3, we have

$$\begin{aligned}&[M^7, \Sigma \mathrm{P}^3 \wedge \mathrm{P}^3] \\ &\quad \cong [M^7, B] \oplus [M^7, M^6] \oplus [M^7, M^6] \oplus [M^8, B \wedge M^6] \oplus [M^8, B \wedge M^6].\end{aligned}$$

By Lemma 3.4, $2[M^7, B] = 0$. By (1.6), $2[M^7, M^6] = 0$. Since $[M^8, B \wedge M^6] \cong [M^8, M^9] \cong \mathbb{Z}_2$, we obtain $2[M^8, B \wedge M^6] = 0$. By Lemma 2.1 (ii), $[M^6, \mathrm{P}^3] \cong [M^6, S^3] \cong (\mathbb{Z}_2)^2$. This shows that $2\Sigma[M^6, \mathrm{P}^3] = 0$. Hence, by Lemma 2.2, we conclude that $2[M^7, \Sigma \mathrm{P}^3] = 0$. Thus the proof of Theorem 1.3 is complete. □

We obtain the following.

Corollary 3.7. *$\pi_6(\Sigma \mathrm{P}^n) \circ \eta_6 p_7 = 0$ for $n \geq 3$, $n \neq 5$ and $\pi_6(\Sigma \mathrm{P}^5) \circ \eta_6 p_7 = \{(\Sigma\gamma_5)\eta_6 p_7\}$.*

Proof. Let $\alpha \in \pi_6(\Sigma \mathrm{P}^n)$ for $n \geq 3$ be a nontrivial element except for $\Sigma\gamma_5$. Then, by Lemmas 3.5 and 3.6, the order of α is 2, and hence α is extendible to $\bar{\alpha} \in [M^7, \Sigma \mathrm{P}^n]$. Thus, by (1.3) and Theorem 1.3, $\alpha \circ (\eta_6 p_7) = 2\bar{\alpha} \in 2[M^7, \Sigma \mathrm{P}^n] = 0$. This completes the proof. □

4. Proof of Theorem 1.1

First of all we show the following.

Lemma 4.1.
(i) $\Sigma[M^6, M^4] = \mathbb{Z}_2\{i_5\eta_4\bar{\eta}_5\} \oplus \mathbb{Z}_2\{\tilde{\eta}_4\eta_6 p_7\}$.
(ii) $\Sigma[M^6, M^3] = \mathbb{Z}_4\{\tilde{\eta}_3\bar{\eta}_5\}$.

Proof. By the fact that $\Sigma(\delta p_6) = \Sigma[\iota_{M^4}, i_4] = 0$ (2.3), (i) is a direct consequence of Lemma 2.1 (iii).

We know that $\tilde{\eta}_3\bar{\eta}_5 \in \Sigma[M^6, M^3]$. We consider the Hopf homomorphism $H : [M^7, M^4] \to [M^7, \Sigma M^3 \wedge M^3]$. By Lemma 3.1 (i),

$$\pi_7(\Sigma M^3 \wedge M^3) = \mathbb{Z}_2\{\tilde{i}_6\eta_6\} \oplus \mathbb{Z}_2\{(\iota_{M^4} \wedge i_3)\tilde{\eta}_5\}.$$

So, by use of $(*)_6$ combining with the fact that $\pi_6(\Sigma M^3 \wedge M^3) = \mathbb{Z}_4\{\tilde{i}_6\}$ and $2\tilde{i}_6 = (i_4 \wedge i_3)\eta_5$ (2.4), we obtain

$$[M^7, \Sigma M^3 \wedge M^3] = \mathbb{Z}_2\{(i_4 \wedge i_3)\bar{\eta}_5\} \oplus \mathbb{Z}_2\{\tilde{i}_6\eta_6 p_7\} \oplus \mathbb{Z}_2\{(\iota_{M^4} \wedge i_3)\tilde{\eta}_5 p_7\}.$$

By [14, Proposition 14],

$$H(\delta) = \pm\tilde{\imath}_6,$$

and so $H(\delta\eta_6 p_7) = \tilde{\imath}_6\eta_6 p_7$.

