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The Penrose transform on conformally Bochner-Kähler manifolds

By

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Abstract

We give a generalization of the Penrose transform on Hermitian manifolds with metrics locally conformally equivalent to Bochner-Kähler metrics. We also give an explicit formula for the inverse transform.

Introduction

The twistor space of a four-dimensional Riemannian manifold was first introduced by Atiyah, Hitchin and Singer in [A.H.S]. The Penrose transform on it was given by Hitchin in [H]. The inverse Penrose transform was given explicitly by Woodhouse in [W].

In [O.R], O'Brian and Rawnsley extended the notion of twistor space to other manifolds such as even dimensional Riemannian manifolds, Hermitian manifolds, Quaternionic Kähler manifolds, etc. In [M], Murray gave the Penrose transform on twistor spaces of even dimensional Riemannian manifolds. An explicit formula for the inverse Penrose transform in this case was given by the author in [I4].

As for twistor spaces of Hermitian manifolds, the author gave the Penrose transform and its inverse transform on the twistor spaces of $\mathbb{C}^n$ in his unpublished work [I2]. They related a certain cohomology group on a twistor space to the space of harmonic forms of the Dolbeault complex. An explicit formula of the inverse transform was also given.

In the present paper, we extend the result of [I2] to Hermitian manifolds whose metrics are locally conformally equivalent to Bochner-Kähler metrics, which we call *conformally Bochner-Kähler manifolds*. Since the Penrose transform involves cohomology groups of positive degree, the integrability condition for the almost complex structure of the twistor space is inevitable. Hence it is a complete generalization of [I2] except low dimensional examples of non-integrable almost Hermitian manifolds, such as S^6 with a well-known almost complex structure.

Since we have no reason to exclude the conformally flat case, all results and proofs in [I2] are contained in this paper.

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Let X be a conformally Bochner-Kähler manifold of dimension $n > 2$. Let k be an integer between 0 and n. Then the twistor space $Z_k(X)$ is a complex manifold ([O.R], [I3]). Let H be the hyperplane bundle as a Grassmannian bundle. Let h be a non-negative integer and V a vector bundle with connection on X such that $H^{-n-h} \otimes p^*V$ has a $(1,0)$-connection where $p : Z_k(X) \to X$ is the projection map. The trivial bundle does not satisfies this condition if the metric of X is not locally conformally equivalent to a flat metric, which is quite different in the Riemannian case. The Penrose transform is a one-to-one correspondence between the cohomology group $H^{k(n-k)}(Z_k(X), \mathcal{O}(H^{-n-h} \otimes p^*V))$ and the solution space of the Laplacian $\mathcal{D}_0$ defined in Definition 2.2 if $h = 0$, or the space of harmonic forms of the Dolbeault complex:

$$\begin{aligned}&\Gamma(\wedge^{0,k-1} X \otimes \hat{S}^{h-1} \wedge^{0,k} X \otimes K_X^{-k} \otimes V) \\ &\qquad \xrightarrow{\bar\partial} \Gamma(\hat{S}^h \wedge^{0,k} X \otimes K_X^{-k} \otimes V) \xrightarrow{\bar\partial} \\ &\qquad\qquad \Gamma(\wedge^{0,k+1} X \otimes \hat{S}^{h-1} \wedge^{0,k} X \otimes K_X^{-k} \otimes V),\end{aligned}$$

if $h > 0$, where $\hat{S}^h \wedge^{0,k}$ is the principal component of $S^h \wedge^{0,k}$ defined in Definition 1.5 (Theorem 3.2 and Theorem 4.7).

The explicit formula for the inverse Penrose transform given in Definition 4.6 has quite similar form as in the Riemannian case ([I4]). Namely put

$$F(x) = \sum_{i=0}^{\infty} \frac{x^i}{(i!)^2},$$

and

$$\begin{aligned}f(x) &= (k+h-1)! F^{(k+h-1)}(x), \\ g(x) &= (n-k+h-1)! F^{(n-k+h-1)}(x).\end{aligned}$$

Let j, D_α, D_β and D_γ be operators defined in Section 4. Then the inverse Penrose transform is given by:

$$\begin{array}{rccc}\mathcal{Q}: & \Gamma(X, \hat{S}^h \wedge^{0,k} X \otimes K_X^{-k} \otimes V) & \longrightarrow & \Omega^{0,k(n-k)}(Z_k(X), H^{-n-h} \otimes p^*V) \\ & \phi & \longmapsto & f(D_\beta + D_\gamma) g(D_\alpha) j(\phi)\end{array}$$

where the expression is well-defined because D_α, D_β and D_γ are even operators and D_β and D_γ are mutually commutative.

Let us explain briefly the contents of this paper. We define in Section 1 the twistor spaces of an almost Hermitian manifold and study their integrability conditions. In Section 2 we give the field equations on conformally Bochner-Kähler manifolds which appear in the Penrose transform. We also give a few formulas about the curvature tensor which are used in Section 4. In Section 3 we define the Penrose transform on conformally Bochner-Kähler manifolds. In Section 4 we prove the surjectivity of the Penrose transform by constructing explicitly the inverse correspondence.

1. The twistor spaces of an almost Hermitian manifold and their integrability conditions

Let X be an n-dimensional almost Hermitian manifold, that is, X is a $2n$-dimensional Riemannian manifold with a compatible almost complex structure. In this section we define twistor spaces $Z_k(X), k = 0, \dots, n$ of X as submanifolds of the Riemannian twistor space $Z(X)$. It is modification of the definition by O'Brian and Rawnsley in [O.R].

Let

$$\Delta = \langle \theta_I \mid I < (1, \dots, n) \rangle_{\mathbb{C}}$$

be a spin module, where $I < (1, \dots, n)$ means that I is a subsequence of $(1, \dots, n)$. We introduce similar notation for multi-indices as in [I4]. We regard multi-indices $I, J, \dots$ as finite sequences of possibly duplicate elements of $\{1, \dots, n\}$, and denote by IJ the composition of sequences I and J. Let us define relations among θ_I's as:

$$\begin{aligned} \theta_{I_1 aa I_2} &= -\theta_{I_1 I_2}, \\ \theta_{I_1 ab I_2} &= -\theta_{I_1 ba I_2}, \quad \text{for } a \neq b. \end{aligned}$$

Then, for any multi-index I, there is a unique subsequence I_0 of $(1, \dots, n)$ such that $\theta_I = \theta_{I_0}$ or $\theta_I = -\theta_{I_0}$. If a multi-index I is regarded as a set, it is meant as the set of numbers contained in I_0. Let $|I|$ denote the length of I_0.

With this notation, we define an action of $\mathbb{R}^{2n} = \langle f_i \mid i = 1, \dots, 2n \rangle_{\mathbb{R}}$ on Δ as follows. For $a = 1, \dots, n$,

$$\begin{aligned} f_a \theta_I &= \theta_{aI}, \\ f_{n+a} \theta_I &= \begin{cases} \sqrt{-1}\theta_{aI}, & \text{if } a \notin I, \\ -\sqrt{-1}\theta_{aI}, & \text{if } a \in I. \end{cases} \end{aligned}$$

This action is canonically extended to a $\mathrm{CLIF}(\mathbb{R}^{2n})$-action on Δ. Since $\mathfrak{spin}\,(2n)$ is a subspace of $\mathrm{CLIF}(\mathbb{R}^{2n})$, we have a $\mathfrak{spin}\,(2n)$-action on Δ. (See [I1] for details.)

If we identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ by the complex structure defined by

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

then $\mathbb{R}^{2n} \otimes \mathbb{C}$ is decomposed into two subspaces:

$$\begin{aligned} \mathbb{C}^n &\simeq \left\langle e_a = \frac{1}{2}(f_a - \sqrt{-1} f_{n+a}) \mid a = 1, \dots, n \right\rangle_{\mathbb{C}}, \\ \bar{\mathbb{C}}^n &\simeq \langle e_{\bar{a}} = \overline{e_a} \mid a = 1, \dots, n \rangle_{\mathbb{C}}. \end{aligned}$$

Actions of these vectors on Δ are computed as:

$$e_a \theta_I = \begin{cases} \theta_{aI}, & \text{if } a \notin I \\ 0, & \text{if } a \in I \end{cases} \qquad e_{\bar{a}} \theta_I = \begin{cases} 0, & \text{if } a \notin I \\ \theta_{aI}, & \text{if } a \in I \end{cases} \tag{1.1}$$

If we consider $\mathfrak{u}(n)$ as a subspace of $\mathrm{CLIF}(\mathbb{R}^{2n})$, we have

$$\mathfrak{u}(n) \subset \langle e_a \cdot e_{\bar{b}} \mid a, b = 1, \ldots, n\rangle_{\mathbb{C}}.$$

Hence the irreducible decomposition of Δ as a $\mathfrak{u}(n)$-module is given by:

$$\Delta = \bigoplus_{k=0}^{n} \Delta^k, \qquad \Delta^k = \langle \theta_I \mid |I| = k\rangle_{\mathbb{C}},$$

where we have

$$\Delta^k \simeq (\textstyle\bigwedge^k \mathbb{C}^n) \otimes (\bigwedge^n \mathbb{C}^n)^{-1/2}. \tag{1.2}$$

We fix an integer k such that $0 \leq k \leq n$. Let Z denote the $\mathrm{PIN}(2n)$-orbit of $[\theta_\emptyset] \in \mathbf{P}(\Delta)$, which is a complex submanifold of $\mathbf{P}(\Delta)$. Now we study the submanifold

$$Z_k = Z \cap \mathbf{P}(\Delta^k).$$

We see that Z_0 and Z_n are one point corresponding to the original complex structure and its complex conjugate structure, respectively. Henceforth we assume $0 < k < n$.

To simplify the expression, we stretch the rules of notation of multi-indices as follows. Let $(Z^I)_{I<(1,\ldots,n)}$ be a coordinate system corresponding to $(\theta_I)_I$. Then, by regarding it as a cospinor, we can define Z^{aI} in a similar way. Since we want to consider Z^{aI} as a homogeneous coordinate of $\mathbf{P}(\Delta^k)$, the length of the multi-index aI is significant. Therefore we write aI (resp. $\bar{a}I$) if $|aI| = |I| + 1$ (resp. $|aI| = |I| - 1$). In the same way, we write $\bar{a}\bar{I}$ (resp. $a\bar{I}$) if $|aI| = |I| + 1$ (resp. $|I| - 1$). These rules are inspired by the formula (1.1) for Clifford multiplication. Owing to it, as we will see below, we can omit the range of summation in most cases.

Let J, K be multi-indices of length $|J| = k + 1$, $|K| = k - 1$. By [I4, Lemma 2.6 (1)], we have

$$\sum_{a \in J \setminus K} Z^{\bar{a}J} Z^{aK} + \sum_{a \in K \setminus J} Z^{aJ} Z^{\bar{a}K} = 0,$$

on Z. Since the second term vanishes on $\mathbf{P}(\Delta^k)$, we have

$$Z^{\bar{a}J} Z^{aK} = 0, \tag{1.3}$$

on Z_k, which is nothing but the Plücker relation. Therefore Z_k coincides with the Grassmannian manifold $G_{k,n}$.

Hence we consider a diagram:

$$\begin{array}{ccc} G_{k,n} & \longrightarrow & \mathbf{P}(\Delta^k) \\ \downarrow & & \downarrow \\ Z & \longrightarrow & \mathbf{P}(\Delta) \end{array} \tag{1.4}$$

where upper (lower) horizontal array is a $\mathrm{U}(n)$- ($\mathrm{O}(2n)$-) equivariant map.

Let X be an n-dimensional almost Hermitian manifold. Let P denote the principal bundle of X. Then we define the kth twistor space as:

$$Z_k(X) = P \times_{\mathrm{U}(n)} G_{k,n}.$$

Let p denote the projection map:

$$p : Z_k(X) \longrightarrow X.$$

If X has a spin structure, we can define the spin-hyperplane bundle H_0 over $Z_k(X)$ by pulling back the hyperplane bundle over $Z(X)$ (or equivalently $\mathbf{P}(\Delta^k(X))$). On the other hand, if we regard $Z_k(X)$ as a Grassmannian bundle $G_k(T^{(1,0)}X)$, it is natural to consider the Grassmannian-hyperplane bundle H, which can be defined even if X does not have a spin structure. Hence we use it to study the Penrose transform and call it the hyperplane bundle of $Z_k(X)$. By (1.2), two line bundles are related by the isomorphism:

$$H \simeq H_0 \otimes p^* K_X^{\frac{1}{2}},$$

where K_X is the canonical bundle of X.

