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Twin positive solutions of boundary value problems for functional differential equations

By

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Abstract

In this paper, the sufficient conditions for the existence of twin positive solutions to boundary value problems for functional differential equations are presented. The results are obtained by using a fixed point theorem (See [1, Theorem 12.3]).

1. Introduction

A great deal of research has been devoted to the existence of solutions for the boundary value problem for first and second order functional differential equations. We refer for instance to [3], [5], [7]–[9] and their references. The methods and techniques employed in these papers involve the use of topological degree theory [9], the upper and the lower solution methods and the fixed point theorems in cones [7], [8]. The existence of multiple positive solutions for ordinary differential equations has also received a great deal of attention. We refer to [2], [4], [6]. These works were done under assumption that the nonlinear term f is nonnegative. On the other hand, Krasnoselskii's fixed point theorem of expansion and compression has been extensively employed in studying the existence of positive solutions for boundary value problems. In this case, the nonlinear terms are usually bounded to satisfy the superlinear or (and) sublinear conditions.

This paper is concerned with the existence of solutions for the functional differential equations (BVP)

$$x' = f(t, x_t), \quad \text{a.e. } t \in [0, T], \quad x_0 = x_T \tag{1.1}$$

and

$$\begin{cases} x'' = f(t, x_t, x'(t)), & \text{a.e. } t \in [0, T], \\ x(T) = \eta, \quad x(t) = \varphi(t), & t \in [-a, 0], \end{cases} \tag{1.2}$$

where $f : J \times C([-a, 0], \mathbb{R}^n) \to \mathbb{R}^n$, $J = [0, T]$ is a compact real interval, $\varphi \in C([-a, 0], \mathbb{R}^n)$ and $\eta \in \mathbb{R}^n$.

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We will choose another strategy of proof which rely essentially on a well known nonzero fixed point theorem (see Lemma 1) and present the existence of at least two positive solutions for BVP(1.1) (and (1.2)) under our weak conditions. One of the key steps is to find a function ψ such that the operator A satisfies the condition $x - Ax \neq \lambda\psi$ in the cited fixed point theorem. In this paper, the nonlinearity needs not nonnegative and superlinear or sublinear conditions are not required. We lead to new existence principles and it seems to be difficult to utilize Krasnoselskii's fixed point theorem of expansion and compression to prove our main results.

For any function $x : [-a, T] \to \mathbb{R}^n$ and $t \in J$, we denote by x_t the element of $C([-a, 0], \mathbb{R}^n)$ defined by

$$x_t(s) = x(t + s), \quad s \in [-a, 0].$$

Here $x_t(\cdot)$ represents the history of the state from time $t - a$ to the time t. Stipulate $x_t = x(t)$ when $a = 0$.

The following notations and definitions are according to [3]. Let $\mathbb{R}^n$ stands for the n dimension real space with Euclidean norm denoted by $|\cdot|$. The partial order of $\mathbb{R}^n$ introduced by $x \leq y$ iff $x_i \leq y_i$ for $i = 1, 2, \dots, n$, where $x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$. $x < y$ if and only if $x \leq y$ and $x \neq y$.

For a fixed $a \geq 0$, let $C([-a, 0], \mathbb{R}^n)$ be endowed with the norm $\|x\| = \sup\{|x(t)| : -a \leq t \leq 0\}$, $\mathbb{R}^n_+ = \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_i \geq 0$ with $i = 1, 2, \dots, n\}$ and $J = [0, T] \subset \mathbb{R}$ with $T > 0$. For $x, y \in C([-a, T], \mathbb{R}^n)$, define $x \leq y$ if and only if $x(t) \leq y(t)$ for each $t \in [-a, T]$, $x < y$ if and only if $x \leq y$ and there exists some $t \in [-a, T]$ such that $x(t) \neq y(t)$.

For a constant $r > 0$, let

$$P_r = \{x \in \mathbb{R}^n : x \geq 0, |x| < r\}, \quad \partial P_r = \{x \in \mathbb{R}^n : x \geq 0 : |x| = r\}.$$

Let $L^1(J, \mathbb{R}^n)$ denote the Banach space of measurable functions $x : J \to \mathbb{R}^n$ which are Lebegsue integrable with norm $\|x\|_1 = \int_0^T |x(t)|dt$. The partial order in $L^1(J, E)$ is defined as $x \leq y \Leftrightarrow x(t) \leq y(t)$ a.e. for $t \in J$.