By use of a generalized version [9] of [20, Proposition 2.2] and by the fact that $H(\nu') = \eta_5$ [20, (5.3)] for $H : \pi_6^3 \to \pi_6^5$, we obtain

$$H(\overline{i_4\nu'}) \circ i_7 = H(i_4\nu') = (i_4 \wedge i_3)\eta_5 = (i_4 \wedge i_3)\bar{\eta}_5 \circ i_7 \in \pi_6(\Sigma M^3 \wedge M^3).$$

Hence, by use of $(*)_6$, we obtain

$$H(\overline{i_4\nu'}) \equiv (i_4 \wedge i_3)\bar{\eta}_5 \bmod \pi_7(\Sigma M^3 \wedge M^3) \circ p_7 = \{\tilde{\imath}_6\eta_6 p_7, (\iota_{M^4} \wedge i_3)\tilde{\eta}_5 p_7\}.$$

Thus (ii) follows from the fact that $H \circ \Sigma = 0$. This completes the proof. $\square$

For the cell complex $\mathrm{P}_3^6 = M^4 \cup_{i_4\bar{\eta}_3} CM^5$, we set

$$i' = i_{M^4,\mathrm{P}_3^6}, \quad i'' = i' \circ i_4, \quad p' = p_{\mathrm{P}_3^6,M^4} \quad \text{and} \quad p'' = p_6 \circ p'.$$

Let $\widetilde{i_5\eta_4 p_5}$ be a coextension of $i_5\eta_4 p_5 = 2\iota_{M^5}$. It is taken as a representative of the Toda bracket

$$\{i', i_4\bar{\eta}_3, i_5\eta_4 p_5\} \subset [M^6, \mathrm{P}_3^6].$$

Then, by the properties of Toda brackets and by the fact that

$$\{2\iota_5, p_5, i_5\} \ni \iota_5 \bmod 2\iota_5,$$

we see that

$$\begin{aligned}
2\widetilde{i_5\eta_4 p_5} &\in \{i', i_4\bar{\eta}_3, i_5\eta_4 p_5\} \circ i_6\eta_5 p_6 \\
&= i' \circ \{i_4\bar{\eta}_3, i_5\eta_4 p_5, i_5\} \circ \eta_5 p_6 \\
&\supset i' \circ \{2\tilde{\eta}_3, p_5, i_5\} \circ \eta_5 p_6 \\
&\supset i'\tilde{\eta}_3 \circ \{2\iota_5, p_5, i_5\} \circ \eta_5 p_6 \\
&\ni i'\tilde{\eta}_3\eta_5 p_6 \bmod i'_*[M^6, M^4] \circ 2\iota_{M^6} + [M^6, \mathrm{P}_3^6] \circ 2\iota_{M^6} \circ 2\iota_{M^6}.
\end{aligned}$$

By (1.3) and Lemma 2.1 (iii), the indeterminacy $i'_*[M^6, M^4] \circ 2\iota_{M^6} + [M^6, \mathrm{P}_3^6] \circ 2\iota_{M^6} \circ 2\iota_{M^6}$ is trivial. That is,

$$2\widetilde{i_5\eta_4 p_5} = i'\tilde{\eta}_3\eta_5 p_6 \in [M^6, \Sigma\mathrm{P}_3^6]. \tag{4.1}$$

Since $p_4 \circ i_4\bar{\eta}_3 = 0$, there exists an extension $\bar{p}_4 \in [\mathrm{P}_3^6, S^4]$ of p_4. We show the following.

Lemma 4.2. $2\iota_{\Sigma\mathrm{P}_3^6} \equiv \pm\Sigma\widetilde{i_5\eta_4 p_5}\Sigma p' \bmod (\Sigma i'')\nu_4\Sigma p''$.

