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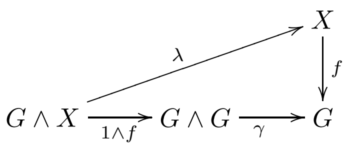
A note on homotopy normality of H-spaces

By

Osamu Nishimura

1. Introduction

Let X be an H-space, G a homotopy associative H-space, and $f\colon X \to G$ an H-map. Let $\gamma\colon G \wedge G \to G$ be the commutator map. (As is well known, G is group-like.) Recall that f is called to be homotopy normal (in the sense of James) if there exists a map $\lambda\colon G \wedge X \to X$ such that $f \circ \lambda$ is homotopic to $\gamma \circ (1 \wedge f)$. (See James [2].)



Localizing spaces and maps concerned at a prime p, we may also consider mod p homotopy normality.

The concept of (mod p) homotopy normality is closely related to that of Samelson products of homotopy groups. In fact, if $f\colon X \to G$ is (mod p) homotopy normal, then all Samelson products (localized at p) from $\pi_k(G) \times f_*(\pi_l(X)) \subset \pi_k(G) \times \pi_l(G)$ to $\pi_{k+l}(G)$ lie in $f_*(\pi_{k+l}(X))$. In [4], Kono and the author studied mod p homotopy normality by using the mod p homology map of the adjoint action on the space of loops, and showed that in many cases for G a compact, 1-connected, simple, exceptional Lie group which has integral p-torsion and H a Lie subgroup of G, the natural inclusion $i\colon H \hookrightarrow G$ is not mod p homotopy normal.

In this paper, we give closer examination for mod p homotopy normality of an H-map $f\colon X \to G$ restricting ourselves to the comparatively manageable case that $p = 3$ and $G = F_4$ where F_4 is the compact, connected, simple, exceptional Lie group of rank 4. We show the following theorem.

Theorem 1.1. *Let X be a mod 3 H-space. If $f\colon X \to F_4$ is a mod 3 homotopy normal H-map and $H^{19}(X;\mathbb{F}_3)$ consists of decomposable elements, then $f^*\colon H^*(F_4;\mathbb{F}_3) \to H^*(X;\mathbb{F}_3)$ is trivial or monomorphic.*

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Here, note that the inclusion of the unit group $* \hookrightarrow F_4$ and the identity map $1_{F_4}\colon F_4 \to F_4$ satisfy the hypothesis in Theorem 1.1 and their mod 3 cohomology maps are trivial and monomorphic respectively.

It is easy to determine $\gamma^*\colon H^*(F_4;\mathbb{F}_3) \to H^*(F_4 \wedge F_4;\mathbb{F}_3)$ and then, it is easy to see that if $f\colon X \to F_4$ is a mod 3 homotopy normal H-map and f^* is neither trivial nor monomorphic, then (there exists an indecomposable element in $H^{19}(X;\mathbb{F}_3)$ by Theorem 1.1 and) $\operatorname{Im} f^*$ is isomorphic to one of the following exterior algebras where $|z_j| = j$:

$$\wedge\,(z_{11}), \tag{1.1}$$

$$\wedge\,(z_{11}, z_{15}), \tag{1.2}$$

$$\wedge\,(z_3, z_{11}), \tag{1.3}$$

$$\wedge\,(z_3, z_{11}, z_{15}), \tag{1.4}$$

$$\wedge\,(z_3, z_7, z_{11}, z_{15}). \tag{1.5}$$

Theorem 1.2. *All these cases are realizable with f's being loop maps.*

The study of this paper is inspired by that of the papers [5], [6] written by Kudou and Yagita. They showed that if $f\colon X \to F_4$ is a mod 3 homotopy normal H-map and there exist no primitive elements in $H^{19}(X;\mathbb{F}_3)$ (this hypothesis is weaker than that in Theorem 1.1, see Milnor-Moore [7]), then one of the following holds: (i) f^* is trivial, (ii) f^* is monomorphic, (iii) $\operatorname{Im} f^*$ is as (1.3) (in other words, $\operatorname{Im} f^* \cong H^*(G_2;\mathbb{F}_3)$, see Mimura [8]), (iv) $\operatorname{Im} f^*$ is as (1.5) (in other words, $\operatorname{Im} f^* \cong H^*(Spin(9);\mathbb{F}_3)$, also see [8]). Here, G_2 is the compact, connected, simple, exceptional Lie group of rank 2. Also they asked whether or not the natural inclusions $G_2 \hookrightarrow F_4$ and $Spin(9) \hookrightarrow F_4$, of which the mod 3 cohomology maps are epimorphic, are mod 3 homotopy normal. Since $H^{19}(G_2;\mathbb{F}_3)$ and $H^{19}(Spin(9);\mathbb{F}_3)$ are trivial, Theorem 1.1 implies the following corollary.