By (1.4), $Z_k(X)$ is a subbundle of $Z(X)$. Since $G_{k,n}$ is a complex submanifold of Z, we can define an almost complex structure of $Z_k(X)$ similarly as in the case of $Z(X)$.

We take a local orthonormal frame $(e_a)_a$ of $T^{(1,0)}X$. Let Z^J denote a corresponding Plücker coordinate of $Z_k(X)$. Then we can define local fiber coordinates of $Z_k(X)$ as follows. Let I be an index of length k. Then

$$w_{ij} = Z^{\bar{\imath}jI}/Z^I, \qquad i \in I, \quad j \notin I$$

are local fiber coordinates on

$$U_I = \{(Z^J)_J \in G_{k,n} \mid Z^I \neq 0\}.$$

We denote the center of the coordinate chart U_I as z_0. For a general multi-index J of length k, let z^J be the function on U_I defined as $z^J = Z^J/Z^I$. Let ω^J_K be the connection form of $\bigwedge^k T^{(1,0)}X$ induced by the Levi-Civita connection. Let $(e^a)_a$ be the dual frame of $\bigwedge^{1,0} X$.

By [I4, Lemma 2.9], the $(1,0)$-forms of $Z_k(X)$ can be explicitly defined as follows.

Definition 1.1. The total space of $H^{*\times} \subset \bigwedge^k T^{(1,0)}X$ has an almost complex structure whose space of $(1,0)$-forms is spanned by

$$\begin{aligned} Z^{aJ} e^{\bar{a}}, &\qquad |J| = k-1, \\ Z^{\bar{a}J} e^{a}, &\qquad |J| = k+1, \\ dZ^J + \omega^J_K Z^K, &\qquad |J| = k. \end{aligned}$$

This induces an almost complex structure on $Z_k(X)$ whose space of $(1,0)$-forms is spanned by

$$\begin{aligned} &z^{aJ}e^{\bar a}, && |J| = k-1, \\ &z^{\bar a J}e^{a}, && |J| = k+1, \\ &\widehat{dw_{ij}}, && i \in I, \quad j \notin I, \end{aligned}$$

where

$$\widehat{dw_{ij}} = dw_{ij} + z^J \omega_J^{\bar\imath j I} - w_{ij} z^J \omega_J^I.$$

Remark. The integrability of $Z_k(X)$ does not implies that of $H^{*\times}$ in general. Hence we should twist the hyperplane bundle by the pull-back of a non-trivial vector bundle on X satisfying a certain condition on the curvature. The condition shall be calculated in the next section.

Since the Penrose transform in the two-dimensional (i.e. real dimension four) case is already given by Hitchin in [H], we consider henceforth the case $n > 2$.

O'Brian and Rawnsley considered in [O.R] an almost complex structure defined as above but use a general connection which makes the almost complex structure of X and the metric tensor of X covariant constant. To avoid unnecessary generality, we restrict ourselves to consider the Levi-Civita connection. Then the extra conditions on the connection do not seem very good, since it is not conformally invariant. We want to avoid this by considering instead the condition that $Z_k(X)$ is an almost complex submanifold of $Z(X)$.

First, we give a definition of Bochner-Kähler manifolds. Let X be a Kähler manifold. Let $(e_a)_a$ be a local orthonormal frame of the holomorphic tangent bundle $T^{(1,0)}X$. Let $(e^a)_a$ and $(e^{\bar a})_{\bar a}$ be the dual frame and its complex conjugate frame, respectively. Let

$$R_a^b e_b = R_{c\bar d a}^b e_b \otimes e^c \wedge e^{\bar d}$$

be the curvature tensor of $T^{(1,0)}X$ induced by the Levi-Civita connection. Put

$$R_{a\bar b c\bar d} = \frac{1}{2} R_{a\bar b c}^d.$$

The Ricci curvature and the scalar curvature are defined as

$$\begin{aligned} \mathcal{R}_{a\bar b} &= \frac{2}{n+2} R_{a\bar b c\bar c}, \\ r &= \frac{1}{n+1} \mathcal{R}_{a\bar a}. \end{aligned}$$

The multiplicative constants in the definitions are just for simplicity.

The curvature tensor of a Kähler manifold has three irreducible components: the scalar curvature, the traceless Ricci curvature and the Bochner curvature. Hence if the Bochner tensor of a manifold vanishes, its curvature tensor can be written explicitly by the scalar and Ricci curvatures. Therefore, we can define a Bochner-Kähler manifold by that explicit formula as in [Ka].

Definition 1.2. A Kähler manifold X is called *a Bochner-Kähler manifold* if its curvature tensor is written as:

$$R_{a\bar{b}c\bar{d}} = \frac{1}{2}(\mathcal{R}_{a\bar{b}}\delta_c^d + \mathcal{R}_{a\bar{d}}\delta_c^b + \mathcal{R}_{c\bar{b}}\delta_a^d + \mathcal{R}_{c\bar{d}}\delta_a^b) - \frac{1}{2}r(\delta_a^b\delta_c^d + \delta_a^d\delta_c^b).$$

Now we return to consider a general almost Hermitian manifold X. We say that an almost Hermitian manifold is *conformally Bochner-Kähler* if it is a complex manifold and its metric is locally conformally equivalent to a Bochner-Kähler metric.

Definition 1.3. The pair (n, k) is called *exceptional* if it is equal to $(3, 1)$, $(3, 2)$ or $(4, 2)$.

Essential part of the following theorem was given by O'Brian and Rawnsley in [O.R].

Theorem 1.4. *Let X be an n-dimensional almost Hermitian manifold with $n > 2$. Let k be an integer between 0 and n. Suppose that $Z_k(X)$ is a complex manifold and also an almost complex submanifold of $Z(X)$.*

(1) *If (n, k) is not exceptional, then X is a conformally Bochner-Kähler manifold.*

(2) *Suppose (n, k) is exceptional and X is a complex manifold. Then X is a conformally Bochner-Kähler manifold.*

Proof. We give the condition at z_0 explicitly.

The condition that $Z_k(X)$ is an almost complex submanifold of $Z(X)$ is

$$\omega_I^{\bar{a}\bar{b}I}, \omega_I^{abI} \in \wedge_{z_0}^{1,0},$$

where ω_K^J is meant as the connection form of $\Delta(X)$ and $\wedge_{z_0}^{1,0}$ is the space of $(1, 0)$-forms at z_0. Let

$$T_{c,b}^{\bar{a}} e^c + T_{\bar{c},b}^{\bar{a}} e^{\bar{c}} = \langle e^{\bar{a}}, \omega(e_b) \rangle$$

be torsion tensors. Note that this is *not* the torsion as a connection of TX, which vanishes because we consider the Levi-Civita connection. Then the condition is rewritten as:

$$T_{c,b}^{\bar{a}} = 0, \qquad a, b, c \in I, \quad a \neq b, \tag{1.5}$$

$$T_{\bar{c},b}^{\bar{a}} = 0, \qquad a, b \in I, \quad c \notin I, \quad a \neq b, \tag{1.6}$$

$$T_{\bar{c},b}^{\bar{a}} = 0, \qquad a, b \notin I, \quad c \in I, \quad a \neq b, \tag{1.7}$$

$$T_{c,b}^{\bar{a}} = 0, \qquad a, b, c \notin I, \quad a \neq b. \tag{1.8}$$

By (1.6) and (1.7), $T_{\bar{c},b}^{\bar{a}} = 0$ for distinct a, b and c. This means that there is a tensor $(t_a)_a$ such that

$$T_{\bar{c},b}^{\bar{a}} = \delta_a^c t_b - \delta_b^c t_a.$$

By (1.5) and (1.8), we have

$$T^{\bar{a}}_{c,b} = 0,$$

if $k \geq 3$ or $n - k \geq 3$, which is satisfied if and only if (n, k) is not exceptional. This tensor is nothing but the Nijenhuis tensor. Hence its vanishing implies precisely that X is a complex manifold.

Therefore, under the assumption of the theorem, for the Kähler form Ω, we have

$$d\Omega = 4t \wedge \Omega,$$

where $t \in \Gamma(T^*X)$ is the *real* section corresponding to $t_a e^a$. Then $dd\Omega = 0$ implies that t is a closed form and there exists a locally defined scalar function s such that $ds = t$. Hence the Hermitian metric obtained from the original metric by multiplying $\exp(-4s)$ is torsion free.

This implies that X is a complex manifold and its metric is locally conformally equivalent to a Kähler metric.

Hence we can assume that X is a Kähler manifold and the connection form of $T^{(1,0)}X$ vanishes at a point $x_0 \in X$. Then we have

$$d\widehat{dw_{ij}}\Big|_{(x_0,z_0)} = \frac{1}{2} R^j_i. \tag{1.9}$$

Hence the integrability condition of the almost complex structure of $Z_k(X)$ at (x_0, z_0) is

$$R^j_{k,\bar{l},i} = 0, \qquad i, k \in I, \quad j, l \notin I.$$

This implies precisely the vanishing of the Bochner tensor by Definition 1.2. $\square$

Example. In the exceptional case, S^6 with the almost complex structure induced by the Cayley algebra has the integrable twistor spaces $Z_k(S^6)$, $k = 1, 2$, which were described in [O.R]. The twistor space $Z_+(S^6)$ is identified with the homogeneous space $\mathrm{SO}(7)/\mathrm{U}(3)$. The twistor space $Z_2(S^6)$ is identified with $G_2/\mathrm{U}(2)$ where G_2 is the subgroup of SPIN(7) which consists with transformations preserving the almost complex structure of S^6.

We give here an explicit description of $Z_2(S^6)$ as a submanifold of $Z_+(S^6)$. The space $Z_+(S^6)$ is identified with the six-dimensional complex hyperquadric:

$$Q_6 = \{[Z^I]_{I<(1,2,3,4)} \mid Z^{\emptyset} Z^{1234} - Z^{12} Z^{34} + Z^{13} Z^{24} - Z^{14} Z^{23} = 0\}$$

(see [I1] for detail). The defining equation of $Z_2(S^6) \subset Q_6$ obviously depends on the choice of the almost complex structure of S^6, or equivalently the embedding $G_2 \subset \mathrm{SPIN}(7)$. For example, if G_2 is the isotropic subgroup at the (co)spinor $\theta^{\emptyset} + \theta^{1234}$, $Z_2(S^6)$ can be written as

$$Z_2(S^6) = \{[Z^I] \in Q_6 \mid Z^{\emptyset} + Z^{1234} = 0\}.$$

Hence it is a five-dimensional complex hyperquadric.

When (n, k) is $(4, 2)$, the product of the six-sphere and the hyperbolic plane with standard almost Hermitian structure gives an exceptional example, which is described in [I5].

The almost complex structure of $Z_k(X)$ can also be defined by using the distribution of some first order differential operator. This gives immediate correspondences between 0-th cohomology groups and some field equations as follows.

Definition 1.5. Let E_1 and E_2 be irreducible $\mathrm{U}(n)$-modules. For $i = 1, 2$, let $v_i \in E_i$ be a highest weight vector with weight λ_i. Then we write $E_1 \hat{\otimes} E_2$ as a unique irreducible submodule of $E_1 \otimes E_2$ having a highest weight vector $v_1 \otimes v_2$ with weight $\lambda_1 + \lambda_2$. Similarly we write $\hat{S}^h E_1$ as a unique irreducible submodule of $S^h E_1$ with highest weight $h\lambda_1$.

Put $E = \wedge^{k,0} \hat{\otimes} \wedge^{1,0} \oplus \wedge^{k,0} \hat{\otimes} \wedge^{0,1}$. Then, by the Littlewood-Richardson rule for the irreducible decomposition of the tensor representation, we have

$$\wedge^{k,0} \otimes (\wedge^{1,0} \oplus \wedge^{0,1}) = \wedge^{k-1,0} \oplus \wedge^{k+1,0} \oplus E.$$

By composition of the covariant derivative and the projection, we define a first order differential operator:

$$\bar{\mathcal{D}} : \Gamma(\wedge^{k,0} X) \longrightarrow \Gamma(E(X)).$$

Then the distribution $V(\bar{\mathcal{D}})$ of $T\mathbf{P}(\wedge^k T^{(1,0)}(X)) \otimes \mathbb{C}$ defined in [A.H.S] and [I1] has minimum rank on $Z_k(X)$ and gives on it the almost complex structure of Definition 1.1.

