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A density formula for the law of time spent on the positive side of one-dimensional diffusion processes

By

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Abstract

We show the existence and continuity of the density and obtain its formula for the law of the *fraction of time spent on the positive side* of a very general class of one-dimensional (generalized or gap) diffusion processes. We also study the asymptotic properties of the density at the extreme values $x = 0$ and $x = 1$, which extend, in particular, some recent results by Kasahara and Yano [8].

0. Introduction

Let $X = (X_t)$ be a standard one-dimensional Brownian motion with $X_0 = 0$ and $A^+(t) = \int_0^t 1_{[0,\infty)}(X_s)ds$ be the time spent on the positive side before t. The well-known *P. Lévy's arcsine law* (cf. e.g. [6, p. 57]) states that, for each fixed $t > 0$, the fraction $\frac{1}{t}A^+(t)$ is arcsine distributed:

$$(0.1) \qquad P\left(\frac{1}{t}A^+(t) \le x\right) = \frac{2}{\pi}\arcsin\sqrt{x}, \qquad 0 \le x \le 1$$

so that this law has a density $f^+(x)$ given by

$$(0.2) \qquad f^+(x) = \frac{1}{\pi}\frac{1}{\sqrt{x(1-x)}}, \qquad 0 < x < 1.$$

A slight generalization is given, in the case of a *skew Brownian motion* $X = (X_t)$ *with the skew parameter* p, $0 < p < 1$, by replacing (0.1) and (0.2) with the following (0.3) and (0.4), respectively:

$$(0.3) \qquad P\left(\frac{1}{t}A^+(t) \le x\right) = \frac{2}{\pi}\arcsin\sqrt{\frac{x}{x + (\frac{p}{1-p})^2(1-x)}}, \qquad 0 \le x \le 1,$$

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(0.4) $$f^+(x) = \frac{1}{\pi} \cdot \frac{p(1-p)}{(1-p)^2 x + p^2(1-x)} \cdot \frac{1}{\sqrt{x(1-x)}}, \qquad 0 < x < 1.$$

Here, we mean by the skew Brownian motion with the skew parameter p, $0 < p < 1$, a diffusion process $X = (X_t)$ on $\mathbb{R}$ uniquely determined by the following three properties: (i) it has the local generator $\frac{1}{2}\frac{d^2}{dx^2}$ on $\mathbb{R} \setminus \{0\}$, (ii) $\int_0^t 1_{\{0\}}(X_s)ds = 0$, a.s. for any $t > 0$, and (iii) the process $Y = (Y_t)$ defined by $Y_t = \frac{1}{p}X_t \vee 0 + \frac{1}{1-p}X_t \wedge 0$ is a martingale.

In the present work, we are concerned with some generalizations of these laws to the class of one-dimensional (generalized or gap) diffusion processes $X = (X_t)$. Before introducing our results, we would review some of such generalizations. The arcsine law for Brownian motions as above was generalized to the case of *skew Bessel diffusion processes* by Barlow, Pitman and Yor [1]. The class of distributions in this case was first introduced by Lamperti [10] and the explicit formula of densities is known, cf. Example 3.1 below. This class of distributions plays an important role in the limit theorem for the law of the fraction $\frac{1}{t}A^+(t)$ when $t \to \infty$ (Watanabe [14]). Recently Kasahara and Yano [8] studied the asymptotic properties of the distribution function at extreme values $x = 0$ and $x = 1$ for each fixed time $t > 0$. To be stated roughly, they showed, among others, that if the speed measure on the positive side behaves near the origin like that of a Bessel diffusion process, then so does the distribution function of the fraction $\frac{1}{t}A^+(t)$.

In all of these works, a fundamental tool is a *double Laplace transform formula* (cf. Theorem A in Section 3), which can be obtained analytically from the Feynman-Kac formula or probabilistically from the excursion theory. In this work, we first try to invert this formula and, in doing this, we obtain a density formula (Theorem 3.1) for the law of the fraction of time. The existence of continuous density can be established under a fairly mild condition on the speed measure of X. We then apply this density formula to study the asymptotic properties of the law at extreme values $x = 0$ and $x = 1$. Some results in [8] can be thereby refined in terms of the density of the law.

We would exhibit here a typical example of applications of our general results: Let $X = (X_t)$ $(X_0 = 0)$ be a diffusion process on an interval (l, r), $-\infty \le l < 0 < r \le \infty$, generated by a second order differential operator $L = a(x)\frac{d^2}{dx^2}$ with suitable boundary conditions when necessary; here we assume that $a(x)$ is positive, continuous except the origin where we assume that finite limits $a_\pm = \lim_{x \to 0\pm} a(x)$ exist such that both $a_\pm > 0$. Then, for each $t > 0$, the law of the fraction $\frac{1}{t}A^+(t)$ has a density $f_t^+(x)$ which is continuous in $(0, 1)$ and

(0.5)
$$f_t^+(x) \sim g_-^*(t)\sqrt{\frac{ta_+}{\pi}}\frac{1}{\sqrt{x}} \quad \text{and} \quad f_t^+(1-x) \sim g_+^*(t)\sqrt{\frac{ta_-}{\pi}}\frac{1}{\sqrt{x}} \qquad \text{as } x \searrow 0$$

with the notation $\sim$ introduced below. Here the constants $g_-^*(t)$ and $g_+^*(t)$ are uniquely determined from the values of the coefficient $a(x)$ on the interval $(l, 0)$

(resp. $(0, r)$) and the boundary condition at $x = l$ (resp. $x = r$). How these constants $g^*_-(t)$ and $g^*_+(t)$ can be determined, will be explained in Section 5.

Finally, we summarize the contents of this paper. Sections 1 and 2 are devoted to give preliminary and basic facts concerning strings and related quantities; particularly, we study the density of the law of the local time at the origin and its inverse of the reflecting diffusion on $[0, \infty)$ associated with a string. The study of the fraction of occupation time on the positive side for a given one-dimensional (generalized) diffusion will start from Section 3; the diffusion is formulated as an (m_+, m_-)-diffusion for a given pair $\{m_+, m_-\}$ of strings. We obtain a density formula of the law of the fraction in Section 3 under a very general condition (the condition **(A)**) on the pair of strings. The continuity of the density will be studied under a more stringent condition (the condition **(P)**) in Section 4. Its asymptotic behavior at the extreme points $x = 0$ and $x = 1$ will be studied in Section 5 by adding another condition (the condition **(G)**). Sufficient conditions and examples for these conditions will be given in Section 6.

Notations. Let $A(x)$ and $B(x)$ be positive functions defined near ∞ [resp. 0]. We write

$$A(x) \sim B(x) \qquad \text{as } x \to \infty\ [0] \qquad \text{if} \qquad \lim_{x\to\infty\ [0]} \frac{A(x)}{B(x)} = 1,$$

and

$$A(x) \asymp B(x) \qquad \text{as } x \to \infty\ [0]$$

if

$$A(x) = O(B(x)) \qquad \text{and} \qquad B(x) = O(A(x)) \qquad \text{as } x \to \infty\ [0].$$

1. Strings and Krein's correspondence

First, we recall a spectral theory of *strings* by M. G. Krein; see, for example, Dym and McKean [3, Chapters 5 and 6], Kasahara [7], Kotani and Watanabe [9] for details.

Let $\mathcal{M}$ be the class of functions $m : [0, \infty) \to [0, \infty]$ which are right-continuous and non-decreasing and which are assumed, for simplicity, to satisfy the conditions $m(0) < \infty$ and $m(\infty-) > 0$. An element m in $\mathcal{M}$ is called a string. Set $l = \sup\{x \geq 0;\ m(x) < \infty\}$ and $m(0-) = 0$. Then we can identify an element m in $\mathcal{M}$ with its Stieltjes measure dm, which is a non-negative Radon measure on $[0, l)$.

