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The Smith sets of finite groups with normal Sylow 2-subgroups and small nilquotients

By

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Abstract

The Smith equivalence of real representations of a finite group has been studied by many mathematicians, e.g. J. Milnor, T. Petrie, S. Cappell-J. Shaneson, K. Pawałowski-R. Solomon. For a given finite group, let the primary Smith set of the group be the subset of real representation ring consisting of all differences of pairs of prime matched, Smith equivalent representations. The primary Smith set was rarely determined for a nonperfect group G besides the case where the primary Smith set is trivial. In this paper we determine the primary Smith set of an arbitrary Oliver group such that a Sylow 2-subgroup is normal and the nilquotient is isomorphic to the direct product of a finite number of cyclic groups of order 2 or 3. In particular, we answer to a problem posed by T. Sumi.

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1. Introduction

Let G be a finite group. Throughout this paper, we mean by a *manifold* a smooth manifold, by a *G-action on a manifold* a smooth G-action on the manifold, and by a *real G-module* a finite dimensional real G-representation space. If M is a G-manifold with a G-fixed point x then the tangent space $T_x(M)$ at x of M has the induced linear G-action which is called the *tangential G-representation* at x. It is interesting to ask how $T_x(M)$ and $T_y(M)$ are similar to each other for x, y in M^G, the G-fixed point set of M. Particularly, the case where M is a sphere has been paid attention since P. Smith [24]. He asked whether the two tangential G-representations of a G-action on a sphere with exactly two G-fixed points are isomorphic. There are well known breakthroughs of the Smith problem. For examples, M. F. Atiyah-R. Bott [1],

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J. W. Milnor [9] studied the problem for semifree actions, G. E. Bredon [2] did for 2-groups, C. U. Sanchez [23] did for groups of odd prime power order, T. Petrie [18] did for odd order, abelian groups having at least 4 noncyclic Sylow subgroups, S. E. Cappell-J. L. Shaneson [3] did for cyclic groups, E. Laitinen-K. Pawałowski [8] did for perfect groups, K. Pawałowski-R. Solomon [16] did related to the Laitinen conjecture. We also have contribution by E. C. Cho, K. H. Dovermann, J. D. Randall, D. Y. Suh, T. Sumi, L. C. Washington and etc. We can refer to articles [21], [16], [17], and [6] for surveys of history of the study of the Smith problem and bibliography.

Two real G-modules V and W are called *Smith equivalent*, and written $V \sim_{\mathfrak{S}} W$, if there exists a homotopy sphere Σ with G-action such that Σ has exactly two G-fixed points and the tangential G-representations at the two points are isomorphic to V and W as real G-modules, namely $\Sigma^G = \{a, b\}$, $T_a(\Sigma) \cong V$ and $T_b(\Sigma) \cong W$. Here a homotopy sphere is a closed manifold which is homotopy equivalent to the standard sphere of same dimension. Let $\mathrm{RO}(G)$ denote the real representation ring. Define the *Smith set* $\mathfrak{S}(G)$ by

$$\mathfrak{S}(G) = \{[V] - [W] \in \mathrm{RO}(G) \mid V \sim_{\mathfrak{S}} W\},$$

where V and W are real G-modules. K. Pawałowski-T. Sumi [17] determined whether $\mathfrak{S}(G)$ is trivial or nontrivial for most finite groups G of order ≤ 2000, while they reported that the Smith sets for the groups $G = SG(1176, 220)$, $SG(1176, 221)$ were not yet determined, where $SG(m, n)$ denotes the 'small group' of order m and type n as specified in GAP [5]. The present paper answers:

Theorem 1.1. *For the small groups* $G = SG(1176, 220)$*,* $SG(1176, 221)$*, the Smith sets* $\mathfrak{S}(G)$ *are trivial, i.e.* $\mathfrak{S}(G) = 0$*.*

This theorem shows that the Laitinen conjecture in [8] fails for $G = SG(1176, 220)$, $SG(1176, 221)$ (see Conjecture 2.1). Thus it is interesting to formulate the theorem above as a generalized criterion. For this purpose, we need to prepare notation. Let $\mathcal{P}(G)$ denote the set of all subgroups P of G such that the order of P is a power of a prime. In particular, the trivial group $\{e\}$ belongs to $\mathcal{P}(G)$. Two real G-modules V and W are said to be *prime matched* if $\mathrm{res}^G_P V \cong \mathrm{res}^G_P W$ as real P-modules for all $P \in \mathcal{P}(G)$. For a subset A of $\mathrm{RO}(G)$ and a set $\mathcal{K}$ of subgroups of G, let

