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Asymptotic expansions for functionals of a Poisson random measure

By

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1. Introduction

The Malliavin calculus for functionals of a Poisson random measure has been developed by many authors. Bismut [2] has generalized the Malliavin calculus for Wiener-Poisson functionals by using the Girsanov theorem. As another method, in Bichteler, Gravereaux and Jacod [1], one can find the study of the Malliavin operator on Wiener-Poisson space and application of it to stochastic differential equations. Both in these works, the authors have given differential operators on Wiener-Poisson space and have proved the integration by parts formulas. These formulation suffers some limitation on an intensity measure, that is, the intensity measure must have a smooth density.

On the other hand, in the Malliavin calculus for Wiener functionals, Wiener chaos expansion of the space of square integrable Wiener functionals can be considered as a Fock space, and the differential operator is regarded as the annihilation operator on a Fock space. This sort of structure can be also found in the case of the space of square integrable functionals of Wiener-Poisson space, see [6]. Nualart and Vives [10], [11], and Picard [13] have studied the annihilation operator and its dual operator (the creation operator) on the space of square integrable functionals of a Poisson random measure. Picard [12] has also given a smoothness criterion by using the duality formula (see Theorem 2.1) for functionals of a Poisson random measure under the Condition 1 (see Section 2) on the intensity measure, and has studied the solution to some stochastic differential equation. This argument of Picard can be generalized for some Wiener-Poisson functionals, see [5]. The Condition 1 differs from that of [1], and allows us to take a intensity measure with some singularity. One can find some interesting examples satisfying Condition 1, for instance, stable processes and CGMY processes (see [3]).

The purpose of this paper is to prove the asymptotic expansion theorem (done in the Wiener space by Watanabe [18]) for functionals of a Poisson random measure. By using the Malliavin operator which we mentioned above, Sakamoto and Yoshida [15] have studied asymptotic expansion formulas of some

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Wiener-Poisson functionals in the statistical view point. In particular, the integration by parts formula has played an important role in [15]. However, as we mentioned above, the intensity measure must have a smooth density. On the other hand, we adopt the framework of Picard [12]. Hence, we can consider functionals of a Poisson random measure with an intensity measure which may have some singularity. Let us roughly explain our main result (Theorem 5.1). We shall introduce Sobolev space $\mathbf{D}_{k,p}$ with norm $|\cdot|_{k,p}$ in Section 2, and give a modification of smoothness criterion of Picard [12] in Section 3. If $F \in \mathbf{D}_\infty := \cap_{k=0}^{\infty} \cap_{p\geq 2} \mathbf{D}_{k,p}$ satisfies the non-degenerate condition, this modification claims that

$$\sup_{|G|_{k,p}=1} |\, \mathbf{E}[Ge^{i\boldsymbol{\xi}\cdot F}]| \leq C_{k,p}(1+|\,\boldsymbol{\xi}\,|^2)^{-(1-\frac{\alpha}{2\beta})\frac{k}{2}},$$

where α, β are some positive constants with $\frac{\alpha}{2} < \beta$. From this inequality, one can show that the function $\psi(\boldsymbol{\xi}) := \mathbf{E}[Ge^{i\boldsymbol{\xi}\cdot F}]$ is a rapidly decreasing function, see Remark 8. In the Malliavin calculus on Wiener space, composites of Schwartz distributions and smooth Wiener functionals are linear continuous functionals on the space of smooth functionals, and can be evaluated by using the integration by parts formula. On the other hand, as we mentioned above, we cannot use the integration by parts formula in our formulation. Hence, to define composites of Schwatz distributions and functionals of a Poisson random measure as linear continuous functionals on $\mathbf{D}_\infty$, we choose the following way; since $\psi(\boldsymbol{\xi}) = \mathbf{E}[Ge^{i\boldsymbol{\xi}\cdot F}]$ is a rapidly decreasing function, we evaluate the composition as follows

$$\langle T \circ F, G\rangle = {}_{\mathcal{S}'}\langle\, \mathcal{F}T\,,\, \psi\,\rangle_{\mathcal{S}},$$

where T is a Schwartz distribution and $\mathcal{F}T$ is the Fourier transform of T. In Section 4, we shall precisely discuss the definition. Now, we shall mention our main result. We shall consider the parametrized functionals $F(\epsilon)$ $\epsilon \in (0,1]$. The asymptotic expansion $F(\epsilon) \sim \sum_{n=0}^{\infty} \epsilon^n f_n$ can be defined similarly as that of Watanabe [18]. If $F(\epsilon)$ has the asymptotic expansion and satisfies the uniformly non-degenerate condition, then the composition $T \circ F(\epsilon)$ has also the asymptotic expansion $\sum_{n=0}^{\infty} \epsilon^n \Phi_n$, where Φ_n's are linear continuous functionals on $\mathbf{D}_\infty$, and are given by the formal Taylor expansion. Hence, roughly speaking, the asymptotic expansion theorem for functionals of a Poisson random measure can be obtained similarly as that of Watanabe [18].

As an application, we shall give the asymptotic expansion of some stochastic differential equation. Our application is the analogue to the study of Kunitomo-Takahashi [7]. Kunitomo-Takahashi [7], [8] have applied the asymptotic expansion of [17] and of [20], [19] to mathematical finance. In [9], they have considered the following stochastic differential equation:

$$dS_t(\epsilon) = rS_t(\epsilon)\,dt + \epsilon\sigma(S_t(\epsilon), t)dW_t,$$

where W_s is a Brownian motion, and have given the asymptotic expansion $S_t(\epsilon) \sim \sum_{j=0}^{\infty} \epsilon^j A_j$. By using this expansion, they have estimated the option price such as $\mathbf{E}[(S_t(\epsilon)-K)_+]$. The authors have also studied jump diffusion case:

$$dS_t(\epsilon) = rS_t(\epsilon)\,dt + \epsilon\sigma(S_t(\epsilon),t)dW_t + \int_{\mathbf{R}} S_{t-}(\epsilon)(e^{\epsilon x}-1)\tilde{N}(dt\times dx)$$

where $\tilde{N}$ is the compensated Poisson random measure whose intensity measure is the Lebesgue measure or the normal distribution by using the formulation of [1]. On the other hand, we shall consider the following stochastic differential equation:

$$dX_t(\epsilon) = b(X_t(\epsilon))dt + \epsilon\int_{\mathbf{R}} \sigma(X_{t-},y)\tilde{N}(dt\times dy),$$

where the intensity measure of $\tilde{N}$ satisfies the Condition1. If the stock price process is given by the stochastic differential equation driven by a Lévy process, to study the asymptotic expansion of $X_t(\epsilon)\sim\sum_{j=0}^{\infty}\epsilon^j f_j$ seems important to evaluate the option price, as in [9]. However, we do not deal with numerical simulation in this work. Financial interpretation of the application can be found in [7]–[9]. As an example, we shall see that geometrical CGMY process satisfies the uniformly non-degenerate condition, although we need some modification on the tail of the density of the Lévy measure.

The reminder of the paper is organized as follows: In the next section, we shall give general notation and introduce the Sobolev space. In section 3, we shall exhibit the preliminary results. In particular, we shall give a modification of the smoothness criterion of Picard for our purpose. Although the proof is essentially due to Picard , we need to give the modification of the proof of his main theorem, to prove the asymptotic expansion theorem. We shall give the proof in Section 8. In Section 4, we shall formulate the composition of the functionals of a Poisson random measure and Schwartz distributions. In section 5, we shall define the asymptotic expansion, and give its elementary properties. The asymptotic expansion theorem will be proven in Section 6. In Section 7, we shall prove the asymptotic expansion of the stochastic differential equation and also give a sufficient condition that satisfies the uniformly non-degenerate condition.

2. Notation

(1). *Probability space and hypothesis.* We set $E=[0,1]\times(\mathbf{R}^d\cap\{0\}^c)$. Let $\mathcal{E}$ be the Borel σ-algebra on E, and $\lambda(du)$ a σ-finite and infinite Radon measure without atoms defined on $(E,\mathcal{E})$. We define

$$\gamma(u)=|x| \quad \text{for } u=(s,x)\in[0,1]\times(\mathbf{R}^{d_0}\cap\{0\}^c)=E.$$

Throughout this paper, we suppose that the measure space $(E,\mathcal{E},\lambda)$ satisfies the following condition:

Condition 1.

Γ-1). $\int_E \gamma^2(u) \wedge 1\, \lambda(du) < \infty$, where $\gamma(u) \wedge 1 = \min\{\gamma(u), 1\}$;

Γ-2). there exists $\alpha \in (0, 2)$ and $c_0 > 0$ such that for each $\rho \in (0, 1]$

$$\Gamma(\rho) := \int_{A(\rho)} \gamma^2(u)\, \lambda(du) \geq c_0 \rho^\alpha,$$

where $A(\rho) = \{u \in E\,;\, \gamma(u) \leq \rho\}$. We shall use the notation α, $\gamma(u)$, and $\Gamma(\rho)$ in the whole paper.

By a point measure on $(E, \mathcal{E})$, we mean a measure ω which has the form $\omega(B) = \sum_j \delta_{u_j}(B)$, where $u_j \in E$, $B \in \mathcal{E}$ and δ_{u_j} is the Dirac point measure at u_j. Define $N(\omega, B) = \omega(B)$ for a point measure ω and for $B \in \mathcal{E}$. We denote by Ω the space of all of point measures on $(E, \mathcal{E})$ such that $\omega(\{u\}) \leq 1$ for each $u \in E$, and by $\mathcal{F}_0$ the smallest σ-algebra such that $N(\cdot, B)$ is measurable for each $B \in \mathcal{E}$. Then it is well-known that there is the probability measure $\mathbf{P}$ on $(\Omega, \mathcal{F}_0)$ such that, under $\mathbf{P}$, $\{N(\cdot, B); B \in \mathcal{E}\}$ is a Poisson random measure with the intensity measure λ:

P1). for $B \in \mathcal{E}$ with $\lambda(B) < \infty$, $N(B)$ is a Poisson random variable with mean $\lambda(B)$;

P2). $N(B) = \infty$, if $\lambda(B) = \infty$;

P3). for $B_1, \ldots, B_k \in \mathcal{E}$, random variables $N(B_1), \ldots, N(B_k)$ are independent, if $B_1, \ldots, B_k$ are pairwise disjoint.

We denote

$$N(B) = N(\omega, B), \quad |N|(du) = N(du) + \lambda(du), \quad \tilde{N}(B) = N(B) - \lambda(B).$$

We write $\mathcal{F}$ for the $\mathbf{P}$-completion of $\mathcal{F}_0$.

(2). *Maps* ε^+, ε^-, *and operators* D, D^*. We shall introduce maps $\varepsilon_u^-, \varepsilon_u^+$ $(u \in E)$ from Ω to Ω. They are defined as follows

$$\varepsilon_u^- \omega(A) = \omega(A \cap \{u\}^c), \quad \varepsilon_u^+ \omega(A) = \varepsilon_u^- \omega(A) + 1_A(u).$$

For a functional $F(\omega)$, we write $(F \circ \varepsilon_u^-)(\omega)$ for $F(\varepsilon_u^- \omega)$, and $(F \circ \varepsilon_u^+)(\omega)$ for $F(\varepsilon_u^+ \omega)$. Remark that these maps are defined for a random parameter ω, and are not well-defined for a random variable; $F = 0$ $\mathbf{P}$-a.s does not always mean that $F \circ \varepsilon_u^+ = 0$ $\mathbf{P}$-a.s. However these maps are well-defined as a stochastic process parametrized by $u \in E$; $F = 0$ $\mathbf{P}$-a.s. implies that $F \circ \varepsilon_u^\pm = 0$ $\lambda \times \mathbf{P}$-a.e.. One can check that for each ω,

$$\varepsilon_u^- \omega = \omega \qquad \lambda(du)\text{-a.e.}, \qquad \varepsilon_u^+ \omega = \omega \qquad N(du)\text{-a.e.}, \tag{2.1}$$

$$\varepsilon_{u_1}^{\theta_1} \circ \varepsilon_{u_2}^{\theta_2} \omega = \varepsilon_{u_2}^{\theta_2} \circ \varepsilon_{u_1}^{\theta_1} \omega, \qquad \varepsilon_u^{\theta_1} \circ \varepsilon_u^{\theta_2} \omega = \varepsilon_u^{\theta_1} \omega, \tag{2.2}$$

where $\theta_1, \theta_2 \in \{+, -\}$. In this paper, we say that the process $\{Z_u; u \in E\}$ is integrable, if

$$\mathbf{E}\left[\int_E |Z_u| \lambda(du)\right] < \infty.$$

and denote by $\mathcal{Z}$ the class of all of integrable processes. In Section 1 in [12], one can find the following property:

$$\tag{2.3} \begin{aligned} \mathbf{E}\left[\int Z_u \circ \varepsilon_u^+ \,\lambda(du)\right] &= \mathbf{E}\left[\int Z_u \, N(du)\right], \\ \mathbf{E}\left[\int Z_u \,\lambda(du)\right] &= \mathbf{E}\left[\int Z_u \circ \varepsilon_u^- \, N(du)\right], \end{aligned}$$

for $Z \in \mathcal{Z}$.

Now, we introduce the operator D and its dual operator D^*. For a functional F, the operator D is defined by

$$D_u F = F \circ \varepsilon_u^+ - F.$$

For an integrable process Z_u, the operator D^* ($\Omega \times E \mapsto \Omega$) is defined by

$$\tag{2.4} D^* Z = \int_E Z_u \circ \varepsilon_u^- \tilde{N}(du).$$

The notion "dual" follows from:

Theorem 2.1. *Let F be in $\mathbf{L}^2(\Omega)$ and Z_u be in $\mathcal{Z}$. Suppose that*

$$\mathbf{E}\left[\int_E |F Z_u| \lambda(du)\right] + \mathbf{E}\left[\int_E |D_u F Z_u| \lambda(du)\right] < \infty.$$

Then, we have

$$\tag{2.5} \mathbf{E}\left[\int_E (D_u F) Z_u \,\lambda(du)\right] = \mathbf{E}[F D^* Z].$$

Remark 1. See Theorem 2 in [11].

The operators D and D^* are frequently said to be the *annihilation* and the *creation* operator, respectively. These notions follow from analysis on the Fock space associated to the Hilbert space $\mathbf{L}^2(E, \mathcal{B}, \lambda)$ (see [10]). In particular, it follows from analysis on a Fock space that the operator D is closable.