$AC(J, \mathbb{R}^n)$ denotes the Banach space of absolutely continuous functions defined on J with values in $\mathbb{R}^n$.

The following fixed point theorem (see [1, Theorem 12.3]) plays a key role in our main results.

Lemma 1. *Let P be a cone of a real Banach space E, Ω_1 and Ω_2 bounded open subset of E with $0 \in \Omega_1$ and $\bar{\Omega}_1 \subset \Omega_2$, and $A : P \cap (\bar{\Omega}_2 \setminus \Omega_1) \to P$ completely continuous. Suppose that one of the following two conditions is satisfied:*

(i) *There exists some $u_0 \in P \setminus \{0\}$ such that $x - Ax \neq tu_0$ for $\forall x \in P \cap \partial\Omega_2$, $t \geq 0$; $Ax \neq \mu x$, for $\forall x \in P \cap \partial\Omega_1, \mu \geq 1$.*

(ii) *There exists some $u_0 \in P \setminus \{0\}$ such that $x - Ax \neq tu_0$ for $\forall x \in P \cap \partial\Omega_1$, $t \geq 0$; $Ax \neq \mu x$, for $\forall x \in P \cap \partial\Omega_2, \mu \geq 1$.*

Then, the operator A has a fixed point $x \in P \cap (\Omega_2 \setminus \bar{\Omega}_1)$.

2. First order boundary value problems

In this section we consider the existence of two positive solutions for first order BVP(1.1). Let us define that a function x is a solution of (1.1) if $x \in C([-a,T],\mathbb{R}^n) \cap AC([0,T],\mathbb{R}^n)$ and satisfies (1.1) a.e. on $[0,T]$.

Let $X = \{x \in C(J,\mathbb{R}^n) : x(0) = x(T)\}$ with the norm

$$\|x\|_J = \sup\{|x(t)| : 0 \le t \le T\}$$

and $X_+ = \{x \in X : x(t) \ge 0 \text{ for } t \in J\}$. It is obvious that X is a Banach space and X_+ is a closed convex cone of X, moreover, $x \in X_+$ if $x(t) \ge 0$ for every $t \in J$. Let us introduce the differential operator $L : AC(J,X) \to L^1(J,\mathbb{R}^n)$ by

$$Lx = x' - \alpha(t)x,$$

where α is given in the following (H2). From the well known results of ordinary differential equations it follows that for any $y \in L^1(J,\mathbb{R}^n)$ the boundary value problem

$$Lx(t) = y(t), \quad x(0) = x(T)$$

has an unique solution $x := Ky \in AC(J,X)$ with the operator K defined by

$$(Ky)(t) = \int_0^T G(t,s)y(s)ds \quad \text{for } t \in J, \tag{2.1}$$

where the Green function $G(t,s)$ satisfies

$$(\tilde{\alpha}(T) - 1)\tilde{\alpha}(t)G(t,s) = \begin{cases} \tilde{\alpha}(T)\tilde{\alpha}(s), & s \le t, \\ \tilde{\alpha}(s), & s > t \end{cases} \tag{2.2}$$

with $\tilde{\alpha}(t) = \exp(-\int_0^t \alpha(s)ds)$. Thus we have that $K = L^{-1}$ and (2.1) guarantees that K is a bounded linear operator from $L^1(J,\mathbb{R}^n)$ into X.

Let us impose the following hypotheses on the map $f : J \times \mathbb{R}^n \to \mathbb{R}^n$.

(H1) $(t,u) \to f(t,u)$ is measurable with respect to t for each $u \in \mathbb{R}^n$, continuous with respect to u for each $t \in J$.

(H2) There exist a function $\alpha \in L^1(J,\mathbb{R}_+)$ such that $|\alpha(t)| > 0$, a function $\beta \in L^1(J \times \mathbb{R}^n,\mathbb{R}^n)$ such that $\beta(t,\cdot)$ is bounded continuous and nondecreasing for each $t \in J$, a function $\psi \in X_+$ and a real number $\lambda > 0$ such that

$$\int_0^T [\beta(s,t\psi(s)) - t\alpha(s)\psi(s)]ds > 0$$

for every $0 < t \le \lambda$. Moreover, we have

$$\beta(t,\phi(0)) \le f(t,\phi) \le \alpha(t)\phi(0)$$

for all $t \in J$, $\phi \in C([-a,0],\mathbb{R}^n_+)$.