Let V be a vector bundle on X with connection. Let h be a non-negative integer. As in [H, §2] and [I1, §9], we have an immediate correspondence between $H^0(Z_k(X), \mathcal{O}(H^h \otimes V))$ and the solution space of the equation $\bar{\mathcal{D}}_h \phi = 0$ where $\bar{\mathcal{D}}_h$ is the differential operator induced by the covariant derivative similarly as $\bar{\mathcal{D}} = \bar{\mathcal{D}}_1$.

$$\bar{\mathcal{D}}_h : \Gamma(\hat{S}^h \wedge^{k,0} X \otimes V) \longrightarrow \Gamma((\wedge^{1,0} X \hat{\otimes} \hat{S}^h \wedge^{k,0} X \oplus \wedge^{0,1} X \hat{\otimes} \hat{S}^h \wedge^{k,0} X) \otimes V).$$

Note that the correspondence is valid even if $Z_k(X)$ is not a complex manifold.

2. Field equations on a conformally Bochner-Kähler manifold

Let n be an integer greater than two. Let X be an n-dimensional conformally Bochner-Kähler manifold. We fix an integer k such that $0 < k < n$. In this section, we introduce field equations on X which appear in the Penrose transform.

Since the construction of the field equations are local, we can assume that the metric of X is a Bochner-Kähler metric. We note that the Kähler condition determines a metric in the conformal class up to a locally constant scalar factor. Therefore the situation is much simpler than that in the case of Riemannian manifolds ([I4]).

Our main concern is a relation between the solution space of a field equation and a cohomology group of positive degree with coefficients in the vector bundle $H^{-n-h} \otimes p^*V$, where h is a non-negative integer and V is a vector bundle on X with connection. Hence the vector bundle $H^{-n-h} \otimes p^*V$ should have a natural holomorphic structure.

Now we give an explicit condition on V and compute the curvature tensor of $\hat{S}^h \bigwedge^{0,k} X \otimes K_X^{-k} \otimes V$, on which the field equation is defined.

Let R_a^b be the curvature tensor of the holomorphic tangent bundle of X. Then the curvature tensor of $\bigwedge^{0,k} X$ is

$$(R\psi)_{\bar{I}} = -R_a^b \psi_{\bar{a}\bar{b}\bar{I}}.$$

By the definition of the Ricci tensor, the curvature tensor of the canonical bundle K_X is

$$-(n+2)\mathcal{R}_{a\bar{b}} e^a \wedge e^{\bar{b}}. \tag{2.1}$$

Now we compute the condition on V. Let R_J^I be the curvature tensor of $\bigwedge^k T^{(1,0)}X$. Then, by Definition 1.2, we have

$$R_I^I = \sum_{a \in I} R_a^a = (k+1)\mathcal{R}_{a\bar{b}} e^a \wedge e^{\bar{b}} + (1,1)\text{-forms at } z_0.$$

If a two-form is of type $(1,1)$ for all almost complex structures corresponding to points of $G_{k,n}$, then it should be a scalar multiple of the Kähler form. Here we use the assumption $n > 2$. Hence we have proved the following lemma.

Lemma 2.1. *Let V' be a vector bundle on X with connection. Then, for an integer l, the vector bundle $H^l \otimes p^*V'$ has a natural holomorphic structure if and only if the curvature of V' is written as*:

$$l(k+1)\mathcal{R}_{a\bar{b}} e^a \wedge e^{\bar{b}} + e^a \wedge e^{\bar{a}} \mathcal{L},$$

where $\mathcal{L}$ is an endomorphism of V'.

Remark. By (2.1), we can use $H^l \otimes p^* K_X^{-l\frac{k+1}{n+2}}$ as a locally defined holomorphic line bundle.

Example. We consider the condition that $H^{-n-h} \otimes p^*V$ is a holomorphic line bundle when $X = \mathbf{H}^l \times \mathbf{P}^{n-l}$ and $V = L(a,b) = p_1^*(H^a) \otimes p_2^*(H^b)$, where p_i is the projection to the ith component. We have

$$K_X \simeq L(-(l+1), -(n-l+1)).$$

Since the curvature tensor of $L(1,-1)$ is a scalar multiple of the Kähler form of X, by (2.1), the condition for V is

$$a + b = -(n+h)(k+1).$$

Hence we have

$$K_X^{-k} \otimes V = L(a', b'), \qquad a' + b' = -h(k+1) - (n-2k).$$

Suppose that the vector bundle V has the curvature such that $H^{-n-h} \otimes p^*V$ has a holomorphic structure. Then, for $\phi \in \Gamma(\hat{S}^h \wedge^{0,k} \otimes K_X^{-k} \otimes V)$, the curvature tensor is written as:

(2.2)

$$(R\phi)_{\bar{I}_1,\dots,\bar{I}_h} = -\sum_{j=1}^{h} R^a_{\bar{b}} \phi_{\bar{I}_1,\dots,\bar{b}a\bar{I}_j,\dots,\bar{I}_h} - (h(k+1)+n-2k)\mathcal{R}_{a\bar{b}} e^a \wedge e^{\bar{b}} \phi_{\bar{I}_1,\dots,\bar{I}_h}$$
$$+ e^a \wedge e^{\bar{a}} (\mathcal{L}\phi)_{\bar{I}_1,\dots,\bar{I}_h},$$

where $\mathcal{L}$ is an endomorphism of V.

We assume for a while that $h \geq 1$. Let $\mathcal{D}_h$ denote the Dirac operator. We obtain its harmonic section from a harmonic form of $\bar{\partial} + \bar{\partial}^*$ by the following commutative diagram:

$$\begin{array}{ccc} \Gamma(\hat{S}^h \wedge^{0,k} \otimes K_X^{-k}) & \xrightarrow{\bar{\partial}+\bar{\partial}^*} & \Gamma((\wedge^{0,k-1} \oplus \wedge^{0,k+1}) \otimes \hat{S}^{h-1} \wedge^{0,k} \otimes K_X^{-k}) \\ \big\downarrow & & \big\downarrow \\ \Gamma(S^h \Delta^{\sigma(k)} X \otimes K_X^{-\frac{h}{2}-k}) & \xrightarrow{\mathcal{D}_h} & \Gamma(\Delta^{\sigma(k+1)} \otimes S^{h-1} \Delta^{\sigma(k)} X \otimes K_X^{-\frac{h}{2}-k}) \end{array}$$

where we omit V and $\sigma(k)$ is $+$ (resp. $-$) if k is even (resp. odd). This diagram also shows that $\bar{\partial}^*$ can be defined as a global operator if we restrict the metric locally chosen to be Kähler. Henceforth we use the operator $\bar{\partial}^*$ in this sense.

For a positive integer a, we consider

$$\nabla^a : \Gamma(V) \longrightarrow \Gamma(V \otimes \textstyle\bigotimes^a T^*X),$$

where the connection of T^*X is the one induced from the Levi-Civita connection. Evaluation by tangent vectors is defined inductively as:

$$\langle \nabla^a \phi, e_a \dots e_1 \rangle = \langle \nabla_{e_1} \nabla^{a-1} \phi, e_a \dots e_2 \rangle.$$

Let ϕ be a harmonic form of $\hat{S}^h \wedge^{0,k} X \otimes K_X^{-k} \otimes V$. Then by considering it as a harmonic section of $S^h \Delta^{\sigma(k)} X \otimes K_X^{-\frac{h}{2}-k} \otimes V$, we have

$$\begin{aligned} \frac{1}{4}(\mathcal{D}_h \mathcal{D}_h \phi)_{\bar{I},\dots,\bar{I}} &= \langle \nabla^2 \phi, e_a e_{\bar{b}} \rangle_{\bar{a}b\bar{I},\dots,\bar{I}} + \langle \nabla^2 \phi, e_{\bar{b}} e_a \rangle_{b\bar{a}\bar{I},\dots,\bar{I}} \\ &= -\langle \nabla^2 \phi, e_a e_{\bar{a}} \rangle_{\bar{I},\dots,\bar{I}} + \langle R\phi, e_{\bar{b}} e_a \rangle_{b\bar{a}\bar{I},\dots,\bar{I}} \\ &= 0. \end{aligned}$$

Hence, by (2.2), we have

$$\begin{aligned} \langle \nabla^2 \phi, e_a e_{\bar{a}} \rangle_{\bar{I},\dots,\bar{I}} = &-2R_{a\bar{b}c\bar{d}} \phi_{\bar{c}db\bar{a}\bar{I},\dots,\bar{I}} - 2(h-1) R_{a\bar{b}c\bar{d}} \phi_{b\bar{a}\bar{I},\bar{c}d\bar{I},\dots,\bar{I}} \\ &- (h(k+1)+n-2k)(S_0\phi)_{\bar{I},\dots,\bar{I}} - (n-k)(\mathcal{L}\phi)_{\bar{I},\dots,\bar{I}}. \end{aligned}$$

Here S_0 is the endomorphism

$$S_0 \phi = \mathcal{R}_{a\bar{b}} \phi_{b\bar{a}\bar{I}_1,\dots,\bar{I}_h} e^{\bar{I}_1,\dots,\bar{I}_h}, \tag{2.3}$$

where $e^{\bar{I}_1,\dots,\bar{I}_h}$ denotes the image of $e^{\bar{I}_1}\otimes\cdots\otimes e^{\bar{I}_h}$ by the projection $\otimes^h \wedge^{0,k} X \to \hat{S}^h \wedge^{0,k} X$. By using Definition 1.2, we compute

$$2R_{a\bar{b}c\bar{d}}\phi_{\bar{c}db\bar{a}\bar{I},\dots,\bar{I}} = 0,$$
$$2R_{a\bar{b}c\bar{d}}\phi_{b\bar{a}\bar{I},\bar{c}d\bar{I},\dots,\bar{I}} = (n-2k)(S_0\phi)_{\bar{I},\dots,\bar{I}} + (n-k)(n-k+1)r\phi_{\bar{I},\dots,\bar{I}}.$$

Hence, by using the irreducibility of $\hat{S}^h \wedge^{0,k}$, we have

$$\begin{aligned}\langle \nabla^2\phi, e_a e_{\bar{a}}\rangle = &-h(n-k+1)S_0\phi - (h-1)(n-k)(n-k+1)r\phi \\ &-(n-k)\mathcal{L}\phi.\end{aligned} \tag{2.4}$$

In order to define the field equation when $h=0$, we use a similar method which was used to define the conformally invariant Laplacian by Hitchin in [H].

Let V be a vector bundle with connection on X such that $H^{-n}\otimes p^*V$ has a holomorphic structure. Let V' be the locally defined line bundle $K_X^{-\frac{k+1}{n+2}}$. Then, by Lemma 2.1, $H\otimes p^*V'$ is a holomorphic line bundle. Let $\psi \in \Gamma(\wedge^{k,0} X\otimes V')$ be a section satisfying the equation $\bar{\mathcal{D}}\psi = 0$. Then we have

$$\begin{aligned}\langle \nabla^2\psi, e_a e_{\bar{a}}\rangle_I &= \sum_{a\notin I}\langle R\psi, e_a e_{\bar{a}}\rangle_I \\ &= (n-k+1)\mathcal{R}_{a\bar{b}}\psi_{\bar{a}bI} + (n-k)(n-k+1)r\psi_I.\end{aligned}$$

Let $\phi \in \Gamma(\wedge^{0,k} X\otimes K_X^{-k}\otimes V\otimes V'^{-1})$ be a solution of the equation $(\bar{\partial}+\bar{\partial}^*)\phi = 0$. Then, since $\nabla\phi$ and $\nabla\psi$ are perpendicular, the contraction $\xi = \phi\psi \in \Gamma(K_X^{-k}\otimes V)$ satisfies

$$\langle \nabla^2\xi, e_a e_{\bar{a}}\rangle = (n-k)(n-k+1)r\xi - (n-k)\mathcal{L}\xi.$$

If we cancel $\mathcal{L}$ by using

$$\langle R\xi, e_a e_{\bar{a}}\rangle = (n+1)(n-2k)r\xi - n\mathcal{L}\xi,$$

we obtain the following field equation.