To a given $m \in \mathcal{M}$ we assign a function h on $(0, \infty)$, called the *spectral characteristic function* of the string m, in the following way. For any fixed

$\lambda \in (0,\infty)$, we consider the following integral equations on $[0,l)$:

$$e_0(x,\lambda) = 1 + \lambda \int_0^x dy \int_{[0,y]} e_0(z,\lambda) dm(z),$$
$$e_1(x,\lambda) = x + \lambda \int_0^x dy \int_{[0,y]} e_1(z,\lambda) dm(z).$$

The equations have unique continuous solutions $e_0(\cdot,\lambda)$ and $e_1(\cdot,\lambda)$ on $[0,l)$ for each $\lambda \in (0,\infty)$. We then define

$$h(\lambda) = \lim_{x \nearrow l} \frac{e_1(x,\lambda)}{e_0(x,\lambda)} = \int_0^l \frac{1}{e_0(x,\lambda)^2} dx.$$

Let h be the spectral characteristic function of a string $m \in \mathcal{M}$. Then $h \neq 0$ and it has the following representation:

$$h(\lambda) = c + \int_{[0,\infty)} \frac{d\sigma(\xi)}{\lambda + \xi} \tag{1.1}$$

where c is a non-negative constant and $d\sigma$, called the *spectral measure associated with the string* m, is a non-negative Borel measure on $[0,\infty)$ such that

$$\int_{[0,\infty)} \frac{d\sigma(\xi)}{1+\xi} < \infty. \tag{1.2}$$

By the representation (1.1), it is easy to see that h can be extended to a holomorphic function on $\mathbb{C} \setminus (-\infty,0]$ where $\mathrm{Im}\, h(\lambda) \leq 0$ for any $\lambda \in \mathbb{C}_+ = \{\lambda \in \mathbb{C};\ \mathrm{Im}\ \lambda > 0\}$. Set $\mathcal{H}$ be the set of all such functions h. The above defined relation between $m \in \mathcal{M}$ and $h \in \mathcal{H}$ is called *Krein's correspondence*. A key result due to M. G. Krein is as follows: *Krein's correspondence* $\mathcal{M} \ni m \mapsto h \in \mathcal{H} \setminus \{0\}$ *is one-to-one and onto and defines a homeomorphism if suitable topologies are introduced on* $\mathcal{M}$ *and* $\mathcal{H}$.

Let $m \in \mathcal{M}$ and $h \in \mathcal{H}$ be in Krein's correspondence and h be in the form (1.1). Then the following relations hold between m and h:

$$\lim_{\lambda \to 0} h(\lambda) = c + \int_{[0,\infty)} \frac{d\sigma(\xi)}{\xi} = l, \tag{1.3}$$

$$\lim_{\lambda \to \infty} h(\lambda) = c = \inf\{x \geq 0;\ m(x) > 0\} = \inf \mathrm{Supp}(dm). \tag{1.4}$$

We remark that the spectral characteristic function h is uniquely determined by c and the spectral measure $d\sigma$, and $d\sigma$ is uniquely determined by its Laplace transform

$$g(x) = \int_{[0,\infty)} e^{-x\xi} d\sigma(\xi). \tag{1.5}$$

For any string $m \in \mathcal{M}$, its right continuous inverse m^* also belongs to $\mathcal{M}$, which is called the *dual string* of m. We write c, l, h etc. which correspond to the dual string m^* as c^*, l^*, h^* etc. Then we have the relation

$$h^*(\lambda) = \frac{1}{\lambda h(\lambda)}. \tag{1.6}$$

By (1.4) and definition, the following are obvious:

$$m(0) = c^* \qquad \text{and} \qquad m(\infty-) = l^*. \tag{1.7}$$

In addition, the following holds:

$$\frac{1}{l} = \lim_{\lambda\to 0} \frac{1}{h(\lambda)} = \lim_{\lambda\to 0} \lambda h^*(\lambda) = d\sigma^*(\{0\}). \tag{1.8}$$

Let $B = (B(t))$ be a standard Brownian motion on $\mathbb{R}$ with $B(0) = 0$ and let $\{l(t,x)\}_{t\geq 0,\ x\in\mathbb{R}}$ be the local time of B:

$$\int_0^t 1_E(B(s))ds = 2\int_E l(t,x)dx$$

for any $t \geq 0$ and for any Borel subset E of $\mathbb{R}$. For fixed $x \in \mathbb{R}$, let $t \mapsto l^{-1}(t,x)$ be the right-continuous inverse of $t \mapsto l(t,x)$. Given a string $m \in \mathcal{M}$, $m \neq 0$, define a right-continuous increasing process $\eta = (\eta(t))$ by

$$\eta(t) = \begin{cases} \displaystyle\int_{[0,l)} l(l^{-1}(t,0),x)dm(x) & \text{if } l^{-1}(t,0) < \tau_l, \\ \infty & \text{if } l^{-1}(t,0) \geq \tau_l, \end{cases}$$

where $\tau_l = \min\{t > 0;\ B(t) = l\}$. The process η is a subordinator i.e. a Lévy process with increasing paths starting at zero with a possibly finite lifetime $\zeta = \inf\{t > 0;\ \eta(t) = \infty\}$. We have

$$E[e^{-\lambda\eta(t)}] = e^{-t\psi(\lambda)} \qquad \lambda > 0$$

with the Lévy–Khinchin exponent $\psi(\lambda)$ given by

$$\psi(\lambda) = \frac{1}{h(\lambda)} = c_1 + c_2\lambda + \int_0^\infty (1 - e^{-\lambda u})n(u)du, \tag{1.9}$$

where $c_1 = 1/l$, $c_2 = c^*$ and the function $n(u)$ is given by

$$n(u) = \int_{(0,\infty)} e^{-u\xi}\xi d\sigma^*(\xi). \tag{1.10}$$

Note that the life time ζ is an exponential time with $E[\zeta] = l$, so that $\zeta = \infty$ almost surely if and only if $l = \infty$.

Example 1.1. For $0 < \alpha < 1$ and $c > 0$, let $m_{\alpha,c} \in \mathcal{M}$ be defined by

$$m_{\alpha,c}(x) = \{D_\alpha c\}^{\frac{1}{\alpha}} x^{\frac{1}{\alpha}-1}, \qquad 0 \leq x < \infty \tag{1.11}$$

where

$$D_\alpha = \{\alpha(1-\alpha)\}^{-\alpha}\Gamma(1+\alpha)/\Gamma(1-\alpha).$$

Then $\psi(\lambda)$ in (1.9) is given by

$$\psi_{\alpha,c}(\lambda) = c\lambda^\alpha, \qquad \lambda > 0, \tag{1.12}$$

so that $\{\eta(t)\}_{t\ge 0}$ is a stable subordinator with index α.

2. Density formulas for the local time and its inverse of a reflecting diffusion process on the half line

We introduce the following condition on a string $m \in \mathcal{M}$:

(A) $m(0) = 0$ and $c = 0$.

That is, the measure dm on $[0, l)$ has no mass at the origin and the left end point of its support is zero. It is obvious that m satisfies **(A)** if and only if its dual string m^* satisfies **(A)**. Also it is easy to see that **(A)** is equivalent to the following:

$$c^* = 0 \qquad \text{and} \qquad \int_{[0,\infty)} d\sigma^*(\xi) = \infty,$$

because, under the condition $c^* = 0$, it holds that

$$\frac{1}{m^*(0)} = \frac{1}{c} = \lim_{\lambda\to\infty} \frac{1}{h(\lambda)} = \lim_{\lambda\to\infty} \lambda h^*(\lambda) = \int_{[0,\infty)} d\sigma^*(\xi).$$

Under the condition **(A)**, we have $\int_0^\infty n(u)du = \int_{(0,\infty)} d\sigma^*(\xi) = \infty$, that is, the total mass of the Lévy measure of the Lévy process $(\eta(t))_{t\ge 0}$ is infinity. Hence, $[0, \zeta) \ni t \mapsto \eta(t) \in [0,\infty)$ is strictly increasing and $\lim_{t\to\infty} \eta(t) = \infty$ almost surely, and therefore its right-continuous inverse $[0,\infty) \ni x \mapsto l(x) := \eta^{-1}(x) \in [0,\infty)$ is continuous. Note that, if $\zeta < \infty$, then $l(x) = l(\zeta)$ for $x \ge \zeta$. It is well-known that the process $(l(x))_{x\ge 0}$ is the *local time at zero* of the reflecting m-diffusion on $[0,\infty)$ so that $(\eta(t))_{t\ge 0}$ is the inverse local time at zero.

We always impose the assumption **(A)** on the strings $m \in \mathcal{M}$ which we consider in what follows, unless otherwise stated.

2.1. Density of the law of the inverse local time

Noting that the function ψ in (1.9) can be extended to a holomorphic function on $\mathbb{C} \setminus (-\infty, 0]$, we set

$$\theta(\lambda) = \mathrm{Re}\ \psi(i\lambda) = \frac{1}{l} + \int_0^\infty (1-\cos\lambda u)n(u)du, \qquad \lambda \in \mathbb{R} \tag{2.1}$$

and

$$\omega(\lambda) = \mathrm{Im}\ \psi(i\lambda) = \int_0^\infty (\sin \lambda u) n(u) du, \qquad \lambda \in \mathbb{R}. \tag{2.2}$$

By a direct calculation using (1.10), we have

$$\theta(\lambda) = \int_{[0,\infty)} \frac{\lambda^2}{\lambda^2 + \xi^2} d\sigma^*(\xi) \qquad \text{and} \qquad \omega(\lambda) = \int_{(0,\infty)} \frac{\lambda \xi}{\lambda^2 + \xi^2} d\sigma^*(\xi). \tag{2.3}$$

Here we used the relation $1/l = d\sigma^*(\{0\})$.