(H3) There exist real numbers $k_2 > k_1 > 0$ such that $\int_0^T f(t, x_t)dt > 0$ with $\|x\|_J \le k_1$ or $\|x\|_J \ge k_2$.

Theorem 1. *Assume that the conditions* $(H1)$–$(H3)$ *hold, then* $BVP(1.1)$ *has at least two positive solutions* x, y *on* $[-a, T]$.

Proof. For any $x \in X_+$, from $x(0) = x(T)$ it follows that x can be uniquely extended to a $T-$periodic function on $\mathbb{R}^n$, written as x^*. Let $\tilde{x} = x^*|_{[-a,T]}$ and $x_t = \tilde{x}_t$ for each $t \in J$. It immediately follows that $x_t \in C([-a, 0], \mathbb{R}^n_+)$, $x_0 = x_T$, $x_t(0) = x(t)$, $\|x_t\| \le \|x\|_J$ and $t \to x_t$ is continuous for $t \in J$.

For any $x \in X_+$, define $(Fx)(t) = f(t, x_t) - \alpha(t)x(t)$ for $t \in J$. By (H2) we have that

$$\beta(t, x(t)) - \alpha(t)x(t) \le (Fx)(t) \le 0. \tag{2.3}$$

This inequality implies that

$$|(Fx)(t)| \le |\beta(t, x(t))| + \alpha(t)|x(t)| \le |\beta(t, x(t))| + \alpha(t)\|x\|_J. \tag{2.4}$$

From this it is easy to see that F is bounded.

Let $A = KF$ be a map from X_+ to X defined by

$$A(t, x) = \int_0^T G(t, s)(Fx)(s)ds$$

for $x \in X_+$ and $t \in J$. From (2.4) it follows that A is bounded. Moreover, for any $x \in X_+$, by (2.3) and $G(t, s) \le 0$, we obtain

$$A(t, x) = \int_0^T G(t, s)(Fx)(s)ds \ge 0.$$

This implies that $A(t, x) \in X_+$. It is easy to see that $A(t, x) \in AC(J, X)$. Thus, $AX_+ \subset AC(J, X) \cap X_+$.

For any bounded set $B \subset X_+$ and any $t, \tau \in [0, T]$ with $t < \tau$ and $x \in B$, let $Q = AB$, $m = \sup_{x\in B} \|x\|_J$, $q = \sup_{z\in Q} \|z\|_J$ and $z(t) = A(t, x) = \int_0^T G(t, s)(Fx)(s)ds$, then $z' = \alpha(t)z + Fx$. By means of (H2), there exists $\beta_1 \in L^1(J, \mathbf{R}_+)$ such that $|\beta(t, x)| \le m\beta_1(t)$ for any $x \in B$. By this and (2.4), we have

$$\begin{aligned} |z(\tau) - z(t)| &\le \int_t^\tau |z'(s)|ds \\ &\le \int_t^\tau [\alpha(s)\|z\|_J + |(Fx)(s)|\,]ds \\ &\le \int_t^\tau [(m + q)\alpha(s) + m\beta_1(s)]ds, \end{aligned}$$

which shows that $A(t, B)$ is equicontinuous on J. In virtue of Arzela-Ascoli lemma, it yields that Q is relatively compact in X, hence, A is a completely continuous map.

We claim that there exists a positive number σ with $\sigma \le \lambda\|\psi\|_J$ such that

$$x - Ax \ne 0 \tag{2.5}$$

for all $x \in X_+$ satisfying $\|x\|_J = \sigma$. Otherwise, for all $\sigma > 0$ there exists an $x \in X_+$ such that $x = Ax$, then the conclusion of Theorem 1 is proved.

Let us show all conditions of Lemma 1 are satisfied in several steps.