$$A_{\mathcal{P}} = \{[V] - [W] \in A \mid V \text{ and } W \text{ are prime matched}\},$$
$$A^{\mathcal{K}} = \{[V] - [W] \in A \mid \dim V^K = 0 = \dim W^K \ \ \forall\, K \in \mathcal{K}\},$$

where V and W are real G-modules. We call $\mathfrak{S}(G)_{\mathcal{P}}$ the *primary Smith set*. It is known that if G does not have an element of order 8 then $\mathfrak{S}(G) = \mathfrak{S}(G)_{\mathcal{P}}$, while if G has a normal subgroup N such that G/N is a cyclic group of order 8 then $\mathfrak{S}(G) \neq \mathfrak{S}(G)_{\mathcal{P}}$. If G is a nontrivial perfect group then $\mathfrak{S}(G)_{\mathcal{P}} = \mathrm{RO}(G)^{\{G\}}_{\mathcal{P}}$. Let G^{nil} denote the smallest normal subgroup N of G such that G/N is nilpotent. In the present paper, we refer to G/G^{nil} as the *nilquotient* of G. A finite group G is called an *Oliver group* if G is not a mod-$\mathcal{P}$ hyperelementary group, namely there never exists a normal series $P \trianglelefteq H \trianglelefteq G$ such that P and G/H are of prime power order and H/P is cyclic. Due to the surveys in [21], [16], [17], [6], the reader can see that $\mathfrak{S}(G)_{\mathcal{P}}$ is rarely determined for a nonperfect group G besides the case $\mathfrak{S}(G)_{\mathcal{P}} = 0$.

Theorem 1.2. *Let G be an Oliver group satisfying the conditions.*
(1) *A Sylow* 2*-subgroup of G is normal in G.*
(2) *$G^{\mathrm{nil}} \neq G$ and G/G^{nil} is isomorphic to a direct product of cyclic groups of order* 3*.*
Then $\mathfrak{S}(G)_{\mathcal{P}}$ coincides with $\mathrm{RO}(G)_{\mathcal{P}}^{\{G^{\mathrm{nil}}\}}$.

We remark that $G = SG(1176, 220)$ and $SG(1176, 221)$ satisfy Conditions (1) and (2) above, $\mathrm{RO}(G)_{\mathcal{P}}^{\{G^{\mathrm{nil}}\}} = 0$, and the groups do not have an element of order 8. Theorem 1.2 follows from Theorem 2.2 in which we treat an arbitrary Oliver group G such that G has a normal Sylow 2-subgroup and

$$G/G^{\mathrm{nil}} \cong C_2 \times \cdots \times C_2 \times C_3 \times \cdots \times C_3,$$

where C_p denotes a cyclic group of order p.

A word should be in order on group actions on disks. For an Oliver group G and two real G-modules V and W, $V \oplus U$ and $V \oplus U$ with some real G-module U can be the tangential representations at the fixed points of a G-action on a disk with exactly two G-fixed points if and only if $[V] - [W] \in \mathrm{RO}(G)_{\mathcal{P}}^{\{G\}}$. Thus local data around G-fixed points of G-actions on spheres are subtly different from those on disks.

2. Generalization and proofs

Let G be a finite group and let $\mathcal{S}(G)$ denote the set of all subgroups of G. For a prime p, let $G^{\{p\}}$ denote the smallest normal subgroup H such that $|G/H|$ is a power of p. Then we have

$$G^{\mathrm{nil}} = \bigcap_p G^{\{p\}},$$

where p runs over the set of all primes dividing $|G|$. We will use the following families of subgroups of G. For a prime p, let

$$\begin{aligned} \mathcal{L}(G,p) &= \{L \in \mathcal{S}(G) \mid L \supseteq G^{\{p\}}\}, \\ \mathcal{L}_p(G) &= \{L \in \mathcal{L}(G) \mid L \trianglelefteq G \text{ and } |G/L| = p\}, \\ \mathcal{L}(G) &= \bigcup_q \mathcal{L}(G,q), \end{aligned}$$

where q runs over the set of all primes dividing $|G|$.

The study of the Smith equivalence in the present paper needs the following lemmas.

Lemma 2.1 ([10, Lemma 2.1]). *If K is a cyclic group of order 2 and M a connected closed manifold of dimension ≥ 1 with K-action, then $|M^K| \neq 1$.*

This lemma has a simple proof and we obtain the next by analogous arguments.

Lemma 2.2. *Let p be an odd prime. If K is a cyclic group of order p and M a connected closed orientable manifold of dimension ≥ 1 with K-action, then $|M^K| \neq 1$.*

In addition, we recall Proposition 3.2 in A. Edmonds-R. Lee [4].