For functionals F and G, we shall use the following rule of D:

$$\tag{2.6} D_u(FG) = F D_u G + G D_u F + D_u F D_u G.$$

$$\tag{2.7} F \circ \varepsilon_u^+ = F + D_u F,$$

$$\tag{2.8} F \circ \varepsilon_{u_1}^+ \circ \varepsilon_{u_2}^+ = D_{u_1} D_{u_2} F + D_{u_1} F + D_{u_2} F + F.$$

By the definition of D and (2.2), we have also

$$\tag{2.9} D_u(F \circ \varepsilon_u^-) = 0.$$

Frequently, we shall consider the product measure space $(E^k, \otimes^k \mathcal{E}, \otimes^k \lambda)$, and use the convention $E^0 = \emptyset$, $\varepsilon_\emptyset^\pm \omega = \omega$, and $D_\emptyset F = F$. For a non-negative integer k, and for $\sigma = (u_1, \ldots, u_k) \in E^k$, we denote

$$\begin{aligned} &\varepsilon_\sigma^+ = \varepsilon_{u_1}^+ \circ \cdots \circ \varepsilon_{u_k}^+, \quad \varepsilon_\sigma^- = \varepsilon_{u_1}^- \circ \cdots \circ \varepsilon_{u_k}^- \\ &D_\sigma F = D_{u_1} \cdots D_{u_k} F. \end{aligned}$$

If we need to express the length of $\sigma = (u_1, \dots, u_k)$, then we denote $D^k_\sigma F = D_\sigma F$. The formula (2.6) can be generalized as follows:

$$D^k_\sigma(FG) = \sum_{\substack{\tau_1,\tau_2\subset\sigma\\ \tau_1\cup\tau_2=\sigma}} c_{|\tau_1|,|\tau_2|}(D_{\tau_1}F)(D_{\tau_2}G), \tag{2.10}$$

where $c_{|\tau_1|,|\tau_2|}$ is a constant which depends on the length of τ_1 and τ_2. If F takes values in $\mathbf{R}^d$, then we define $D_\sigma F = (D_\sigma F_1, \dots, D_\sigma F_d)$.

(3). *Spaces of random variables.* Here, we shall introduce some spaces of random variables. We denote by $\mathbf{L}^p(\Omega)$ the L^p-space on $(\Omega, \mathcal{F}, \mathbf{P})$ and by $\|\cdot\|_p$ the L^p-norm. If a functional $F = (F_1, \dots, F_d)$ values in $\mathbf{R}^d$ then we define

$$\|F\|_p = \mathbf{E}[|F|^p]^{\frac{1}{p}},$$
$$\mathbf{L}^p(\Omega;\mathbf{R}^d) = \{\, F\,; \mathbf{R}^d\text{-valued random variable and } \|F\|_p < \infty\}.$$

For a non-negative integer k, we extend the function γ and the measure $\lambda(du)$ by putting

$$\gamma(\sigma) = \gamma(u_1)\cdots\gamma(u_k), \quad \lambda(d\sigma) = \otimes^k\lambda(d\sigma) = \lambda(du_1)\dots\lambda(du_k).$$

In the case $k = 0$, we also use the convention $\gamma(\emptyset) = 1$. For a random variable F which takes values in $\mathbf{R}^d$, for $\rho \in (0, 1)$, for $p \geq 2$, and for a non-negative integer k, we define

$$|F|_{\mathbf{D}_{k,p,\rho}(\mathbf{R}^d)} = \left[\mathbf{E}[|F|^p] + \sum_{j=1}^{k} \operatorname*{ess\,sup}_{\sigma\in A^j(\rho)} \mathbf{E}\left[\left|\frac{D^j_\sigma F}{\gamma(\sigma)}\right|^p\right]\right]^{\frac{1}{p}}, \tag{2.11}$$

where the essential supremum is relative to the measure $\lambda(d\sigma)$.

Remark 2. This norm corresponds to the condition (a) in Theorem 2.1 of Picard [12]: for each p, k, $\|D^k_\sigma F\|_p \leq C_{p,k}\gamma(\sigma) \quad \lambda(d\sigma)$-a.e..

We shall use the convention

$$|F|_{\mathbf{D}_{0,p,\rho}(\mathbf{R}^d)} = \|F\|_p, \quad |F|_{k,p,\rho} = |F|_{\mathbf{D}_{k,p,\rho}(\mathbf{R}^1)}, \quad |F|_{\mathbf{D}_{k,p}(\mathbf{R}^d)} = |F|_{\mathbf{D}_{k,p,1}(\mathbf{R}^d)}.$$

Definition 2.1. We denote by $\mathbf{D}_{k,p}(\mathbf{R}^d)$ the set of all $F \in \mathbf{L}^p(\Omega;\mathbf{R}^d)$ for which there is a sequence $\{F_n : n = 1, 2, \dots\} \subset \mathbf{L}^p(\Omega;\mathbf{R}^d)$ going to F in $\mathbf{L}^p(\Omega;\mathbf{R}^d)$, such that $|F_n|_{\mathbf{D}_{k,p}(\mathbf{R}^d)} < \infty$ and $|F_n - F_m|_{\mathbf{D}_{k,p}(\mathbf{R}^d)} \to 0$ as $n, m \to \infty$. We define $D_\sigma F = \lim_{n\to\infty} D_\sigma F_n$. We denote $\mathbf{D}_\infty(\mathbf{R}^d) = \bigcap_{p\geq 2}\bigcap_{k=0}^{\infty} \mathbf{D}_{k,p}(\mathbf{R}^d)$. In the case $d = 1$, we write $\mathbf{D}_{k,p}$ for $\mathbf{D}_{k,p}(\mathbf{R})$, and $\mathbf{D}_\infty$ for $\bigcap_{p\geq 2}\bigcap_{k=0}^{\infty} \mathbf{D}_{k,p}$.

Remark 3. We shall show the closability of D with respect to the norm $|\cdot|_{\mathbf{D}_{k,p}}$. For this, we shall show the closability of $D : \mathbf{L}^2(\Omega) \mapsto \mathbf{L}^2(\Omega \times A(1))$ in the first place. Let $\mathcal{P}_n$ be the set of permutations of $\{1, 2, \dots, n\}$. We define

$$(\mathbf{L}^2(E))^{\odot n} := \{f \in \mathbf{L}^2(E^n)\,; f(u_{\sigma(1)}, \dots, u_{\sigma(n)}) = f(u_1, \dots, u_n), \forall\sigma \in \mathcal{P}_n\}.$$

Recall that $\mathbf{L}^2(\Omega)$ can be considered as the Fock space $\bigoplus_{n=0}^{\infty}\mathcal{C}_n$, where $\mathcal{C}_n$ is the Wiener chaos of n-th order:

$$\mathcal{C}_n := \left\{ I_n(f)\,;\, I_n(f) = \int_{\{\sigma=(u_1,\ldots,u_n)\,;\,u_i\neq u_j\}} f(\sigma)\,\tilde{N}(d\sigma)\,,\; f\in(\mathbf{L}^2(E))^{\odot n}\right\}$$

As we mentioned above, D can be considered as the annihilation operator: $D_uI_n(f) = nI_{n-1}(f(\cdot,u))$. Hence, we have

$$\begin{aligned}\mathbf{E}\left[\int_{A(1)}|D_uI_n(f)|^2\lambda(du)\right] &\leq \mathbf{E}\left[\int |D_uI_n(f)|^2\lambda(du)\right]\\ &= n^2\,\mathbf{E}\left[\int |I_{n-1}(f(\cdot,u)|^2\lambda(du)\right]\\ &= nn!\int_{E^n}|f(\sigma)|^2\lambda(d\sigma) = nn!\,\mathbf{E}[|I_n(f)|^2].\end{aligned}$$

This means that the restriction of the operator D on each chaos $\mathcal{C}_n$ is closed. Hence, the operator $D:\mathbf{L}^2(\Omega)=\bigoplus_{n=0}^{\infty}\mathcal{C}_n\mapsto\mathbf{L}^2(\Omega\times A(1))$ is closed. Now, suppose that $F_n\to 0$ $(n\to\infty)$ in $\mathbf{L}^2(\Omega)$ and that $\operatorname{ess\,sup}_{u\in A(1)}\left\|\frac{D_uF_n-Z_u}{\gamma(u)}\right\|_2\to 0$ $(n\to\infty)$ for some process Z_u. Then, we have

$$\begin{aligned}\mathbf{E}\left[\int_{A(1)}|D_uF_n-Z_u|^2\lambda(du)\right]^{\frac{1}{2}} &\leq \left(\int_{A(1)}\gamma^2(u)\lambda(du)\right)^{\frac{1}{2}}\operatorname*{ess\,sup}_{u\in A(1)}\left\|\frac{D_uF_n-Z_u}{\gamma(u)}\right\|_2\\ &\to 0,\end{aligned}$$

as $n\to\infty$. Since $D:\mathbf{L}^2(\Omega)\mapsto\mathbf{L}^2(\Omega\times A(1))$ is closable, we have $Z_u=0$, $\mathbf{P}\times\lambda(du)$-*a.e.*. This means D is also closable with respect to the norm $|\cdot|_{\mathbf{D}_{k,p}}$. Therefore, the space $\mathbf{D}_{k,p}(\mathbf{R}^d)$ is a Banach space for $p\geq 2$.

One can check that, for $k\leq k'$ and for $p\leq p'$.

$$|F|_{\mathbf{D}_{k,p,\rho}(\mathbf{R}^d)}\leq|F|_{\mathbf{D}_{k',p',\rho}(\mathbf{R}^d)},\tag{2.12}$$

and that $\mathbf{D}_\infty$ is an algebra, that is, $F,G\in\mathbf{D}_\infty$ implies $FG\in\mathbf{D}_\infty$

For a complex valued process Z_u ($u\in E$), we define

$$|Z|^{\sim}_{k,p,\rho} := \left[\sum_{j=0}^{k}\operatorname*{ess\,sup}_{(u,\sigma)\in A(\rho)}\mathbf{E}\left[\left|\frac{D^j_\sigma Z_u}{\gamma(\sigma)\gamma(u)}\right|^p\right]\right]^{\frac{1}{p}},$$

where we used the convention $|Z|^{\sim}_{0,p,\rho}=\operatorname{ess\,sup}_{u\in A(\rho)}\mathbf{E}\left[\left|\frac{Z_u}{\gamma(u)}\right|^p\right]^{\frac{1}{p}}$. We define

$$\mathbf{D}^{\sim}_{k,p}=\{Z_u\,;\,|Z|^{\sim}_{k,p,\rho}<\infty\},\quad \mathbf{D}^{\sim}_{\infty}=\bigcap_{p\geq 2}\bigcap_{k=0}^{\infty}\mathbf{D}^{\sim}_{k,p}.$$

Lemma 2.1. *Let p_1, p_2, and r be positive numbers satisfying $p_1, p_2, r \geq 1$ and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r} \leq 1$. For any non-negative integer k, suppose that $|F^{(i)}|_{k,p_i,\rho} < \infty$, $|Z^{(i)}|^{\sim}_{k,p_i,\rho} < \infty$ for $i = 1, 2$. Then, there is a positive constant $C = C(p_1, p_2, r, k)$ such that the following inequalities hold*

$$|F^1 F^2|_{k,r,\rho} \leq C|F^1|_{k,p_1,\rho}|F^2|_{k,p_2,\rho}, \tag{2.13}$$

$$|Z^1 F^2|^{\sim}_{k,r,\rho} \leq C|Z^1|^{\sim}_{k,p_1,\rho}|F^2|_{k,p_2,\rho}, \tag{2.14}$$

$$|Z^1 Z^2|^{\sim}_{k,r,\rho} \leq C\rho|Z^1|^{\sim}_{k,p_1,\rho}|Z^2|^{\sim}_{k,p_2,\rho} \tag{2.15}$$

$$\left|\int_{A(\rho)} Z^1_u Z^2_u \,\lambda(du)\right|_{k,r,\rho} \leq C\Gamma(\rho)|Z^1|^{\sim}_{k,p_1,\rho}|Z^2|^{\sim}_{k,p_2,\rho}, \tag{2.16}$$

where Γ is given in Condition 1.

Proof. We prove (2.15) only, because (2.13), (2.14), (2.16) can be proved in a similar way. Applying the formula (2.10), we have for $\sigma \in A^l(\rho)$ $(l \leq k)$,

$$|D^l_\sigma(Z^{(1)}_u Z^{(2)}_u)| \leq C \sum |(D_{\tau_1} Z^{(1)}_u)(D_{\tau_2} Z^{(2)}_u)|.$$

where the sum is given by (2.10). Because $\gamma(u) \leq \rho$ on $A(\rho)$, we have $\gamma(\tau_1)\gamma(\tau_2) \leq \gamma(\sigma)$. Hence, we get

$$\begin{aligned}\operatorname*{ess\,sup}_{(u,\sigma)\in A^{l+1}(\rho)} \left\| \frac{(D^i_{\tau_1} Z^{(1)}_u)(D^j_{\tau_2} Z^{(2)}_u)}{\gamma(u)\gamma(\sigma)} \right\|_r &\leq \rho \operatorname*{ess\,sup}_{(u,\sigma)\in A^{l+1}(\rho)} \left\| \frac{D^i_{\tau_1} Z^{(1)}_u}{\gamma(\tau_1)\gamma(u)} \frac{D^j_{\tau_2} Z^{(2)}_u}{\gamma(\tau_2)\gamma(u)} \right\|_r \\ &\leq \rho \operatorname*{ess\,sup}_{(u,\tau_1)\in A^{i+1}(\rho)} \left\| \frac{D^i_{\tau_1} Z^{(1)}_u}{\gamma(\tau_1)\gamma(u)} \right\|_{p_1} \operatorname*{ess\,sup}_{(u,\tau_2)\in A^{j+1}(\rho)} \left\| \frac{D^j_{\tau_2} Z^{(2)}_u}{\gamma(\tau_2)\gamma(u)} \right\|_{p_2},\end{aligned}$$

where, in the last inequality, we have used Hölder's inequality with $p_1^{'} = \frac{p_1}{r}, p_2^{'} = \frac{p_2}{r}$. This proves (2.15). □

Next, we denote $\mathbf{D}^*_{k,p}$ by the (analytical) adjoint space of $\mathbf{D}_{k,p}$, that is, the space of all of linear continuous functionals on $\mathbf{D}_{k,p}$. It is well-known that $\mathbf{D}^*_{k,p}$ is a Banach space with respect to the norm

$$|\Phi|^*_{k,p} := \sup_{|G|_{k,p}\leq 1} |\langle \Phi, G\rangle| \quad (\Phi \in \mathbf{D}^*_{k,p}).$$

We define

$$\mathbf{D}^*_\infty := \bigcup_{k=0} \bigcup_{p\geq 2} \mathbf{D}^*_{k,p}, \quad \widehat{\mathbf{D}}^*_\infty := \bigcup_{k=0} \bigcap_{p>2} \mathbf{D}^*_{k,p}.$$

Composites of Schwartz distributions and functionals of a Poisson random measure will be defined as elements in $\widehat{\mathbf{D}}^*_\infty$. Note that we do not choose the space $\bigcup_{k=0} \bigcap_{p\geq 2} \mathbf{D}^*_{k,p}$ instead of $\widehat{\mathbf{D}}^*_\infty$, see Remark 9.