Corollary 1.1. *The natural inclusions $Spin(9) \hookrightarrow F_4$ and $G_2 \hookrightarrow F_4$ are not mod 3 homotopy normal.*

This was first proved in [4].

This paper is organized as follows. In Section 2, we study the mod 3 cohomology map $\tilde{\gamma}^*$ where $\tilde{\gamma}\colon F_4 \wedge F_4 \to \tilde{F}_4$ is a lift of γ to $\tilde{F}_4$, the 3-connective cover of F_4. In Section 3, we study the mod 3 cohomology map $\acute{\gamma}^*$ where $\acute{\gamma}\colon F_4 \wedge F_4 \to \acute{F}_4$ is a lift of γ to $\acute{F}_4$, the homotopy fiber of a representative of the homotopy class corresponding to the generator $x_8 \in H^8(F_4;\mathbb{F}_3) \cong [F_4, K(\mathbb{F}_3, 8)]$. In Section 4, We use the results in Section 2 and Section 3 to prove Theorem 1.1. In Section 5, we prove Theorem 1.2.

All spaces and maps are localized at 3. Homology and cohomology are mod 3 unless otherwise stated. Let $\mathrm{I}H^*(-)$ denote the module which consists of the positive degree elements of $H^*(-)$. Let $\mathrm{D}H^*(-) = \mathrm{I}H^*(-)\cdot\mathrm{I}H^*(-)$, the decomposable module. Let $\mathrm{Q}H^*(-) = \mathrm{I}H^*(-)/\mathrm{D}H^*(-)$, the indecomposable module. If X is an H-space, then $\mathrm{Q}H^*(X)$ is dual to $\mathrm{P}H_*(X)$, the module which consists of the primitive elements of $H_*(X)$. The subscript integer of an element of a graded module designates the degree.

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2. A lift of the commutator map to the 3-connective cover

First we fix the notation and recall the data concerning F_4. Let $\mu\colon F_4 \times F_4 \to F_4$, $\iota\colon F_4 \to F_4$, and $\Delta\colon F_4 \to F_4 \times F_4$ denote the multiplication, the inverse, and the diagonal map of F_4 respectively. Also let $q\colon F_4 \times F_4 \to F_4 \wedge F_4$ and $T\colon F_4 \wedge F_4 \to F_4 \wedge F_4$ denote the natural projection and the switching map respectively.

Recall that $H^*(F_4) = \mathbb{F}_3[x_8]/(x_8^3) \otimes \wedge(x_3, x_7, x_{11}, x_{15})$. The cohomology operations in $H^*(F_4)$ are given by $\wp^1 x_3 = x_7, \beta x_7 = x_8, \wp^1 x_{11} = x_{15}$, and others. Let $\bar{\mu}^*$ be the reduced coproduct of $H^*(F_4)$: $\bar{\mu}^*(x) = \mu^*(x) - x \otimes 1 - 1 \otimes x$ for $x \in H^*(F_4)$. The coalgebra structure of $H^*(F_4)$ is given by $\bar{\mu}^*(x_j) = 0$ for $j = 3, 7, 8$ and $\bar{\mu}^*(x_j) = x_8 \otimes x_{j-8}$ for $j = 11, 15$. (For the detail of the above, see Mimura [8].)

Recall that the commutator map $\gamma\colon F_4 \wedge F_4 \to F_4$ is given by

$$\gamma \circ q = \mu \circ (\mu \times 1) \circ (\mu \times 1 \times 1) \circ (1 \times 1 \times \iota \times \iota) \circ (1 \times T \times 1) \circ (\Delta \times \Delta).$$

By the usual computation, we can easily show that $\gamma^*(x_j) = 0$ for $j = 3, 7, 8$ and $\gamma^*(x_j) = x_8 \otimes x_{j-8} - x_{j-8} \otimes x_8$ for $j = 11, 15$.