Theorem 2.1. *For each $t > 0$, the law of the inverse local time $\eta(t)$ on $(0, \infty)$, which is a defective distribution with total mass $e^{-t/l}$ when $l < \infty$, has a density $p(t, x)$: $P(\eta(t) \in dx) = p(t, x)dx$ for $x \in (0, \infty)$, equivalently,*

$$\int_0^\infty e^{-\lambda x} p(t, x) dx = e^{-t\psi(\lambda)}, \qquad t > 0,\ \lambda > 0. \tag{2.4}$$

Furthermore, $p(t, x)$ may be chosen to be continuous in $(t, x) \in (0, \infty) \times (0, \infty)$.

Proof. The analytic continuation implies

$$E[e^{-i\lambda\eta(t)}] = e^{-t\psi(i\lambda)}.$$

So the density $p(t, x)$ of the law of $\eta(t)$, when exists, should be given by the Fourier inversion formula:

$$p(t, x) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{ix\xi} e^{-t\psi(i\xi)} d\xi.$$

Note that $p(t, x)$ is real, θ is even and ω is odd. Then this formula is equivalent to

$$p(t, x) = \frac{1}{\pi} \int_0^\infty e^{-t\theta(\xi)} \cos(\xi x - t\omega(\xi)) d\xi. \tag{2.5}$$

We show that the improper integral in the right hand side (RHS) of (2.5) converges for fixed $(t, x) \in (0, \infty) \times (0, \infty)$ and the convergence is uniform on any compact set in $(0, \infty) \times (0, \infty)$. We can then conclude the assertion of the theorem by applying, for example, a standard Lévy's inversion formula for characteristic functions.

For this, we first note the following facts obtained immediately by direct calculations on formulas (2.3) combined with properties of the spectral measure of the dual string under the assumption **(A)**: $\int_{[0,\infty)} d\sigma^*(\xi) = \infty$ and $\int_{[0,\infty)} \frac{d\sigma^*(\xi)}{1+\xi} < \infty$.

(i) $\theta(\lambda) \nearrow \infty$ as $\lambda \nearrow \infty$.

(ii) $\theta'(\lambda) = \int_{(0,\infty)} \frac{2\lambda\xi^2}{(\lambda^2+\xi^2)^2} d\sigma^*(\xi), \quad \omega'(\lambda) = \int_{(0,\infty)} \frac{(\xi^2-\lambda^2)\xi}{(\lambda^2+\xi^2)^2} d\sigma^*(\xi),$ so that $\theta'(\lambda) \to 0$ and $\omega'(\lambda) \to 0$ as $\lambda \to \infty$.

(iii) $\omega''(\lambda) = \int_{(0,\infty)} \frac{2\lambda^3\xi - 6\lambda\xi^3}{(\lambda^2+\xi^2)^3} d\sigma^*(\xi)$ so that $\omega''(\xi) \in L^1([\xi_0,\infty), d\xi)$ for every $\xi_0 > 0$.

Let $0 < a_1 < a_2$ and $b > 0$ be given. Then, by (ii), we can choose $\xi_0 > 0$ large enough and $\alpha > 0$ small enough so that the map $\xi \in (0,\infty) \mapsto \rho(\xi) := x\xi - t\omega(\xi) \in \mathbb{R}$ satisfies

$$\rho'(\xi) = x - t\omega'(\xi) \geq \alpha \qquad \text{for all } (t,x) \in (0,a_2] \times [b,\infty) \text{ and } \xi \geq \xi_0.$$

Take any $(t,x) \in (0,a_2] \times [b,\infty)$ and fix it. The map $\xi \in [\xi_0,\infty) \mapsto \rho(\xi) \in [\rho(\xi_0),\infty)$ is a diffeomorphism with $\rho'(\xi) \geq \alpha$. Denote its inverse by $\rho^{-1} : [\rho(\xi_0),\infty) \ni \tau \mapsto \rho^{-1}(\tau) \in [\xi_0,\infty)$. Then for $\xi > \xi_0$,

(2.6)
$$\begin{aligned}\int_{\xi_0}^{\xi} \cos(\xi x - t\omega(\xi)) d\xi &= \int_{\rho(\xi_0)}^{\rho(\xi)} \cos\tau \cdot \frac{d\tau}{\rho'(\rho^{-1}(\tau))} = \int_{\rho(\xi_0)}^{\rho(\xi)} \frac{\cos\tau}{x - t\omega'(\rho^{-1}(\tau))} d\tau \\ &= \frac{1}{x}\int_{\rho(\xi_0)}^{\rho(\xi)} \cos\tau \cdot d\tau + \int_{\rho(\xi_0)}^{\rho(\xi)} \cos\tau \cdot \frac{t\omega'(\rho^{-1}(\tau))}{x\{x - t\omega'(\rho^{-1}(\tau))\}} d\tau.\end{aligned}$$

It is easy to deduce from (iii) that $\frac{d}{d\tau}\left[\frac{\omega'(\rho^{-1}(\tau))}{x - t\omega'(\rho^{-1}(\tau))}\right]$ is in $L^1([\rho(\xi_0),\infty), d\tau)$ and hence that the integral in (2.6) is uniformly bounded in $(t,x) \in (0,a_2] \times [b,\infty)$ and $\xi \geq \xi_0$. We can then conclude by a standard calculus that the integral in (2.5) is convergent, the convergence being bounded in $(t,x) \in (0,a_2] \times [b,\infty)$ and uniform in $(t,x) \in [a_1,a_2] \times [b,\infty)$. □

Corollary 2.1. *For any $t_0 > 0$ and $x_0 > 0$,*

$$\sup_{t\in(0,t_0]} \sup_{x\in[x_0,\infty)} p(t,x) < \infty. \tag{2.7}$$

By Theorem 2.1 we immediately obtain the following relations.

Corollary 2.2. *Let $g(x)$ be given by (1.5). Then, for any $x \in (0,\infty)$,*

$$g(x) = \int_0^\infty p(t,x)dt, \tag{2.8}$$

and

$$E[l(x)] = \int_0^x g(y)dy = \int_{[0,\infty)} \frac{1-e^{-x\xi}}{\xi} d\sigma(\xi) < \infty. \tag{2.9}$$

Proof. We deduce (2.8) by noting that the Laplace transform of both sides coincides with $1/\psi(\lambda)$. Then (2.9) follows from (2.8) as

$$E[l(x)] = \int_0^\infty P(t \le l(x))dt = \int_0^\infty P(\eta(t) \le x)dt = \int_0^x g(y)dy.$$

□

Remark 1. By (1.5), $x \mapsto g(x)$ is continuous (indeed, C^∞), decreasing and $\lim_{x\to 0} g(x) = \infty$. Note that $\int_{[0,\infty)} d\sigma(\xi) = \infty$ because of the assumption **(A)**.

2.2. Density of the law of the local time

For $(t, x) \in (0,\infty) \times (0,\infty)$, we set

$$q(t,x) = \int_0^x p(t,y)g^*(x-y)dy, \tag{2.10}$$

where $g^*(x)$ is the Laplace transform of the spectral measure $d\sigma^*$ of the dual string (cf. (1.5)):

$$g^*(x) = \int_{[0,\infty)} e^{-x\xi} d\sigma^*(\xi). \tag{2.11}$$

Noting (1.8) and (1.10), we also have the following expression

$$g^*(x) = d\sigma^*(\{0\}) + \int_{(0,\infty)} e^{-x\xi} d\sigma^*(\xi) = \frac{1}{l} + \int_x^\infty n(u)du. \tag{2.12}$$

The integral in (2.10) is finite and defines a continuous function in (t, x) on $(0,\infty) \times (0,\infty)$. Indeed, we can apply the following lemma to the function $q(t, x)$.

Lemma 2.1. *Let $f(x)$ be a non-negative continuous function on $(0,\infty)$ which is integrable on $(0, a)$ for any $a > 0$. Then the function $F(t, x)$ defined by*

$$F(t,x) = \int_0^x p(t,y)f(x-y)dy$$

is finite and continuous in (t, x) on $(0,\infty) \times (0,\infty)$. Moreover, $F(t, x)$ is bounded on $(0, t_0] \times [x_0, x_1]$ for any $0 < t_0 < \infty$ and $0 < x_0 < x_1 < \infty$.