Lemma 2.3 (A. Edmonds-R. Lee). *Let K be a finite group with a normal Sylow 2-subgroup and M a smooth K-manifold fulfilling $\widetilde{H}^*(M;\mathbb{Z}_2) = 0$. Then M^K is stably complex, and hence every connected component of M^K is orientable.*

Let a_G denote the number of all real conjugacy classes $(g)^{\pm} = (g) \cup (g^{-1})$ of elements $g \in G$ such that the order of g is not a power of a prime. It is known (and easily shown) that

$$\operatorname{rank} \mathrm{RO}(G)^{\{G\}}_{\mathcal{P}} = a_G - 1$$

if $a_G > 0$. The Laitinen conjecture in [8] implies the next.

Conjecture 2.1 (A_G-conjecture). If G is an Oliver group with $a_G \geq 2$ then $\mathfrak{S}(G) \neq 0$.

A real G-module is called a *gap module* if the following two conditions are fulfilled.

(1) $V^L = 0$ for all $L \in \mathcal{L}(G)$.

(2) $\dim V^P > 2 \dim V^H$ for all $P \in \mathcal{P}(G)$ and all $H \in \mathcal{S}(G)$ such that $H \supsetneq P$.

Lemma 2.4. *For an arbitrary finite group G, $\mathfrak{S}(G) \subseteq RO(G)^{\mathcal{L}_2(G)}$. If $G/G^{\{2\}} \cong C_2 \times \cdots \times C_2$, where C_2 is a cyclic group of order 2, then $\mathfrak{S}(G) \subseteq \mathrm{RO}(G)^{\{G^{\{2\}}\}}$.*

Proof. Let $[V] - [W] \in \mathfrak{S}(G)$. We may suppose $V^G = 0 = W^G$. By [10, Proposition 2.2],

$$V^N \cong W^N \text{ as real } G\text{-modules for any } N \triangleleft G \text{ such that } |G/N| = 2. \tag{2.1}$$

Thus $\mathfrak{S}(G) \subseteq RO(G)^{\mathcal{L}_2(G)}$.

Let $L = G^{\{2\}}$ and suppose $G/L = C_2 \times \cdots \times C_2$. By the representation theory, it holds that

$$V^L = \bigoplus_{N \triangleleft G:\ |G/N|=2} V^N \quad \text{and} \quad W^L = \bigoplus_{N \triangleleft G:\ |G/N|=2} W^N$$

as real G-modules. Thus (2.1) implies $V^L \cong W^L$ as real G-modules. Thus we get

$$\mathfrak{S}(G) \subseteq \mathrm{RO}(G)^{\{G^{\{2\}}\}}.$$

□

Similarly we obtain the next lemma.

Lemma 2.5. *If G is a finite group with a normal Sylow 2-subgroup then*

$$\dim V^N = \dim W^N$$

holds for arbitrary $[V]-[W] \in \mathfrak{S}(G)$ and arbitrary $N \lhd G$ such that $|G/N|$ is a prime. In particular,

$$\mathfrak{S}(G) \subseteq \mathrm{RO}(G)^{\mathcal{L}_2(G)\cup\mathcal{L}_3(G)}.$$

If furthermore $G/G^{\{3\}} \cong C_3 \times \cdots \times C_3$, where C_3 is a cyclic group of order 3, then $\mathfrak{S}(G) \subseteq \mathrm{RO}(G)^{\{G^{\{3\}}\}}$.

Proof. Let Σ be a homotopy sphere with G-action such that $\Sigma^G = \{a, b\}$, $a \neq b$. Set $V = T_a(\Sigma)$ and $W = T_b(\Sigma)$. Let N be a normal subgroup of G with $|G/N| = p$. We are going to prove $\dim V^N = \dim W^N$ by showing the contrary assumption that $\dim V^N \neq \dim W^N$ implies a contradiction. Let Σ_a^N and Σ_b^N denote the connected components of Σ^N containing a and b, respectively. The assumption implies $\dim \Sigma_a^N > 0$ or $\dim \Sigma_b^N > 0$. In particular, we get $\dim \Sigma > 0$. If $\Sigma_a^N = \Sigma_b^N$ then $\dim V^N = \dim W^N$. Thus, we have $\Sigma_a^N \neq \Sigma_b^N$. Each connected component of Σ^N is a connected component of $\Sigma^N \smallsetminus \{a\}$ or $\Sigma^N \smallsetminus \{b\}$. Since Σ is a homotopy sphere of dimension ≥ 1, $\Sigma \smallsetminus \{a\}$ and $\Sigma \smallsetminus \{b\}$ are homeomorphic to the Euclidean space. By Lemma 2.3, each connected component of $\Sigma^N \smallsetminus \{a\}$ and $\Sigma^N \smallsetminus \{b\}$ is orientable. Thus Σ_a^N and Σ_b^N are orientable. Note $(\Sigma_a^N)^{G/N} = \{a\}$ and $(\Sigma_b^N)^{G/N} = \{b\}$. But by Lemma 2.2, this can not happen. Namely we have encountered with a contradiction, and hence proved $\dim V^N = \dim W^N$.