Let $x_3^{\mathbb{Z}} \in H^3(F_4; \mathbb{Z})$ be the integral class of x_3, which is also regarded as an element in $[F_4, K(\mathbb{Z}, 3)]$ through the identification $H^3(F_4; \mathbb{Z}) \cong [F_4, K(\mathbb{Z}, 3)]$. Let $\tilde{F}_4$ be the homotopy fiber of a representative of the homotopy class $x_3^{\mathbb{Z}}$, which is the 3-connective cover of F_4, and $\tilde{\pi}\colon \tilde{F}_4 \to F_4$ the projection. Recall that the cohomology class $x_3^{\mathbb{Z}}$ is universally transgressive so that $\tilde{F}_4$ is a loop space with the classifying space $B\tilde{F}_4$, the 4-connective cover of BF_4. Let $\tilde{\mu}\colon \tilde{F}_4 \times \tilde{F}_4 \to \tilde{F}_4$, $\tilde{\iota}\colon \tilde{F}_4 \to \tilde{F}_4$, and $\tilde{\Delta}\colon \tilde{F}_4 \to \tilde{F}_4 \times \tilde{F}_4$ denote the multiplication, the inverse, and the diagonal map of $\tilde{F}_4$ respectively. Recall that by the Serre spectral sequence of $\mathbb{C}P^\infty \to \tilde{F}_4 \xrightarrow{\tilde{\pi}} F_4$, we have $H^*(\tilde{F}_4) = \mathbb{F}_3[\tilde{y}_{18}] \otimes \wedge(\tilde{x}_{11}, \tilde{x}_{15}, \tilde{y}_{19}, \tilde{y}_{23})$ where $\tilde{x}_j = \tilde{\pi}^*(x_j)$. Then γ can be lifted to $\tilde{\gamma}\colon F_4 \wedge F_4 \to \tilde{F}_4$.

Let J be the ideal of $H^*(F_4 \wedge F_4)$ generated by $\mathrm{I}H^*(F_4) \otimes \mathrm{D}H^*(F_4)$ and $\mathrm{D}H^*(F_4) \otimes \mathrm{I}H^*(F_4)$. (Here we think of the identification $\mathrm{I}H^*(F_4 \wedge F_4) \cong \mathrm{I}H^*(F_4) \otimes \mathrm{I}H^*(F_4)$.) It is clear that $\tilde{\gamma}^*(\tilde{y}_{19})$ is a linear combination of $x_8 \otimes x_{11}$ and $x_{11} \otimes x_8$ mod J. We can easily see that

$$\tilde{\gamma} \circ T \simeq \tilde{\iota} \circ \tilde{\gamma}\colon F_4 \wedge F_4 \to \tilde{F}_4. \tag{2.1}$$

Also we can easily see that $\tilde{\iota}^*(\tilde{y}_{19}) = -\tilde{y}_{19}$ and that $T^*(J) \subset J$. It follows from these that if we put

$$\tilde{\gamma}^*(\tilde{y}_{19}) \equiv \alpha x_8 \otimes x_{11} + \alpha' x_{11} \otimes x_8 \mod J$$

where $\alpha, \alpha' \in \mathbb{F}_3$ and apply the mod 3 cohomology maps of (2.1) to $\tilde{y}_{19}$, we have

$$\alpha x_{11} \otimes x_8 + \alpha' x_8 \otimes x_{11} \equiv -\alpha x_8 \otimes x_{11} - \alpha' x_{11} \otimes x_8 \mod J$$

and hence we have $\alpha' = -\alpha$. Thus, we may put

$$\tilde{\gamma}^*(\tilde{y}_{19}) \equiv \alpha(x_8 \otimes x_{11} - x_{11} \otimes x_8) \mod J.$$

Let $a_j \in \mathrm{P}H_*(F_4)$, $\tilde{a}_j \in \mathrm{P}H_*(\tilde{F}_4)$, and $\tilde{b}_j \in \mathrm{P}H_*(\tilde{F}_4)$ be the dual elements of $\{x_j\} \in \mathrm{Q}H^*(F_4)$, $\{\tilde{x}_j\} \in \mathrm{Q}H^*(\tilde{F}_4)$, and $\{\tilde{y}_j\} \in \mathrm{Q}H^*(\tilde{F}_4)$ respectively. By [9] and by considering the natural inclusion $F_4 \hookrightarrow E_6$ (or by [10] and by considering the homology suspension $\sigma\colon H_*(\Omega\tilde{F}_4) \to H_*(\tilde{F}_4)$), we can choose $\tilde{y}_{19}$ so that $\tilde{\mathrm{ad}}_*(a_8 \otimes \tilde{a}_{11}) = \tilde{b}_{19}$ where $\tilde{\mathrm{ad}}\colon F_4 \times \tilde{F}_4 \to \tilde{F}_4$ covers the adjoint action $\mathrm{ad}\colon F_4 \times F_4 \to F_4$. (See Kono-Kozima [3] and Hamanaka-Hara [1].) We can easily see that