3. Preliminaries

In this section, we shall exhibit preliminary results on functionals of a Poisson random measure which play an important role in the whole paper. In particular, we shall give a modification of the smoothness criterion of Picard. To this end, we shall study the domain of D^*.

The positive constant numbers will be denoted by C and may vary from line to line. If they depend on some parameters, then this is emphasized by index.

(1). *Remarks on the domain of the operator* D^* . We introduced norms $\|\cdot\|_{\mathbf{D}_{k,p}}$ and $\|\cdot\|_{\mathbf{D}_{k,p}}^{\sim}$ in the previous section. To study the domain of D^*, we shall introduce other norms

$$
\begin{aligned}
|F|_{\mathbf{W}_{k,p}} &= \left[\mathbf{E}[|F|^p] + \sum_{j=1}^{k} \mathbf{E}\left[\int_{A(1)^j} \left|\frac{D^j_\sigma F}{\gamma(\sigma)}\right|^p \lambda_0(d\sigma)\right]\right]^{\frac{1}{p}}, \\
|Z|_{\mathbf{W}_{k,p}}^{\sim} &= \left[\sum_{j=0}^{k} \mathbf{E}\left[\int_{A(1)^{j+1}} \left|\frac{D^j_\sigma Z_u}{\gamma(\sigma)}\right|^p \lambda_0(d\sigma)\lambda_0(du),\right]\right]^{\frac{1}{p}}
\end{aligned}
\tag{3.1}
$$

where $\lambda_0(du)$ is the probability measure

$$
\lambda_0(du) = \frac{1_{A(1)}(u)\gamma(u)^2}{\int_{A(1)} \gamma^2(u)\lambda(du)}\lambda(du). \tag{3.2}
$$

Then, the following inequalities are true

$$
|F|_{\mathbf{W}_{k,p}} \le |F|_{\mathbf{D}_{k,p}}, \quad |Z|_{\mathbf{W}_{k,p}}^{\sim} \le |Z|_{\mathbf{D}_{k,p}}^{\sim}, \tag{3.3}
$$

for a functional $F \in \mathbf{D}_{k,p}$ and a process $Z_u \in \mathbf{D}_{k,p}^{\sim}$. Let $\mathbf{S}$ be the collection of random variable X written as

$$
X = f\left(\int \phi_1(u)\tilde{N}(du), \dots, \int \phi_n(u)\tilde{N}(du)\right)
$$

where $f(x_1, \dots, x_n)$ is a bounded smooth function in $x_1, \dots x_n$ in $\mathbf{R}^d$, $n \in \mathbf{N}$, and $\phi_1, \dots, \phi_n$ are smooth functions on E with compact support. Note that $|X|_{k,p} < \infty$. From (3.3), we have $|X|_{\mathbf{W}_{k,p}} < \infty$ if $X \in \mathbf{S}$. Let $\mathbf{W}_{k,p}$ be the completion of $\mathbf{S}$ by $\|\cdot\|_{\mathbf{W}_{k,p}}$. We denote $\mathbf{W}_\infty = \bigcap_{k=0}^\infty \bigcap_{p\ge 2} \mathbf{W}_{k,p}$, and $\mathbf{W}_\infty^{\sim} = \bigcap_{k=0}^\infty \bigcap_{p\ge 2} \mathbf{W}_{k,p}^{\sim}$. In [5], these norms (3.1), and spaces $\mathbf{W}_{k,p}$ and $\mathbf{W}_{k,p}^{\sim}$ were introduced and discussed.

Theorem 3.1. *Let Z_u be a stochastic process belonging to $\mathcal{Z} \cap \mathbf{W}_\infty^{\sim}$, and p an even number, and k a non-negative integer. Then, there is a constant $C = C(p,k) > 0$ such that*

$$
|D^*(Z1_{A(1)})|_{\mathbf{W}_{k,p}} \le C|Z1_{A(1)}|_{\mathbf{W}_{k+p,k+p}}^{\sim} \tag{3.4}
$$

holds.

Remark 4. Ishikawa-Kunita have shown slightly stronger estimation than (3.4) for Wiener-Poisson functional. For the proof, see Theorem 3.2 in Ishikawa-Kunita [5].

Note that the dual operator

$$D^* \ : \ \mathbf{L}^2(\Omega \times E_0) \mapsto \mathbf{L}^2(\Omega)$$

is closable.

Lemma 3.1. *Suppose that $|Z|_{\tilde{\mathbf{W}}_{k,p}} < \infty$ for any k, p, and that $Z_u = 0$ on $A(1)^c$. Then, Z is in the domain of D^*. Further, for any non-negative integer k and even number p, there is a constant $C = C(p,k) > 0$ such that*

$$|D^*(Z))|_{\mathbf{W}_{k,p}} \leq C|Z|_{\tilde{\mathbf{W}}_{k+p,k+p}} \tag{3.5}$$

holds.

Remark 5. Recall that the operator D^* is defined for elements in $\mathcal{Z}$ by (2.4). In Lemma 3.1, we said that the domain of D^* is that of the dual operator of D. Hence, (2.4) does not necessarily hold for the process Z which satisfies the condition of Lemma 3.1. However, it can be deduced from the proof of Lemma 3.1 that

$$D^*(Z) = \lim_{l \to \infty} \int_{A(1) \cap A(\frac{1}{l})^c} Z_u \circ \varepsilon_u^- \tilde{N}(du), \quad \text{in } \mathbf{W}_{k,p}.$$

From this, one can deduce that (2.5) holds for Z satisfying the condition of Lemma 3.1 and for $F \in \mathbf{L}^2(\Omega)$ satisfying $\mathbf{E}[\int_E |D_u F|^2 \lambda(du)] < \infty$.

Proof. For a natural number l, we set $Z_u^l = Z_u 1_{A(1) \cap A(\frac{1}{l})^c}(u)$. Note that $\lambda(\,A(\frac{1}{l})\,) < \infty$. Hence it holds that the process Z_u^l is integrable. Indeed, applying Schwarz's inequality, we have

$$\mathbf{E}\left[\int |Z_u^l| \lambda(du)\right] \leq \Gamma\left(\frac{1}{l}\right)^{\frac{1}{2}} \mathbf{E}\left[\int \left|\frac{Z_u}{\gamma(u)}\right|^2 \lambda_0(du)\right]^{\frac{1}{2}},$$

where $c = \int_{A(1)} \gamma^2(u)\lambda(du)$. Note that $\lambda_0(A(\frac{1}{l})) = \Gamma(\frac{1}{l}) \to 0$ as $l \to \infty$. It also holds that $Z^l \to Z$ in $\tilde{\mathbf{W}}_{k,p}$ for any non-negative integer k and $p \geq 2$, since Schwartz's inequality yields that

$$\begin{aligned}
&\mathbf{E}\left[\int_{A(1)} \int_{A(1)^k} \left|\frac{D_\sigma(Z_u^l - Z_u)}{\gamma(\sigma)}\right|^p \lambda_0(d\sigma)\lambda_0(du)\right] \\
&\quad = \mathbf{E}\left[\int_{A(\frac{1}{l})} \int_{A(1)^k} \left|\frac{D_\sigma Z_u}{\gamma(\sigma)}\right|^p \lambda_0(d\sigma)\lambda_0(du)\right] \leq \Gamma\left(\frac{1}{l}\right)^{\frac{1}{2}} |Z|_{\tilde{\mathbf{W}}_{k,2p}} \to 0,
\end{aligned}$$

as $l \to \infty$. Applying Theorem 3.1, we get

$$|D^*(Z^{l_1}) - D^*(Z^{l_2})|_{\mathbf{W}_{k,p}} \le C_{k,p}|Z^{l_1} - Z^{l_2}|_{\mathbf{W}^{\sim}_{k+p,k+p}} \to 0 \tag{3.6}$$

as $l_1, l_2 \to \infty$, for any non-negative integer k and even number p. In particular $\|D^*(Z^{l_1}) - D^*(Z^{l_2})\|_2 \to 0$. As we noted above, D^* is a closable operator. This shows that Z is in the domain of D^*. Because $\mathbf{W}_{k,p}$ is a Banach space, one can can check that $D^*(Z)$ is in $\mathbf{W}_{k,p}$ for any k, p. We have

$$\begin{aligned} |D^*(Z)|_{\mathbf{W}_{k,p}} &\le |D^*(Z) - D^*(Z^l)|_{\mathbf{W}_{k,p}} + |D^*(Z^l)|_{\mathbf{W}_{k,p}} \\ &\le |D^*(Z) - D^*(Z^l)|_{\mathbf{W}_{k,p}} + C|Z1_{A(1)\cap A(\frac{1}{l})^c}|_{\mathbf{W}_{k+p,k+p}}. \end{aligned}$$

Letting l tends to infinity, we get the inequality (3.5). □

Remark 6. From this lemma, we can also define D^*Z if Z is in $\mathbf{D}^{\sim}_{k,p}$ for any k, p.

Pick a system $\mathbf{Z} = \{Z_u^{(j)}; j = 1, 2, \ldots\} \subset \mathbf{D}^{\sim}_{\infty}$ which satisfies $Z_u^{(j)} = 0$ on $u \in A^c(1)$ for any j. For any $G \in \mathbf{D}_\infty$, it follows from Lemma 2.1 that $GZ^{(1)}$ is in $\mathbf{D}^{\sim}_{\infty}$. Hence, Lemma 3.1 shows that $D^*(ZG)$ can be defined and is in $\mathbf{W}_\infty$. By the iteration of this argument, and by Lemma 2.1 in [5], one can define

$$D^{*(n)}_{\mathbf{Z}}(G) := D^*(Z^{(n)}D^*(Z^{(n-1)} \ldots D^*(Z^{(1)}G)\ldots)), \tag{3.7}$$

and show that $D^{*(n)}_{\mathbf{Z}}(G) \in \mathbf{W}_\infty$. Repeating the argument of the proof of Lemma 3.1, we have:

Lemma 3.2. *$\mathbf{Z} = \{Z_u^{(j)}; j = 1, 2, \ldots\} \subset \mathbf{W}^{\sim}_{\infty}$ which satisfy $Z_u^{(j)} = 0$ on $u \in A^c(1)$ for any j. Put $Z_u^{(j),l} = Z_u^{(j)}1_{A^c(\frac{1}{l})}(u)$ for a natural number l. Then, for $G \in \mathbf{W}_\infty$, we have*

$$\lim_{l\to\infty} |D^{*(n)}_{\mathbf{Z}^l}(G) - D^{*(n)}_{\mathbf{Z}}(G)|_{\mathbf{W}_{k,p}} \to 0,$$

where $D^{*(n)}_{\mathbf{Z}^l}(G)$ is defined by (3.7) with $Z_u^{(j)} = Z_u^{(j),l}$.

(2). *Smoothness criterion of Picard.* The following theorem is a modification of Theorem 2.1 of Picard [12]. Recall that Condition 1 in Section 2 holds, and that we use the notation $\alpha, \gamma(u), \Gamma(\rho)$ in the whole paper.

Theorem 3.2. *Suppose that $F \in \mathbf{D}_\infty(\mathbf{R}^d)$ satisfies the following condition:*
(**ND**). *there exists some $\beta \in (\frac{\alpha}{2}, 1]$ such that for any $p \in (1, \infty)$, any $\rho \in (0, 1)$, and any non-negative integer k,*

$$\sup_{\substack{\mathbf{v}\in\mathbf{R}^d \\ |\mathbf{v}|=1}} \operatorname*{ess\,sup}_{\tau\in A^k(\rho)} \Big\| \Big(\int_{A(\rho)} |\mathbf{v} \cdot D_u F|^2 1_{\{|\mathbf{v}\cdot D_u F| \le \rho^\beta\}} \lambda(du) \Big)^{-1} \circ \varepsilon^+_\tau \Big\|_p \le C_{p,k}\Gamma(\rho)^{-1},$$

where $C_{p,k}$ does not depend on ρ. Then, for any natural number n, for $r > 2$, for $G \in \mathbf{D}_\infty$, and for $\boldsymbol{\xi} \in \mathbf{R}^d$, we have

$$\sup_{|G|_{n,r}=1} \left|\mathbf{E}[Ge^{i\boldsymbol{\xi}\cdot F}]\right| \leq C(1+|\boldsymbol{\xi}|^2)^{-(1-\frac{\alpha}{2\beta})\frac{n}{2}} \tag{3.8}$$

where C is a constant which depends on n, r and F.

Remark 7. One can weaken the non-degenerate condition (ND) to the following form;

$$\sup_{\substack{\mathbf{v}\in\mathbf{R}^d \\ |\mathbf{v}|=1}} \operatorname*{ess\,sup}_{\tau\in A^k(\rho)} \left\| \Big(\int_{A(\rho)} |e^{\mathbf{v}\cdot D_u F} - 1|^2 \lambda(du) \Big)^{-1} \circ\varepsilon_\tau^+ \right\|_p \leq C_{p,k}\Gamma(\rho)^{-1},$$

because, in [12], the author used the non-degenerate condition (ND) to estimate the random variable $\Big(\int_{A(\rho)} |e^{\mathbf{v}\cdot D_u F} - 1|^2 \lambda(du) \Big)^{-1} \circ\varepsilon_\tau^+$. See Section 2 in [12].

Remark 8. Suppose that $F = (F_1, \ldots, F_d) \in \mathbf{D}_\infty(\mathbf{R}^d)$ satisfies the condition (ND). We set $\psi_F^G(\boldsymbol{\xi}) := \mathbf{E}[Ge^{i\boldsymbol{\xi} F}]$. Because $\mathbf{D}_\infty$ is an algebra, $F_1^{m_1} \cdots F_d^{m_d}$ is also in $\mathbf{D}_\infty$ for each multi-index $\mathbf{m} = (m_1, \ldots, m_d)$. From the inequality (3.8), we obtain

$$\left| \frac{\partial^{m_1}}{\partial\xi_1^{m_1}} \cdots \frac{\partial^{m_d}}{\partial\xi_d^{m_d}} \psi_F^G(\xi) \right| = \left| \mathbf{E}[GF_1^{m_1} \cdots F_d^{m_d} e^{i\xi F}] \right| \leq C(1+|\xi|^2)^{-(1-\frac{\alpha}{2\beta})\frac{n}{2}},$$

where C is a constant which depends on n, m, F, and G. This means that ψ_F^G is a rapidly decreasing function, hence so is the Fourier inversion of ψ_F^G. In particular, F has the density function which is rapidly decreasing. We denote by $p_F(x)$ the rapidly decreasing density function of F. Then, because one can write $\psi_F^G(\xi) = \int e^{i\xi x}\mathbf{E}[G|F=x]p_F(x)\,dx$, the function $\mathcal{F}\psi_F^G := \mathbf{E}[G|F=x]p_F(x)$ is also rapidly decreasing.