Definition 2.2. Let $\mathcal{D}_0$ be the differential operator on $\Gamma(K_X^{-k}\otimes V)$:

$$\mathcal{D}_0\xi = \langle \nabla^2\xi, k e_a e_{\bar{a}} + (n-k)e_{\bar{a}}e_a\rangle - (n+2)k(n-k)r\xi.$$

This can also be considered as a global operator on a conformally Bochner-Kähler manifold. Then (2.4) is valid even in the case $h=0$ if ϕ satisfies the field equation $\mathcal{D}_0\phi = 0$.

3. The Penrose transform

In this section we give a generalization of the Penrose transform, which gives a relation between the field equations given in the previous section and cohomology groups of positive degree on twistor spaces.

Lemma 3.1. *We have an isomorphism*

$$H^{k(n-k)}(G_{k,n}, \mathcal{O}(H^{-n-h})) \simeq \hat{S}^h \wedge^{0,k} \otimes (\wedge^{n,0})^{-k}$$

as a representation space of $\mathrm{U}(n)$.

Proof. If we consider the both spaces as $\mathrm{SU}(n)$-modules, the equivariance is immediate from the Bott-Borel-Weil-Kostant theorem (the BBWK-theorem). Under this identification, we can show easily that the actions of a scalar matrix coincide. $\square$

As in the previous section, X is an n-dimensional conformally Bochner-Kähler manifold with $n > 2$ and V is a vector bundle on X with connection such that $W = H^{-n-h} \otimes p^*V$ has a holomorphic structure. We define the Penrose transform:

$$\begin{array}{rcl} \mathcal{P} : H^{k(n-k)}(Z_k(X), \mathcal{O}(W)) & \longrightarrow & \Gamma(X, \cup_{x \in X} H^{k(n-k)}(Z_k(X)_x, \mathcal{O}(W))) \\ & & \Vert \\ & & \Gamma(X, \hat{S}^h \wedge^{0,k} X \otimes K_X^{-k} \otimes V) \end{array}$$

Theorem 3.2. *If $h = 0$, then an image of $\mathcal{P}$ is a solution of the equation $\mathcal{D}_0 \phi = 0$. If $h > 0$, an image of $\mathcal{P}$ is a solution of the equation $(\bar{\partial} + \bar{\partial}^*)\phi = 0$, that is, a harmonic form of the complex on X:*

$$\begin{aligned} &\Gamma(\wedge^{0,k-1} X \otimes \hat{S}^{h-1} \wedge^{0,k} X \otimes K_X^{-k} \otimes V) \\ &\qquad \xrightarrow{\bar{\partial}} \Gamma(\hat{S}^h \wedge^{0,k} X \otimes K_X^{-k} \otimes V) \xrightarrow{\bar{\partial}} \\ &\qquad\qquad \Gamma(\wedge^{0,k+1} X \otimes \hat{S}^{h-1} \wedge^{0,k} X \otimes K_X^{-k} \otimes V). \end{aligned}$$

Furthermore $\mathcal{P}$ is injective.

Proof. By the definition of the almost complex structure, for $x \in X$, $Z_k(X)_x$ is a complex submanifold of $Z_k(X)$. The normal bundle is a homogeneous vector bundle $N \simeq N_1 \oplus N_2$:

$$N_1 = \mathrm{SU}(n) \times_{\kappa_1} \mathbb{C}^k, \qquad N_2 = \mathrm{SU}(n) \times_{\kappa_2} \mathbb{C}^{n-k},$$

where κ_1 and κ_2 are representations of $S(\mathrm{U}(k) \times \mathrm{U}(n-k))$:

$$\kappa_1(A, B) = \bar{A}, \qquad \kappa_2(A, B) = B.$$

Let κ_3 be the representation:

$$\kappa_3(A, B) = \det(A)^{-1}.$$

Then the associated line bundle is the hyperplane bundle.

Let $T = S(\mathrm{U}(1) \times \cdots \times \mathrm{U}(1))$ be the subgroup of diagonal matrices of $\mathrm{SU}(n)$. By restricting the action of $\mathrm{SU}(n)$ or $S(\mathrm{U}(k) \times \mathrm{U}(n-k))$ to T, we consider weights of representations. For an integer i such that $1 \leq i \leq n-1$, let λ_i be the highest weight with respect to the representation $\bigwedge^i \mathbb{C}^n$. As stated in [B.E], when we apply the BBWK-theorem, it is convenient to consider the lowest weights of irreducible representations of $S(\mathrm{U}(k) \times \mathrm{U}(n-k))$. We can show, by simple computation, that the irreducible representations $\bigwedge^a \kappa_1$, $1 \leq a \leq k$, $\bigwedge^b \kappa_2$, $1 \leq b \leq n-k$ and κ_3 are characterized by their lowest weights $-\lambda_a$, $-\lambda_{n-b}$ and $-\lambda_k$, respectively.

In order to prove that the Penrose transform is injective, let $[\omega]$ be an element of $H^{k(n-k)}(\mathcal{O}(W))$ such that the restriction $[\omega_x] \in H^{k(n-k)}(Z_k(X)_x, \mathcal{O}(W))$ vanishes for each $x \in X$. Let $\omega_0 = \omega \in \Omega^{0,k(n-k)}(W)$ be a representative of $[\omega]$. Inductively, for a non-negative integer l, let ω_l be a representative of $[\omega]$ having the horizontal degree at least l. We fix $x \in X$ for a while. We have an isomorphism induced from the standard Hermitian metric as a homogeneous vector bundle:

$$\textstyle\bigwedge_H^{0,1} Z_k(X) \simeq \bar{N}_1^* \oplus \bar{N}_2^* \simeq N_1 \oplus N_2,$$

where $\bigwedge_H^{0,1} Z_k(X)$ is the horizontal part of $\bigwedge^{0,1} Z_k(X)$. For non-negative integers a and b such that $a+b=l$, we can define an element

$$\omega_{a,b}(x) \in \Omega_{G_{k,n}}^{0,k(n-k)-l}(W \otimes \textstyle\bigwedge^a N_1 \otimes \bigwedge^b N_2)$$

by the canonical projection of ω_l. The $\bar{\partial}$-closedness of ω_l implies that $\omega_{a,b}(x)$ is $\bar{\partial}$-closed. If $l=0$, then by the assumption, there is a section

$$s_{0,0}(x) \in \Omega_{G_{k,n}}^{0,k(n-k)-1}(W)$$

such that $\bar{\partial} s_{0,0}(x) = \omega_{0,0}(x)$. By the BBWK-theorem, we have

$$H^{k(n-k)-1}(G_{k,n}, H^{-n-h}) = 0.$$

Hence $s_{0,0}(x)$ is unique and smooth with respect to x by the elliptic regularity. In the case $l > 0$, by the BBWK-theorem, we have

$$H^{k(n-k)-l-i}(G_{k,n}, H^{-n-h} \otimes \textstyle\bigwedge^a N_1 \otimes \bigwedge^b N_2) = 0, \qquad i = 0, 1.$$

This means, for each $x \in X$, there exists a unique section

$$s_{a,b}(x) \in \Omega_{G_{k,n}}^{0,k(n-k)-l-1}(W \otimes \textstyle\bigwedge^a N_1 \otimes \bigwedge^b N_2)$$

such that $\bar{\partial} s_{a,b}(x) = \omega_{a,b}(x)$. Again, by the elliptic regularity, $s_{a,b}(x)$ is smooth with respect to x.

Hence if we consider $s_l = \sum_{a+b=l} s_{a,b}(x)$ as a section of $\Omega^{0,k(n-k)-1}(W)$, $\omega_l - \bar{\partial} s_l$ has horizontal degree at least $l+1$. Hence, by induction on l, we have shown that the Penrose transform is injective.

Now we show that $\mathcal{P}\alpha$ satisfies the field equation. Put $Z = Z_k(X)$. As stated in [I3], by a theorem of Kodaira ([Ko]), there exists a double fibration:

$$\begin{tikzcd} & Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ Z & & X_{\mathbb{C}} \end{tikzcd}$$

where $X_{\mathbb{C}}$ parametrizes submanifolds of Z isomorphic to $G_{k,n}$ whose normal bundles are isomorphic to N. The space $X_{\mathbb{C}}$ contains X as the set of real fibers. The manifold Y is defined as:

$$Y = \{(z, x) \in Z \times X_{\mathbb{C}} \mid z \text{ is a point of the submanifold corresponding to } x.\}.$$

Hence $p_1 \circ p_2^{-1}(x)$ is the submanifold corresponding to $x \in X_{\mathbb{C}}$, and the projection $p_2 : Y \to X_{\mathbb{C}}$ is a holomorphic fiber bundle with fiber $G_{k,n}$. Moreover the tangent space $T_x X_{\mathbb{C}}$ is canonically isomorphic to the space of global sections of the normal bundle of $p_1 \circ p_2^{-1}(x)$. For $x \in X$, let $Z_x = p_2^{-1}(x)$. We compare two embeddings $Z_x \subset Y$ and $p_1(Z_x) \subset Z$. Let $\mathcal{O}_Y^l$ and $\mathcal{O}_Z^l$ denote the lth order neighborhood sheaves. Then the l-jet of $\mathcal{P}\alpha$ lies in the image of

$$\begin{array}{ccc} H^{k(n-k)}(p_1(Z_x), \mathcal{O}_Z^l(W)) & \longrightarrow & H^{k(n-k)}(Z_x, \mathcal{O}_Y^l(p_1^* W)) \\ & & \| \\ & & J_l(\hat{S}^h \wedge^{0,k} X \otimes K_X^{-k} \otimes V)_x \end{array}$$

In general, let B be a submanifold of A with normal bundle N and E be a holomorphic vector bundle over A. Then we have an exact sequence of sheaves:

$$0 \longrightarrow \mathcal{O}_B(S^l N^* \otimes E) \longrightarrow \mathcal{O}_A^l(E) \longrightarrow \mathcal{O}_A^{l-1}(E) \longrightarrow 0.$$

Applying this sequence to $Z_x \subset Y$ and $p_1(Z_x) \subset Z$, we have exact sequences:

$$\begin{array}{ccccc} 0 & \longrightarrow & H^{k(n-k)}(\mathcal{O}(S^l N^* \otimes W)) & \longrightarrow & H^{k(n-k)}(\mathcal{O}_Z^l(W)) \\ & & \big\downarrow{\scriptstyle \sigma^*} & & \big\downarrow{\scriptstyle p_1^*} \\ 0 & \longrightarrow & (S^l T^* X \otimes \hat{S}^h \wedge^{0,k} \otimes K_X^{-k} \otimes V)_x & \longrightarrow & J_l(\hat{S}^h \wedge^{0,k} \otimes K_X^{-k} \otimes V)_x \end{array}$$

$$\begin{array}{cccc} \longrightarrow & H^{k(n-k)}(\mathcal{O}_Z^{l-1}(W)) & \longrightarrow & 0 \\ & \big\downarrow{\scriptstyle p_1^*} & & \\ \longrightarrow & J_{l-1}(\hat{S}^h \wedge^{0,k} \otimes K_X^{-k} \otimes V)_x & \longrightarrow & 0 \end{array}$$

where $H^{k(n-k)-1}$ terms disappear because of the BBWK-theorem.

If $l = 0$, p_1^* is an isomorphism by definition. For all l the map σ^* is injective as can be shown by the BBWK-theorem. Hence by induction p_1^* is injective for all l and maps $H^{k(n-k)}(\mathcal{O}_Z^l(W))$ into some subspace of $J_l(\hat{S}^h \wedge^{0,k} X \otimes K_X^{-k} \otimes V)_x$.

If $h \geq 1$ and $l = 1$, then the 1-jet bundle does not depend on the Hermitian metric of X. Hence, by calculating in the flat case, we see that this is the subspace corresponding to the equation $(\bar{\partial} + \bar{\partial}^*)\mathcal{P}\alpha = 0$.

When $h = 0$, p_1^* gives an isomorphism so $\mathcal{P}\alpha$ satisfies no first order equation. Passing to the second order neighborhood, however, we do obtain a proper subspace of $J_2(K_X^{-k} \otimes V)$. Thus $\mathcal{P}\alpha$ satisfies a second order equation $\mathcal{D}_* \mathcal{P}\alpha = 0$.