$$\tilde{\gamma} \circ q \circ (1 \times \tilde{\pi}) \simeq \tilde{\mu} \circ (\tilde{\mathrm{ad}} \times \tilde{\iota}) \circ (1 \times \tilde{\Delta})\colon F_4 \times \tilde{F}_4 \to \tilde{F}_4.$$

Applying the mod 3 homology maps of these to $a_8 \otimes \tilde{a}_{11}$, we have from the left hand side $\tilde{\gamma}_*(a_8 \otimes a_{11})$ and from the right hand side

$$\tilde{\mu}_* \circ (\tilde{\mathrm{ad}}_* \otimes \tilde{\iota}_*)(a_8 \otimes \tilde{a}_{11} \otimes 1 + a_8 \otimes 1 \otimes \tilde{a}_{11}) = \tilde{\mu}_*(\tilde{b}_{19} \otimes 1) = \tilde{b}_{19}.$$

(Recall that $\tilde{\mathrm{ad}}_*(a_8 \otimes 1) = 0$ by the general property of the adjoint action.) Thus, we have $\tilde{\gamma}_*(a_8 \otimes a_{11}) = \tilde{b}_{19}$. Taking the pairing of this with $\tilde{y}_{19}$, we have from the left hand side

$$\langle \tilde{y}_{19}, \tilde{\gamma}_*(a_8 \otimes a_{11})\rangle = \langle \tilde{\gamma}^*(\tilde{y}_{19}), a_8 \otimes a_{11}\rangle = \langle \alpha(x_8 \otimes x_{11} - x_{11} \otimes x_8), a_8 \otimes a_{11}\rangle = \alpha$$

and from the right hand side $\langle \tilde{y}_{19}, \tilde{b}_{19}\rangle = 1$. (Note that the pairing of $a_8 \otimes a_{11}$ with an element of J vanishes.) Thus, we have $\alpha = 1$ and hence we have

$$\tilde{\gamma}^*(\tilde{y}_{19}) \equiv x_8 \otimes x_{11} - x_{11} \otimes x_8 \mod J. \tag{2.2}$$

We use this result later.

Remark 1. Making more efforts, we can determine $\tilde{\gamma}^*\colon H^*(\tilde{F}_4) \to H^*(F_4 \wedge F_4)$ completely. In fact, we have $\tilde{\gamma}^*(\tilde{x}_j) = \gamma^*(x_j)$, and in the same way as above, we can determine $\tilde{\gamma}^*(\tilde{y}_j)$ mod J. Then, by algebraic computation based on the property of the commutator map, we can determine $\tilde{\gamma}^*(\tilde{y}_j)$ without mod J. We omit the detail.

3. A lift of the commutator map to another space

Recall that we have the cohomology class $x_8 \in H^8(F_4)$, which is also regarded as an element in $[F_4, K(\mathbb{F}_3, 8)]$ through the identification $H^8(F_4) \cong [F_4, K(\mathbb{F}_3, 8)]$. Let $\acute{F}_4$ be the homotopy fiber of a representative of the homotopy class x_8 and $\acute{\pi}\colon \acute{F}_4 \to F_4$ the projection. Let $u_3 \in H^3(K(\mathbb{Z}, 3))$ be the fundamental class. Then we have $\beta\wp^1 u_3 \in H^8(K(\mathbb{Z}, 3)) \cong [K(\mathbb{Z}, 3), K(\mathbb{F}_3, 8)]$

and $\beta\wp^1 u_3 \circ x_3^{\mathbb{Z}} = \beta\wp^1 x_3 = x_8$ where we regard the elements as the appropriate homotopy classes. Also note that the cohomology class x_8 as well as $x_3^{\mathbb{Z}}$ is universally transgressive. Thus, we have the following homotopy commutative diagram of loop spaces and loop maps. (Representatives of $x_3^{\mathbb{Z}}$, x_8, and $\beta\wp^1 u_3$ are denoted by the same symbols respectively, and the maps π, $\tilde{h}$, and $\acute{h}$ are defined in a obvious way.)