Proof. By use of the canonical bijection

$$[C\mathrm{P}_3^6, \mathrm{P}_3^6; \Sigma\mathrm{P}_3^6, M^5] \cong [\Sigma\mathrm{P}_3^6, M^7] = \{\Sigma p'\},$$

we obtain the exact sequence

$$[\Sigma \mathrm{P}_3^6, M^5] \xrightarrow{(\Sigma i')_*} [\Sigma \mathrm{P}_3^6, \Sigma \mathrm{P}_3^6] \xrightarrow{(\Sigma p')_*} [\Sigma \mathrm{P}_3^6, M^7] \longrightarrow 0.$$

Since $(\Sigma p')_*(2\iota_{\Sigma \mathrm{P}_3^6} - \Sigma \widetilde{i_5 \eta_4 p_5} \Sigma p') = 0$, we get that

$$2\iota_{\Sigma \mathrm{P}_3^6} - \Sigma \widetilde{i_5 \eta_4 p_5} \Sigma p' \in (\Sigma i')_*[\Sigma \mathrm{P}_3^6, M^5].$$

Making use of the exact sequence induced from the cofiber sequence starting with $i_5\bar{\eta}_4 : M^6 \to M^5$:

$$[M^6, M^5] \xleftarrow{(i_5\bar{\eta}_4)^*} [M^5, M^5] \xleftarrow{(\Sigma i')^*} [\Sigma \mathrm{P}_3^6, M^5] \xleftarrow{(\Sigma p')^*} [M^7, M^5] \xleftarrow{(i_6\bar{\eta}_5)^*} [M^6, M^5],$$

together with (1.3) and Lemma 2.1 (i), we obtain

$$[\Sigma \mathrm{P}_3^6, M^5] = \{i_5\eta_4\Sigma\bar{p}_4, \tilde{\eta}_4\eta_6\Sigma p'', i_5\nu_4\Sigma p''\} \cong (\mathbb{Z}_2)^3.$$

So, by the fact that $i' \circ i_4\eta_3 = i' \circ i_4\bar{\eta}_3 \circ i_5 = 0$ and by (4.1), we obtain

$$(\Sigma i')_*[\Sigma \mathrm{P}_3^6, M^5] = \{2\Sigma\widetilde{i_5\eta_4 p_5}\Sigma p', (\Sigma i'')\nu_4\Sigma p''\}.$$

This leads to the relation, completing the proof. □

In fact we can show the following.

Remark 4.3. $[\Sigma \mathrm{P}_3^6, \Sigma \mathrm{P}_3^6] = \mathbb{Z}_8\{\iota_{\Sigma \mathrm{P}_3^6}\} \oplus \mathbb{Z}_2\{(\Sigma i'')\nu_4\Sigma p''\}$.

Now we prove Theorem 1.4.

Proof of Theorem 1.4.
We consider the exact sequence $(i = \Sigma i_{4,6}, p = \Sigma p_{6,4})$

$$[\Sigma \mathrm{P}^4, \Sigma \mathrm{P}^6] \xleftarrow{i^*} [\Sigma \mathrm{P}^6, \Sigma \mathrm{P}^6] \xleftarrow{p^*} [M^7, \Sigma \mathrm{P}^6].$$

By use of the commutative diagram:

$$\begin{array}{ccc} \mathrm{P}^4 & \xrightarrow{i_{4,6}} & \mathrm{P}^6 \\ \downarrow{\scriptstyle p_{4,2}} & & \downarrow{\scriptstyle p_{6,2}} \\ M^4 & \xrightarrow{i'} & \mathrm{P}_3^6 \end{array}$$

and by (2.7),

$$\overline{(\Sigma i_{1,4})\eta_2\bar{\eta}_3} \circ \Sigma p_{6,2} \circ \Sigma i_{4,6} = (\Sigma i_{1,4})\eta_2\bar{\eta}_3\Sigma p_{4,2} = 4\iota_{\Sigma \mathrm{P}^4}.$$

So, by the relation

$$\Sigma i_{4,6} \circ 4\iota_{\Sigma \mathrm{P}^4} = 4\iota_{\Sigma \mathrm{P}^6} \circ \Sigma i_{4,6} \in [\Sigma \mathrm{P}^4, \Sigma \mathrm{P}^6],$$

we have Theorem 1.4 (i).