Step 1. We proceed to prove

$$x - Ax \ne t\psi \quad \text{for all } x \in \partial P_\sigma \text{ and } t \ge 0. \tag{2.6}$$

Suppose, on the contrary, there exist a $y \in \partial P_\sigma$ and $t_1 \ge 0$ such that $y - Ay = t_1\psi$. From (2.5) it follows that $t_1 > 0$. Clearly, $y = t_1\psi + Ay \ge t_1\psi$. This yields $t_1\|\psi\|_J \le \|y\|_J = \sigma \le \lambda\|\psi\|_J$, that is, $t_1 \le \lambda$. On the other hand, $Ay = y - t_1\psi$ guarantees that $L(y - t_1\psi) = Fy$, i.e.,

$$(y(t) - t_1\psi(t))' = f(t, y_t) - t_1\alpha(t)\psi(t).$$

By integrating this expression with respect to t, together with (H2), we obtain

$$\begin{aligned} 0 &= \int_0^T f(s, y_s)ds - t_1 \int_0^T \alpha(s)\psi(s)ds \\ &\ge \int_0^T \beta(s, y(s))ds - t_1 \int_0^T \alpha(s)\psi(s)ds \\ &\ge \int_0^T \beta(s, t_1\psi(s))ds - t_1 \int_0^T \alpha(s)\psi(s)ds \\ &= \int_0^T [\beta(s, t_1\psi(s)) - t_1\alpha(s)\psi(s)]ds > 0. \end{aligned}$$

This is a contradiction. Therefore, (2.6) holds.

Step 2. In virtue of (H3), for any positive number $\xi \le k_1$ we have

$$\int_0^T f(s, x_s)ds > 0 \tag{2.7}$$

for all $x \in P_\xi$. Given $0 < r < \min\{k_1, \sigma\}$, we proceed to prove that

$$Ax \ne \tau x \quad \text{for all } x \in \partial P_r \text{ and } \tau \ge 1. \tag{2.8}$$

Suppose, on the contrary, there exist $z \in \partial P_r$ and some $\tau_0 \ge 1$ such that $Az = \tau_0 z$. If $\tau_0 = 1$, then z is a fixed point of A. Otherwise, $\tau_0 > 1$, then $\tau_0 Lz = Fz$, i.e.,

$$z'(t) = \left(1 - \frac{1}{\tau_0}\right)\alpha(t)z(t) + \frac{1}{\tau_0}f(t, z_t).$$

By integrating this expression with respect to t we obtain

$$0 = \left(1 - \frac{1}{\tau_0}\right)\int_0^T \alpha(s)z(s)ds + \frac{1}{\tau_0}\int_0^T f(s, z_s)ds,$$

that is

$$\int_0^T f(s, z_s)ds = (1-\tau_0)\int_0^T \alpha(s)z(s)ds \le 0.$$

This contradicts (2.7). Hence, (2.8) is true. Consequently, Lemma 1 guarantees that A has a fixed point $x_1 \in P_\sigma \setminus \bar{P}_r$.

Step 3. By means of (H3) again, for any positive number $l \ge k_2$ and any $x \in \partial P_l$, we have $\int_0^T f(s, x_s)ds > 0$. Given $R > \max\{k_2, \sigma\}$, similar to step 2, we can prove

$$Ax \ne \tau x \quad \text{for all } x \in \partial P_R \text{ and } \tau \ge 1. \tag{2.9}$$

Hence, Lemma 1 guarantees that A has a fixed point $x_2 \in P_R \setminus \bar{P}_\sigma$. Obviously, x_1, x_2 are positive solutions of BVP(1.1) and this proof is completed. □

Example 1. Let $J = [0, 1]$, $h \in L^1(J, \mathbb{R}_+)$ with $h(t) > 0$, $g : \mathbb{R} \to [\varepsilon,\ \rho]$ with $0 < \varepsilon < \rho$ be a continuous function and

$$f(t,\phi) = \begin{cases} |\phi(0)|h(t)g(\phi(-a)), & |\phi(0)| \ge 1, \\ \dfrac{1}{3}(1+\phi(0))\left(\dfrac{1}{2}+\phi(0)\right)h(t)g(\phi(-a)), & -1 < \phi(0) < 1 \end{cases}$$

for $t \in J,\ \phi \in C([-a, 0], \mathbb{R}) =: C$, then f satisfies the condition (H1). Take

$$\alpha(t) = \rho h(t),\ \beta(t,\phi) = \begin{cases} \varepsilon h(t), & \phi(0) \ge 1, \\ \dfrac{\varepsilon}{6}h(t), & 0 \le \phi(0) < 1, \\ -\dfrac{\varepsilon}{16}h(t), & \text{others.} \end{cases}$$