$$\begin{array}{ccccccc}
\mathbb{C}P^\infty & \xrightarrow{\tilde{h}} & \tilde{F}_4 & \xrightarrow{\tilde{\pi}} & F_4 & \xrightarrow{x_3^{\mathbb{Z}}} & K(\mathbb{Z},3) \\
{\scriptstyle *}\downarrow & & {\scriptstyle \pi}\downarrow & & \| & & \downarrow{\scriptstyle \beta\wp^1 u_3} \\
K(\mathbb{F}_3,7) & \xrightarrow[\acute{h}]{} & \acute{F}_4 & \xrightarrow[\acute{\pi}]{} & F_4 & \xrightarrow[x_8]{} & K(\mathbb{F}_3,8) \\
\| & & \uparrow{\scriptstyle \acute{h}} & & \uparrow{\scriptstyle *} & & \| \\
K(\mathbb{F}_3,7) & = & K(\mathbb{F}_3,7) & \xrightarrow[*]{} & * & \xrightarrow[*]{} & K(\mathbb{F}_3,8)
\end{array}$$

Here the horizontal arrows form homotopy fiber sequences.

Let $u_j \in H^*(K(\mathbb{F}_3, j))$ be the fundamental classes for $j = 7, 8$. We can easily see that $\wp^3 u_7$ is a permanent cycle in the Serre spectral sequence of $K(\mathbb{F}_3, 7) \xrightarrow{\acute{h}} \acute{F}_4 \xrightarrow{\acute{\pi}} F_4$. Hence we can take $\acute{y}_{19} \in H^*(\acute{F}_4)$ so that $\acute{h}^*(\acute{y}_{19}) = \wp^3 u_7$. Moreover, it is easy to see that $\acute{y}_{19}$ can be taken also to be transgressive and $\tau(\acute{y}_{19}) = \wp^3 u_8$ in the Serre spectral sequence of $\acute{F}_4 \xrightarrow{\acute{\pi}} F_4 \xrightarrow{x_8} K(\mathbb{F}_3, 8)$. Here, note that $\pi^*(\acute{y}_{19})$ is a scalar multiple of $\tilde{y}_{19}$ and observe that $\tilde{y}_{19}$ is transgressive and $\tau(\tilde{y}_{19}) = \pm\beta\wp^3\wp^1 u_3 = \pm\wp^3\beta\wp^1 u_3 \neq 0$ in the Serre spectral sequence of $\tilde{F}_4 \xrightarrow{\tilde{\pi}} F_4 \xrightarrow{x_3^{\mathbb{Z}}} K(\mathbb{Z}, 3)$. Hence $\pi^*(\acute{y}_{19})$ is transgressive and $\tau(\pi^*(\acute{y}_{19}))$ is a scalar multiple of $\wp^3\beta\wp^1 u_3$. Moreover, $\pi^*(\acute{y}_{19})$ is non-zero if and only if $\tau(\pi^*(\acute{y}_{19}))$ is non-zero. Then, $\tau(\pi^*(\acute{y}_{19}))$ is the image of $\tau(\acute{y}_{19}) = \wp^3 u_8$ under the mod 3 cohomology map of $\beta\wp^1 u_3\colon K(\mathbb{Z}, 3) \to K(\mathbb{F}_3, 8)$, and hence is $\wp^3\beta\wp^1 u_3 \neq 0$. Thus, we have $\pi^*(\acute{y}_{19}) = \pm\tilde{y}_{19}$.