By definition, we can assume that the highest order part of $\mathcal{D}_*$ is the same as for $\mathcal{D}_0$. Since the problem is local, we can assume that there is a line bundle V' on X such that $H \otimes p^*V'$ is a holomorphic line bundle. We consider the natural product map

$$H^0(p_1(Z_x), \mathcal{O}_Z^2(H \otimes p^*V')) \otimes H^{k(n-k)}(p_1(Z_x), \mathcal{O}_Z^2(W \otimes H^{-1} \otimes p^*{V'}^{-1})) \\ \longrightarrow H^{k(n-k)}(p_1(Z_x), \mathcal{O}_Z^2(W)) \longrightarrow J_2(K_X^{-k} \otimes V)_x.$$

Here every holomorphic section of $H \otimes p^*V'$ on the first order neighborhood of $p_1(Z_x) \subset Z$ extends uniquely to the second order neighborhood by the BBWK-theorem. Hence the image is non-zero and, by the computation in the previous section, it is annihilated by both $\mathcal{D}_*$ and $\mathcal{D}_0$. Thus $\mathcal{D}_* - \mathcal{D}_0$ is a first order operator whose homomorphism annihilates the image of

$$H^0(p_1(Z_x), \mathcal{O}_Z^1(H \otimes p^*V')) \otimes H^{k(n-k)}(p_1(Z_x), \mathcal{O}_Z^1(W \otimes H^{-1} \otimes p^*{V'}^{-1}))$$

in $J_1(K_X^{-k} \otimes V)_x$. Again, since the 1-jet bundle is not intervened by the Hermitian metric of X, by computing in the flat case, this is shown to be the whole space. Hence we conclude that $\mathcal{D}_* = \mathcal{D}_0$.

Therefore, for every $h \geq 0$, $\mathcal{P}\alpha$ satisfies the desired equation. □

4. The inverse Penrose transform

In this section, we prove that the Penrose transform in the previous section is surjective. This is done by constructing explicitly a Dolbeault representative corresponding to a solution of the field equation on the base manifold.

The idea is essentially same as in the case of Riemannian manifolds. But the presence of the curvature tensor makes the proof of well-definedness far more complicated.

Since the problem is local, we can assume that X is an n-dimensional Bochner-Kähler manifold with $n > 2$. Let V be a vector bundle on X with connection satisfying the condition that $H^{-n-h} \otimes p^*V$ is a holomorphic vector bundle.

First, we give a lemma about the curvature tensor of a Bochner-Kähler manifold. Let $(e_a)_a$ be a local orthonormal frame of the holomorphic tangent bundle of X. We use notation of curvature tensors defined in Section 1.

By a theorem of Kamishima ([Ka]), a Bochner-Kähler manifold is locally isomorphic to (i) $\mathbb{C}^n$ with a flat metric, or (ii) $\mathbf{H}^l \times \mathbf{P}^{n-l}$ for some integer l such that $0 \leq l \leq n$. This means that the Bochner-flat condition also strongly restricts the Ricci tensor. In this paper, we only need the following lemma, which is deduced immediately from that theorem.

Lemma 4.1. *The Ricci tensor of a Bochner-Kähler manifold X is covariantly constant and satisfies*

$$\mathcal{R}_{a\bar{b}}\mathcal{R}_{b\bar{c}} = r\mathcal{R}_{a\bar{c}} + s\delta_a^c,$$

where s is a scalar function.

Let $\nabla e_a = \omega_a^b e_b$ be the connection form of the Levi-Civita connection. Since a connection defines horizontal lifts of vector fields on X to those on $Z_k(X)$, e_a is also considered to be a vector field on $Z_k(X)$. By restricting the $\mathrm{SO}(2n)$-action on Z, we have a $\mathrm{U}(n)$-action on $G_{k,n}$. It is convenient however to consider the complexified $\mathrm{GL}(n;\mathbb{C})$-action on $G_{k,n}$. It defines a $\mathbb{C}$-linear map:

$$\mathcal{F} : \mathfrak{gl}\,(n;\mathbb{C}) \longrightarrow H^0(G_{k,n}, \mathcal{O}(T^{(1,0)}G_{k,n})).$$

With respect to the Lie algebra structures, we have:

$$\mathcal{F}([a,b]) = -[\mathcal{F}(a), \mathcal{F}(b)]. \tag{4.1}$$

Let $(E_b^a)_{a,b}$ denote the standard basis of $\mathfrak{gl}\,(n;\mathbb{C})$ such that $E_b^a e_c = \delta_c^a e_b$. We define

$$\mathcal{F}_b^a = -\mathcal{F}(\mathrm{E}_b^a). \tag{4.2}$$

Before giving an explicit description of $\mathcal{F}_b^a$, we give here a few relations between the Plüker coordinates induced from the Plüker relation.

Lemma 4.2.
(1) *Let J, K be multi-indices of length k. Then we have, on $G_{k,n}$*

$$Z^{a\bar{c}J} Z^{\bar{b}cK} = -Z^J Z^{\bar{b}aK} + Z^{\bar{b}aJ} Z^K.$$

(2) *For a multi-index J of length k, we have, on $G_{k,n}$*

$$Z^{\bar{a}b\bar{c}dJ} Z^J = Z^{\bar{a}bJ} Z^{\bar{c}dJ} + Z^{\bar{a}dJ} Z^{b\bar{c}J}.$$

Proof. When $a \neq b$, by using (1.3), we compute:

$$\begin{aligned} Z^{a\bar{c}J} Z^{\bar{b}cK} &= Z^{a\bar{c}J} Z^{\bar{b}cK} - Z^{\bar{c}aJ} Z^{c\bar{b}K} \\ &= Z^{a\bar{a}J} Z^{\bar{b}aK} - Z^{\bar{a}aJ} Z^{a\bar{b}K} + Z^{a\bar{b}J} Z^{\bar{b}bK} - Z^{\bar{b}aJ} Z^{b\bar{b}K} \\ &= -Z^J Z^{\bar{b}aK} + Z^{\bar{b}aJ} Z^K. \end{aligned}$$

The case $a = b$ can be proved in the same way and we have proved (1).
By putting $J = \bar{d}cK$ in (1), we have (2). □

For a multi-index I of length k, we take local fiber coordinates as in Section 1. That is,

$$w_{ij} = Z^{\bar{i}jI}/Z^I, \qquad i \in I, \quad j \notin I$$

are local coordinates on

$$U_I = \{(Z^J)_J \in G_{k,n} \mid Z^I \neq 0\}$$

with the center z_0. For a general multi-index J of length k, put $z^J = Z^J/Z^I$.

Lemma 4.3. *The vector field $\mathcal{F}^a_b$ is written in the local coordinates as*:

$$\mathcal{F}^a_b = -z^{a\bar{\imath}I} z^{\bar{b}jI} \frac{\partial}{\partial w_{ij}}.$$

Proof. When $a \neq b$, we compute the infinitesimal action of the one-parameter subgroup:

$$A(t) = \exp(tE^a_b).$$

Let $(e_J)_J$ be the standard basis of $\wedge^k \mathbb{C}^n$. Then the action of the matrix is written as:

$$A(t)e_J = e_J + te_{\bar{a}bJ}.$$

Hence the action on a Plücker coordinate is written as

$$Z^J(t) = Z^J + tZ^{\bar{b}aJ}.$$

Therefore, by using Lemma 4.2 (2), we have

$$\frac{d}{dt} w_{ij}(t)\bigg|_{t=0} = z^{\bar{b}a\bar{\imath}jI} - z^{\bar{b}aI} z^{\bar{\imath}jI} = z^{a\bar{\imath}I} z^{\bar{b}jI},$$

for $i \in I$ and $j \notin I$.

When $a = b$, let

$$B(t) = \exp(t\sqrt{-1}E^a_a),$$

where we remark that the index a is fixed and Einstein's convention for summation is not applied. Then, in a similar way, we compute

$$\frac{d}{dt} w_{ij}(t)\bigg|_{t=0} = \sqrt{-1} z^{a\bar{\imath}I} z^{\bar{a}jI}.$$

Hence we complete the proof. □

Let $(e^a)_a$ be the dual frame of $\wedge^{1,0} X$. As in the case of $e_{\bar{a}}$, let $e^{\bar{a}}$, $\mathcal{F}^{\bar{a}}_{\bar{b}}$, $R^{\bar{a}}_{\bar{b}}$, and $\omega^{\bar{a}}_{\bar{b}}$ be the complex conjugate of e^a, $\mathcal{F}^a_b$, R^a_b and ω^a_b, respectively. Put $W = H^{-n-h} \otimes p^*V$. Now we define operators acting on $\Omega^*(Z_k(X), W)$:

$$T_0 = -i(\mathcal{F}^b_a)R^a_b - i(\mathcal{F}^{\bar{b}}_{\bar{a}})R^{\bar{a}}_{\bar{b}}, \tag{4.3}$$

$$\tilde{L}_a = L_{e_a} - \omega^b_a i(e_b) - [i(e_a), T_0], \tag{4.4}$$

$$\tilde{L}_{\bar{a}} = L_{e_{\bar{a}}} - \omega^{\bar{b}}_{\bar{a}} i(e_{\bar{b}}) - [i(e_{\bar{a}}), T_0], \tag{4.5}$$

$$D^a = e^b i(\mathcal{F}^{\bar{b}}_{\bar{a}}), \tag{4.6}$$

$$D^{\bar{a}} = -e^{\bar{b}} i(\mathcal{F}^{\bar{a}}_{\bar{b}}), \tag{4.7}$$

where L denotes the Lie derivative. Vector fields and forms on X are considered to be those on $Z_k(X)$ in a natural way. Note that $D^{\bar{a}}$ is not the complex conjugate of D^a.

Lemma 4.4.

(1) *Let $e'_a = e_b h^b_a$ be another local orthonormal frame of $T^{(1,0)}X$. Let $\tilde{L}'_a$, $\tilde{L}'_{\bar{a}}$, D'^a and $D'^{\bar{a}}$ be operators defined as above with respect to the frame $(e'_a)_a$. Then we have*

$$\tilde{L}'_a = h^b_a \tilde{L}_b, \quad \tilde{L}'_{\bar{a}} = \overline{h^b_a}\tilde{L}_{\bar{b}}, \quad D'^a = D^b (h^{-1})^a_b, \quad D'^{\bar{a}} = D^{\bar{b}}\overline{(h^{-1})^a_b}.$$

(2) *Operators $\tilde{L}_a$, $\tilde{L}_{\bar{a}}$, D^a and $D^{\bar{a}}$ preserve the double grading as differential forms. That is, for non-negative integers l and l', they map $\Omega^{l,l'}(W)$ to itself.*

Proof. The transformation rules in (1) are immediate by

$$\begin{aligned} L_{fv} &= fL_v + df \wedge i(v), \\ \omega'^a_b &= (h^{-1})^a_c dh^c_b + (h^{-1})^a_c \omega^c_d h^d_b, \end{aligned}$$

where v is a vector field and f is a function.

The operators D^a and $D^{\bar{a}}$ preserve the grading because we can write them as:

$$\begin{aligned} D^a &= -z^{a\bar{d}\bar{I}}(z^{\bar{b}c\bar{I}} e^b) i\left(\frac{\partial}{\partial \bar{w}_{cd}}\right), \\ D^{\bar{a}} &= z^{\bar{a}c\bar{I}}(z^{b\bar{d}\bar{I}} e^{\bar{b}}) i\left(\frac{\partial}{\partial \bar{w}_{cd}}\right). \end{aligned}$$

We can assume that the connection forms of $T^{(1,0)}X$ and V vanish at x_0 and we compute values at (x_0, z_0). By (1.9), we have

$$\begin{aligned} [L_{e_a}, \widehat{dw_{ij}}] &= [i(e_a), R^j_i], \\ [L_{e_a}, \widehat{d\bar{w}_{ij}}] &= [i(e_a), R^{\bar{j}}_{\bar{i}}], \end{aligned}$$

where $\widehat{d\bar{w}_{ij}}$ is the complex conjugate of $\widehat{dw_{ij}}$. On the other hand, we have

$$\begin{aligned} [-\omega^b_a i(e_b) - [i(e_a), T_0], \widehat{dw_{ij}}] &= -[i(e_a), R^j_i], \\ [-\omega^b_a i(e_b) - [i(e_a), T_0], \widehat{d\bar{w}_{ij}}] &= -[i(e_a), R^{\bar{j}}_{\bar{i}}]. \end{aligned}$$

Consequently we have

$$[\tilde{L}_a, \widehat{dw_{ij}}] = [\tilde{L}_a, \widehat{d\bar{w}_{ij}}] = 0.$$

With respect to horizontal forms, we compute

$$\begin{aligned} [\tilde{L}_a, z^{\bar{b}\bar{J}} e^b] &= [\tilde{L}_a, z^{bJ} e^{\bar{b}}] = 0, \qquad |J| = k-1, \\ [\tilde{L}_a, z^{b\bar{J}} e^{\bar{b}}] &= [\tilde{L}_a, z^{\bar{b}J} e^b] = 0, \qquad |J| = k+1. \end{aligned}$$

For a function f, we have

$$[\tilde{L}_a, f] = e_a(f).$$

Hence $\tilde{L}_a$ preserves the grading. The proof for $\tilde{L}_{\bar{a}}$ can be done in the same way. □

Now we define operators which are used to construct the inverse Penrose transform.

$$D_\alpha = D^a \tilde{L}_a, \tag{4.8}$$

$$D_\beta = D^{\bar{a}} \tilde{L}_{\bar{a}}, \tag{4.9}$$

$$D_\gamma = -\mathcal{R}_{a\bar{b}} D^{\bar{b}} D^a. \tag{4.10}$$

These operators are globally well-defined and preserve the grading of forms by the above lemma.