Clearly, $\beta(t, \phi(0)) \le f(t, \phi) \le \alpha(t)\phi(0)$ for $t \in J$ and $\phi \in C([-a, 0], \mathbb{R}_+)$. Now, we take

$$\lambda = \frac{\varepsilon}{\rho}, \quad \psi(t) = \frac{1}{12}, \quad \text{for } t \in J.$$

Then $\psi \in X_+$. For any $0 < t_1 \le \lambda$, we have

$$\begin{aligned} &\int_0^1 [\beta(t, t_1\psi(t)) - t_1\alpha(t)\psi(t)]dt \\ &\quad \ge \int_0^1 \left[\frac{\varepsilon}{6}h(t) - \frac{\varepsilon}{12}h(t)\right]dt = \frac{\varepsilon}{12}\int_0^1 h(t)dt > 0. \end{aligned}$$

So, condition (H2) holds. Finally, we check that condition (H3) holds. In fact, it is easy to see that

$$\inf_{\phi \in C, |\phi(0)| \ge 1} f(t,\phi) > 0, \qquad \inf_{\phi \in C, |\phi(0)| \le 1/4} f(t,\phi) > 0.$$

By Fatou's theorem, we infer that condition (H3) holds with $k_1 = \frac{1}{4}$ and $k_2 = 1$.

Remark 1. Example 1 shows that f need not nonnegative.

3. Second order boundary value problems

In this section, we consider existence of solutions for BVP(1.2). A function $x \in C([-a,T],\mathbb{R}^n)$ is called the solution if $x_0 = \varphi$, $x(T) = \eta$, for any $t \in J$, $x'(t)$ exists and is absolutely continuous and BVP(1.2) is satisfied.

Let $X = \{x \in C^1(J,\mathbb{R}^n) : x(0) = x(T) = 0,\ x'(0) = x'(T)\}$ with the norm $\|x\|_X = \|x\|_J + \|x'\|_J$, $X_+ = \{x \in X : x \geq 0\}$, $Y = L^1(J,\mathbb{R}^n)$, $Z = \{x \in X : x'$ is absolutely continuous$\}$.

The following hypotheses will be used.

(H'1) The mapping $(t,\chi,u) \to f(t,\chi,u)$ is measurable with respect to t for each $(\chi,u) \in C([-a,0],\mathbb{R}^n) \times \mathbb{R}^n$, continuous with respect to (χ,u) for each $t \in J$.

(H'2) There exist $\alpha > 0$, a function $\beta \in L^1(J \times \mathbb{R}^n,\mathbb{R}^n)$ such that $\beta(t,\cdot)$ is bounded continuous and nondecreasing for each $t \in J$, a function $\psi \in X_+$ and real numbers $\lambda > 0$ such that

$$\int_0^T [\beta(s,t\psi(s)) - t\alpha(\psi(s)]ds > 0$$

for every $0 < t \leq \lambda$. Moreover, we have

$$\beta(t,\phi(0) - \xi(t)) \leq f(t,\phi,y) \leq \alpha[\phi(0) - \xi(t)]$$

for any $t \in J,\ \phi \in C,\ y \in \mathbb{R}^n,\ \phi(0) \geq \xi(t)$. Where $\xi(t) = \varphi(0) + \frac{t}{T}[\eta - \varphi(0)]$.

Theorem 2. *If the conditions* (*H'*1), (*H'*2) *and* (*H*3) *hold, then BVP*(1.2) *has at least two solutions* $x_2 > x_1$ *on* $[-a,T]$ *with* $x_1(t), x_2(t) \geq \xi(t)$ *(*$t \in J$*).*

Proof. First, let $z = x - \xi,\ \mu = \frac{\eta - \varphi(0)}{T}$, then BVP(1.2) is transformed into

$$z''(t) = f(t, z_t + \xi_t, z'(t) + \mu) := \tilde{f}(t, z_t, z'(t)) \quad t \in J,$$
$$z_0 = \varphi - \xi_0 := \hat{\varphi}, \quad z(T) = 0.$$

Here $\hat{\varphi} \in C$ and $\hat{\varphi}(0) = 0$. The condition (H'2) implies that $\tilde{f}(t,\phi,y) = f(t,\phi + \xi_t, y + \mu)$ satisfies

$$\beta(t,\phi(0)) \leq \tilde{f}(t,\phi,y) \leq \alpha\phi(0) \quad \phi \in C,\ \psi(0) \geq 0.$$

Since $x(t) \geq \xi(t)$ is equivalent to $z(t) \geq 0$, for the sake of convenience, we assume that $\varphi(0) = \eta = 0$, which shows that $\xi(t) \equiv 0,\ \mu = 0$.