Remark 9. It seems difficult to obtain the inequality (3.8) for $r = 2$. We shall prove Theorem 4.1 by using (3.8) and define the composites of Schwartz distributions and functionals of a Poisson random measure. This is the reason why we define $\widehat{\mathbf{D}}_\infty^*$ as in the previous section.

Theorem 3.3. *Let $Z_u^{(j)}(\boldsymbol{\xi})\,;\, j = 1, 2, \ldots$ be processes parametrized by $(u, \boldsymbol{\xi}) \in E \times \mathbf{R}^d$. Suppose that there exists some $\frac{\alpha}{2} < \beta \leq 1$ such that $Z_u^{(j)}(\boldsymbol{\xi}) = 0$ if $u \in A(|\boldsymbol{\xi}|^{-\frac{1}{\beta}})$, and that, for any non-negative integer j, k, for $p \geq 2$ and for any $\boldsymbol{\xi} \in \{\boldsymbol{\xi} \in \mathbf{R}^d\,;\, |\boldsymbol{\xi}| \geq 1\}$,*

$$|Z^{(j)}|_{k,p,|\boldsymbol{\xi}|^{-\frac{1}{\beta}}} \leq C_{k,p,j}(\,|\boldsymbol{\xi}|\Gamma(|\boldsymbol{\xi}|^{-\frac{1}{\beta}})\,)^{-1}$$

where $C_{k,p,j}$ does not depend on $\boldsymbol{\xi}$. Then for each $G \in \mathbf{D}_\infty$ and $r > 2$, it holds that

$$\sup_{|G|_{n,r}=1} \|D^{*(n)}_{\mathbf{Z}(\boldsymbol{\xi})}(G)\|_2 \leq C(1+|\boldsymbol{\xi}|^2)^{-(1-\frac{\alpha}{2\beta})\frac{n}{2}},$$

where $D^{*(n)}_{\mathbf{Z}(\boldsymbol{\xi})}(G)$ is defined by (3.7) with $Z_u^{(j)} = Z_u^{(j)}(\boldsymbol{\xi})$.

Remark 10. The proof is essentially due to Picard, although slight modifications are needed. We shall prove this theorem in Section 8.

Here, applying Theorem 3.3, we prove Theorem 3.2. Put

$$Z_u(\boldsymbol{\xi}) = \frac{(e^{-i\boldsymbol{\xi}\cdot D_uF}-1)1_{A(|\boldsymbol{\xi}|^{-\frac{1}{\beta}})}(u)}{\int_{A(|\boldsymbol{\xi}|^{-\frac{1}{\beta}})}|e^{i\boldsymbol{\xi}\cdot D_vF}-1|^2\lambda(dv)}. \tag{3.9}$$

We shall use the following Lemma of Picard (see proof of Lemma 2.8 in [12]);

Lemma 3.3. *Let F be in $\mathbf{D}_\infty(\mathbf{R}^d)$. Then it holds that, for any non-negative integer k for $p \geq 2$ and for any $\boldsymbol{\xi} \in \{\boldsymbol{\xi} \in \mathbf{R}^d\,;\,|\boldsymbol{\xi}| \geq 1\}$,*

$$\operatorname*{ess\,sup}_{(u,\sigma)A^{1+k}(|\boldsymbol{\xi}|^{-\frac{1}{\beta}})}\left\|\frac{D_\sigma(e^{i\boldsymbol{\xi}\cdot D_uF}-1)}{\gamma(u)\gamma(\sigma)}\right\|_p \leq C_{p,k}|\boldsymbol{\xi}|, \tag{3.10}$$

and that

$$|Z(\cdot,\boldsymbol{\xi})|^{\sim}_{k,p} \leq C_{k,p,j}\left(|\boldsymbol{\xi}|\int_{A(|\boldsymbol{\xi}|^{-\frac{1}{\beta}})}\gamma^2(u)\lambda(du)\right)^{-1}. \tag{3.11}$$

Remark 11. Lemma 2.8 in [12] claims only that the inequality (3.11) holds, but in the proof, it is shown that (3.10) also holds. We shall use the inequality (3.10) in Section 6.

Note that

$$D_ue^{i\boldsymbol{\xi}\cdot F} = (e^{i\boldsymbol{\xi}\cdot D_uF}-1)e^{i\boldsymbol{\xi}\cdot F},$$

hence, we have

$$e^{i\boldsymbol{\xi}\cdot F} = \int Z_u(\boldsymbol{\xi})D_ue^{i\boldsymbol{\xi}F}\,\lambda(du).$$

It follows from Lemma 3.1 that

$$\mathbf{E}[Ge^{i\boldsymbol{\xi}\cdot F}] = \mathbf{E}\left[G\int Z_u(\boldsymbol{\xi})D_ue^{i\boldsymbol{\xi}F}\,\lambda(du)\right] = \mathbf{E}[D^*(Z.(\boldsymbol{\xi})G)e^{i\boldsymbol{\xi}\cdot F}].$$

Repeating this argument, we get

$$\mathbf{E}[Ge^{i\boldsymbol{\xi}\cdot F}] = \mathbf{E}[D^*(Z(\boldsymbol{\xi})D^*(Z(\boldsymbol{\xi})\ldots D^*(Z(\boldsymbol{\xi})G)\ldots))e^{i\boldsymbol{\xi}\cdot F}].$$

The absolute value of the right hand side is dominated by

$$\|D^*(Z(\boldsymbol{\xi})D^*(Z(\boldsymbol{\xi})\ldots D^*(Z(\boldsymbol{\xi})G)\ldots))\|_2.$$

Therefore, applying Theorem 3.3 with $Z_u^{(j)} = Z_u(\boldsymbol{\xi})$, we complete the proof of Theorem 3.2.

Remark 12. As we mentioned above, the proof is essentially due to Picard. However, the process $Z(\boldsymbol{\xi})$ is differ from that of Picard [12]; instead of $Z(\boldsymbol{\xi})$, the author used the following integrable process

$$\frac{(e^{-i\boldsymbol{\xi}\cdot D_uF}-1)1_{A(|\boldsymbol{\xi}|^{-\frac{1}{\beta}})\cap A(\zeta)^c}(u)}{\int_{A(|\boldsymbol{\xi}|^{-\frac{1}{\beta}})\cap A(\zeta)^c}|e^{i\boldsymbol{\xi}\cdot D_vF}-1|^2\lambda(dv)},$$

where $\zeta>0$ small enough. The process $Z(\boldsymbol{\xi})$ defined by (3.9) is not necessarily integrable. However, to prove asymptotic expansion theorem, we have to show Theorem 3.3 for $Z^{(j)}$'s which are not necessarily integrable.

4. Composites of functionals of a Poisson random measure and Schwarz distributions

In this section, we shall formulate composites of functionals of a Poisson random measure and Schwartz distributions. Our formulation slightly differs from that of the case for Wiener functionals (see Watanabe [17]). Take a natural number d and be fixed. We shall consider Schwartz distributions on $\mathbf{R}^d$. We denote by $\mathcal{S}$ the space of rapidly decreasing functions on $\mathbf{R}^d$. Recall that $\mathcal{S}$ is a Fréchet space. We denote by $\mathcal{S}'$ the space of continuous linear functionals on $\mathcal{S}$, that is the space of tempered distributions. For $\phi\in\mathcal{S}$, we write $\mathcal{F}\phi(\boldsymbol{\xi})$, $\check{\mathcal{F}}\phi(\mathbf{x})$ for its Fourier transform and its inverse Fourier transform, respectively:

$$\mathcal{F}\phi(\boldsymbol{\xi})=\frac{1}{(2\pi)^d}\int_{\mathbf{R}^d}e^{-i\boldsymbol{\xi}\cdot\mathbf{x}}\phi(\mathbf{x})\,d\mathbf{x}\ ,\qquad \check{\mathcal{F}}\phi(\mathbf{x})=\int_{\mathbf{R}^d}e^{i\mathbf{x}\cdot\boldsymbol{\xi}}\phi(\boldsymbol{\xi})\,d\boldsymbol{\xi}.$$

For $\phi\in\mathcal{S}$ and $s\in\mathbf{R}$, we define

$$\|\phi\|_{\mathbf{H}_s}:=\Big[\int_{\mathbf{R}^d}(1+|\boldsymbol{\xi}|^2)^s|\,\mathcal{F}\phi(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi}\,\Big]^{\frac{1}{2}},$$

and denote by $\mathbf{H}_s$ the completion of $\mathcal{S}$ by $\|\cdot\|_{\mathbf{H}_s}$. We also denote $\mathbf{H}_\infty=\cap_{s>0}\mathbf{H}_s$, and $\mathbf{H}_{-\infty}=\cup_{s>0}\mathbf{H}_{-s}$. Then it is obvious that for $s<r$

$$\|\phi\|_{\mathbf{H}_s}\le\|\phi\|_{\mathbf{H}_r}\ ,\qquad \mathcal{S}\subset\mathbf{H}_r\subset\mathbf{H}_s\subset\mathcal{S}'. \tag{4.1}$$

For a functional F, we define $A_F\phi=\phi(F)$ $(\phi\in\mathcal{S})$. Since for any k, the map $\mathbf{D}_{k,p}\ni G\mapsto\mathbf{E}[GA_F\phi]$ is a linear continuous functional on $\mathbf{D}_{k,p}$, we can regard A_F as the operator

$$A_F:\mathcal{S}\ni\phi\mapsto A_F\phi\in\mathbf{D}^*_{k,p}. \tag{4.2}$$

Theorem 4.1. *For $F\in\mathbf{D}_\infty(\mathbf{R}^d)$, we suppose that F satisfies the condition* (**ND**) *in Theorem* 3.2. *Then for any $s>0$, there exists a natural number n such that*

$$|A_F(\phi)|_{D^*_{n,p}}\le C\|\phi\|_{\mathbf{H}_{-s}},\quad \textit{for all }\phi\in\mathcal{S},\ \textit{and any }p>2, \tag{4.3}$$

where C depends on F,n,p and s.

Remark 13. This theorem corresponds to Theorem 1.12 in Watanabe [17] for Wiener functionals.

Proof. For $s > 0$, take an integer n so that $n \geq \frac{1 \vee (\frac{2s}{1+d})}{1-\frac{\alpha}{2\beta}}(1+d)$. Then, one can check that $\int_{\mathbf{R}^d}(1+|\boldsymbol{\xi}|^2)^{-(1-\frac{\alpha}{2\beta})\frac{n}{2}}\,d\boldsymbol{\xi} < \infty$. Note that

$$\phi(F) = (\check{\mathcal{F}}\mathcal{F}\phi)(F) = \int_{\mathbf{R}^d} e^{iF\cdot\boldsymbol{\xi}}\mathcal{F}\phi(\boldsymbol{\xi})\,d\boldsymbol{\xi}.$$

By Theorem 3.2 and using Schwarz's inequality, we have

$$\begin{aligned}
|A_F(\phi)|_{D^*_{n,p}} &= \sup_{|G|_{n,p}=1} |\mathbf{E}[G\phi(F)]| \\
&= \sup_{|G|_{n,p}=1} \Big|\, \mathbf{E}\,\Big[\int_{\mathbf{R}^d} Ge^{iF\cdot\boldsymbol{\xi}}\mathcal{F}\phi(\boldsymbol{\xi})\,d\boldsymbol{\xi}\,\Big]\Big| \\
&\leq \int_{\mathbf{R}^d} \sup_{|G|_{n,p=1}} \big|\,\mathbf{E}[Ge^{iF\cdot\boldsymbol{\xi}}]\,\big|\,|\mathcal{F}\phi(\boldsymbol{\xi})|\,d\boldsymbol{\xi} \\
&\leq C\int_{\mathbf{R}^d}(1+|\boldsymbol{\xi}|^2)^{-(1-\frac{\alpha}{2\beta})\frac{n}{2}}|\mathcal{F}\phi(\boldsymbol{\xi})|\,d\boldsymbol{\xi} \\
&\leq C\,\Big[\int_{\mathbf{R}^d}(1+|\boldsymbol{\xi}|^2)^{-(1-\frac{\alpha}{2\beta})\frac{n}{2}}\,d\boldsymbol{\xi}\,\Big]^{\frac{1}{2}}\,\|\phi\|_{\mathbf{H}^{-(1-\frac{\alpha}{2\beta})\frac{n}{2}}} \\
&= C'\,\|\phi\|_{\mathbf{H}_{-(1-\frac{\alpha}{2\beta})\frac{n}{2}}} \leq C'\,\|\phi\|_{\mathbf{H}_{-s}},
\end{aligned}$$

where, in the last inequality, we used the fact that $-(1-\frac{\alpha}{2\beta})\frac{n}{2} \leq -s$ and (4.1). □

Because $\mathcal{S}$ is dense in $\mathbf{H}_s$, the inequality (4.3) shows that the linear operator (4.2) has the unique continuous extension:

$$A_F : \mathbf{H}_s \ni T \mapsto A_F T \in \widehat{\mathbf{D}}^*_\infty. \tag{4.4}$$

From this fact, we define composites of Schwartz distributions and functionals of a Poisson random measure:

Definition 4.1. Suppose that $F = (F_1, \ldots, F_d)$ satisfies the condition (ND) in Theorem 3.2. For any $T \in \mathbf{H}_{-\infty}$, we say that the linear continuous functional $A_F T$ is composite of $T \in \mathbf{H}_{-\infty}$ and F, and denote $T \circ F = A_F$.

In the Malliavin calculus on Wiener space, composites of smooth Wiener functionals and Schwartz distributions can be evaluated by *integration by parts formula* (see Watanabe [17]). On the other hand, as we mentioned in Section 1, we cannot apply the integration by parts formula in our formulation. However, from Theorem 3.2, we know that the function $\psi_F^G(\boldsymbol{\xi}) = \mathbf{E}[Ge^{i\boldsymbol{\xi}\cdot F}]$ is in $\mathcal{S}$. Hence, the inequality (4.3) yields

$$T \circ F : G \mapsto \langle\, T \circ F\,,\, G\,\rangle = {}_{\mathcal{S}'}\langle \mathcal{F}T\,,\, \psi_F^G\rangle_{\mathcal{S}}.$$

Note that $\mathcal{F}\psi_F^G(x) = \mathbf{E}[G|F = x]p_F(x)$, where p_F is the rapidly decreasing density function of F. Hence, the following equality also holds

$$\langle\, T \circ F\,,\, G\,\rangle = {}_{\mathcal{S}'}\langle\, T\,,\, \mathcal{F}\psi_F^G\,\rangle_{\mathcal{S}}. \tag{4.5}$$

Next, we define the product $H \in \mathbf{D}_\infty$ and $T \circ F$ by

$$\langle\, HT \circ F\,,\, G\,\rangle := {}_{\mathcal{S}'}\langle\, \mathcal{F}T\,,\, \psi_F^{G,H}\,\rangle_{\mathcal{S}} \quad (G \in \mathbf{D}_\infty),$$

where $\psi_F^{G,H}(\boldsymbol{\xi}) = \mathbf{E}[GHe^{i\boldsymbol{\xi}\cdot F}]$. From Theorem 3.2, $\psi_F^{G,H}$ is in $\mathcal{S}$. By the same argument as in the proof of Theorem 4.1, one can deduce that $HT \circ F \in \widehat{\mathbf{D}}_\infty^*$.