Proof. Let $0 < t_0 < t_1 < \infty$ and $0 < x_0 < x_1 < \infty$. We change the variables in the integral to have

$$F(t,x) = x\int_0^1 p(t,xy)f(x(1-y))dy.$$

Divide the interval $(0,1)$ into three parts $(0,\delta)$, $(\delta,1-\varepsilon)$ and $(1-\varepsilon,1)$. By the boundedness of $p(t,x)$ and $f(x)$ on compact subsets, we can make the integral on $(0,\delta)$ and $(1-\varepsilon,1)$ arbitrarily small uniformly in (t,x) on $[t_0,t_1]\times[x_0,x_1]$ if we take $\delta>0$ and $\varepsilon>0$ small enough. We can see that the function $[t_0,t_1]\times[x_0,x_1]\ni(t,x)\mapsto\int_\delta^{1-\varepsilon}p(t,xy)f(x(1-y))dy$ is continuous in (t,x) on $[t_0,t_1]\times[x_0,x_1]$, and hence we conclude the first assertion. We can easily deduce by Corollary 2.1 the second assertion in a similar way. □

Theorem 2.2. *For every $t>0$ and $x>0$, it holds that*

$$\int_0^x p(t,y)dy=\int_t^\infty q(s,x)ds. \tag{2.13}$$

Thus, the law of the local time $l(x)$ at a fixed time $x>0$ has a continuous density $(0,\infty)\ni t\mapsto q(t,x)$:

$$P(l(x)\in dt)=q(t,x)dt \qquad \text{for } t\in(0,\infty). \tag{2.14}$$

Proof. Define $\eta^x(t):=x+\eta(t)$ for $t\ge 0$ and $x\ge 0$. Then the family $\{\eta^x\}_{x\ge 0}$ defines a strong Markov process on $[0,\infty)$ (with the point infinity as the terminal point when the lifetime is finite) with the transition semigroup

$$T_tf(x)=E[f(\eta^x(t))]=\int_0^\infty f(x+y)p(t,y)dy$$

which is a Feller semigroup on $\hat{C}([0,\infty))$, the space of continuous functions $[0,\infty]\to\mathbb{R}$ which vanish at infinity. Its Hille–Yosida generator $\mathcal{A}$ has the space $\hat{C}^1([0,\infty)):=\{f\in\hat{C}([0,\infty));\ f'\in\hat{C}([0,\infty))\}$ as its core and is given on this space as

$$\mathcal{A}f(x)=\int_0^\infty\{f(x+y)-f(x)\}n(y)dy-\frac{1}{l}f(x), \qquad \text{cf. e.g. [12].}$$

Hence, for $f\in\hat{C}^1([0,\infty))$, we have

$$T_tf(x)-f(x)=\int_0^t T_s\mathcal{A}f(x)ds,$$

that is,

$$\begin{aligned}&\int_0^\infty f(x+y)p(t,y)dy-f(x)\\&=\int_0^t ds\int_0^\infty p(s,z)dz\left\{\int_0^\infty\{f(x+z+y)-f(x+z)\}n(y)dy-\frac{1}{l}f(x+z)\right\}.\end{aligned}$$

Let $a>0$. We approximate the function $1_{[0,a]}$ by functions in $\hat{C}^1([0,\infty))$ so that the convergence occurs at every point boundedly. Then we obtain, for $0<x<a$,

$$\int_0^{a-x}p(t,y)dy-1=-\int_0^t ds\int_0^{a-x}p(s,z)\left\{\int_{a-(x+z)}^\infty n(y)dy+\frac{1}{l}\right\}dz.$$

By (2.12), we have

$$\int_x^\infty n(y)dy + \frac{1}{l} = g^*(x),$$

so that, we obtain, for every $t > 0$ and $x > 0$,

$$\int_0^x p(t,y)dy = 1 - \int_0^t q(s,x)ds.$$

Letting $t \to \infty$ in this formula, we obtain

$$\int_0^\infty q(s,x)ds = 1 \tag{2.15}$$

and then, (2.13) follows immediately.

The LHS of (2.13) is equal to $P(\eta(t) \le x)$ and also to $P(l(x) \ge t)$, which proves (2.14). □

Example 2.1. For $m_{\alpha,c} \in \mathcal{M}$ of Example 1.1, the densities $p(t,x)$ and $q(t,x)$ are given by

$$p_{\alpha,c}(t,x) = \frac{\alpha ct}{x^{1+\alpha}}\phi_\alpha\left(\frac{ct}{x^\alpha}\right) \qquad \text{and} \qquad q_{\alpha,c}(t,x) = \frac{c}{x^\alpha}\phi_\alpha\left(\frac{ct}{x^\alpha}\right) \tag{2.16}$$

where the function $\phi_\alpha(x)$, $0 < x < \infty$, called the *Mittag-Leffler density* ([4]), is defined through its Laplace transform or through its moments by

$$\int_0^\infty e^{-\lambda x}\phi_\alpha(x)dx = \sum_{k=0}^\infty \frac{(-1)^k}{\Gamma(1+k\alpha)}\lambda^k, \qquad 0 < \lambda < \infty, \tag{2.17}$$

or

$$\int_0^\infty x^k \phi_\alpha(x)dx = \frac{k!}{\Gamma(1+k\alpha)}, \qquad k = 0,1,2,\dots. \tag{2.18}$$

It is known ([11]) that $\phi_\alpha(x)$ can be extended to an entire function given by

$$\phi_\alpha(z) = \frac{1}{\pi\alpha}\sum_{j=1}^\infty \frac{(-1)^j}{j!}\sin(\pi\alpha j)\Gamma(\alpha j + 1)z^{j-1}. \tag{2.19}$$

3. (m_+, m_-)-diffusions and a density formula for the fraction of time spent on the positive side

Let $m_+, m_- \in \mathcal{M}$ with $m_-(0) = 0$. We write h, m^* etc. which correspond to the string $m_\pm$ as $h_\pm, m^*_\pm$ etc. respectively. The notion of (m_+, m_-)-diffusion process has been introduced in [9] (cf. also [14]): It is a strong Markov process (gap diffusion process) $X = (X_t, P^x)$ on the state space $I \subset (-\infty, \infty)$ with the

Feller generator $\frac{d}{dm}\frac{d}{dx}$. Here I =(Supp$(dm) \cup \{-l_-\} \cup \{l_+\}) \cap (-\infty, \infty)$ and dm is a non-negative Radon measure on $(-l_-, l_+)$ defined by

$$dm(x) = \begin{cases} dm_+(x) & \text{on } [0, l_+) \\ d\check{m}_-(x) & \text{on } (-l_-, 0) \end{cases}$$

where $d\check{m}_-(x)$ is the image measure of dm_- under the reflection $x \mapsto -x$. When $-l_-$ or l_+ is in I, it is a trap for X and is identified with the terminal point. We know well (cf. e.g. [14, p. 160]) that X can be obtained by a time change from a one-dimensional Brownian motion B.

Set

$$A^+(t) = \int_0^t 1_{[0,\infty)}(X_s)ds \qquad \text{and} \qquad A^-(t) = \int_0^t 1_{(-\infty,0)}(X_s)ds.$$

Then the following *double Laplace transform formula* is known, which we can find in Barlow, Pitman and Yor [1], Truman and Williams [13] and Watanabe [14]. See also Kasahara and Yano [8]. Note that the assumption **(A)** is *not* needed in this formula.

Theorem A.

$$\int_0^\infty e^{-\mu t} dt E^0[e^{-\lambda A^+(t)}] = \frac{\psi_+(\lambda+\mu)/(\lambda+\mu) + \psi_-(\mu)/\mu}{\psi_+(\lambda+\mu) + \psi_-(\mu)}. \tag{3.1}$$

If $m_+(0) = 0$, then almost all paths of the diffusion (X_t) do not have a holding time at the origin, and thus we also have the formula for $A^-(t) = t - A^+(t)$ by interchanging $+$ and $-$ in the right hand side of (3.1).

Now we are in a position to invert the above formula. We assume that both m_+ and m_- satisfy the condition **(A)**.