Put $\acute{\gamma} = \pi \circ \tilde{\gamma}\colon F_4 \wedge F_4 \to \acute{F}_4$, which is a lift of γ to $\acute{F}_4$. By (2.2), we have

$$\begin{aligned} \acute{\gamma}^*(\acute{y}_{19}) &= \tilde{\gamma}^* \circ \pi^*(\acute{y}_{19}) \\ &= \pm\tilde{\gamma}^*(\tilde{y}_{19}) \\ &\equiv \pm(x_8 \otimes x_{11} - x_{11} \otimes x_8) \mod J. \end{aligned} \tag{3.1}$$

4. Proof of Theorem 1.1

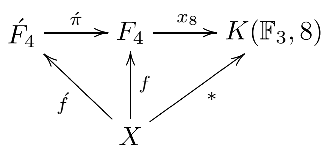
First, for any map $g\colon Y \to F_4$, we have $\wp^1 g^*(x_3) = g^*(x_7)$, $\beta g^*(x_7) = g^*(x_8)$, and $\wp^1 g^*(x_{11}) = g^*(x_{15})$. Thus, $g^*(x_3) = 0$ implies $g^*(x_7) = 0$, $g^*(x_7) = 0$ implies $g^*(x_8) = 0$, and $g^*(x_{11}) = 0$ implies $g^*(x_{15}) = 0$.

Next, let $f\colon X \to F_4$ be a mod 3 homotopy normal H-map. We have a map $\lambda\colon F_4 \wedge X \to X$ such that $f \circ \lambda \simeq \gamma \circ (1 \wedge f)\colon F_4 \wedge X \to F_4$. For $j = 11, 15$, we have

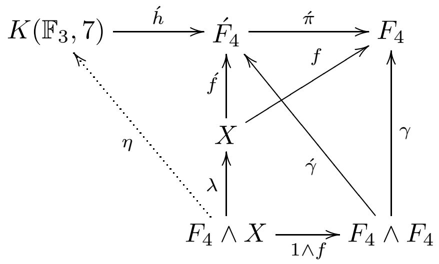
$$\begin{aligned}\lambda^* \circ f^*(x_j) &= (1 \wedge f)^* \circ \gamma^*(x_j) \\ &= (1 \wedge f)^*(x_8 \otimes x_{j-8} - x_{j-8} \otimes x_8) \\ &= x_8 \otimes f^*(x_{j-8}) - x_{j-8} \otimes f^*(x_8).\end{aligned}$$

Thus, $f^*(x_j) = 0$ implies $f^*(x_{j-8}) = 0$ for $j = 11, 15$. Hence we can see that f^* is trivial if and only if $f^*(x_{11}) = 0$. Moreover, considering the elementary theory of Hopf algebras (see Milnor-Moore [7]), we can see that f^* is monomorphic if and only if $f^*(x_j) \neq 0$ for any j if and only if $f^*(x_8) \neq 0$.

Now, suppose that $H^{19}(X) = \mathrm{D}H^{19}(X)$ (in other words, $\mathrm{Q}H^{19}(X) = 0$), $f^*(x_{11}) \neq 0$, and $f^*(x_8) = 0$. We show a contradiction. Since $f^*(x_8) = 0$, we have a lift $\acute{f}\colon X \to \acute{F}_4$ of f.



Then, since we have two lifts $\acute{f} \circ \lambda$ and $\acute{\gamma} \circ (1 \wedge f)$ of $f \circ \lambda \simeq \gamma \circ (1 \wedge f)$ to $\acute{F}_4$, there exists a map $\eta\colon F_4 \wedge X \to K(\mathbb{F}_3, 7)$ such that $\acute{\gamma} \circ (1 \wedge f) \simeq \acute{f} \circ \lambda + \acute{h} \circ \eta$.



In particular, we have

$$(4.1) \qquad (1 \wedge f)^* \circ \acute{\gamma}^*(\acute{y}_{19}) = \lambda^* \circ \acute{f}^*(\acute{y}_{19}) + \eta^* \circ \acute{h}^*(\acute{y}_{19}).$$

Let J' be the ideal of $H^*(F_4 \wedge X)$ generated by $\mathrm{I}H^*(F_4) \otimes \mathrm{D}H^*(X)$ and $\mathrm{D}H^*(F_4) \otimes \mathrm{I}H^*(X)$. Note that $(1 \wedge f)^*(J) \subset J'$. Hence by (3.1) and by $f^*(x_8) = 0$, we have

$$(4.2) \qquad (1 \wedge f)^* \circ \acute{\gamma}^*(\acute{y}_{19}) \equiv \pm(x_8 \otimes f^*(x_{11})) \mod J'.$$

Also note that $\lambda^*(\mathrm{D}H^*(X)) \subset \mathrm{D}H^*(F_4 \wedge X) \subset J'$. Since $\acute{f}^*(\acute{y}_{19}) \in H^{19}(X) = \mathrm{D}H^{19}(X)$, we have

$$(4.3) \qquad \lambda^* \circ \acute{f}^*(\acute{y}_{19}) \in J'.$$

We may put $\eta^*(u_7) = x_3 \otimes \zeta \in H^3(F_4) \otimes H^4(X) = H^7(F_4 \wedge X)$. Since $\acute{h}^*(\acute{y}_{19}) = \wp^3 u_7$, we have