Example 4.1. If T is the Dirac point mass δ_x at x, then it follows from (4.5) that

$$\langle T \circ F, G\rangle = \mathbf{E}[G|F = x]p_F(x).$$

If T is the Heaviside function $1_{[0,\infty)}$, then we have also

$$\langle T \circ F, G\rangle = \mathbf{E}[G; F \geq 0].$$

5. Asymptotic expansions

In this section, we shall consider a family of functionals $F(\epsilon)$ ($\epsilon \in (0, 1)$) depending on the parameter ϵ.

Definition 5.1.
A) We say that $F(\epsilon)$ has the asymptotic expansion $F(\epsilon) \sim \sum_{j=0}^{\infty} \epsilon^j f_j$ in $\mathbf{D}_\infty(\mathbf{R}^d)$, if the following conditions hold:

A1) $F(\epsilon), f_0, f_1, \ldots, \in \mathbf{D}_\infty(\mathbf{R}^d)$ for each $\epsilon \in (0, 1)$.

A2) For each non-negative integer m, k and for $p \geq 2$,

$$\limsup_{\epsilon\to 0} \left| \frac{F(\epsilon) - \sum_{n=0}^{m} \epsilon^j f_j}{\epsilon^{m+1}} \right|_{\mathbf{D}_{k,p}(\mathbf{R}^d)} < \infty.$$

B). We say that $\Phi(\epsilon) \in \widehat{\mathbf{D}}_\infty^*$ has the asymptotic expansion $\Phi(\epsilon) \sim \sum_{j=0}^{\infty} \epsilon^j \Phi_j$ in $\widehat{\mathbf{D}}_\infty^*$, if the following conditions hold:

$B1$) $\Phi(\epsilon), \Phi_0, \Phi_1, \ldots, \in \widehat{\mathbf{D}}_\infty^*$ for each $\epsilon \in (0, 1)$.

$B2$) For each non-negative integer m, there exists a $k = k(m)$ such that $\Phi(\epsilon), \Phi_0, \Phi_1, \ldots, \Phi_m \in \bigcap_{p>2} \mathbf{D}_{k,p}^*$ and

$$\limsup_{\epsilon\to 0} \frac{|\Phi(\epsilon) - \sum_{j=0}^{m} \epsilon^j \Phi_j|_{k,p}^*}{\epsilon^{m+1}} < \infty.$$

Remark 14. In condition $B2$), we restricted p to be in $(2, \infty)$. This restriction arise from the definition of $\widehat{\mathbf{D}}_\infty^*$.

We set

$$U^d := \{\boldsymbol{\xi} \in \mathbf{R}^d \, : \, |\,\boldsymbol{\xi}\,| \geq 1\}.$$

We shall consider complex valued random variables has the form $F(\epsilon, \boldsymbol{\xi})$ where $(\epsilon, \boldsymbol{\xi}) \in (0,1] \times U^d$, and give another definition of asymptotic expansions. Let $q(\boldsymbol{\xi})$ be a positive functions defined on U^d, $\rho(\boldsymbol{\xi})$ be a function defined on U^d and taking values in $(0,1]$. Let $F(\epsilon, \boldsymbol{\xi})$ $(\epsilon, \boldsymbol{\xi}) \in (0,1] \times U^d$ be a family of elements in $\mathbf{D}_\infty$. For a non-negative integer m, we denote

$$F(\epsilon, \boldsymbol{\xi}) \sim O(\epsilon^m q(\boldsymbol{\xi})) \quad \text{on } A(\rho(\boldsymbol{\xi}))$$

if it holds that, for any $p \geq 2$ and for any non-negative integer k,

$$\limsup_{\epsilon \to 0} \sup_{\boldsymbol{\xi} \in U^d} \frac{|F(\epsilon, \boldsymbol{\xi})|_{k,p,\rho(\boldsymbol{\xi})}}{\epsilon^m q(\boldsymbol{\xi})} < \infty. \tag{5.1}$$

Definition 5.2.

$A^{'}$). We say that a complex valued random variable $F(\epsilon, \boldsymbol{\xi})$ has the asymptotic expansion $F(\epsilon, \boldsymbol{\xi}) \sim \sum_{j=0}^{\infty} \epsilon^j f_j(\boldsymbol{\xi})$ in $\mathbf{D}_\infty(q(\boldsymbol{\xi}), A(\rho(\boldsymbol{\xi})))$, if the following conditions hold:

$A^{'}1)$ for any $(\epsilon, \boldsymbol{\xi}) \in (0,1) \times U^d$, $F(\epsilon, \boldsymbol{\xi}), f_0(\boldsymbol{\xi}), f_1(\boldsymbol{\xi}) \ldots, \in \mathbf{D}_\infty$.

$A^{'}2)$ For any non-negative integer m

$$F(\epsilon, \boldsymbol{\xi}) - \sum_{j=0}^{m} \epsilon^j f_j(\boldsymbol{\xi}) \sim O(\epsilon^{m+1} q(\boldsymbol{\xi})) \quad \text{on } A(\rho(\boldsymbol{\xi})).$$

$A^{'}3)$ For any non-negative integer j, k, for any $p \geq 2$ and for any $\boldsymbol{\xi} \in U^d$,

$$|f_j(\boldsymbol{\xi})|_{k,p,\rho(\boldsymbol{\xi})} \leq C_{j,p,k} q(\boldsymbol{\xi}).$$

$C^{'}$). We say that a complex valued process $Z_u(\epsilon, \boldsymbol{\xi})$ has the asymptotic expansion $Z_u(\epsilon, \boldsymbol{\xi}) \sim \sum_{j=0}^{\infty} \epsilon^j z_u^{(j)}(\boldsymbol{\xi})$ in $\mathbf{D}_\infty^{\sim}(q(\boldsymbol{\xi}), A(\rho(\boldsymbol{\xi})))$, if the following conditions hold:

$C^{'}1)$ $Z_u(\epsilon, \boldsymbol{\xi}), z_u^{(0)}(\boldsymbol{\xi}), z_u^{(1)}(\boldsymbol{\xi}), \ldots, \in \mathbf{D}_\infty^{\sim}$, for any $(\epsilon, \boldsymbol{\xi}) \in (0,1) \times U^d$.

$C^{'}2)$ For any non-negative integer m,

$$Z(\epsilon, \boldsymbol{\xi}) - \sum_{j=0}^{m} \epsilon^j z^{(j)}(\boldsymbol{\xi}) \sim O(\epsilon^{m+1} q(\boldsymbol{\xi})) \quad \text{on } A(\rho(\boldsymbol{\xi})).$$

$C^{'}3)$ For any non-negative integer j, k, for any $p \geq 2$ and for any $\boldsymbol{\xi} \in U^d$,

$$|z^{(j)}(\boldsymbol{\xi})|^{\sim}_{k,p,\rho(\boldsymbol{\xi})} \leq C_{p,k} q(\boldsymbol{\xi}).$$

$C^{'}4)$ For any j, $Z_u(\epsilon, \boldsymbol{\xi}) = z_u^{(j)}(\boldsymbol{\xi}) = 0$ if $u \in A^c(\rho(\boldsymbol{\xi}))$.

Remark 15. One can check that the coefficient of the asymptotic expansion is uniquely determined.

For a multi-index $\mathbf{n} = (n_1, \ldots, n_d)$ and for $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbf{R}^d$, we denote

$$|\mathbf{n}| = n_1 + \cdots + n_d, \quad \mathbf{n}! = n_1! \ldots n_d!,$$
$$\partial^{\mathbf{n}} = \frac{\partial^{n_1}}{\partial x_1^{n_1}} \cdots \frac{\partial^{n_d}}{\partial x_d^{n_d}}, \quad \mathbf{x}^{\mathbf{n}} = x_1^{n_1} \cdots x_d^{n_d}.$$

Definition 5.3. Let $F(\epsilon)$ ($\epsilon \in (0,1]$) be a parametrized process such that $F(\epsilon) \in \mathbf{D}_\infty(\mathbf{R}^d)$ for any $\epsilon \in (0,1]$. We say that $F(\epsilon)$ satisfies the uniformly non-degenerate condition if

(UN). there exists some $\frac{\alpha}{2} < \beta \leq 1$ such that for any $p \in (1, \infty)$, and any non-negative integer k,

$$\limsup_{\epsilon \to 0} \sup_{\rho \in (0,1)} \sup_{\substack{\mathbf{v} \in \mathbf{R}^d \\ |\mathbf{v}|=1}} \operatorname*{ess\,sup}_{\tau \in A^k(\rho)} \left\| \Gamma(\rho)(K(\epsilon, \rho, \mathbf{v}) \circ \varepsilon_\tau^+)^{-1} \right\|_p < \infty,$$

where $K(\epsilon, \rho, \mathbf{v}) = \int_{A^k(\rho)} |\mathbf{v} \cdot D_u F(\epsilon)|^2 1_{\{|D_u F(\epsilon)| \leq \rho^\beta\}} \lambda(du)$.

The following theorem is our main result:

Theorem 5.1. *Suppose that $F(\epsilon)$ ($\epsilon \in (0,1]$) satisfies the uniformly non-degenerate condition (UN) and that $F(\epsilon) \sim \sum_{j=0}^\infty \epsilon^j f_j$ in $\mathbf{D}_\infty(\mathbf{R}^d)$. Then, for any distribution $T \in \mathbf{H}_{-\infty}$, $T \circ F(\epsilon) \in \widehat{\mathbf{D}}_\infty^*$ has the asymptotic expansion in $\widehat{\mathbf{D}}_\infty^*(\mathbf{R}^d)$:*

$$T \circ F(\epsilon) \sim \sum_{m=0}^\infty \sum_{|\mathbf{n}|=m} \frac{1}{\mathbf{n}!} (\partial^{\mathbf{n}} T) \circ f_0 \cdot (F(\epsilon) - f_0)^{\mathbf{n}}$$
$$\sim \Phi + \epsilon \Phi_1 + \epsilon^2 \Phi_2 + \cdots,$$

where $\Phi, \Phi_1, \Phi_2, \ldots,$ are given by the formal Taylor expansion

$$\Phi_0 = T \circ f_0, \quad \Phi_1 = \sum_{i=1}^d f_1^i \left(\frac{\partial}{\partial x^i} T \right) \circ f_0,$$
$$\Phi_2 = \sum_{i=1}^d f_2^i \left(\frac{\partial}{\partial x^i} T \right) \circ f_0 + \frac{1}{2!} \sum_{i,j=1}^d f_1^i f_1^j \left(\frac{\partial^2}{\partial x^i \partial x^j} T \right) \circ f_0$$
$$\Phi_3 = \sum_{i=1}^d f_3^i \left(\frac{\partial}{\partial x^i} T \right) \circ f_0 + \frac{2}{2!} \sum_{i,j=1}^d f_1^i f_2^j \left(\frac{\partial^2}{\partial x^i \partial x^j} T \right) \circ f_0$$
$$+ \frac{1}{3!} \sum_{i,j,k=1}^d f_1^i f_1^j f_1^k \left(\frac{\partial^3}{\partial x^i \partial x^j \partial x^k} T \right) \circ f_0, \ldots.$$

Remark 16. We shall give the proof of Theorem 5.1 in Section 6.

By using Lemma 2.1, one can easily prove the following proposition.

Proposition 5.1. *Let $q_i(\boldsymbol{\xi}), q_i^{'}(\boldsymbol{\xi})$ $(i = 1,2)$ be non-negative functions defined on U^d, and $\rho(\boldsymbol{\xi})$ be a function defined on U^d which values in $(0,1]$. Suppose that for $i = 1,2$*

$$F_i(\epsilon,\boldsymbol{\xi}) \sim \sum_{j=0}^{\infty} \epsilon^j f_j^i(\boldsymbol{\xi}) \quad \text{in } \mathbf{D}_\infty(\, q_i(\boldsymbol{\xi}), A(\rho(\boldsymbol{\xi}))\,) \quad (i = 1,2),$$

$$Z_u^i(\epsilon,\boldsymbol{\xi}) \sim \sum_{j=0}^{\infty} \epsilon^j z_u^{(j),i}(\boldsymbol{\xi}) \quad \text{in } \mathbf{D}_\infty^{\sim}(\, q_i^{'}(\boldsymbol{\xi}), A(\rho(\boldsymbol{\xi}))\,) \quad (i = 1,2).$$

Then, it holds that

i).

$$F_1(\epsilon,\boldsymbol{\xi})F_2(\epsilon,\boldsymbol{\xi}) \sim \sum_{j=0}^{\infty} \epsilon^j h_j(\boldsymbol{\xi}) \quad \text{in } \mathbf{D}_\infty^{\sim}(\, q_1(\boldsymbol{\xi})q_2(\boldsymbol{\xi}), A(\rho(\boldsymbol{\xi}))\,),$$

where $h_j(\boldsymbol{\xi}) = \sum_{j_1+j_2=j} f_{j_1}^1(\boldsymbol{\xi}) f_{j_2}^2(\boldsymbol{\xi})$;

ii).

$$Z^1(\epsilon,\boldsymbol{\xi})F_1(\epsilon,\boldsymbol{\xi}) \sim \sum_{j=0}^{\infty} \epsilon^j x^{(j)} \quad \text{in } \mathbf{D}_\infty^{\sim}(\, q_1(\boldsymbol{\xi})q_1^{'}(\boldsymbol{\xi}), A(\rho(\boldsymbol{\xi}))\,),$$

where $x_u^{(j)}(\boldsymbol{\xi}) = \sum_{j_1+j_2=j} z_u^{(j_1),1}(\boldsymbol{\xi}) f_{j_2}^1(\boldsymbol{\xi})$;

iii).

$$Z^1(\epsilon,\boldsymbol{\xi})Z^2(\epsilon,\boldsymbol{\xi}) \sim \sum_{j=0}^{\infty} \epsilon^j z^{(j)}(\boldsymbol{\xi}) \quad \text{in } \mathbf{D}_\infty^{\sim}(\, \rho(\boldsymbol{\xi})q_1^{'}(\boldsymbol{\xi})q_2^{'}(\boldsymbol{\xi}), A(\rho(\boldsymbol{\xi}))\,)$$

where $z_u^{(j)}(\boldsymbol{\xi}) = \sum_{j_1+j_2=j} z_u^{(j_1),1}(\boldsymbol{\xi}) z_u^{(j_2),2}(\boldsymbol{\xi})$;

iv).

$$\int Z_u^1(\epsilon,\boldsymbol{\xi})Z_u^2(\epsilon,\boldsymbol{\xi})\lambda(du) \sim \sum_{j=0}^{\infty} \epsilon^j g_j(\boldsymbol{\xi}), \quad \text{in } \mathbf{D}_\infty(\, \Gamma(\rho(\boldsymbol{\xi}))q_1^{'}(\boldsymbol{\xi})q_2^{'}(\boldsymbol{\xi}), A(\rho(\boldsymbol{\xi}))\,),$$

where $g_j(\boldsymbol{\xi}) = \sum_{j_1+j_2=j} \int z_u^{(j_1),1}(\boldsymbol{\xi}) z_u^{(j_2),2}(\boldsymbol{\xi})\, \lambda(du)$.