Put

$$F(x) = \sum_{i=0}^{\infty} \frac{x^i}{(i!)^2}.$$

This function and its derivatives play an important role by the following property.

Lemma 4.5. *Let l be a non-negative integer. Then the lth derivative of F satisfies*

$$xF^{(l+2)}(x) + (l+1)F^{(l+1)}(x) - F^{(l)}(x) = 0.$$

Let $\wedge_V^{0,k(n-k)}$ denote the line subbundle of $\wedge^{0,k(n-k)} Z_k(X)$ spanned by vertical forms, which are defined by the Levi-Civita connection. If we identify H^{-1} with $\bar{H}$ by the standard Hermitian metric, we have

$$\wedge^{0,k(n-k)} Z_k(X) \otimes H^{-n-h} \supset \wedge_V^{0,k(n-k)} \otimes H^{-n-h} \simeq (H^h \otimes K_X^k)^*,$$

where the canonical bundle appears because we consider the action of $\mathrm{U}(n)$ rather than $\mathrm{SU}(n)$. On the other hand, we have an isomorphism:

$$H^0(G_{k,n}, \mathcal{O}(H^h)) \simeq \hat{S}^h \wedge^{k,0} X$$

by the BBWK-theorem. Hence we obtain

$$j : \Gamma(X, \hat{S}^h \wedge^{0,k} X \otimes K_X^{-k} \otimes V) \longrightarrow \Omega^{0,k(n-k)}(Z_k(X), W).$$

We put

$$f(x) = (k+h-1)! F^{(k+h-1)}(x),$$
$$g(x) = (n-k+h-1)! F^{(n-k+h-1)}(x).$$

Definition 4.6. We define

$$\begin{array}{rccc} \mathcal{Q}: & \Gamma(X, \hat{S}^h \wedge^{0,k} X \otimes K_X^{-k} \otimes V) & \longrightarrow & \Omega^{0,k(n-k)}(Z_k(X), W) \\ & \phi & \longmapsto & f(D_\beta + D_\gamma) g(D_\alpha) j(\phi). \end{array}$$

Remark. This is well-defined since the differential operators in the formula are even operators and D_β and D_γ are mutually commutative. By the proof of Lemma 4.4, the operators D^i_α, D^i_β and D^i_γ vanish for sufficiently large i.

Theorem 4.7. *The form $\mathcal{Q}(\phi)$ is $\bar\partial$-closed if ϕ satisfies the differential equation stated in Theorem* 3.2. *The restriction of $\mathcal{Q}$ to the space of harmonic forms gives the inverse of the Penrose transform.*

The remainder of this section is devoted to a proof of this theorem. Since the operators D_α, D_β and D_γ decrease the vertical grading of forms, the last statement of the theorem follows immediately if we show the first statement.

Let $(e_a)_a$ be a local frame of $T^{(1,0)}X$ such that the connection form of the Levi-Civita connection vanishes at x_0. Let $(e^{\bar I})_{\bar I}$ denote the associated frame of $\wedge^{0,k} X$. Instead of determining an explicit basis of $\hat S^h \wedge^{0,k} X$, we treat it as a subbundle of $\otimes^h \wedge^{0,k} X$. On the other hand, we consider an element of $\otimes^h \wedge^{0,k} X$ as an element of $\hat S^h \wedge^{0,k} X$ by the canonical projection. Let $I_1, \dots, I_h$ be multi-indices of length k. Put

$$e^{\bar I_1,\dots,\bar I_h} = e^{\bar I_1} \otimes \cdots \otimes e^{\bar I_h},$$

which is considered as an element of $\hat S^h \wedge^{0,k} X$ as mentioned above. Since we have taken a local frame of $T^{(1,0)}X$, we have a canonical trivialization of K_X. We also take a local frame of V such that the connection form vanishes at x_0. Since the choice of the local frame of V does not affect the appearance of the computation, we omit its index. In fact, the information of V which is needed for computation is already encoded in (2.2) and (2.4). Thus we can consider $e^{\bar I_1,\dots,\bar I_h}$ as a section of $\hat S^h \wedge^{0,k} X \otimes K_X^{-k} \otimes V$. Then we put

$$s^{\bar I_1,\dots,\bar I_h} = j(e^{\bar I_1,\dots,\bar I_h}).$$

Define

$$E^a = [\bar\partial, D^a] + \omega^a_b D^b, \tag{4.11}$$

$$E^{\bar a} = [\bar\partial, D^{\bar a}] + \omega^{\bar a}_{\bar b} D^{\bar b}. \tag{4.12}$$

They satisfy the same transformation rules as D^a and $D^{\bar a}$, respectively.

Lemma 4.8. *For local coordinates $(w_{ij})_{i,j}$, we have*

$$\frac{\partial z^J}{\partial w_{ij}} = -z^{i\bar jJ}.$$

Proof. It is obvious if $|J \setminus I| \le 1$. If $|J \setminus I| \ge 2$, let $i_k \in I \setminus J$, $j_k \in J \setminus I$, $k = 1, 2$, be numbers such that $i_1 \ne i_2$, $j_1 \ne j_2$. Then, by Lemma 4.2 (2), we have

$$Z^J = \frac{1}{Z^{\bar j_1 i_1 \bar j_2 i_2 J}} (Z^{\bar j_1 i_1 J} Z^{\bar j_2 i_2 J} + Z^{\bar j_1 i_2 J} Z^{i_1 \bar j_2 J}).$$

Hence we obtain inductively the desired formula. □

Lemma 4.9. *We have*

$$E^a s^{\bar I_1,\dots,\bar I_h} \equiv -(n-k+h)e^a s^{\bar I_1,\dots,\bar I_h} - \sum_{j=1}^h e^b s^{\bar I_1,\dots,\bar b a \bar I_j,\dots,\bar I_h},$$

$$E^{\bar a} s^{\bar I_1,\dots,\bar I_h} \equiv -(k+h)e^{\bar a} s^{\bar I_1,\dots,\bar I_h} - \sum_{j=1}^h e^{\bar b} s^{\bar I_1,\dots,b\bar a \bar I_j,\dots,\bar I_h},$$

where we consider equivalence modulo $(1,0)$*-forms.*

Proof. Let $\rho^{\bar I}$ be the image of the standard trivialization of $\bar H$ by the isomorphism $\bar H \simeq H^{-1}$. Let $K^{\bar I}$ be the standard trivialization of $\wedge_V^{0,k(n-k)}$. Then, by definition, we have

$$s^{\bar I_1,\dots,\bar I_h} = \rho^{\bar I_1} \otimes \cdots \otimes \rho^{\bar I_h} \otimes {\rho^{\bar I}}^{\otimes(n)} \otimes K^{\bar I}. \tag{4.13}$$

Put

$$N = \sum_J \left| \frac{z^J}{z^I} \right|^2 .$$

First, we compute:

$$\begin{aligned}
L_{\mathcal{F}^{\bar a}_{\bar b}} \rho^{\bar I} &= \nabla_{\mathcal{F}^{\bar a}_{\bar b}} \rho^{\bar I} = -\frac{\mathcal{F}^{\bar a}_{\bar b}(N)}{N}\rho^{\bar I} \\
&= -\frac{z^J z^{\bar\imath j \bar J} z^{\bar a i \bar I} z^{b \bar\jmath \bar I}}{N}\rho^{\bar I} && \text{[By Lemma 4.8]} \\
&= -\frac{z^J z^{\bar a i \bar I} z^{b \bar\imath \bar J}}{N}\rho^{\bar I} && \text{[By Lemma 4.2 (1)]} \\
&= \left(\frac{z^J z^{b \bar a \bar J}}{N} - z^{b \bar a \bar I} \right)\rho^{\bar I} && \text{[By Lemma 4.2 (1)]}
\end{aligned}$$

By changing indices, we obtain:

$$L_{\mathcal{F}^{\bar a}_{\bar b}} \rho^{\bar I} = \left(\delta^b_a + z^{\bar a b \bar I} + \frac{z^{\bar a b J} z^{\bar J}}{N} \right) \rho^{\bar I}.$$

Second, we compute:

$$L_{\mathcal{F}^{\bar a}_{\bar b}} d\bar w_{ij} = d\mathcal{F}^{\bar a}_{\bar b}(\bar w_{ij}) = (-\delta^b_i z^{\bar a i \bar I} + \delta^j_a z^{b \bar\jmath \bar I}) d\bar w_{ij} + \dots$$

Hence we have

$$L_{\mathcal{F}^{\bar a}_{\bar b}} K^{\bar I} = (-(n-k) z^{\bar a b \bar I} + k z^{b \bar a \bar I}) K^{\bar I} = -(k\delta^b_a + n z^{\bar a b \bar I}) K^{\bar I}.$$

Thus, we have

$$\begin{aligned}
L_{\mathcal{F}^{\bar a}_{\bar b}} s^{\bar I_1,\dots,\bar I_h} = (n-k+h)\delta^b_a s^{\bar I_1,\dots,\bar I_h} + \sum_{j=1}^h s^{\bar I_1,\dots,\bar a b \bar I_j,\dots,\bar I_h} \\
+ (n+h)\frac{z^{\bar a b J} z^{\bar J}}{N} s^{\bar I_1,\dots,\bar I_h}.
\end{aligned}$$

Consequently we have the first equality. By changing indices we have:

$$L_{\mathcal{F}^{\bar{a}}_{\bar{b}}} s^{\bar{I}_1,\dots,\bar{I}_h} = -(k+h)\delta^b_a s^{\bar{I}_1,\dots,\bar{I}_h} - \sum_{j=1}^{h} s^{\bar{I}_1,\dots,b\bar{a}\bar{I}_j,\dots,\bar{I}_h}$$
$$- (n+h)\frac{z^{b\bar{a}J} z^{\bar{J}}}{N} s^{\bar{I}_1,\dots,\bar{I}_h},$$

from which the second equality follows. □

Lemma 4.10. *We have*

$$[E^a, D^b] = -e^a D^b - e^b D^a,$$
$$[E^{\bar{a}}, D^{\bar{b}}] = -e^{\bar{a}} D^{\bar{b}} - e^{\bar{b}} D^{\bar{a}},$$
$$[E^{\bar{a}}, D^b] = \delta^b_a B_0,$$

where

$$B_0 = e^a D^{\bar{a}} = D^a e^{\bar{a}}. \tag{4.14}$$

Proof. These formulas follow from

$$[\mathcal{F}^a_b, \mathcal{F}^c_d] = \delta^a_d \mathcal{F}^c_b - \delta^c_b \mathcal{F}^a_d,$$

which is an immediate consequence of (4.1). □

We define

$$d_\alpha = e^a \tilde{L}_a, \tag{4.15}$$
$$d_\beta = e^{\bar{a}} \tilde{L}_{\bar{a}}, \tag{4.16}$$
$$d_\gamma = -i(\mathcal{F}^{\bar{b}}_{\bar{a}}) R^{\bar{a}}_{\bar{b}}. \tag{4.17}$$

Obviously these operators do not preserve the grading of forms. But we see, by Lemma 4.4, they do not decrease the first grading. So we can consider them as operators acting on

$$\frac{\Omega^*(Z_k(X), W)}{\bigoplus_{i=1}^{\infty} \Omega^{i,*}(Z_k(X), W)} \simeq \Omega^{0,*}(Z_k(X), W).$$

Now we compute $\bar{\partial}\mathcal{Q}(\phi)$. We split it into two steps

$$\bar{\partial} g(D_\alpha) j(\phi) = (d_\beta + d_\gamma) g(D_\alpha) j(\phi), \tag{I}$$
$$[\bar{\partial}, f(D_\beta + D_\gamma)] g(D_\alpha) j(\phi) = -f(D_\beta + D_\gamma)(d_\beta + d_\gamma) g(D_\alpha) j(\phi), \tag{II}$$

from which the theorem follows immediately.