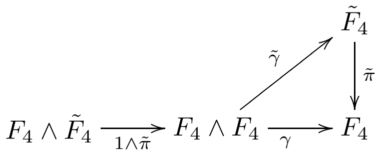
$$(4.4) \qquad \eta^* \circ \acute{h}^*(\acute{y}_{19}) = \wp^3 \eta^*(u_7) = \wp^3(x_3 \otimes \zeta) = x_7 \otimes \zeta^3 \in J'.$$

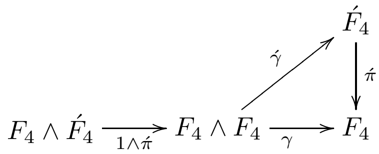
Thus by (4.1), (4.2), (4.3), and (4.4), we have $x_8 \otimes f^*(x_{11}) \in J'$. However, $f^*(x_{11}) \neq 0$ is primitive (recall that f is an H-map and that $f^*(x_8) = 0$) and hence is indecomposable (see Milnor-Moore [7]). It follows that $x_8 \otimes f^*(x_{11}) \notin J'$. This is a contradiction. □

5. Proof of Theorem 1.2

Let $f\colon X \to F_4$ be a mod 3 homotopy normal H-map. Because of the argument in the first two paragraphs of Section 4, and of the elementary theory of Hopf algebras (see Milnor-Moore [7]), we can see that if $f^*(x_{11}) \neq 0$ and $f^*(x_8) = 0$, then $\operatorname{Im} f^*$ is isomorphic to one of (1.1)–(1.5) where $z_j = f^*(x_j) \neq 0$. We exhibit an example for each case with f being a loop map.

For (1.2) and (1.5), we have $\tilde{\pi}\colon \tilde{F}_4 \to F_4$ and $\acute{\pi}\colon \acute{F}_4 \to F_4$, respectively, which are loop maps. These are homotopy normal since γ has a lift $\tilde{\gamma}$ to $\tilde{F}_4$ and a lift $\acute{\gamma}$ to $\acute{F}_4$. (See Kudou-Yagita [6].)



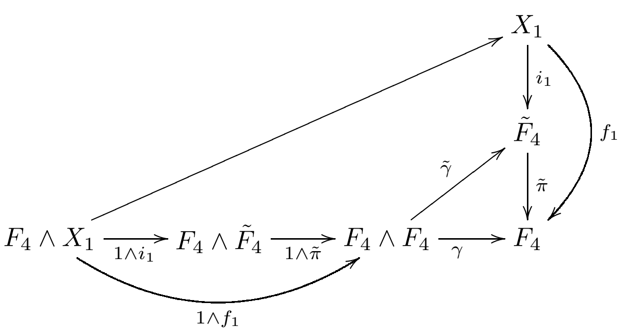


We know that $\tilde{\pi}^*(x_j) = 0$ for $j = 3, 7, 8$ and that $\tilde{\pi}^*(x_j) = \tilde{x}_j \neq 0$ for $j = 11, 15$. By definition, we have $\acute{\pi}^*(x_8) = 0$ and by the Serre spectral sequence of $K(\mathbb{F}_3, 7) \xrightarrow{\acute{h}} \acute{F}_4 \xrightarrow{\acute{\pi}} F_4$, we can easily see that $\acute{\pi}^*(x_j) \neq 0$ for $j = 3, 7, 11, 15$. Thus, $\operatorname{Im} \tilde{\pi}^*$ is as (1.2) and $\operatorname{Im} \acute{\pi}^*$ is as (1.5).