The following theorem is a sufficient condition to satisfy the uniformly non-degenerate condition (UN), and is a version of Theorem 3.1 of Picard [12].

Theorem 5.2. *Suppose that $\lambda(ds \times dx) = ds \times \nu(dx)$. Define the $d \times d$ matrix $V(\rho)$ by $V_{i,j}(\rho) = \int_{\{|x| \le \rho\}} x_i x_j \nu(dx)$. Suppose also that the ratio between the largest and smallest eigenvalues of $V(\rho)$ is bounded as $\rho \to 0$, and that*

$$\liminf_{\rho \to 0} \rho^{-\alpha} \int_{\{|\mathbf{x}| \le \rho\}} |\mathbf{x}|^2 \nu(dx) > 0.$$

The random variable $F(\epsilon) \in \mathbf{D}_\infty$ *satisfies the uniformly non-degenerate condition if the following conditions* (a) *and* (b) *hold*:
(a). *for any* $p > 1$ *and for any non-negative integer* k

$$\sup_{\epsilon} \left\| \operatorname*{ess\,sup}_{\tau \in A^k(1)} \left| \frac{D^k_\tau F(\epsilon)}{\gamma(\tau)} \right| \right\|_p < \infty,$$

(b). *there exists a matrix-valued process* $\psi_t(\epsilon)$ *such that for* $|x| \le 1$ *and for any* $p \ge 1$,

$$\sup_{\epsilon} \|D_{t,x} F(\epsilon) - \psi_t(\epsilon) x\|_p \le C_p |x|^r,$$

for some $r > 1$, *and*

$$\limsup_{\epsilon \to 0} \left\| \left(\det \int_0^1 \psi_t(\epsilon) \psi_t^*(\epsilon) dt \right)^{-1} \right\|_p < \infty, \tag{5.2}$$

where $\psi_t^*(\epsilon)$ *is the transpose of the matrix* $\psi_t(\epsilon)$.

Remark 17. Note that the conditions (a) and (b) are uniform in ϵ. Therefore, Theorem 5.2 can be proven by a similar argument as Theorem 3.1 in [12], so we omit the proof.

6. Proof of Theorem 5.1

Let $\beta \in (\frac{\alpha}{2}, 1)$ be fixed. We define functions $\rho(\boldsymbol{\xi}), q_1(\boldsymbol{\xi}), q_2(\boldsymbol{\xi})$ on U^d by

$$\rho(\boldsymbol{\xi}) = |\,\boldsymbol{\xi}\,|^{-\frac{1}{\beta}}, \quad q_1(\boldsymbol{\xi}) = |\,\boldsymbol{\xi}\,|, \quad q_2(\boldsymbol{\xi}) = |\,\boldsymbol{\xi}\,|^2 \Gamma(\rho(\boldsymbol{\xi})).$$

Lemma 6.1. *Suppose that* $G(\epsilon, \boldsymbol{\xi}) \sim \sum_{j=0}^\infty \epsilon^j g_j(\boldsymbol{\xi})$ *in* $\mathbf{D}_\infty(q_2(\boldsymbol{\xi}), A(\rho(\boldsymbol{\xi})))$, *and that for any* $p \ge 2$ *for any non-negative integer* k, *and for* $\boldsymbol{\xi} \in U^d$,

$$\limsup_{\epsilon \to 0} \sup_{\boldsymbol{\xi} \in U^d} \operatorname*{ess\,sup}_{\tau \in A^k(\rho(\boldsymbol{\xi}))} \|q_2(\boldsymbol{\xi}) (G(\epsilon, \boldsymbol{\xi}) \circ \varepsilon^+_\tau)^{-1}\|_p < \infty. \tag{6.1}$$

Then, $G^{-1}(\epsilon, \boldsymbol{\xi})$ *has the asymptotic expansion*

$$G^{-1}(\epsilon, \boldsymbol{\xi}) \sim \sum_{j=0}^\infty \epsilon^j g_j^{'}(\boldsymbol{\xi}) \ \textit{in}\ \mathbf{D}_\infty(q_2^{-1}(\boldsymbol{\xi}), A(\rho(\boldsymbol{\xi}))), \tag{6.2}$$

where the coefficients are given by the formal expansion:

$$\frac{1}{G(\epsilon, \boldsymbol{\xi})} = \frac{1}{g_0(\boldsymbol{\xi})} \frac{1}{1 + \frac{G(\epsilon, \boldsymbol{\xi}) - g_0(\boldsymbol{\xi})}{g_0(\boldsymbol{\xi})}} \sim \frac{1}{g_0(\boldsymbol{\xi})} \sum_{j=0}^\infty (-1)^j \left(\frac{G(\epsilon, \boldsymbol{\xi}) - g_0(\boldsymbol{\xi})}{g_0(\boldsymbol{\xi})} \right)^j$$
$$\sim \sum_{j=0}^\infty \epsilon^j g_j^{'}(\boldsymbol{\xi}).$$

Remark 18. In the proof below, we shall show that

$$\operatorname*{ess\,sup}_{\tau\in A^k(\rho(\boldsymbol{\xi}))} \|g_0^{-1}\circ\varepsilon_\tau^+\|_p \le C_{p,k} q_2^{-1}(\boldsymbol{\xi}). \tag{6.3}$$

The argument of the proof of iii) in Lemma 2.7 of Picard [12] show that (6.1), and (6.3) yield

(6.4)
$$\limsup_{\epsilon\to 0}\sup_{\boldsymbol{\xi}\in U^d} |q_2(\boldsymbol{\xi})G^{-1}(\epsilon,\boldsymbol{\xi})|_{k,p,\rho(\boldsymbol{\xi})} < \infty, \quad |g_0^{-1}(\boldsymbol{\xi})|_{k,p,\rho(\boldsymbol{\xi})} \le C_{k,p} q_2^{-1}(\boldsymbol{\xi}),$$

respectively.

Proof. Firstly, we shall prove (6.3). For any $r>0$, applying Chebychev's and Schwarz's inequality, we get

$$\begin{aligned}
&\operatorname*{ess\,sup}_{\tau\in A^k(\rho(\boldsymbol{\xi}))} \mathbf{P}\left(\left|\frac{g_0(\boldsymbol{\xi})\circ\varepsilon_\tau^+}{q_2(\boldsymbol{\xi})}\right|\le r\right)\\
&\le \operatorname*{ess\,sup}_{\tau\in A^k(\rho(\boldsymbol{\xi}))} \mathbf{P}\left(\left|\frac{G(\epsilon,\boldsymbol{\xi})\circ\varepsilon_\tau^+}{q_2(\boldsymbol{\xi})}\right|\le r+\left|\frac{G(\epsilon,\boldsymbol{\xi})-g_0(\boldsymbol{\xi})}{q_2(\boldsymbol{\xi})}\right|\circ\varepsilon_\tau^+\right)\\
&\le C_p \operatorname*{ess\,sup}_{\tau\in A^k(\rho(\boldsymbol{\xi}))} \left\|\left|\left(r+\left|\frac{G(\epsilon,\boldsymbol{\xi})-g_0(\boldsymbol{\xi})}{q_2(\boldsymbol{\xi})}\right|\circ\varepsilon_\tau^+\right)\right|\cdot\left|\frac{G(\epsilon,\boldsymbol{\xi})}{q_2(\boldsymbol{\xi})}\circ\varepsilon_\tau^+\right|^{-1}\right\|_p^p\\
&\le C_p\Bigl(r^p+\operatorname*{ess\,sup}_{\tau\in A^k(\rho(\boldsymbol{\xi}))}\left\|\frac{(G(\epsilon,\boldsymbol{\xi})-g_0(\boldsymbol{\xi}))}{q_2(\boldsymbol{\xi})}\circ\varepsilon_\tau^+\right\|_{2p}^p\Bigr)\operatorname*{ess\,sup}_{\tau\in A^k(\rho(\boldsymbol{\xi}))}\left\|\frac{q_2(\boldsymbol{\xi})}{|G(\epsilon,\boldsymbol{\xi})\circ\varepsilon_\tau^+|}\right\|_{2p}^p
\end{aligned}$$

By iteration of (2.8) and by the assumption, we have

$$\lim_{\epsilon\to 0}\sup_{\boldsymbol{\xi}\in U^d}\operatorname*{ess\,sup}_{\tau\in A^k(\rho(\boldsymbol{\xi}))}\left\|\frac{(G(\epsilon,\boldsymbol{\xi})-g_0(\boldsymbol{\xi}))}{q_2(\boldsymbol{\xi})}\circ\varepsilon_\tau^+\right\|_{2p}^p=0$$

Hence, from (6.1), we have

$$\sup_{\boldsymbol{\xi}\in U^d}\operatorname*{ess\,sup}_{\tau\in A^k(\rho(\boldsymbol{\xi}))}\mathbf{P}\left(\left|\frac{g_0(\boldsymbol{\xi})\circ\varepsilon_\tau^+}{q_2(\boldsymbol{\xi})}\right|\le r\right)\le C_{p,k}r^p.$$

This implies (6.3).

We shall show (6.2). On the event $B:=\left\{\left|\frac{G(\epsilon,\boldsymbol{\xi})-g_0(\boldsymbol{\xi})}{g_0(\boldsymbol{\xi})}\right|<1\right\}$, we have

$$\frac{1}{G(\epsilon,\boldsymbol{\xi})}=\frac{1}{g_0(\boldsymbol{\xi})}\frac{1}{1+\frac{G(\epsilon,\boldsymbol{\xi})-g_0(\boldsymbol{\xi})}{g_0(\boldsymbol{\xi})}}=\frac{1}{g_0(\boldsymbol{\xi})}\sum_{j=0}^{\infty}(-1)^j\left(\frac{G(\epsilon,\boldsymbol{\xi})-g_0(\boldsymbol{\xi})}{g_0(\boldsymbol{\xi})}\right)^j.$$

Hence, we can write

$$\frac{1}{G(\epsilon,\boldsymbol{\xi})}-\frac{1}{g_0(\boldsymbol{\xi})}\sum_{j=0}^{m}(-1)^j\left(\frac{G(\epsilon,\boldsymbol{\xi})-g_0(\boldsymbol{\xi})}{g_0(\boldsymbol{\xi})}\right)^j =\frac{(-1)^{m+1}}{G(\epsilon,\boldsymbol{\xi})}\left(\frac{G(\epsilon,\boldsymbol{\xi})-g_0(\boldsymbol{\xi})}{g_0(\boldsymbol{\xi})}\right)^{m+1}.$$

By the assumption, we see that, for any $p \geq 2$ and k, there exists $\epsilon_0 > 0$ such that $|G(\epsilon, \boldsymbol{\xi}) - g_0(\boldsymbol{\xi})|_{k,p,\rho(\boldsymbol{\xi})} \leq C_{p,k}\epsilon q_2(\boldsymbol{\xi})$ holds for any $\epsilon \in (0, \epsilon_0)$. Hence, it follows from (2.10) and (6.4) that

(6.5)
$$\operatorname*{ess\,sup}_{\sigma \in A^k(\rho(\boldsymbol{\xi}))} \left\| \left(\frac{D_\sigma}{\gamma(\sigma)} \left(\frac{1}{G(\epsilon, \boldsymbol{\xi})} - \frac{1}{g_0(\boldsymbol{\xi})} \sum_{j=0}^{m} (-1)^j \left(\frac{G(\epsilon, \boldsymbol{\xi}) - g_0(\boldsymbol{\xi})}{g_0(\boldsymbol{\xi})} \right)^j \right) \right) 1_B \right\|_p \leq C_{p,k}\epsilon^{m+1} q_2(\boldsymbol{\xi})^{-1},$$

for $\epsilon > 0$ small enough. On the other hand, applying Chebychev's inequality, we have

$$P(B^c)^{\frac{1}{2p}} \leq C_{p,m} \left\| \frac{G(\epsilon, \boldsymbol{\xi}) - g_0(\boldsymbol{\xi})}{g_0(\boldsymbol{\xi})} \right\|_{2p(m+1)}^{m+1} \leq C_{p,m}\epsilon^{m+1}, \tag{6.6}$$

for $\epsilon > 0$ small enough. One can check that

$$\begin{aligned} &\operatorname*{ess\,sup}_{\sigma \in A^k(\rho(\boldsymbol{\xi}))} \left\| \frac{D_\sigma}{\gamma(\sigma)} \left(\frac{1}{G(\epsilon, \boldsymbol{\xi})} - \frac{1}{g_0(\boldsymbol{\xi})} \sum_{j=0}^{m} (-1)^j \left(\frac{G(\epsilon, \boldsymbol{\xi}) - g_0(\boldsymbol{\xi})}{g_0(\boldsymbol{\xi})} \right)^j \right); B^c \right\|_p \\ &\quad \leq \operatorname*{ess\,sup}_{\sigma \in A^k(\rho(\boldsymbol{\xi}))} \left\| \frac{D_\sigma}{\gamma(\sigma)} \frac{1}{G(\epsilon, \boldsymbol{\xi})}; B^c \right\|_p \\ &\qquad + \sum_{j=0}^{m} \operatorname*{ess\,sup}_{\sigma \in A^k(\rho(\boldsymbol{\xi}))} \left\| \frac{D_\sigma}{\gamma(\sigma)} \left(\frac{1}{g_0(\boldsymbol{\xi})} (-1)^j \left(\frac{G(\epsilon, \boldsymbol{\xi}) - g_0(\boldsymbol{\xi})}{g_0(\boldsymbol{\xi})} \right)^j \right); B^c \right\|_p. \end{aligned}$$

It follows from Schwarz's inequality, (6.4) and (6.6) that the right hand side is bounded by $C_{p,k}\epsilon^{m+1}q_2(\boldsymbol{\xi})^{-1}$. Combining (6.5), we have

$$\left\| \frac{1}{G(\epsilon, \boldsymbol{\xi})} - \frac{1}{g_0(\boldsymbol{\xi})} \sum_{j=0}^{m} (-1)^j \left(\frac{G(\epsilon, \boldsymbol{\xi}) - g_0(\boldsymbol{\xi})}{g_0(\boldsymbol{\xi})} \right)^j \right\|_{k,p,\rho(\boldsymbol{\xi})} \leq C_{m,k} \frac{\epsilon^{m+1}}{q_2(\boldsymbol{\xi})},$$

for $\epsilon > 0$ small enough. Further, Proposition 5.1 yields that, for any j,

$$\left(\frac{G(\epsilon, \boldsymbol{\xi}) - g_0(\boldsymbol{\xi})}{g_0(\boldsymbol{\xi})} \right)^j \sim \sum_{n=j}^{\infty} \epsilon^j g_n^{(j)}(\boldsymbol{\xi}) \quad \text{in } \mathbf{D}_\infty(1, A(\rho(\boldsymbol{\xi}))).$$

This completes the proof. □

Lemma 6.2. *Suppose that* $F(\epsilon) \sim \sum_{n=0}^{\infty} \epsilon^n f_n$ *in* $\mathbf{D}_\infty(\mathbf{R}^d)$. *We set*

$$X_u(\epsilon, \boldsymbol{\xi}) = (e^{i\boldsymbol{\xi} \cdot D_u F(\epsilon, \boldsymbol{\xi})} - 1) 1_{A(\rho(\boldsymbol{\xi}))}(u).$$

Then, we have $X_u(\epsilon, x) \sim \sum_{j=0}^{\infty} \epsilon^j x_u^{(j)}(\boldsymbol{\xi})$ *in* $\mathbf{D}_\infty^\sim(q_1(\boldsymbol{\xi}), A(\rho(\boldsymbol{\xi})))$. *In particular,* $x_u^{(0)}(\boldsymbol{\xi})$ *is given by* $(e^{i\boldsymbol{\xi} \cdot D_u f_0(\boldsymbol{\xi})} - 1) 1_{A(\rho(\boldsymbol{\xi}))}(u)$.