Theorem 3.1. *For* $(t, x) \in (0, \infty) \times (0, 1)$*, define*

(3.2)
$$f_t^\pm(x) = t\left\{\int_0^\infty p_\mp(s, t(1-x))q_\pm(s, tx)ds + \int_0^\infty q_\mp(s, t(1-x))p_\pm(s, tx)ds\right\}.$$

Then the law of the fraction $\frac{1}{t}A^\pm(t)$ *of time spent on the half line for fixed* $t > 0$ *under the probability* P^0 *has a lower semicontinuous density* $f_t^\pm$ *with respect to the Lebesgue measure. That is,*

$$P^0\left(\frac{1}{t}A^\pm(t) \in dx\right) = f_t^\pm(x)dx \qquad \textit{for } x \in (0,1).$$

Proof. We only show the claim for A^+; the case of A^- follows readily from the relation $f_t^-(x) = f_t^+(1-x)$. We prove the claim by showing that the double Laplace transform of $f_t^+(x)$ coincides with the right hand side of (3.1).

By definition the double Laplace transform of $f_t^+(x)$ is equal to

$$\int_0^\infty e^{-\mu t}dt \int_0^1 e^{-\lambda tx} f_t^+(x)dx = \int_0^\infty e^{-\mu t}dt \int_0^t e^{-\lambda x} f_t^+(x/t)/tdx = I_1 + I_2$$

where

$$I_1 = \int_0^\infty e^{-\mu t}dt \int_0^t e^{-\lambda x}dx \int_0^\infty p_-(s,t-x)q_+(s,x)ds,$$
$$I_2 = \int_0^\infty e^{-\mu t}dt \int_0^t e^{-\lambda x}dx \int_0^\infty q_-(s,t-x)p_+(s,x)ds.$$

By changing the order of and then translating the integration, we have

$$I_1 = \int_0^\infty ds \int_0^\infty e^{-(\lambda+\mu)x}q_+(s,x)dx \int_0^\infty e^{-\mu t}p_-(s,t)dt,$$
$$I_2 = \int_0^\infty ds \int_0^\infty e^{-\mu t}q_-(s,t)dt \int_0^\infty e^{-(\lambda+\mu)x}p_+(s,x)dx.$$

So the assertion holds if we can show the following:

$$I_1 = \frac{1}{\lambda+\mu} \cdot \frac{\psi_+(\lambda+\mu)}{\psi_+(\lambda+\mu)+\psi_-(\mu)},$$
$$I_2 = \frac{1}{\mu} \cdot \frac{\psi_-(\mu)}{\psi_+(\lambda+\mu)+\psi_-(\mu)}.$$

We show these equalities by proving the following:

$$\text{(3.3)} \qquad \int_0^\infty ds \int_0^\infty e^{-\lambda x}q_i(s,x)dx \int_0^\infty e^{-\mu t}p_j(s,t)dt = \frac{1}{\lambda} \cdot \frac{\psi_i(\lambda)}{\psi_i(\lambda)+\psi_j(\mu)}$$

where i and j are either $+$ or $-$.

By (2.4), the LHS of (3.3) is equal to

$$\int_0^\infty e^{-s\psi_j(\mu)}ds \int_0^\infty e^{-\lambda x}q_i(s,x)dx = \int_0^\infty e^{-\lambda x}dx \int_0^\infty e^{-s\psi_j(\mu)}q_i(s,x)ds.$$

We can further continue this identity as follows by an integration by parts combined with (2.13) and (2.4):

$$= \int_0^\infty e^{-\lambda x}dx \left\{1 - \psi_j(\mu)\int_0^\infty e^{-s\psi_j(\mu)}ds \int_s^\infty q_i(u,x)du\right\}$$
$$= \int_0^\infty e^{-\lambda x}dx \left\{1 - \psi_j(\mu)\int_0^\infty e^{-s\psi_j(\mu)}ds \int_0^x p_i(s,y)dy\right\}$$
$$= \frac{1}{\lambda} - \frac{\psi_j(\mu)}{\lambda}\int_0^\infty e^{-s(\psi_i(\lambda)+\psi_j(\mu))}ds = \text{the RHS of (3.3)}.$$

□

Example 3.1. A skew Bessel diffusion process mentioned in Introduction is the case of (m_+, m_-)-diffusion where, using the notation in Example 1.1,

$$m_+ = m_{\alpha,p} \quad \text{and} \quad m_- = m_{\alpha,1-p}, \qquad 0 < \alpha < 1,\ 0 < p < 1.$$

Here, p is called its *skew parameter* and $2-2\alpha$ ($\in (0,2)$) is called its *dimension*, so that the case $\alpha = 1/2$, i.e. the dimension $2-2\alpha = 1$, and $p = 1/2$ is a Brownian motion up to a multiplicative constant. The law of fraction $\frac{1}{t}A_+(t)$ for the skew Bessel diffusion process is independent of t and its density $f^+_{\alpha,p}(x)$, $0 < x < 1$, is given explicitly by

(3.4)
$$f^+_{\alpha,p}(x) = \frac{\sin\alpha\pi}{\pi} \frac{p(1-p)x^{\alpha-1}(1-x)^{\alpha-1}}{p^2(1-x)^{2\alpha} + (1-p)^2x^{2\alpha} + 2p(1-p)x^\alpha(1-x)^\alpha \cos\alpha\pi},$$

cf. [10], [14], [8]. It is also given by our density formula (3.2). If we substitute $p_\pm$ and $q_\pm$ in this formula by $p_{\alpha,\frac{1}{2}\pm(p-\frac{1}{2})}$ and $q_{\alpha,\frac{1}{2}\pm(p-\frac{1}{2})}$ given by (2.16), and if we compare this with (3.4), we immediately obtain the following formula for the Mittag-Leffler density ϕ_α:

(3.5)
$$\int_0^\infty \phi_\alpha(As)\phi_\alpha(Bs)sds = \frac{\sin\alpha\pi}{\alpha\pi}\frac{1}{A^2+B^2+2AB\cos\alpha\pi} \qquad \text{for any } A, B > 0.$$

Pinned case

Recently Y. Yano has studied in [15] the asymptotic behavior of the distribution function of the occupation time of one-dimensional *pinned diffusion processes*. She also studied *pinned skew Bessel diffusion processes* and obtained the formula of the distribution functions expressed by the Riemann–Liouville fractional integrals. In her paper she obtained the following double Laplace transform formula by means of Kac's formula.

Theorem B ([15]).
$$\int_0^\infty e^{-\mu t}dtE^0[e^{-\lambda A^+(t)}|X_t = 0]p(t,0,0) = \frac{1}{\psi_+(\lambda+\mu)+\psi_-(\mu)}$$
where $p(t,x,y)$ denotes the transition density with respect to the speed measure $dm(y)$ of the (m_+, m_-)-diffusion.

Define a function $h \in \mathcal{H}$ by
$$\frac{1}{h(\lambda)} = \frac{1}{h_+(\lambda)} + \frac{1}{h_-(\lambda)}$$
and represent the function h in the form (1.1) by a non-negative constant c and a spectral measure $d\sigma$. Then the following holds (cf. [9]):
$$p(t,0,0) = \int_{[0,\infty)} e^{-t\xi}d\sigma(\xi).$$

In our context we can also obtain the density formula for the law of the occupation time of one-dimensional pinned diffusion processes.

Theorem 3.2. *For $(t,x) \in (0,\infty) \times (0,1)$, define*

$$k_t^{\pm}(x) = \frac{t}{p(t,0,0)} \int_0^\infty p_{\pm}(s,tx)p_{\mp}(s,t(1-x))ds.$$

Then the law of the fraction $\frac{1}{t}A^{\pm}(t)$ for fixed $t > 0$ under the pinned diffusion measure has a density $k_t^{\pm}$ with respect to the Lebesgue measure. That is,

$$P^0\left(\frac{1}{t}A^{\pm}(t) \in dx | X_t = 0\right) = k_t^{\pm}(x)dx.$$

Proof. We give a proof only in the case for A_t^+ for simplicity.

$$\begin{aligned}
&\int_0^\infty e^{-\mu t}dt \int_0^1 e^{-\lambda t x} k_t^+(x)dx \cdot p(t,0,0) \\
&= \int_0^\infty e^{-\mu t}dt \int_0^t e^{-\lambda x} k_t^+(x/t)/tdx \cdot p(t,0,0) \\
&= \int_0^\infty e^{-\mu t}dt \int_0^t e^{-\lambda x}dx \int_0^\infty p_+(s,x)p_-(s,t-x)ds \\
&= \int_0^\infty ds \int_0^\infty e^{-(\lambda+\mu)x}p_+(s,x)dx \int_0^\infty e^{-\mu t}p_-(s,t)dt \\
&= \int_0^\infty e^{-s(\psi_+(\lambda+\mu)+\psi_-(\mu))}ds \\
&= \frac{1}{\psi_+(\lambda+\mu)+\psi_-(\mu)}.
\end{aligned}$$