For (1.1), let X_1 be the homotopy fiber of a representative of the homotopy class $\tilde{x}_{15} \in H^{15}(\tilde{F}_4) \cong [\tilde{F}_4, K(\mathbb{F}_3, 15)]$ and $i_1\colon X_1 \to \tilde{F}_4$ the projection. Put $f_1 = \tilde{\pi} \circ i_1\colon X_1 \to F_4$, which is a loop map. (Note that the cohomology class $\tilde{x}_{15}$ is universally transgressive.) Since

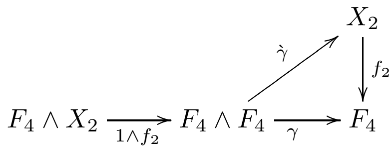
$$(1 \wedge f_1)^* \circ \tilde{\gamma}^*(\tilde{x}_{15}) = (1 \wedge i_1)^* \circ (1 \wedge \tilde{\pi})^*(x_8 \otimes x_7 - x_7 \otimes x_8) = 0,$$

the map $\tilde{\gamma} \circ (1 \wedge f_1)\colon F_4 \wedge X_1 \to \tilde{F}_4$, which is a lift of $\gamma \circ (1 \wedge f_1)\colon F_4 \wedge X_1 \to F_4$, has a lift to X_1.



Thus, f_1 is homotopy normal. It is easy to see that $f_1^*(x_j) = 0$ for $j \neq 11$ and that $f_1^*(x_{11}) \neq 0$. Thus, $\operatorname{Im} f_1^*$ is as (1.1).

For (1.4), let X_2 be the homotopy fiber of a representative of the homotopy class $x_7 \in H^7(F_4) \cong [F_4, K(\mathbb{F}_3, 7)]$ and $f_2\colon X_2 \to F_4$ the projection, which is a loop map since the cohomology class x_7 is universally transgressive. Then $\gamma^*(x_7) = 0$ implies that γ has a lift $\grave{\gamma}\colon F_4 \wedge F_4 \to X_2$. Thus f_2 is homotopy normal.

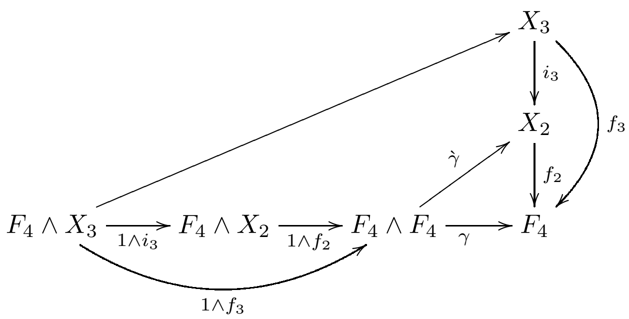


By the Serre spectral sequence of the fibering $K(\mathbb{F}_3, 6) \xrightarrow{h} X_2 \xrightarrow{f_2} F_4$, we can see that $f_2^*(x_j) = 0$ for $j = 7, 8$ and that $f_2^*(x_j) \neq 0$ for $j = 3, 11, 15$. Thus, $\operatorname{Im} f_2^*$ is as (1.4). We can also see that $H^*(X_2) = \wedge(f_2^*(x_3)) \otimes \mathbb{F}_3[\xi_{10}]$ for $* \leq 10$ where $h^*(\xi_{10}) \neq 0$.

Finally, for (1.3), let X_3 be the homotopy fiber of a representative of the homotopy class $f_2^*(x_{15}) \in H^{15}(X_2) \cong [X_2, K(\mathbb{F}_3, 15)]$ and $i_3\colon X_3 \to X_2$ the projection. Put $f_3 = f_2 \circ i_3\colon X_3 \to F_4$. By the Serre spectral sequence of $X_2 \to * \to BX_2$, we know that $H^*(BX_2) = \mathbb{F}_3[\tau(f_2^*(x_3))]$ for $* \leq 10$. Then we know that the cohomology class $f_2^*(x_{11}) \in H^*(X_2)$ is universally transgressive, and hence so is $f_2^*(x_{15}) = \wp^1 f_2^*(x_{11})$. It follows that i_3 is a loop map, and hence so is f_3. By

$$(1 \wedge f_3)^* \circ \grave{\gamma}^* \circ f_2^*(x_{15}) = (1 \wedge i_3)^* \circ (1 \wedge f_2)^* \circ \gamma^*(x_{15}) = 0$$

and by the same argument as that for f_1, the loop map f_3 is homotopy normal.



It is easy to see that $f_3^*(x_j) = 0$ for $j = 7, 8, 15$ and that $f_3^*(x_j) \neq 0$ for $j = 3, 11$. Thus, $\operatorname{Im} f_3^*$ is as (1.3). $\square$

Department of Mathematics
Faculty of Science
Kyoto University
e-mail: osamu@math.kyoto-u.ac.jp

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