Note that if $|G/N| = 3$ then $V^N \cong W^N$ follows from $\dim V^G = \dim W^G$ and $\dim V^N = \dim W^N$. By arguments similar to ones in Proof of Lemma 2.4, we obtain $\mathfrak{S}(G) \subseteq \mathrm{RO}(G)^{\mathcal{L}_3(G)}$. By Lemma 2.4, we get $\mathfrak{S}(G) \subseteq \mathrm{RO}(G)^{\mathcal{L}_2(G)\cup\mathcal{L}_3(G)}$. □

The next theorem immediately follows.

Theorem 2.1. *If G is a finite group with a normal Sylow 2-subgroup and*

$$G/G^{\mathrm{nil}} \cong C_2 \times \cdots \times C_2 \times C_3 \times \cdots \times C_3$$

then $\mathfrak{S}(G) \subseteq \mathrm{RO}(G)^{\mathcal{L}(G)}$

A finite group G is called a *gap group* if there exists a gap real G-module. It is known that any Oliver group G satisfying one of the following conditions is a gap group.

(1) $G = G^{\{2\}}$ ([7, Theorem 2.3]).

(2) $|G/G^{\mathrm{nil}}|$ is divisible by (at least 2) distinct odd primes ([7, Theorem 2.3]).

(3) A Sylow 2-subgroup of G is normal in G ([13, Proposition 4.3]).

If G is a nilpotent Oliver group then Conditions (2) and (3) above are satisfied.

We can show the next fact by computation using data available from GAP [5] and the details are left to the reader.

Fact 2.1. *If G is either $SG(1176, 220)$ or $SG(1176, 221)$ then the following properties hold.*

(1) *G is an Oliver group, i.e. not a mod $\mathcal{P}$ hyperelementary group.*

(2) *Any element of G is not of order 8.*

(3) *A Sylow 2-subgroup of G is normal in G.*

(4) *$G^{\{3\}} = [G, G]$ and $|G/G^{\{3\}}| = 3$.*

(5) *$G = G^{\{p\}}$ for any prime $p \neq 3$.*

(6) *$G/G^{\mathrm{nil}} \cong C_3$ (the cyclic group of order 3).*

(7) *$\mathrm{RO}(G)_{\mathcal{P}}^{\mathcal{L}(G)} = \mathrm{RO}(G)_{\mathcal{P}}^{G^{\{3\}}} = 0$.*

(8) *G is a gap group.*

Hence if G is either $SG(1176, 220)$ or $SG(1176, 221)$ then

$$\mathfrak{S}(G) = \mathfrak{S}(G)_{\mathcal{P}} \subseteq \mathrm{RO}_{\mathcal{P}}^{G^{\{3\}}} = 0.$$

We have proved Theorem 1.1.

Recall Realization Theorem in [16].

Lemma 2.6 (K. Pawałowski-R. Solomon)**.** *If G is a gap Oliver group then $\mathrm{RO}(G)_{\mathcal{P}}^{\mathcal{L}} \subseteq \mathfrak{S}(G)_{\mathcal{P}}$.*

We, however, do not know whether there exists an Oliver group G such that $\mathrm{RO}(G)_{\mathcal{P}}^{\mathcal{L}} \not\subseteq \mathfrak{S}(G)_{\mathcal{P}}$. Since any Oliver group with a normal Sylow 2-subgroup is a gap group, we obtain the next.

Theorem 2.2. *If G is an Oliver group with a normal Sylow 2-subgroup and*

$$G/G^{\mathrm{nil}} \cong C_2 \times \cdots \times C_2 \times C_3 \times \cdots \times C_3$$

then

$$\mathfrak{S}(G)_{\mathcal{P}} = \mathrm{RO}(G)_{\mathcal{P}}^{\mathcal{L}(G)} \quad (= \mathrm{RO}(G)_{\mathcal{P}}^{\mathcal{L}_2(G) \cup \mathcal{L}_3(G)}).$$

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References

[1] M. F. Atiyah and R. Bott, *A Lefschetz fixed point formula for elliptic complexes: II. Applications*, Ann. of Math. **88** (1968), 451–491.