Proof of (I). Let $\phi = \phi_{\bar{I}_1,\dots,\bar{I}_h} e^{\bar{I}_1,\dots,\bar{I}_h}$. Then we have

$$j(\phi) = \phi_{\bar{I}_1,\dots,\bar{I}_h} s^{\bar{I}_1,\dots,\bar{I}_h}$$

and

$$\bar{\partial} j(\phi) \equiv d\phi_{\bar{I}_1,\ldots,\bar{I}_h} \wedge s^{\bar{I}_1,\ldots,\bar{I}_h} + \phi_{\bar{I}_1,\ldots,\bar{I}_h} ds^{\bar{I}_1,\ldots,\bar{I}_h}.$$

The value of the first term at (x_0, z_0) is

$$d\phi_{\bar{I}_1,\ldots,\bar{I}_h} \wedge s^{\bar{I}_1,\ldots,\bar{I}_h} = (d_\alpha + d_\beta) j(\phi).$$

By (1.9), we have

$$dK^{\bar{I}}|_{(x_0,z_0)} = d_\gamma K^{\bar{I}}.$$

Hence, by (4.13), we have

$$\bar{\partial} j(\phi) = (d_\alpha + d_\beta + d_\gamma) j(\phi).$$

By an inductive argument, we have

$$D^i_\alpha j(\phi) = D^{a_1} \ldots D^{a_i} j\langle \nabla^i \phi, e_{a_i} \ldots e_{a_1} \rangle.$$

Hence we compute

$$\bar{\partial} D^i_\alpha j(\phi) = [\bar{\partial}, D^{a_1} \ldots D^{a_i}] j\langle \nabla^i \phi, e_{a_i} \ldots e_{a_1} \rangle + (d_\alpha + d_\beta + d_\gamma) D^i_\alpha j(\phi).$$

Therefore we obtain

$$(\bar{\partial} - d_\beta - d_\gamma) D^i_\alpha j(\phi) = [\bar{\partial}, D^{a_1} \ldots D^{a_i}] j\langle \nabla^i \phi, e_{a_i} \ldots e_{a_1} \rangle + d_\alpha D^i_\alpha j(\phi).$$

In order to compute the first term of the right-hand side, put

$$E_\alpha = E^a \tilde{L}_a. \tag{4.18}$$

Then, by Lemma 4.9 and the assumption on ϕ, we have

$$E_\alpha j(\phi) = -(n - k + h) d_\alpha j(\phi).$$

By Lemma 4.10, we compute

$$[E_\alpha, D_\alpha] D^i_\alpha j(\phi) = -2 d_\alpha D^{i+1}_\alpha j(\phi),$$

where we use

$$[d_\alpha, D_\alpha] D^i_\alpha j(\phi) = 0,$$

which follows from the fact that the curvature form is of type $(1,1)$ with respect to the original complex structure of X. This also means

$$[[E_\alpha, D_\alpha], D_\alpha] D^i_\alpha j(\phi) = 0.$$

Hence we obtain

$$E_\alpha D^i_\alpha j(\phi) = -2i d_\alpha D^i_\alpha j(\phi) - (n - k + h) d_\alpha D^i_\alpha j(\phi).$$

Therefore we compute

$$[\bar{\partial}, D^{a_1} \dots D^{a_i}] j \langle \nabla^i \phi, e_{a_i} \dots e_{a_1} \rangle = \sum_{j=1}^{i} D_\alpha^{j-1} E_\alpha D_\alpha^{i-j} j(\phi)$$
$$= -i(i-1) d_\alpha D_\alpha^{i-1} j(\phi)$$
$$- i(n-k+h) d_\alpha D_\alpha^{i-1} j(\phi).$$

Thus, by Lemma 4.5, we have

$$(\bar{\partial} - d_\beta - d_\gamma) g(D_\alpha) j(\phi) = d_\alpha(-g''(D_\alpha) D_\alpha - (n-k+h) g'(D_\alpha) + g(D_\alpha)) j(\phi)$$
$$= 0,$$

which completes the proof. □

Proof of (II). We have

$$D_\beta^i D_\gamma^j j(\phi) = D_\gamma^j D^{\bar{a}_1} \dots D^{\bar{a}_i} j \langle \nabla^i \phi, e_{\bar{a}_i} \dots e_{\bar{a}_1} \rangle.$$

Hence, by putting

$$E_\beta = E^{\bar{a}} \tilde{L}_{\bar{a}}, \tag{4.19}$$

$$E_\gamma = [\bar{\partial}, D_\gamma], \tag{4.20}$$

we obtain

$$[\bar{\partial}, D_\beta + D_\gamma](D_\beta + D_\gamma)^i g(D_\alpha) j(\phi) =$$
$$(E_\beta + [d_\alpha, D_\beta] + E_\gamma)(D_\beta + D_\gamma)^i g(D_\alpha) j(\phi).$$

Now, as in the proof of (I), we compute the commutation relation between $E_\beta + [d_\alpha, D_\beta] + E_\gamma$ and $D_\beta + D_\gamma$. By Lemma 4.10, we have

$$[E_\beta, D_\beta](D_\beta + D_\gamma)^i g(D_\alpha) j(\phi) = -2 D_\beta d_\beta (D_\beta + D_\gamma)^i g(D_\alpha) j(\phi)$$

and

$$[E_\beta, D_\gamma](D_\beta + D_\gamma)^i g(D_\alpha) j(\phi) =$$
$$(D_\beta B_1 - D_\gamma d_\beta - B_0 C_1)(D_\beta + D_\gamma)^i g(D_\alpha) j(\phi),$$

where

$$B_1 = \mathcal{R}_{a\bar{b}} D^a e^{\bar{b}}, \tag{4.21}$$

$$C_1 = \mathcal{R}_{a\bar{b}} D^{\bar{b}} \tilde{L}_{\bar{a}}. \tag{4.22}$$

We compute

$$[d_\alpha, D_\beta] D_\beta^{i+1} D_\alpha^j j(\phi)$$
$$= e^a D^{\bar{b}} D^{\bar{c}} D^{\bar{c}_1} \dots D^{\bar{c}_i} D^{d_1} \dots D^{d_j} j \langle R \nabla^{i+j+1} \phi, e_{d_j} \dots e_{d_1} e_{\bar{c}_i} \dots e_{\bar{c}_1} e_{\bar{c}} e_{\bar{b}} e_a \rangle$$
$$= e^a D^{\bar{b}} D^{\bar{c}} D^{\bar{c}_1} \dots D^{\bar{c}_i} D^{d_1} \dots D^{d_j} j \langle -R_{\bar{c}}^{\bar{d}} \nabla_{e_{\bar{d}}} \nabla^{i+j} \phi, e_{d_j} \dots e_{d_1} e_{\bar{c}_i} \dots e_{\bar{c}_1} e_{\bar{b}} e_a \rangle$$
$$+ D_\beta [d_\alpha, D_\beta] D_\beta^i D_\alpha^j j(\phi).$$

Obviously we have

$$[[d_\alpha, D_\beta], D_\gamma] D_\beta^i D_\gamma^j g(D_\alpha) j(\phi) = 0.$$

Therefore we obtain

$$\begin{aligned}
&[[d_\alpha, D_\beta], D_\beta](D_\beta + D_\gamma)^i g(D_\alpha) j(\phi) = 2T_1 (D_\beta + D_\gamma)^i g(D_\alpha) j(\phi), \\
&[[d_\alpha, D_\beta], D_\gamma](D_\beta + D_\gamma)^i g(D_\alpha) j(\phi) = 0,
\end{aligned}$$

where

$$T_1 = R_{a\bar{b}c\bar{d}} e^a D^{\bar{b}} D^{\bar{d}} \tilde{L}_{\bar{c}}. \tag{4.23}$$

We compute

$$\begin{aligned}
&[E_\gamma, D_\beta](D_\beta + D_\gamma)^i g(D_\alpha) j(\phi) \\
&\qquad = -\mathcal{R}_{a\bar{b}} (D^a [E^{\bar{b}}, D_\beta] + D^{\bar{b}} [E^a, D_\beta])(D_\beta + D_\gamma)^i g(D_\alpha) j(\phi) \\
&\qquad = (D_\beta B_1 - D_\gamma d_\beta - B_0 C_1)(D_\beta + D_\gamma)^i g(D_\alpha) j(\phi),
\end{aligned}$$

and

$$\begin{aligned}
&[E_\gamma, D_\gamma](D_\beta + D_\gamma)^i g(D_\alpha) j(\phi) \\
&\qquad = -\mathcal{R}_{a\bar{b}} (D^a [E^{\bar{b}}, D_\gamma] + D^{\bar{b}} [E^a, D_\gamma])(D_\beta + D_\gamma)^i g(D_\alpha) j(\phi) \\
&\qquad = 2D_\gamma (B_1 + B_2 - rB_0)(D_\beta + D_\gamma)^i g(D_\alpha) j(\phi) \\
&\qquad = -2D_\gamma d_\gamma (D_\beta + D_\gamma)^i g(D_\alpha) j(\phi),
\end{aligned}$$

where we put

$$B_2 = \mathcal{R}_{a\bar{b}} e^a D^{\bar{b}}, \tag{4.24}$$

and we use Lemma 4.1 and

$$\begin{aligned}
D^{\bar{a}} D^a &= 0, \\
B_1 + B_2 - rB_0 &= -d_\gamma.
\end{aligned} \tag{4.25}$$

The first equation is immediately obtained by (1.3). Therefore, by using

$$T_1 + D_\beta B_1 - B_0 C_1 = -D_\beta d_\gamma,$$

we obtain

$$\begin{aligned}
&[[\bar{\partial}, D_\beta + D_\gamma], D_\beta + D_\gamma](D_\beta + D_\gamma)^i g(D_\alpha) j(\phi) \\
&\qquad\qquad = -2(d_\beta + d_\gamma)(D_\beta + D_\gamma)^{i+1} g(D_\alpha) j(\phi).
\end{aligned}$$

Since

$$[[[\bar{\partial}, D_\beta + D_\gamma], D_\beta + D_\gamma], D_\beta + D_\gamma](D_\beta + D_\gamma)^i g(D_\alpha) j(\phi) = 0,$$

by using Lemma 4.5, we finally obtain

$$\begin{aligned}[\bar{\partial}, f(D_\beta + D_\gamma)]g(D_\alpha)j(\phi) &= f'(D_\beta + D_\gamma)(E_\beta + [d_\alpha, D_\beta] + E_\gamma)g(D_\alpha)j(\phi) \\ &\quad - f''(D_\beta + D_\gamma)(D_\beta + D_\gamma)(d_\beta + d_\gamma)g(D_\alpha)j(\phi) \\ &= -f(D_\beta + D_\gamma)(d_\beta + d_\gamma)g(D_\alpha)j(\phi) \\ &\quad + f'(D_\beta + D_\gamma)T_2 g(D_\alpha)j(\phi),\end{aligned}$$

where

$$T_2 = E_\beta + [d_\alpha, D_\beta] + E_\gamma + (k+h)(d_\beta + d_\gamma). \tag{4.26}$$

Hence we complete the proof by showing

$$T_2 g(D_\alpha)j(\phi) = 0. \tag{II'}$$

Again this can be proved by computing commutation relations. We compute

$$\begin{aligned}E_\beta D_\alpha^{i+1} j(\phi) &= E^{\bar{a}} D^b D^{c_1} \dots D^{c_i} j \langle \nabla^{i+2}\phi, e_{c_i} \dots e_{c_1} e_b e_{\bar{a}} \rangle \\ &= B_0 D^{c_1} \dots D^{c_i} j \langle \nabla^{i+2}\phi, e_{c_i} \dots e_{c_1} e_a e_{\bar{a}} \rangle \\ &\quad + D^b E^{\bar{a}} D^{c_1} \dots D^{c_i} j \langle R\nabla^i \phi, e_{c_i} \dots e_{c_1} e_b e_{\bar{a}} \rangle + D_\alpha E_\beta D_\alpha^i j(\phi).\end{aligned}$$