By Lemma 4.2, we see that

$$\begin{aligned} 2\overline{(\Sigma i_{1,4})\eta_2\bar{\eta}_3} &= \overline{(\Sigma i_{1,4})\eta_2\bar{\eta}_3} \circ 2\iota_{\Sigma \mathrm{P}^6_3} \\ &\equiv \pm\overline{(\Sigma i_{1,4})\eta_2\bar{\eta}_3} \circ \Sigma\widetilde{i_5\eta_4 p_5}\Sigma p' \\ &\quad \bmod \overline{(\Sigma i_{1,4})\eta_2\bar{\eta}_3} \circ (\Sigma i'')\nu_4\Sigma p''. \end{aligned}$$

By the relation $\eta_3\nu_4 = \nu'\eta_6$ and (3.4),

$$\begin{aligned} \overline{(\Sigma i_{1,4})\eta_2\bar{\eta}_3} \circ (\Sigma i'')\nu_4 &= (\Sigma i_{1,4})\eta_2\eta_3\nu_4 \\ &= (\Sigma i_{1,4})\eta_2\nu'\eta_6 = 0 \in \pi_7(\Sigma \mathrm{P}^4). \end{aligned}$$

By (4.1), (2.1) and (3.4),

$$\begin{aligned} 2(\overline{(\Sigma i_{1,4})\eta_2\bar{\eta}_3} \circ \Sigma\widetilde{i_5\eta_4 p_5}) &= \overline{(\Sigma i_{1,4})\eta_2\bar{\eta}_3} \circ (\Sigma i')\tilde{\eta}_4\eta_6 p_7 \\ &= (\Sigma i_{1,4})\eta_2\nu'\eta_6 p_7 = 0 \in [M^7, \Sigma \mathrm{P}^4]. \end{aligned}$$

Hence we conclude that

$$2\overline{(\Sigma i_{1,4})\eta_2\bar{\eta}_3} = \overline{(\Sigma i_{1,4})\eta_2\bar{\eta}_3} \circ \Sigma\widetilde{i_5\eta_4 p_5}\Sigma p' \in [\Sigma \mathrm{P}^6_3, \Sigma \mathrm{P}^4].$$

By [20, Proposition 1.7], we obtain

$$\begin{aligned} &\overline{(\Sigma i_{1,4})\eta_2\bar{\eta}_3} \circ \Sigma\widetilde{i_5\eta_4 p_5} \in \{(\Sigma i_{1,4})\eta_2\bar{\eta}_3, i_5\bar{\eta}_4, i_6\eta_5 p_6\}_1 \\ &\bmod (\Sigma i_{1,4})\eta_2\bar{\eta}_3 \circ \Sigma[M^6, M^4] + [M^7, \Sigma \mathrm{P}^4] \circ i_7\eta_6 p_7. \end{aligned}$$

By (1.3) and Theorem 1.3, $[M^7, \Sigma \mathrm{P}^4] \circ i_7\eta_6 p_7 = 2[M^7, \Sigma \mathrm{P}^4] = 0$. By Lemma 4.1 (i), (2.1), (3.4) and by the relation $(\Sigma i_{1,4})\eta_2^3 = 0$, we obtain

$$(\Sigma i_{1,4})\eta_2\bar{\eta}_3 \circ \Sigma[M^6, M^4] = \{(\Sigma i_{1,4})\eta_2^3\bar{\eta}_5, (\Sigma i_{1,4})\eta_2\nu'\eta_6 p_7\} = 0.$$

Hence the indeterminacy is trivial and we get that

$$\overline{(\Sigma i_{1,4})\eta_2\bar{\eta}_3} \circ \Sigma\widetilde{i_5\eta_4 p_5} = \{(\Sigma i_{1,4})\eta_2\bar{\eta}_3, i_5\bar{\eta}_4, i_6\eta_5 p_6\}_1 \in [M^7, \Sigma \mathrm{P}^4].$$

This leads to Theorem 1.4 (ii).