Proof. Put $x_u^{(0)}(\boldsymbol{\xi}) = (e^{i\boldsymbol{\xi}\cdot D_u f_0(\boldsymbol{\xi})} - 1)1_{A(\rho(\boldsymbol{\xi}))}(u)$. Then, one can write

$$X_u(\epsilon, \boldsymbol{\xi}) - x_u^{(0)}(\boldsymbol{\xi}) = (e^{i\boldsymbol{\xi}\cdot D_u(F(\epsilon)-f_0)} - 1)e^{i\boldsymbol{\xi}\cdot D_u f_0(\boldsymbol{\xi})}1_{A(\rho(\boldsymbol{\xi}))}(u).$$

It follows from (3.10) that $|x^{(0)}|^{\sim}_{k,p,\rho(\boldsymbol{\xi})} \leq C_{p,k}q_1(\boldsymbol{\xi})$ and $|e^{i\boldsymbol{\xi}\cdot D.f_0(\boldsymbol{\xi})}|^{\sim}_{k,p,\rho(\boldsymbol{\xi})} \leq C_{p,k}q_1(\boldsymbol{\xi})$. Hence, if we can show that $(e^{i\boldsymbol{\xi}\cdot D_u(F(\epsilon)-f_0)} - 1)1_{A(\rho(\boldsymbol{\xi}))}(u)$ has the asymptotic expansion $\sum_{j=1}^{\infty}\epsilon^j h_u^{(j)}$ in $\mathbf{D}^{\sim}_{\infty}(q_1(\boldsymbol{\xi}), A(\rho(\boldsymbol{\xi})))$, then the assertion follows from (2.15) and the fact that $\rho(\boldsymbol{\xi})(q_1(\boldsymbol{\xi}))^2 = |\boldsymbol{\xi}|^{2-\frac{1}{\beta}} \leq q_1(\boldsymbol{\xi})$.

We set $H_u = D_u(F(\epsilon) - f_0)$. It follows from the proof of Lemma 3.3 that

$$\sup_{\mu\in(0,1]} \left\|e^{i\mu\boldsymbol{\xi}\cdot H_u} - 1\right\|_{k,p,\rho(\boldsymbol{\xi})} \leq C_{p,k}\epsilon q_1(\boldsymbol{\xi}), \tag{6.7}$$

for $\epsilon > 0$ small enough. Note that

(6.8)
$$\left\|\frac{D_\sigma\left(e^{i\boldsymbol{\xi}\cdot H_u(\epsilon)} - 1 - \sum_{j=1}^{m}\frac{(\boldsymbol{\xi}\cdot H_u(\epsilon))^j}{j!}\right)}{\gamma(\sigma)\gamma(u)}\right\|_p$$
$$= \left\|D_\sigma\frac{(\boldsymbol{\xi}\cdot H_u(\epsilon))^m}{\gamma(\sigma)\gamma(u)m!}\int_0^1\int_0^{\mu_1}\cdots\int_0^{\mu_{m-1}}(e^{-i\mu_m\boldsymbol{\xi}\cdot H_u(\epsilon)} - 1)\,d\mu_m d\mu_{m-1}\cdots d\mu_1\right\|_p$$
$$\leq \sup_{\mu\in(0,1)}\ \operatorname*{ess\,sup}_{(u,\sigma)\in A(\rho(\boldsymbol{\xi}))}\left\|D_\sigma\frac{(\boldsymbol{\xi}\cdot H_u(\epsilon))^m(e^{-i\mu\boldsymbol{\xi}\cdot H_u(\epsilon)} - 1)}{\gamma(\sigma)\gamma(u)}\right\|_p.$$

As we mentioned above, $\rho(\boldsymbol{\xi})(q_1(\boldsymbol{\xi}))^2 \leq q_1(\boldsymbol{\xi})$ holds. Hence, it follows from (2.15) that the right hand side is bounded by $\epsilon^{m+1}q_1(\boldsymbol{\xi})$ for $\epsilon > 0$ small enough. From Proposition 5.1, we know that $(\boldsymbol{\xi}\cdot H_u(\epsilon))^j$ has the asymptotic expansion $\sum_{n=j}^{\infty}e^n h_u^{j,(n)}$ in $\mathbf{D}^{\sim}_{\infty}(q_1(\boldsymbol{\xi}); A(\rho(\boldsymbol{\xi})))$. Therefore, $e^{i\boldsymbol{\xi}\cdot D_u(F(\epsilon)-f_0)} - 1$ has the asymptotic expansion $\sum_{j=1}^{\infty}\epsilon^j h_u^{(j)}$ in $\mathbf{D}^{\sim}_{\infty}(q_1(\boldsymbol{\xi}), A(\rho(\boldsymbol{\xi})))$. □

Lemma 6.3. *Suppose that $F(\epsilon) \sim \sum_{j=0}^{\infty}\epsilon^j f_j$ in $A(1)$, and $F(\epsilon)$ satisfies the uniformly non-degenerate condition. Put*

$$G(\epsilon, \boldsymbol{\xi}) := \int_{A(\rho(\boldsymbol{\xi}))}|e^{i\boldsymbol{\xi}\cdot D_u F(\epsilon)} - 1|^2\lambda(du). \tag{6.9}$$

Then, we have

$$\frac{1}{G(\epsilon, \boldsymbol{\xi})} \sim \sum_{n=0}^{\infty}\epsilon^n g_n^{'}(\boldsymbol{\xi}) \quad \text{in} \quad \mathbf{D}_{\infty}(q_2(\boldsymbol{\xi}), A(\rho(\boldsymbol{\xi}))).$$

In particular, we have

$$\operatorname*{ess\,sup}_{\tau\in A^k(\rho(\boldsymbol{\xi}))}\left\|\left(\int_{A(\rho(\boldsymbol{\xi}))}|e^{i\boldsymbol{\xi}\cdot f_0} - 1|^2\lambda(du)\right)^{-1}\circ\varepsilon_\tau^+\right\|_p \leq C_{p,k}q_2(\boldsymbol{\xi})^{-1}. \tag{6.10}$$

Proof. Note that

$$|G(\epsilon,\boldsymbol{\xi})\circ\varepsilon_\tau^+| \geq |\boldsymbol{\xi}|^2 \left(\int_{A(\rho(\boldsymbol{\xi}))} \left|\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\cdot D_v F(\epsilon)\right|^2 1_{\{|D_u F(\epsilon)|\leq|\boldsymbol{\xi}|^{-1}\}}(u)\lambda(du)\right)\circ\varepsilon_\tau^+.$$

Hence, uniformly non-degenerate condition yields that

$$\limsup_{\epsilon\to 0+}\sup_{\boldsymbol{\xi}\in U^d}\operatorname*{ess\,sup}_{\tau\in A(\rho(\boldsymbol{\xi}))}\left\|q_2(\boldsymbol{\xi})\left(G(\epsilon,\boldsymbol{\xi})\circ\varepsilon_\tau^+\right)^{-1}\right\|_p<\infty.$$

Because, $G(\epsilon,\boldsymbol{\xi})=\int_{A(\rho(\boldsymbol{\xi}))}(e^{i\boldsymbol{\xi}\cdot D_uF(\epsilon)}-1)(e^{-i\boldsymbol{\xi}\cdot D_uF(\epsilon)}-1)\lambda(du)$, the assertion follows from Lemma 6.2, iv) in Proposition 5.1, and Lemma 6.1. □

The following theorem is easily deduced from Lemma 6.3, Lemma 6.2 and ii) in Proposition 5.1.

Theorem 6.1. *Suppose that $F(\epsilon)\sim\sum_{j=0}^\infty\epsilon^j f_j$ in $A(1)$, and $F(\epsilon)$ satisfies the uniformly non-degenerate condition.*

We set

$$Z_u(\epsilon,\boldsymbol{\xi})=\frac{(e^{-i\boldsymbol{\xi}\cdot D_uF(\epsilon)}-1)1_{A(\rho(\boldsymbol{\xi}))}(u)}{\int_{A(\rho(\boldsymbol{\xi})}|e^{i\boldsymbol{\xi}\cdot D_vF(\epsilon)}-1|^2\lambda(dv)}. \tag{6.11}$$

Then, we have $Z(\epsilon,u,\boldsymbol{\xi})\sim\sum_{j=0}^\infty\epsilon^j z_j(u,\boldsymbol{\xi})$ in $\mathbf{D}_\infty^\sim(q_0(\boldsymbol{\xi});A(\rho(\boldsymbol{\xi})))$, where

$$q_0(\boldsymbol{\xi})=(|\boldsymbol{\xi}|\Gamma(\rho(\boldsymbol{\xi})))^{-1}.$$

Lemma 6.4. *Suppose that $F(\epsilon)\sim\sum_{j=0}^\infty\epsilon^j f_j$, and $F(\epsilon)$ satisfies the uniformly non-degenerate condition. We set*

$$R_m(\boldsymbol{\xi},\epsilon)=e^{-i\boldsymbol{\xi}\cdot F(\epsilon)}-\sum_{|\mathbf{n}|\leq m}\frac{e^{-i\boldsymbol{\xi}\cdot g_0}}{\mathbf{n}!}(-i\boldsymbol{\xi}\cdot(F(\epsilon)-f_0))^{\mathbf{n}}.$$

Then, for any natural number n for any $r>2$ and for $\boldsymbol{\xi}\in U^d$, we have

(6.12)
$$\limsup_{\epsilon\to 0}\sup_{\boldsymbol{\xi}\in U^d}\sup_{|G|_{n,r}=1}|\epsilon^{-(m+1)}(1+|\boldsymbol{\xi}|^2)^{-(m+1)+(1-\frac{\alpha}{2\beta})\frac{n}{2}}\mathbf{E}[GR_m(\boldsymbol{\xi},\epsilon)]|<\infty.$$

Proof. It follows from Proposition 5.1 that $(\xi\cdot(F(\epsilon)-f_0))^j$ has the asymptotic expansion $\sum_{k=j}^\infty\epsilon^k h_k^j(\boldsymbol{\xi})$ in $\mathbf{D}(|\boldsymbol{\xi}|^j;A(\rho(\boldsymbol{\xi})))$. From Lemma 6.3, the inequality (6.10) holds. Hence, from Theorem 3.2 (see Remark 7), we have

$$\sup_{|G|_{n,r}=1}\left|\mathbf{E}\left[G\frac{e^{i\boldsymbol{\xi}\cdot f_0}}{k!}(\xi\cdot(F(\epsilon)-f_0))^k\right]-\mathbf{E}\left[G\frac{e^{i\boldsymbol{\xi}\cdot f_0}}{k!}\sum_{j=k}^m\epsilon^j h_j^k(\boldsymbol{\xi})\right]\right|$$
$$\leq C_{m,k,n}\epsilon^{m+1}|\boldsymbol{\xi}|^{k-(1-\frac{\alpha}{2\beta})\frac{n}{2}},$$

for $\epsilon > 0$ small enough. We can regard $G \to \mathbf{E}[G\frac{e^{i\boldsymbol{\xi}\cdot f_0}}{k!}\epsilon^j h_j^k(\boldsymbol{\xi})]$ as a linear continuous functional on $\mathbf{D}_{n,r}$. Therefore, there is a sequence $\{l_n(\boldsymbol{\xi},\cdot)\,;\, n = 0,1,2,\ldots\} \subset \mathbf{D}^*_{n,r}$ such that, for $\epsilon > 0$ small enough,

$$\begin{aligned}(6.13)\qquad &\sup_{|G|_{n,r}=1}\left|\sum_{k=0}^{m}\mathbf{E}\left[G\frac{e^{i\boldsymbol{\xi}\cdot f_0}}{k!}(\xi\cdot(F(\epsilon)-f_0))^k\right]-\sum_{k=0}^{m}\epsilon^k l_k(\boldsymbol{\xi},G)\right| \\ &\quad\le C\epsilon^{m+1}|\,\boldsymbol{\xi}\,|^{m+1-(1-\frac{\alpha}{2\beta})\frac{n}{2}}.\end{aligned}$$

By the Taylor expansion, we have

$$(6.14)\qquad e^{i\boldsymbol{\xi}\cdot F(\epsilon)}=\sum_{k=0}^{m}\frac{e^{i\boldsymbol{\xi}\cdot f_0}}{k!}i(\boldsymbol{\xi}\cdot(F(\epsilon)-f_0))^k+O((\epsilon|\,\boldsymbol{\xi}\,||F(\epsilon)-f_0|)^{m+1}).$$

Hence, it holds that for each m

$$(6.15)\qquad \sup_{|G|_{n,r}=1}\left|\mathbf{E}[Ge^{i\boldsymbol{\xi}\cdot F(\epsilon)}]-\sum_{k=0}^{m}\epsilon^k l_k(\boldsymbol{\xi},G)\right|\le C\epsilon^{m+1}|\,\boldsymbol{\xi}\,|^{m+1},$$

for $\epsilon > 0$ small enough.