□

Example 3.2. Let $X = (X_t, P^x)$ be the skew Brownian motion with the skew parameter p, $(0 < p < 1)$, as given in Introduction: Under a change of coordinates so that the new coordinate is a canonical scale, X is given by (m_+, m_-)-diffusion process on $\mathbb{R}$ where

$$m_+(x) = m_{\frac{1}{2},p}(x) = p^2x \qquad \text{and} \qquad m_-(x) = m_{\frac{1}{2},1-p}(x) = (1-p)^2x.$$

Then

$$\psi_+(\lambda) = p\sqrt{\lambda}, \qquad \psi_-(\lambda) = (1-p)\sqrt{\lambda}$$

and

$$p_+(t,x) = \frac{pt}{2\sqrt{\pi x^3}} \exp -\frac{p^2t^2}{4x}, \qquad p_-(t,x) = \frac{(1-p)t}{2\sqrt{\pi x^3}} \exp -\frac{(1-p)^2t^2}{4x}.$$

Since $\int_0^\infty e^{-\lambda t} p(t,0,0)dt = (\psi_+(\lambda)+\psi_-(\lambda))^{-1} = 1/\sqrt{\lambda}$, we have

$$p(t,0,0) = \frac{1}{\sqrt{\pi t}}.$$

We can now compute the integral in Theorem 4 to obtain

$$k_t^+(x) = \frac{p(1-p)}{2}\{p^2(1-x)+(1-p)^2 x\}^{-\frac{3}{2}}, \qquad 0<x<1.$$

When $p=1/2$, X is a Brownian motion and we have $k_t^+(x) \equiv 1$. Thus we recovered a well-known result by P. Lévy (cf. [6, p. 58]) that the law is uniform for a pinned Brownian motion.

4. Continuity of the density

By (2.7) in Corollary 2.1, $p(t,x)$ has the following property: For any $t_0>0$ and $x_0>0$, there exists a constant $C>0$ such that

$$p(t,x) \le C \qquad \text{for any } (t,x) \in (0,t_0]\times[x_0,\infty). \tag{4.1}$$

(4.1) can be a little improved as follows:

Lemma 4.1. *For any $t_0>0$ and $0<x_0<x_1$, there exists a constant $C>0$ such that*

$$p(t,x) \le Ct \qquad \text{for any } (t,x)\in(0,t_0]\times[x_0,x_1]. \tag{4.2}$$

Proof. Let $n(x)$ be given in (1.10) and set $k(x) = xn(x)$. Then the following relation holds:

$$xp(t,x) = t\int_0^x p(t,y)k(x-y)dy \qquad \text{for any } t>0 \text{ and } x>0. \tag{4.3}$$

This formula can be proved easily by differentiating (2.4) in λ (cf. [12, p. 385]). By (1.10), $k(x)$ is continuous in $(0,\infty)$ and

$$\int_0^a k(x)dx = \int_{(0,\infty)} \frac{1-e^{-a\xi}-a\xi e^{-a\xi}}{\xi} d\sigma^*(\xi) < \infty$$

for every $a>0$. Applying Lemma 2.1, we see that $\int_0^x p(t,y)k(x-y)dy$ is bounded in $(t,x)\in(0,t_0]\times[x_0,x_1]$ and this completes the proof. □

We now introduce the following assumption on $m \in \mathcal{M}$ through the condition on the density $p(t,x)$ of the law of $\eta(t)$ which corresponds to m.

(P) For any $0<x_0<x_1$, there exist constants $t_0>0$ and $C>0$ such that

$$\sup_{x\in[x_0,x_1]} p(t,x) \le C \qquad \text{for all } t \ge t_0. \tag{4.4}$$

In view of (4.1), we see that the following holds: When m satisfies **(P)**, then for every $0 < x_0 < x_1$, we have

$$\sup_{t>0,\ x\in[x_0,x_1]} p(t,x) < \infty. \tag{4.5}$$

Obviously, (4.5) and (4.2) imply the following: If m satisfies **(P)**, then, for any $0 < x_0 < x_1$,

$$\sup_{t>0} \left(\frac{1}{t} \vee 1\right) \sup_{x\in[x_0,x_1]} p(t,x) < \infty. \tag{4.6}$$

Lemma 4.2. *Suppose that m satisfies the condition* **(P)**. *Then $q(t,x)$ given by* (2.10) *has the following property: for every $0 < x_0 < x_1$,*

$$\sup_{t>0,\ x\in[x_0,x_1]} q(t,x) < \infty. \tag{4.7}$$

Proof. Take any $0 < x_0 < x_1$. Then, if $x \in [x_0, x_1]$,

$$\begin{aligned} q(t,x) &= \int_0^x p(t,y)g^*(x-y)dy \\ &= \int_0^{\frac{x}{2}} p(t,y)g^*(x-y)dy + \int_{\frac{x}{2}}^x p(t,y)g^*(x-y)dy \\ &=: I_1 + I_2. \end{aligned}$$

Then, for any $t > 0$,

$$I_1 \le \sup_{x\in[x_0/2,x_1]} g^*(x)$$

and, noting (4.5),

$$I_2 \le \sup_{x\in[x_0/2,x_1]} p(t,x) \int_{\frac{x}{2}}^x g^*(x-y)dy \le C' \int_0^{x_1} g^*(y)dy$$

for some $C' > 0$ which is independent of t. Note finally

$$\int_0^{x_1} g^*(y)dy = \int_{[0,\infty)} \frac{1-e^{-x_1\xi}}{\xi} d\sigma^*(\xi) < \infty.$$

□

Theorem 4.1. *Assume that both m_+ and $m_- \in \mathcal{M}$ satisfy the condition* **(P)**. *Then the function $f_t^{\pm}(x)$ is continuous in $(t,x) \in (0,\infty)\times(0,1)$. In particular, the density function $(0,1) \ni x \mapsto f_t^{\pm}(x)$ is continuous for each fixed $t > 0$.*

Proof. The formula (3.2) for $f_t^+(x)$ can be written as

$$f_t^+(x) = t\{E[p_-(l_+(tx), t(1-x))] + E[p_+(l_-(t(1-x)), tx)]\}.$$

Take any $0 < t_0 < t_1$ and $0 < x_0 < x_1 < 1$. If $t \in [t_0, t_1]$ and $x \in [x_0, x_1]$, then both tx and $t(1-x)$ are in the interval $[t_0(x_0 \wedge (1-x_1)), t_1(x_1 \vee (1-x_0))]$, so that, by (4.5), we have

$$\sup_{t\in[t_0,t_1],\ x\in[x_0,x_1]} p_-(l_+(tx), t(1-x)) \le C$$

and

$$\sup_{t\in[t_0,t_1],\ x\in[x_0,x_1]} p_+(l_-(t(1-x)), tx) \le C$$

for some constant $C > 0$. The assertion follows at once from the dominated convergence theorem. □

Pinned case

We introduce another assumption on a string $m \in \mathcal{M}$ which is stronger than **(P)**: Let $\alpha \ge 0$.

(P)$^\alpha$ For every $0 < x_0 < x_1$,

$$\limsup_{t\to\infty} t^\alpha \sup_{x\in[x_0,x_1]} p(t,x) < \infty.$$

Obviously, if m satisfies **(P)**$^\alpha$, then there exists a constant $C > 0$ such that

$$\sup_{x\in[x_0,x_1]} p(t,x) \le \frac{C}{1+t^\alpha} \qquad \text{for all } t > 0. \tag{4.8}$$

Theorem 4.2. *Assume that m_+ and $m_- \in \mathcal{M}$ satisfy* **(P)**$^{\alpha_+}$ *and* **(P)**$^{\alpha_-}$ *respectively and that $\alpha_+ + \alpha_- > 1$. Then the function $k_t^\pm(x)$ is continuous in $(t,x) \in (0,\infty)\times(0,1)$. In particular, the density function $(0,1) \ni x \mapsto k_t^\pm(x)$ is continuous for each fixed $t > 0$.*

The proof is immediate by Theorem 3.2, so we omit it.

5. Asymptotic behavior at the extreme points

Our goal is to establish the following asymptotic behavior for the density of fractions $\frac{1}{t}A^\pm(t)$ at each fixed time $t > 0$:

$$f_t^+(x) \sim t g_-^*(t) g_+(tx) \qquad \text{as } x \searrow 0 \tag{5.1}$$

and

$$f_t^-(x) \sim t g_+^*(t) g_-(tx) \qquad \text{as } x \searrow 0. \tag{5.2}$$

Noting the relation

$$f_t^{\pm}(x) = f_t^{\mp}(1-x), \qquad 0 < x < 1,$$

the asymptotic behaviors (5.1) and (5.2) are equivalent to the following, respectively:

$$f_t^{+}(x) \sim t g_+^*(t) g_-(t(1-x)) \qquad \text{as } x \nearrow 1 \tag{5.3}$$

and

$$f_t^{-}(x) \sim t g_-^*(t) g_+(t(1-x)) \qquad \text{as } x \nearrow 1. \tag{5.4}$$

We introduce the following assumption on a string $m \in \mathcal{M}$; recall that $g(x) = \int_0^\infty p(t,x)dt$.