Since $(\Sigma p_{5,4})(\Sigma\gamma_5) = 2\iota_6$ by (1.2), we obtain

$$\Sigma\gamma_5 \in \{\Sigma i_{4,5}, \Sigma\gamma_4, 2\iota_5\}.$$

By the properties of Toda brackets,

$$(\Sigma\gamma_5)\eta_6 p_7 \in \{\Sigma i_{4,5}, \Sigma\gamma_4, 2\iota_5\} \circ \eta_6 p_7 = \Sigma i_{4,5} \circ \{\Sigma\gamma_4, 2\iota_5, \eta_5 p_6\}.$$

The indeterminacy of $\Sigma i_{4,5} \circ \{\Sigma\gamma_4, 2\iota_5, \eta_5 p_6\}$ is $(\Sigma i_{4,5} \circ \Sigma\gamma_4) \circ [M^7, S^5] + \Sigma i_{4,5} \circ \pi_6(\Sigma \mathrm{P}^4) \circ \eta_6 p_7 = 0$ by (1.1) and Corollary 3.7. Therefore

$$(\Sigma\gamma_5)\eta_6 p_7 = \Sigma i_{4,5} \circ \{\Sigma\gamma_4, 2\iota_5, \eta_5 p_6\}.$$

By the fact that $\Sigma[M^6, S^4] = [M^7, S^5]$,

$$\{\Sigma\gamma_4, 2\iota_5, \eta_5 p_6\} = \{\Sigma\gamma_4, 2\iota_5, \eta_5 p_6\}_1,$$

and hence

$$(\Sigma\gamma_5)\eta_6 p_7 = \Sigma i_{4,5} \circ \{\Sigma\gamma_4, 2\iota_5, \eta_5 p_6\}_1. \tag{4.2}$$

By (2.6), (1.4), (2.5) and by the relation $i_4\eta_3^2 = i_4\bar{\eta}_3 \circ i_5\bar{\eta}_4 \circ i_6$, we obtain

$$\begin{aligned}\{\Sigma\gamma_4, 2\iota_5, \eta_5 p_6\}_1 &\subset \{(\Sigma i_{3,4})h'', 2\tilde{\eta}_3, \eta_5 p_6\}_1 \\ &= \{(\Sigma i_{3,4})h'', i_4\eta_3^2, \eta_5 p_6\}_1 \\ &\supset \{(\Sigma i_{1,4})\eta_2\bar{\eta}_3, i_5\bar{\eta}_4, i_6\eta_5 p_6\}_1 \\ &\qquad \operatorname{mod} (\Sigma i_{3,4})h'' \circ \Sigma[M^6, M^3] + \pi_6(\Sigma \mathrm{P}^4) \circ \eta_6 p_7.\end{aligned}$$

By Corollary 3.7, $\pi_6(\Sigma\mathrm{P}^4) \circ \eta_6 p_7 = 0$. By Lemma 4.1 (ii) and (2.6),

$$(\Sigma i_{3,4})h'' \circ \Sigma[M^6, M^3] = \{(\Sigma\gamma_4)\bar{\eta}_5\}.$$

So we obtain

$$\{\Sigma\gamma_4, 2\iota_5, \eta_5 p_6\}_1 \equiv \{(\Sigma i_{1,4})\eta_2\bar{\eta}_3, i_5\bar{\eta}_4, i_6\eta_5 p_6\}_1 \operatorname{mod} (\Sigma\gamma_4)\bar{\eta}_5.$$

Thus, by (1.1) and (4.2), we obtain

$$(\Sigma\gamma_5)\eta_6 p_7 = (\Sigma i_{4,5}) \circ \{(\Sigma i_{1,4})\eta_2\bar{\eta}_3, i_5\bar{\eta}_4, i_6\eta_5 p_6\}_1.$$

This leads to Theorem 1.4 (iii), completing the proof of Theorem 1.4. □

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