[2] G. E. Bredon, *Representations at fixed points of smooth actions of compact groups*, Ann. of Math. (2) **89** (1969), 515–532.

[3] S. E. Cappell and J. L. Shaneson, *Representations at fixed points*, Group Actions on Manifolds (Boulder, Colo., 1983), Contemp. Math. **36**, Amer. Math. Soc., Providence, RI, 1985, pp. 151–158.

[4] A. L. Edmonds and Ronnie Lee, *Fixed point sets of group actions on Euclidean space*, Topology **14** (1975), 339–345.

[5] GAP, *Groups, Algorithms and Programming, a System for Computational Discrete Algebra*, Release 4.3, 06 May 2002, URL: http://www.gap-system.org.

[6] X.M. Ju, *The Smith isomorphism question: A review and new results*, RIMS Kokyuroku no. 1569 (2007), Res. Inst. Math. Sci., Kyoto Univ., 43–51.

[7] E. Laitinen and M. Morimoto, *Finite groups with smooth one fixed point actions on spheres*, Forum Math. **10** (1998), 479–520.

[8] E. Laitinen and K. Pawałowski, *Smith equivalence of representations for finite perfect groups*, Proc. Amer. Math. Soc. **127** (1999), 297–307.

[9] J. W. Milnor, *Whitehead torsion*, Bull. Amer. Math. Soc. **72** (1966), 358–426.

[10] M. Morimoto, *The Smith problem and a counterexample to Laitinen's conjecture*, RIMS Kokyuroku no. 1517 (2006), Res. Inst. Math. Sci., Kyoto Univ., 25–31.

[11] ______, *Construction of smooth actions on spheres for Smith equivalent representations*, RIMS Kokyuroku no. 1569 (2007), Res. Inst. Math. Sci., Kyoto Univ., 52–58.

[12] ______, *Smith equivalent* $\mathrm{Aut}(A_6)$*-representations are isomorphic*, accepted by Proc. Amer. Math. Soc.

[13] M. Morimoto, T. Sumi and M. Yanagihara, *Finite groups possessing gap modules*, in: Geometry and Topology, Aarhus 1998, Contemp. Math. **258**, Amer. Math. Soc., Providence, 2000, pp. 329–342.

[14] B. Oliver, *Fixed point sets and tangent bundles of actions on disks and Euclidean spaces*, Topology **35** (1996), 583–615.

[15] R. Oliver, *Fixed point sets of groups on finite acyclic complexes*, Comment. Math. Helv. **50** (1975), 155–177.

[16] K. Pawałowski and R. Solomon, *Smith equivalence and finite Oliver groups with Laitinen number* 0 *or* 1, Algebr. Geom. Topol. **2** (2002), 843–895.

[17] K. Pawałowski and T. Sumi, *Finite groups with Smith equivalent, nonisomorphic representations*, in Proc. 33rd Symposium on Transformation Groups, in Yokohama 2006, (ed. T. Kawakami), Wing Co. Ltd., Wakayama, Japan, 2007, pp. 68–76.

[18] T. Petrie, *Three theorems in transformation groups*, Algebraic Topology (Aarhus 1978), Lecture Notes in Math. **763**, Springer Verlag, Berlin-Heidelberg-New York, 1979, pp. 549–572.

[19] ______, *The equivariant J homomorphism and Smith equivalence of representations*, Current Trends in Algebraic Topology (London, Ont., 1981), CMS Conf. Proc. 2, Part 2, Amer. Math. Soc., Providence, RI, 1982, pp. 223–233.

[20] ______, *Smith equivalence of representations*, Math. Proc. Cambridge Philos. Soc. **94** (1983), 61–99.

[21] T. Petrie and J. Randall, *Transformation Groups on Manifolds*, Marcel Dekker, Inc., New York and Basel, 1984.

[22] ______, *Spherical isotropy representations*, Publ. Math. IHES **62** (1985), 5–40.

[23] C. U. Sanchez, *Actions of groups of odd order on compact orientable manifolds*, Proc. Amer. Math. Soc. **54** (1976), 445–448.

[24] P. A. Smith, *New results and old problems in finite transformation groups*, Bull. Amer. Math. Soc. **66** (1960), 401–415.

[25] T. Sumi, *Finite groups possessing Smith equivalent, nonisomorphic representations*, RIMS Kokyuroku no. 1569 (2007), Res. Inst. Math. Sci., Kyoto Univ., 170–179.