(G) For every $\varepsilon > 0$, it holds that $\lim_{x \searrow 0} g^\varepsilon(x)/g(x) = 0$, where

$$g^\varepsilon(x) = \int_\varepsilon^\infty p(t,x)dt. \tag{5.5}$$

Theorem 5.1. *Assume that* m_- *(resp.* m_+*) satisfies* **(P)** *and* m_+ *(resp.* m_-*) satisfies* **(G)**. *Then the asymptotic behavior* (5.1) *(resp.* (5.2)*) is valid.*

Proof. We only prove for f_t^+, since the case of f_t^- is just the same. Let $t > 0$ be fixed. We divide f_t^+ into three parts:

$$I_1 = t\int_0^\infty p_-(s, t(1-x)) q_+(s, tx) ds,$$
$$I_2 = t\int_\varepsilon^\infty q_-(s, t(1-x)) p_+(s, tx) ds.$$
$$I_3 = t\int_0^\varepsilon q_-(s, t(1-x)) p_+(s, tx) ds.$$

By Theorem 2.2, we have

$$I_1 = tE[p_-(l_+(tx), t(1-x))].$$

By the condition **(P)** of m_-, (4.6) holds for p_-, so there exists a constant $C > 0$ such that $I_1 \le CtE[l_+(tx)]$. Hence we see that I_1 converges to zero as $x \searrow 0$.

By the uniform boundedness (4.7) of q_-, we see that

$$I_2 \le tC' \int_\varepsilon^\infty p_+(s, tx) ds = tC' g_+^\varepsilon(tx)$$

for some constant $C' < \infty$, if x is small enough. Thus the quantity $I_2/g_+(tx)$ converges to zero as $x \searrow 0$ by the condition **(G)** of m_+.

Now we consider the leading term I_3. By the definition of $q(t,x)$, we have

$$\lim_{s \searrow 0} q_-(s, t(1-x)) = g_-^*(t(1-x))$$

uniformly in $x \in (0, x_0]$ for $0 < x_0 < 1$. So we have

$$\begin{aligned} I_3 &= t\{g_-^*(t(1-x)) + o(1)\} \int_0^\varepsilon p_+(s, tx) ds \\ &= t\{g_-^*(t(1-x)) + o(1)\}\{g_+(tx) - g_+^\varepsilon(tx)\}, \end{aligned}$$

where we can make $o(1)$ arbitrarily small uniformly in $x \in (0, x_0]$ if we take $\varepsilon > 0$ small enough. Therefore, using the condition **(G)** of m_+ again, we obtain

$$\frac{I_3}{g_+(tx)} \to t g_-^*(t)$$

as $x \searrow 0$, which completes the proof. □

6. Sufficient conditions and examples

We would show that Theorem 4.1 and Theorem 5.1 can be applied to a very general class of (m_+, m_-)-diffusion processes so that the fraction $\frac{1}{t}A^+(t)$ has a continuous density $f_t^+(x)$ in $(0,1)$ and the asymptotics (5.1) and (5.3) hold at extreme values $x = 0$ and $x = 1$. For this, we need to find conditions on a string $m \in \mathcal{M}$ under which the conditions **(P)** and **(G)** are satisfied. As we shall see, our main tools in this problem are Tauberian theorems for Stieltjes transforms (cf. Appendix) and asymptotic results for Krein's correspondence (cf. [7], [9]).

The following condition on a string $m \in \mathcal{M}$ covers a large class of strings.

(M) There exist $0 < \alpha < 1$ and a slowly varying function $K(x)$ at $x = 0$ such that

$$m(x) \asymp x^{\frac{1}{\alpha}-1} K(x) \qquad \text{as } x \searrow 0. \tag{6.1}$$

Remark 2. The condition **(M)**, in particular, the condition

$$m(x) \asymp x^{\frac{1}{\alpha}-1} \qquad \text{as } x \searrow 0, \tag{6.2}$$

is satisfied by a class of fractal measures (cf. [5]). As a typical example, if $m(x)$ is the *Cantor function* or the *de Rham function of parameter* p, then (6.2) holds with

$$\frac{1}{\alpha} - 1 = \frac{\log 2}{\log 3} \text{ or } \frac{\log 2}{\log \frac{1}{p}} \qquad \text{accordingly.}$$

Theorem 6.1. *If a string* $m \in \mathcal{M}$ *satisfies the condition* **(M)**, *then it satisfies both the conditions* **(P)**, **(G)** *and, more strongly,* $\mathbf{(P)}^\alpha$ *for any* $\alpha > 0$.

Therefore by Theorem 5.1 we can conclude the following. (The similar conclusion holds for the asymptotic of $f_t^-(x) = f_t^+(1-x)$ as $x \searrow 0$ when the conditions on m_+ and m_- are exchanged.)

Theorem 6.2. *If a string m_- satisfies the condition* **(P)** *and m_+ satisfies* **(M)** *with $0 < \alpha < 1$ and $K(x)$, then the asymptotic behavior* (5.1) *holds. The behavior of $g_+(x)$ is given by*

$$g_+(x) \asymp x^{\alpha-1}L(x) \qquad \text{as } x \searrow 0$$

where $L(x)$ is a slowly varying function at $x = 0$ such that $t^\alpha L(t)$ is the inverse of $x^{\frac{1}{\alpha}}K(x)$. In particular, if

$$m_+(x) \sim x^{\frac{1}{\alpha}-1}K(x) \qquad \text{as } x \searrow 0, \tag{6.3}$$

then the behavior of $g_+(x)$ is explicitly given by

$$g_+(x) \sim C(\alpha)x^{\alpha-1}L(x) \qquad \text{as } x \searrow 0 \tag{6.4}$$

where the constant $C(\alpha)$ is given by

$$C(\alpha) = \frac{\alpha}{\{\alpha(1-\alpha)\}^\alpha \Gamma(1-\alpha)}.$$

To assure that the string m_- satisfy the condition **(P)**, it is sufficient to assume, for example, that m_- satisfies **(M)** or (6.7) below.

Note that the asymptotics (0.5) given in Introduction is a particular example where both m_+ and m_- satisfy the condition **(M)**, more precisely, (6.3) with

$$\alpha_+ = \alpha_- = \frac{1}{2}, \qquad K_+(x) = \frac{1}{a_+} \qquad \text{and} \qquad K_-(x) = \frac{1}{a_-}.$$

For the proof of Theorem 6.1 the following is essential.

Lemma 6.1. *Suppose that*

$$\liminf_{\lambda\to\infty} \frac{\theta(\lambda)}{\lambda^\alpha} > 0 \qquad \text{for some } \ \alpha > 0. \tag{6.5}$$

Then the function $p(t,x)$, extended on $(0,\infty)\times\mathbb{R}$ by setting $p(t,x) = 0$ for $x \in (-\infty, 0]$, satisfies the following:

(i) *The function $p(t,x)$ is C^∞ on $(0,\infty)\times\mathbb{R}$.*

(ii) $\displaystyle\sup_{t\in[t_0,\infty)} t^j \sup_{x\in\mathbb{R}} p(t,x) < \infty \qquad$ *for any $t_0 > 0$ and $j \geq 0$.*

Proof of Lemma 6.1. The assertions (i) and (ii) are immediately obtained from the Fourier inversion formula (2.5) by noting that

$$e^{-t\theta(\xi)}\xi^j \in L^1((0,\infty), d\xi)$$

for any $t > 0$ and $j \geq 0$ and that

$$\theta(\lambda) = O(\lambda) \qquad \text{and} \qquad \omega(\lambda) = O(\lambda) \qquad \text{as } \lambda \to \infty.$$

□

Proof of Theorem 6.1. By asymptotic results for Krein's correspondence (cf. [9] Theorem 2.3), it is known that (6.1) is equivalent, in terms of the corresponding spectral characteristic functions, to

$$h(\lambda) \asymp \lambda^{-\alpha} L(\lambda) \qquad \text{as } \lambda \to \infty$$

or to

$$h^*(\lambda) \asymp \lambda^{-(1-\alpha)} L(\lambda)^{-1} \qquad \text{as } \lambda \to \infty \tag{6.6}$$

where $L(\lambda)$ is some slowly varying function at ∞.