Hence we obtain

$$\begin{aligned}[E_\beta, D_\alpha] D_\alpha^i j(\phi) &= B_0 D^{c_1} \dots D^{c_i} j \langle \nabla^{i+2}\phi, e_{c_i} \dots e_{c_1} e_a e_{\bar{a}} \rangle \\ &\quad + D^b E^{\bar{a}} D^{c_1} \dots D^{c_i} j \langle R\nabla^i \phi, e_{c_i} \dots e_{c_1} e_b e_{\bar{a}} \rangle \\ &= 2B_0 D^b D^{c_1} \dots D^{c_{i-1}} j \langle R\nabla^i \phi, e_{c_{i-1}} \dots e_{c_1} e_a e_b e_{\bar{a}} \rangle \\ &\quad + D^b D^d E^{\bar{a}} D^{c_1} \dots D^{c_{i-1}} j \langle -R_d^e \nabla_{e_e} \nabla^{i-1}\phi, e_{c_{i-1}} \dots e_{c_1} e_b e_{\bar{a}} \rangle \\ &\quad + D_\alpha [E_\beta, D_\alpha] D_\alpha^{i-1} j(\phi).\end{aligned}$$

Moreover we compute

$$\begin{aligned}&[[E_\beta, D_\alpha], D_\alpha] D_\alpha^i j(\phi) \\ &\qquad = 2B_0 D^b D^{c_1} \dots D^{c_i} j \langle R\nabla^{i+1}\phi, e_{c_i} \dots e_{c_1} e_a e_b e_{\bar{a}} \rangle \\ &\qquad\quad + 2R_{b\bar{a}d\bar{e}} D^b D^d E^{\bar{a}} D^{c_1} \dots D^{c_i} j \langle \nabla^{i+1}\phi, e_{c_i} \dots e_{c_1} e_e \rangle \\ &\qquad = 2B_0 D^b D^d D^{c_1} \dots D^{c_{i-1}} j \langle -R_d^e \nabla_{e_e} \nabla^i \phi, e_{c_{i-1}} \dots e_{c_1} e_a e_b e_{\bar{a}} \rangle \\ &\qquad\quad + 2R_{b\bar{a}d\bar{e}} D^b D^d B_0 D^{c_1} \dots D^{c_{i-1}} j \langle \nabla^{i+1}\phi, e_{c_{i-1}} \dots e_{c_1} e_a e_e \rangle \\ &\qquad\quad + D_\alpha [[E_\beta, D_\alpha], D_\alpha] D_\alpha^{i-1} j(\phi).\end{aligned}$$

Hence we obtain

$$\begin{aligned}&[[[E_\beta, D_\alpha], D_\alpha], D_\alpha] D_\alpha^i j(\phi) = \\ &\qquad\qquad 6B_0 R_{b\bar{a}d\bar{e}} D^b D^d D^{c_1} \dots D^{c_i} j \langle \nabla^{i+2}\phi, e_{c_i} \dots e_{c_1} e_a e_e \rangle.\end{aligned}$$

Thus we have

$$[[[[E_\beta, D_\alpha], D_\alpha], D_\alpha], D_\alpha]D_\alpha^i j(\phi) = 0.$$

Therefore we obtain

$$[E_\beta, g(D_\alpha)]j(\phi) = (g'(D_\alpha)S_1 + g''(D_\alpha)S_2 + g'''(D_\alpha)B_0S_3)(\phi),$$

where

$$S_1(\phi) = B_0 j\langle\nabla^2\phi, e_a e_{\bar a}\rangle + D^a E^{\bar b} j\langle R\phi, e_a e_{\bar b}\rangle, \tag{4.27}$$

$$S_2(\phi) = B_0 D^a j\langle R\nabla\phi, e_b e_a e_{\bar b}\rangle + R_{a\bar b c\bar d} D^a D^c E^{\bar b} j(\nabla_{e_d}\phi), \tag{4.28}$$

$$S_3(\phi) = R_{a\bar b c\bar d} D^a D^c j\langle\nabla^2\phi, e_b e_d\rangle. \tag{4.29}$$

Define

$$C_2 = \mathcal{R}_{a\bar b} D^a \tilde L_b, \tag{4.30}$$

$$T_3 = R_{a\bar b c\bar d} e^{\bar b} D^a D^c \tilde L_d, \tag{4.31}$$

$$T_4 = -2R_{a\bar b c\bar d} e^a D^{\bar b} D^c \tilde L_d. \tag{4.32}$$

Then, in the same way, we compute

$$\begin{aligned}
[E_\gamma, g(D_\alpha)]j(\phi) &= g'(D_\alpha)(D_\alpha B_2 - D_\gamma d_\alpha - B_0 C_2)j(\phi),\\
[d_\beta, g(D_\alpha)]j(\phi) &= (g'(D_\alpha)[d_\beta, D_\alpha] + g''(D_\alpha)T_3)j(\phi),\\
[d_\gamma, g(D_\alpha)]j(\phi) &= 0,\\
[[d_\alpha, D_\beta], g(D_\alpha)]j(\phi) &= g'(D_\alpha)T_4 j(\phi).
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
[T_2, g(D_\alpha)]j(\phi) = g'(D_\alpha)&(S_1 + (k+h)[d_\beta, D_\alpha]j - B_0(2C_2 - rD_\alpha)j)(\phi)\\
&+ g''(D_\alpha)(S_2 + (k+h)T_3 j)(\phi) + g'''(D_\alpha)B_0S_3(\phi),
\end{aligned}$$

where we use

$$T_4 = D_\gamma d_\alpha - B_0C_2 - D_\alpha(B_2 - rB_0).$$

We have

$$S_3(\phi) = D_\alpha(2C_2 - rD_\alpha)j(\phi).$$

Thus, by assuming the relations:

(II"a)

$$(S_2 + (k+h)T_3 j - (n-k+h+1)(2C_2 - rD_\alpha)B_0 j)(\phi) = -D_\alpha T_2 j(\phi),$$

(II"b)
$$(S_1 + (k+h)[d_\beta, D_\alpha]j)(\phi) = -(n-k+h)T_2 j(\phi),$$

we have

$$[T_2, g(D_\alpha)]j(\phi) = -g(D_\alpha)T_2 j(\phi).$$

Therefore we have (II') and the theorem follows.

To prove (II"a) and (II"b), we compute $T_2 j(\phi)$ first. By using Lemma 4.9 and Lemma 4.10, we compute

$$E_\beta j(\phi) = -(k+h)d_\beta j(\phi),$$
$$E_\gamma j(\phi) = (h(S_4+S_5) + (k+h)B_1 j + (n-k+h)B_2 j - (n+1)rB_0 j)(\phi),$$

where

$$S_4(\phi) = \mathcal{R}_{a\bar{b}} D^a e^{\bar{c}} \phi_{b\bar{c}\bar{I}_1,\dots,\bar{I}_h} s^{\bar{I}_1,\dots,\bar{I}_h}, \tag{4.33}$$

$$S_5(\phi) = \mathcal{R}_{a\bar{b}} D^{\bar{b}} e^{c} \phi_{\bar{a}c\bar{I}_1,\dots,\bar{I}_h} s^{\bar{I}_1,\dots,\bar{I}_h}. \tag{4.34}$$

We compute

$$\begin{aligned}[d_\alpha, D_\beta] j(\phi) = h(-S_5(\phi) + B_0 j(S_0\phi)) - (n-2k)B_2 j(\phi) \\ + h(n-k)rB_0 j(\phi) + B_0 j(\mathcal{L}\phi),\end{aligned}$$

where we use (2.2) and

$$\mathcal{R}_{a\bar{b}} e^a D^{\bar{c}} \phi_{\bar{c}b\bar{I}_1,\dots,\bar{I}_h} s^{\bar{I}_1,\dots,\bar{I}_h} = -B_2 j(\phi).$$

Hence, by (4.25), we have

$$T_2 j(\phi) = h(S_4(\phi) + B_0 j(S_0\phi)) + (h-1)(n-k+1)rB_0 j(\phi) + B_0 j(\mathcal{L}\phi).$$

In order to prove (II"a), we compute first

$$\begin{aligned}R_{a\bar{b}c\bar{d}} D^a D^c E^{\bar{b}} j(\nabla_{e_d}\phi) &+ (k+h)T_3 j(\phi) \\ &= -hR_{a\bar{b}c\bar{d}} D^a D^c e^{\bar{e}} (\nabla_{e_d}\phi)_{b\bar{e}\bar{I}_1,\dots,\bar{I}_h} s^{\bar{I}_1,\dots,\bar{I}_h} \\ &= -h(D_\alpha S_4 - B_0 C_2 j + rD_\alpha B_0 j)(\phi),\end{aligned}$$

where we use

$$\mathcal{R}_{a\bar{d}} D^a D^b e^{\bar{e}} (\nabla_{e_d}\phi)_{b\bar{e}\bar{I}_1,\dots,\bar{I}_h} s^{\bar{I}_1,\dots,\bar{I}_h} = -B_0 C_2 j(\phi).$$

Next we compute using (2.2)

$$\begin{aligned}D^a j\langle R\nabla\phi, e_b e_a e_{\bar{b}}\rangle &= D^a(-j\langle R^c_b \nabla_{e_c}\phi, e_a e_{\bar{b}}\rangle + j\langle \nabla_{e_b} R\phi, e_a e_{\bar{b}}\rangle) \\ &= (h + 2(n-k+1))C_2 j(\phi) - hD_\alpha j(S_0\phi) \\ &\quad - h(n-k+1)rD_\alpha j(\phi) - D_\alpha j(\mathcal{L}\phi),\end{aligned}$$

where we use

$$\mathcal{R}_{a\bar{b}} D^a (\nabla_{e_c}\phi)_{\bar{c}b\bar{I}_1,\dots,\bar{I}_h} s^{\bar{I}_1,\dots,\bar{I}_h} = 0,$$
$$\mathcal{R}_{a\bar{b}} D^c (\nabla_{e_b}\phi)_{\bar{a}c\bar{I}_1,\dots,\bar{I}_h} s^{\bar{I}_1,\dots,\bar{I}_h} = 0.$$

Therefore we obtain (II"a).

As for (II"b), we have

$$D^a E^{\bar{b}} j\langle R\phi, e_a \bar{e_b}\rangle + (k+h)[d_\beta, D_\alpha] j(\phi) = -hS_6(\phi),$$

where

$$S_6(\phi) = D^a e^{\bar{b}} \langle R\phi, e_a e_{\bar{c}}\rangle_{c\bar{b}\bar{I}_1,\dots,\bar{I}_h} s^{\bar{I}_1,\dots,\bar{I}_h}. \tag{4.35}$$

We compute using (2.2)

$$\begin{aligned} S_6(\phi) = (&2R_{a\bar{b}c\bar{d}} D^a e^{\bar{e}} \phi_{\bar{c}db\bar{e}\bar{I}_1,\dots,\bar{I}_h} + 2(h-1) R_{a\bar{b}c\bar{d}} D^a e^{\bar{e}} \phi_{b\bar{e}\bar{I}_1,\bar{c}d\bar{I}_2,\dots,\bar{I}_h}) s^{\bar{I}_1,\dots,\bar{I}_h} \\ &+ (h(k+1) + n - 2k) S_4(\phi) + B_0 j(\mathcal{L}\phi). \end{aligned}$$

By using (1.3) and Definition 1.2, we compute

$$2R_{a\bar{b}c\bar{d}} D^a e^{\bar{e}} \phi_{\bar{c}db\bar{e}\bar{I}_1,\dots,\bar{I}_h} s^{\bar{I}_1,\dots,\bar{I}_h} = 0$$

and

$$\begin{aligned} 2R_{a\bar{b}c\bar{d}} D^a e^{\bar{e}} &\phi_{b\bar{e}\bar{I}_1,\bar{c}d\bar{I}_2,\dots,\bar{I}_h} s^{\bar{I}_1,\dots,\bar{I}_h} \\ &= -kS_4(\phi) + B_0 j(S_0\phi) + (n-k+1) r B_0 j(\phi). \end{aligned}$$

Thus we have

$$S_6(\phi) = (n-k+h) S_4(\phi) + (h-1) B_0 \left(j(S_0\phi) + (n-k+1) r j(\phi)\right) + B_0 j(\mathcal{L}\phi).$$

Hence, by using (2.4), we have (II"b). □

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