Defining

$$L : Z \to Y, \quad x \to x'' - \alpha x,$$

where α is given in (H'2). Similar to the proof of Theorem 1, there exists the operator $K = L^{-1}$ defined by

$$(Ky)(t) = \int_0^T G(t,s)y(s)ds \quad \text{for } t \in J,\ y \in Y,$$

where Green's function $G(t,s)$ satisfies

$$G(t,s)\sqrt{\alpha}\sinh\sqrt{\alpha} = \begin{cases} \sinh\sqrt{\alpha}(t-T)\sinh\sqrt{\alpha}\,s, & s \le t, \\ \sinh\sqrt{\alpha}(s-T)\sinh\sqrt{\alpha}\,t, & s > t. \end{cases}$$

Next, for $x \in X_+$, $t \in J$, let

$$x_t(s) = \begin{cases} x(t+s), & \max\{-a,-t\} \le s \le 0, \\ \varphi(t+s), & -a \le s \le -t. \end{cases}$$

Since $x(0) = \varphi(0) = 0$, we have that $x_t \in C$ and $\|x_t\| \le \|x\|_J + \|\varphi\|$, $t \to x_t$ is continuous for $(t \in J)$.

For $x \in X_+$, define $F(t,x) = f(t, x_t, x'(t))) - \alpha x(t)$ for $t \in J$. (H'1) guarantees that $F(t,x)$ is measurable with respect to $t \in J$. From the condition (H'2) it follows that

$$\beta(t,x(t)) - \alpha x(t) \le F(t,x)) \le 0.$$

This implies that $|F(t,x)| \le |\beta(t,x(t))| + \alpha\|x\|_X$. This shows that $F : X_+ \to Y$ is bounded. Let $A = KF$ be a map from X_+ to X defined by

$$A(t,x) = KH(t,x) = \int_0^T G(t,s)F(s,x(s))ds$$

for $x \in X_+$ and $t \in J$. It is clear that A is bounded. Similar to Theorem 1 we can prove that $AX_+ \subset AC(J,X) \cap X_+$ and A is continuous and completely continuous.

Finally, we prove that A satisfies the all conditions of Lemma 1. We first prove that (2.6) holds. Suppose, on the contrary, there exist a $y \in \partial P_\sigma$ and $t_1 \ge 0$ such that $y - Ay = t_1\psi$. From (2.5) it follows that $t_1 > 0$. Similar to the proof of Theorem 1 we have

$$(y(t) - t_1\psi(t))'' = f(t, y_t, y'(t)) - t_1\alpha\psi(t).$$

By integrating this expression with respect to t, together with (H'2), we obtain

$$\begin{aligned} 0 &= \int_0^T f(s, y_s, y'(s))ds - t_1\int_0^T \alpha\psi(s))ds \\ &\ge \int_0^T \beta(s, y(s))ds - t_1\int_0^T \alpha\psi(s)ds \\ &\ge \int_0^T \beta(s, t_1\psi(s))ds - t_1\int_0^T \alpha\psi(s)ds \\ &= \int_0^T [\beta(s, t_1\psi(s)) - t_1\alpha\psi(s)]ds > 0. \end{aligned}$$

This is a contradiction. Therefore, (2.6) holds. In what follows, we prove that A satisfies (2.8). Suppose that this is not the case, then there exist $\tau_0 \geq 1,\ z \in \partial P_r$ such that $Az = \tau_0 z$. We can assume $\tau_0 > 1$, then $\tau_0 Lz = Fz$, i.e.,

$$z''(t) = \left(1 - \frac{1}{\tau_0}\right)\alpha z(t) + \frac{1}{\tau_0} f(t, z_t, z'(t)).$$

By integrating this expression with respect to t we obtain

$$0 = \left(1 - \frac{1}{\tau_0}\right)\int_0^T \alpha z(s)ds + \frac{1}{\tau_0}\int_0^T f(s, z_s, z'(s))ds,$$

that is

$$\int_0^T f(s, z_s, z'(s))ds = (1 - \tau_0)\int_0^T \alpha z(s)ds \leq 0.$$

On the other hand, (H3) shows $\int_0^T f(s, z_s, z'(s))ds > 0$ for $\|z\|_J \leq k_1$. This is a contradiction. In the same way, (2.9) is satisfied. By Lemma 1, A has two fixed points $x_1, x_2 \in X_+$, which is a solution to BVP(1.2). The proof is completed. $\square$