By Lemma 6.3 given below, the asymptotic (6.6) is equivalent to

$$\sigma^*(\lambda) \asymp \lambda^{\alpha} L(\lambda)^{-1} \qquad \text{as } \lambda \to \infty.$$

Recalling that the function $\theta(\lambda)$ defined in Section 2.1 satisfies

$$\theta(\lambda) = \lambda^2 \int_{(0,\infty)} \frac{d\sigma^*(\xi)}{\lambda^2 + \xi^2} = 2\lambda^2 \int_{(0,\infty)} \frac{\xi \sigma^*(\xi)}{(\lambda^2 + \xi^2)^2} d\xi,$$

we can easily deduce that

$$\theta(\lambda) \asymp \lambda^{\alpha} L(\lambda)^{-1} \qquad \text{as } \lambda \to \infty.$$

Hence (6.5) holds and we can apply Lemma 6.1.

By (ii) of Lemma 6.1 the conditions **(P)** and $\mathbf{(P)}^{\alpha}$ for any $\alpha > 0$ are obviously satisfied.

To prove that the condition **(G)** is also satisfied, we note the following:

$$\begin{aligned} g^{\varepsilon}(x) &:= \int_{\varepsilon}^{\infty} p(s,x) ds \\ &= \int_0^{\infty} ds \int_0^x p(\varepsilon, x-y) p(s,y) dy \\ &= \int_0^x p(\varepsilon, x-y) g(y) dy. \end{aligned}$$

By (ii) of Lemma 6.1 again this implies that

$$|g^{\varepsilon}(x)| \leq C \int_0^x g(y) dy \to 0 \qquad \text{as } x \searrow 0$$

for some constant C and hence the condition **(G)** is satisfied. □

The sufficient condition on strings for **(P)** and **(G)** given in Theorem 6.1 is far from being necessary. To see this by examples, we consider another type of conditions on strings. Assume that $m \in \mathcal{M}$ satisfies the condition: there exist constants $0 < \alpha \le 1$, $0 < c_1 < c_2$ and $\delta > 0$ such that

$$\exp\left\{-\frac{c_2}{x^\alpha}\right\} \le m(x) \le \exp\left\{-\frac{c_1}{x^\alpha}\right\} \qquad \text{for all } x \in (0,\delta). \tag{6.7}$$

This condition is equivalent to

$$m^*(x) \asymp \left(\log \frac{1}{x}\right)^{-\frac{1}{\alpha}} \qquad \text{as } x \searrow 0$$

and also (cf. [9]) to

$$\psi(\lambda) = \lambda h^*(\lambda) \asymp (\log \lambda)^{\frac{1}{\alpha}} \qquad \text{as } \lambda \to \infty.$$

Then we can deduce in the same way as above that

$$\theta(\lambda) \asymp (\log \lambda)^{\frac{1}{\alpha}} \qquad \text{as } \lambda \to \infty.$$

We can conclude that m satisfies both **(P)** and **(G)** if $0 < \alpha < 1$, and that m satisfies **(P)** if $\alpha = 1$.

Appendix: Tauberian theorems

Let $\nu(\xi)$ be a right continuous increasing function on $[0,\infty)$ such that $\nu(0) = 0$, $\nu(\xi) \to \infty$ as $\xi \to \infty$ and $\int_{[0,\infty)} (\xi+1)^{-1} d\nu(\xi) < \infty$. Let $\phi(\lambda)$, $\lambda > 0$ be its Stieltjes transform:

$$\phi(\lambda) = \int_{[0,\infty)} \frac{d\nu(\xi)}{\lambda + \xi} = \int_0^\infty \frac{\nu(\xi)}{(\lambda+\xi)^2} d\xi.$$

The following is well-known as a Tauberian theorem of Hardy and Littlewood (cf. e.g. [2]):

Lemma 6.2. *Let $0 < \beta \le 1$ and $L(\lambda)$ be a slowly varying function at ∞. Then*

$$\phi(\lambda) \sim \lambda^{-\beta} L(\lambda) \qquad \text{as } \lambda \to \infty$$

if and only if

$$\nu(\lambda) \sim \lambda^{1-\beta} L(\lambda) \qquad \text{as } \lambda \to \infty.$$

We also need its variant in the form of Lemma 6.3 below which may be deduced, for example, from Theorem 2.10.2 in [2]; Here we give its direct proof for completeness.

Lemma 6.3. *Let $0 < \beta \leq 1$ and $L(\lambda)$ be a slowly varying function at ∞. Then*

$$\phi(\lambda) \asymp \lambda^{-\beta} L(\lambda) \qquad \text{as } \lambda \to \infty$$

if and only if

$$\nu(\lambda) \asymp \lambda^{1-\beta} L(\lambda) \qquad \text{as } \lambda \to \infty.$$

Proof. The proof of "if" part is easy and omitted. We prove the "only if" part. So we assume $\phi(\lambda) \asymp \lambda^{-\beta} L(\lambda)$ as $\lambda \to \infty$. We first note that

$$\phi(\lambda) \asymp \int_\lambda^\infty \frac{\nu(\xi)}{\xi^2} d\xi =: G(\lambda) \qquad \text{as } \lambda \to \infty.$$

Indeed,

$$\begin{aligned} \int_\lambda^\infty \frac{\nu(\xi)}{(\lambda+\xi)^2} d\xi \leq \phi(\lambda) &= \int_0^\lambda \frac{\nu(\xi)}{(\lambda+\xi)^2} d\xi + \int_\lambda^\infty \frac{\nu(\xi)}{(\lambda+\xi)^2} d\xi \\ &\leq \frac{1}{2\lambda} \nu(\lambda) + \int_\lambda^\infty \frac{\nu(\xi)}{(\lambda+\xi)^2} d\xi \\ &\leq 2 \int_\lambda^\infty \frac{\nu(\xi)}{(\lambda+\xi)^2} d\xi \end{aligned}$$

and, noting $1/(2\xi) \leq 1/(\lambda+\xi) \leq 1/\xi$ for $\xi \geq \lambda$, we see readily that $\phi(\lambda) \asymp G(\lambda)$ as $\lambda \to \infty$. Hence $G(\lambda) \asymp \lambda^{-\beta} L(\lambda)$ as $\lambda \to \infty$. So we can find $0 < k < K < \infty$ and $\lambda_0 > 0$ such that

$$k\lambda^{-\beta} L(\lambda) \leq G(\lambda) \leq K\lambda^{-\beta} L(\lambda) \qquad \text{for all } \lambda \geq \lambda_0.$$

Then, noting $G(\lambda) \geq \nu(\lambda) \int_\lambda^\infty \frac{d\xi}{\xi^2}$, we have

$$\nu(\lambda) \leq K\lambda^{1-\beta} L(\lambda) \qquad \text{for all } \lambda \geq \lambda_0. \tag{6.8}$$

Let $c = (3K/k)^{1/\beta} > 1$ and choose $\lambda_1 \geq \lambda_0$ such that

$$\frac{2}{3} \leq \frac{L(c\lambda)}{L(\lambda)} \leq \frac{3}{2} \qquad \text{for all } \lambda \geq \lambda_1.$$

Then

$$k - \frac{K}{c^\beta} \frac{L(c\lambda)}{L(\lambda)} \geq k - \frac{3K}{2c^\beta} = \frac{k}{2} \qquad \text{for } \lambda \geq \lambda_1$$

and hence

$$\begin{aligned} \frac{k}{2} \lambda^{-\beta} L(\lambda) &\leq k\lambda^{-\beta} L(\lambda) - K(c\lambda)^{-\beta} L(c\lambda) \\ &\leq G(\lambda) - G(c\lambda) = \int_\lambda^{c\lambda} \frac{\nu(\xi)}{\xi^2} d\xi \leq \frac{\nu(c\lambda)}{\lambda}, \end{aligned}$$

that is,

$$\nu(c\lambda) \geq \frac{k}{2}\lambda^{1-\beta}L(\lambda) \geq \frac{k}{3c^{1-\beta}}(c\lambda)^{1-\beta}L(c\lambda).$$

Then, setting $\lambda_2 = c\lambda_1$, we have

$$\nu(\lambda) \geq \frac{k}{3c^{1-\beta}}\lambda^{1-\beta}L(\lambda) \qquad \text{for } \lambda \geq \lambda_2. \tag{6.9}$$

By (6.8) and (6.9), it holds that

$$\nu(\lambda) \asymp \lambda^{1-\beta}L(\lambda) \qquad \text{as } \lambda \to \infty.$$

□

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