Example 2. Let $J = [0, 1]$. Consider the following BVP

$$\tag{3.1} \begin{cases} x'' = f(t, x_t, x'(t)), & a.e. t \in J, \\ x(1) = 0, \quad x(t) = te^t, & t \in [-a, 0] \end{cases}$$

with

$$f(t, \phi, u) = \begin{cases} |\phi(0)|h(t)g(\phi(-a), u), & |\phi(0)| \geq 1, \\ \dfrac{1}{3}(1 + \phi(0))\left(\dfrac{1}{2} + \phi(0)\right)h(t)g(\phi(-a), u), & -1 < \phi(0) < 1 \end{cases}$$

for $t \in J,\ \phi \in C([-a, 0], \mathbb{R}) =: C$ and $u \in \mathbb{R}$, where, $h \in C^1(J, \mathbb{R}_+)$ with $0 < h(t) < H$, $g : \mathbb{R} \times \mathbb{R} \to [\varepsilon,\ \rho]$ with $0 < \varepsilon < \rho$, H, ε, ρ are constants and g is a continuous function.

Conclusion. BVP (3.1) has at least two solutions x_1, x_2 with $x_2 > x_1 \geq 0$ on $[-a, 1]$.

Proof. f satisfies the condition (H'1). Trivial. Take

$$\alpha(t) = \rho H,\ \beta(t, \phi) = \begin{cases} \varepsilon t(1 - t)h(t), & \phi(0) \geq 1, \\ \dfrac{\varepsilon}{6} t(1 - t)h(t), & 0 \leq \phi(0) < 1, \\ -\dfrac{\varepsilon}{16} t(1 - t)h(t), & \text{others.} \end{cases}$$

Clearly, $\xi(t) = 0$, hence, $\beta(t, \phi(0)) \leq f(t, \phi, u) \leq \alpha\phi(0)$ for $t \in J$ and $\phi \in C([-a, 0], \mathbb{R}_+)$. Now, we take

$$\lambda = \frac{\varepsilon}{\rho}, \quad \psi(t) = \frac{1}{4H} t(1 - t)h(t), \quad \text{for } t \in J.$$

Then $\psi \in X_+$. For any $0 < t_1 \leq \lambda$, we have

$$\begin{aligned}&\int_0^1 [\beta(t, t_1\psi(t)) - t_1\alpha\psi(t)]dt \\ &\quad \geq \int_0^1 \left[\frac{\varepsilon}{6}h(t) - \frac{\varepsilon}{12}h(t)\right] t(1-t)dt = \frac{\varepsilon}{12}\int_0^1 t(1-t)h(t)dt > 0.\end{aligned}$$

So, condition (H'2) holds. Finally, similar to Example 1, we can check that condition (H3) holds. Theorem 2 guarantees that the conclusion is true. □

Corollary 1. *Let $a = 0$ and $F : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. If F satisfies the conditions (H'1) and (H3). In addition, There exist $\alpha > 0$, a function $\beta \in L^1(J \times \mathbb{R}^n, \mathbb{R}^n)$ such that $\beta(t, \cdot)$ is bounded continuous and nondecreasing for each $t \in J$, a function $\psi \in X_+$ and real numbers $b > 0$ such that*

$$\int_0^T [\beta(s, t\psi(s)) - t\alpha(\psi(s)]ds > 0$$

for every $0 < t \leq b$. Moreover, we have

$$\beta(t, x - \xi(t)) \leq f(t, x, y) \leq \alpha[x - \xi(t)],$$

where $t \in J$, $x, y \in \mathbb{R}^n$ with $x \geq \xi(t)$, $\xi(t) = \eta + \frac{t}{T}[\mu - \eta]$. Then second order ordinary differential equations

$$\begin{cases} x''(t) = f(t, x(t), x'(t)) \ (t \in J), \\ x(0) = \eta, \ x(T) = \mu, \end{cases}$$

has at least two solutions $x_1, x_2 \in C^1(J, \mathbb{R}^n)$, with $x_1(t), x_2(t) \geq \xi(t)$ on J.

□

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