On the other hand, we define $Z_u(\epsilon,\boldsymbol{\xi})$ by (6.11). Then, as we saw in Section 3, we have

$$\mathbf{E}\left[Ge^{i\xi\cdot F(\epsilon)}\right]=\mathbf{E}\left[D^{*(n)}_{\mathbf{Z}(\epsilon,\boldsymbol{\xi})}(G)e^{i\xi\cdot F(\epsilon)}\right],$$

where $D^{*(n)}_{\mathbf{Z}(\epsilon,\boldsymbol{\xi})}(G)$ is defined by (3.7) with $Z_u^{(j)} = Z_u^{(j)}(\epsilon,\boldsymbol{\xi})$. It follows from Theorem 6.1 that $Z_u(\epsilon,\boldsymbol{\xi})$ has the asymptotic expansion $\sum_{j=0}^{\infty}\epsilon^j z_u^{(j)}(\boldsymbol{\xi})$ in $\mathbf{D}^{\sim}_{\infty}(q_0(\boldsymbol{\xi}),A(\rho(\boldsymbol{\xi})))$. Hence, we have

$$\begin{aligned}&\mathbf{E}\left[D^{*(n_0)}_{Z(\epsilon,\boldsymbol{\xi})}(G)e^{i\xi\cdot F(\epsilon)}\right]\\ (6.16)\qquad &=\mathbf{E}\left[D^*\left(\left(Z(\epsilon,\boldsymbol{\xi})-\sum_{j=0}^{m}\epsilon^j z^{(j)}(\xi)\right)\,D^{*(n-1)}_{Z(\epsilon,\boldsymbol{\xi})}(G)\right)e^{i\xi\cdot F(\epsilon)}\right]\\ &\quad+\sum_{j=0}^{m}\epsilon^j\,\mathbf{E}\left[D^*_{z_j(\xi)}\left(D^{*(n-1)}_{Z(\epsilon,\boldsymbol{\xi})}(G)\right)e^{i\xi\cdot F(\epsilon)}\right].\end{aligned}$$

From Theorem 3.3, we have

$$\begin{aligned}&\sup_{|G|_{n,r}=1}\left|\mathbf{E}\left[D^*\left(\left(Z(\epsilon,\boldsymbol{\xi})-\sum_{j=0}^{m}\epsilon^j z_j(\xi)\right)\,D^{*(n-1)}_{\mathbf{Z}(\epsilon,\boldsymbol{\xi})}(G)\right)e^{i\xi\cdot F(\epsilon)}\right]\right|\\ &\quad\le C_n\epsilon^{m+1}|\,\boldsymbol{\xi}\,|^{-(1-\frac{\alpha}{2\beta})\frac{n}{2}}.\end{aligned}$$

Similarly, by expanding the sum $\sum_{j=0}^{m}\epsilon^j\,\mathbf{E}\left[D^*_{z_j(\xi)}\left(D^{*(n-1)}_{\mathbf{Z}(\epsilon)}(G)\right)e^{i\xi\cdot F(\epsilon)}\right]$ as in (6.16), we get

$$\sup_{|G|_{n,r}=1}\left|\mathbf{E}\left[Ge^{i\xi\cdot F(\epsilon)}\right]-\sum_{j=0}^{m}\epsilon^j l'_j(\epsilon,\boldsymbol{\xi},G)\right|\le C_{m,n}\epsilon^{m+1}|\,\boldsymbol{\xi}\,|^{-(1-\frac{\alpha}{2\beta})\frac{n}{2}},$$

where $l_j^{'}(\epsilon, \boldsymbol{\xi}, G)$ is a linear continuous functional on $\mathbf{D}_{n,r}$ and is given by

$$l_j^{'}(\epsilon, \boldsymbol{\xi}, G) = \sum_{j_1+\cdots+j_n=j} \mathbf{E}\left[D^*_{z^{j_1}(\boldsymbol{\xi})}(D^*_{z^{j_2}(\boldsymbol{\xi})}(\cdots(D^*_{z^{j_n}(\boldsymbol{\xi})}(G)\cdots))e^{i\xi\cdot F(\epsilon)}\right].$$

Again, from Theorem 3.3, we have $\sup_{|G|_{n,r}=1}|l_j^{'}(\epsilon, \boldsymbol{\xi}, G)| \le C|\boldsymbol{\xi}|^{-(1-\frac{\alpha}{2\beta})\frac{n}{2}}$. Therefore, by the Taylor expansion (6.14), we can pick the sequence $\{l_k^{\sim}(\boldsymbol{\xi}, \cdot)\,;\, k = 0, 1, 2, \ldots\}$ of linear functional on $\mathbf{D}_{n,r}$ such that, for each m,

$$\text{(6.17)} \qquad \sup_{|G|_{n,r}=1}\left|\mathbf{E}[Ge^{i\boldsymbol{\xi}\cdot F(\epsilon)}] - \sum_{k=0}^{m} l_k^{\sim}(\boldsymbol{\xi}, G)\right| \le C_{n,m}\epsilon^{m+1}|\boldsymbol{\xi}|^{m+1-(1-\frac{\alpha}{2\beta})\frac{n}{2}},$$

for $\epsilon > 0$ small enough. If we compare (6.15) with (6.17), then we get $l_k(\boldsymbol{\xi}, \cdot) = l_k^{\sim}(\boldsymbol{\xi}, \cdot)$ for any k. Therefore, from (6.13) and (6.17), we get

$$\begin{aligned}\sup_{|G|_{n,r}=1}|\mathbf{E}[GR_m(\boldsymbol{\xi}, \epsilon)]| &\le \sup_{|G|_{n,r}=1}\left|\mathbf{E}[Ge^{i\boldsymbol{\xi}\cdot F(\epsilon)}] - \sum_{k=0}^{m} l_k^{\sim}(\boldsymbol{\xi}, G)\right| \\ &\quad + \sup_{|G|_{n,r}=1}\left|\sum_{k=0}^{m}\mathbf{E}[G\frac{e^{i\boldsymbol{\xi}\cdot f_0}}{k!}(\xi\cdot(F(\epsilon)-f_0))^k] - \sum_{k=0}^{m} l_k(\boldsymbol{\xi}, G)\right| \\ &\le C_m\epsilon^{m+1}|\boldsymbol{\xi}|^{m+1-(1-\frac{\alpha}{2\beta})\frac{n}{2}},\end{aligned}$$

for $\epsilon > 0$ small enough. □

Proof of Theorem 5.1. By the definition of the composition, we have

$$\begin{aligned}&\left\langle \sum_{|\mathbf{n}|\le m}\left(\frac{1}{\mathbf{n}!}\partial^{\mathbf{n}}T\right)\circ f_0\ (F(\epsilon)-f_0)^{\mathbf{n}}\,,\, G\right\rangle \\ &\quad = \sum_{|\mathbf{n}|\le m}\frac{1}{\mathbf{n}!}{}_{\mathcal{S}'}\left\langle\, \mathcal{F}\partial^{\mathbf{n}}T\,,\, \mathbf{E}[G(F(\epsilon)-f_0)^{\mathbf{n}}e^{i\boldsymbol{\xi}\cdot f_0}]\,\right\rangle_{\mathcal{S}_\xi} \\ &\quad = {}_{\mathcal{S}'}\left\langle \mathcal{F}T\,,\, \sum_{|\mathbf{n}|\le m}\frac{(-i\boldsymbol{\xi})^{\mathbf{n}}}{\mathbf{n}!}\mathbf{E}[G(F(\epsilon)-f_0)^{\mathbf{n}}e^{i\boldsymbol{\xi}\cdot f_0}]\right\rangle_{\mathcal{S}_{\boldsymbol{\xi}}},\end{aligned}$$

Taking $s_0 > 0$ so that $T \in \mathbf{H}^{-s_0}$ and n so that $n \ge \frac{d-2(s+m)-1}{(2-\frac{\alpha}{\beta})}$. Then, we have

$\int(1+|\boldsymbol{\xi}|^2)^{s_0+m+1-(1-\frac{\alpha}{2\beta})\frac{n}{2}}\, d\boldsymbol{\xi} < \infty$. We define $R_m(\epsilon, \boldsymbol{\xi})$ by (6.12). We get

$$\begin{aligned}
&\Big\| \, T \circ F(\epsilon) - \sum_{|\mathbf{n}|\le m} \frac{1}{\mathbf{n}!} (\partial^{\mathbf{n}} T) \circ f_0 \, (F(\epsilon) - f_0)^{\mathbf{n}} \, \Big\|^*_{n,r} \\
&= \sup_{|G|_{n,r}=1} \Big| \, \langle T \circ F(\epsilon) \, , \, G \rangle - \Big\langle \sum_{|\mathbf{n}|\le m} \frac{1}{\mathbf{n}!} (\partial^{\mathbf{n}} T) \circ f_0 \, (F(\epsilon) - f_0)^{\mathbf{n}} \, , \, G \Big\rangle \, \Big| \\
&= \sup_{|G|_{n,r}=1} \Big| \, {}_{\mathcal{S}'} \langle \mathcal{F} T \, , \, \mathbf{E}[G R_m(\boldsymbol{\xi}, \epsilon)] \, \rangle_{\mathcal{S}_{\boldsymbol{\xi}}} \, \Big| \\
&\le \sup_{|G|_{n,r}=1} \|T\|_{\mathbf{H}_{-s_0}} \Big[\int_{\mathbf{R}^d} (1+|\boldsymbol{\xi}|^2)^{s_0} \left| \mathbf{E} \left[\, G R_m(\boldsymbol{\xi}, \epsilon) \, \right] \right|^2 d\boldsymbol{\xi} \, \Big]^{\frac{1}{2}} \\
&\le \|T\|_{\mathbf{H}_{-s_0}} \Big[\int_{\mathbf{R}^d} (1+|\boldsymbol{\xi}|^2)^{s_0} \sup_{|G|_{n,r}=1} \left| \mathbf{E} \left[G R_m(\boldsymbol{\xi}, \epsilon) \right] \right|^2 d\boldsymbol{\xi} \, \Big]^{\frac{1}{2}} .
\end{aligned}$$

The assertion follows from Lemma 6.4. □

7. Applications to stochastic differential equations

In this section, we suppose that $E = (0,1] \times \mathbf{R}$ and $\lambda(du) = ds \times \nu(du)$. As we mentioned in Section 1, we shall consider the following real valued stochastic differential equation:

$$X_t(\epsilon) = x_0 + \int_0^t b(X_{s-}(\epsilon)) ds + \epsilon \int_{0+}^t \int a(X_{s-}(\epsilon), y) \, \tilde{N}(ds \times dy) \tag{7.1}$$

where $\epsilon \in (0,1)$, and give the asymptotic expansion:

$$X_1(\epsilon) \sim \sum_{n=0}^{\infty} \epsilon^n f_n \quad \text{in } \mathbf{D}_\infty. \tag{7.2}$$

For a function $h(x,y)$ on $\mathbf{R}^2$, we denote by $h^{(n)}(x,y)$ the n-times derivatives with respect to x, if it is differentiable. The regularity and boundedness assumptions for functions a and b are the following:

Assumption 1. Suppose that there exist functions $\bar{a}(x)$ and $\tilde{a}(x,y)$ such that

$$a(x,y) = \bar{a}(x) y + \tilde{a}(x,y). \tag{7.3}$$

The functions $\bar{a}(x)$ and $b(x)$ are infinitely differentiable with bounded derivative. The function $\tilde{a}(x,y)$ is also infinitely differentiable with respect to x and satisfy the following conditions: for some positive constant $r_0 > 1$ such that $\int_{\{|x|<1\}} |x|^{r_0} \nu(dx) < \infty$, the following inequalities

$$|\tilde{a}(x,y)| \le C(1+|x|)|y|^{r_0}, \quad |\tilde{a}^{(n)}(x,y)| \le C|y|^{r_0} \qquad \text{on } \{y \,;\, |y| \le 1\}, \tag{7.4}$$

hold. Further, suppose that the following inequalities hold, for any $p \geq 2$ and for $n \geq 1$,

$$\int |a(x,y)|^p \nu(dy) \leq C(1+|x|)^p, \quad \sup_x \int |a^{(n)}(x,y)|^p \nu(dy) < \infty. \tag{7.5}$$

Remark 19. Under Assumption 1, the stochastic differential equation (7.1) has the solution. Moreover, one can find several properties of the solution to the stochastic differential equation (7.1) which Picard [12] has obtained. For example, one can deduce from the proof of Theorem 4.1 in [12] that, for any k, p,

$$\sup_\epsilon \mathbf{E}\left[\operatorname*{ess\,sup}_{\sigma \in A^k(1)} \left|\frac{D^k_\sigma X_t(\epsilon)}{\gamma(\sigma)}\right|^p\right] < \infty,$$

in particular $X_t(\epsilon) \in \mathbf{D}_\infty$. Using the argument of the proof of Theorem 4.1, we can also show that

$$\sup_\epsilon \sup_{0<t\leq 1} |X_t(\epsilon)|_{k,p} < \infty. \tag{7.6}$$

For convenience, we extend the region of the parameter ϵ to the open interval $(-1,1)$. We denote by $\mathcal{F}_s$ the least σ-field which $N(A)$ $(A \subset (0,s] \times \mathbf{R}\})$ are measurable.

Lemma 7.1. *Let $0 \leq s_0 < 1$ be fixed, and $Y_{s_0,t}(\epsilon)$ $(0 < s_0 < t \leq 1)$ be a semimartingale having the following form*

$$Y_{s_0,t}(\epsilon) = Y_{s_0,s_0}(\epsilon) + \int_{s_0}^t g_r(\epsilon)\, dr + \int_{s_0}^t \int h_r(\epsilon, y) \tilde{N}(ds \times dy).$$

where $Y_{s_0,s_0}(\epsilon)$ is $\mathcal{F}_{s_0}$-measurable, and $g_r(\epsilon)$ $h_r(\epsilon,y)$ are predictable processes for each fixed $(\epsilon, y) \in (-1,1) \times \mathbf{R}$. Suppose that, for any $p \geq 2$,

$$\sup_{\epsilon \in (-1,1)} \mathbf{E}[|Y_{s_0,s_0}(\epsilon)|^p] + \sup_{\epsilon \in (-1,1)} \mathbf{E}[\sup_{s_0 < r \leq 1} |Y_{s_0,r}(\epsilon)|^2] < \infty. \tag{7.7}$$

Moreover, we suppose that there exists a predictable process $\eta_t(\epsilon)$ $(s_0 < t \leq 1)$ such that, for any $t \in (s_0, 1]$

$$\begin{aligned} &|g_t(\epsilon)| \leq C(\sup_{s_0 < r \leq t} |Y_{r-}(\epsilon)| + |\eta_t(\epsilon)|), \\ &\int |h_t(\epsilon,y)|^p \nu(dy) \leq C_p(\sup_{s_0<r\leq t} |Y_{r-}(\epsilon)| + |\eta_t(\epsilon)|)^p, \end{aligned} \tag{7.8}$$

then we have

$$\| \sup_{s_0 \leq r \leq 1} |Y_r(\epsilon)| \, \|_p \leq C_p \left(\|Y_{s_0,s_0}\|_p + \mathbf{E}\left[\int_{s_0}^1 |\eta_r(\epsilon)|^p \, dr\right]^{\frac{1}{p}} \right). \tag{7.9}$$

Remark 20. The proof of the inequality (7.9) is essentially due to [4], although the argument in [4] is for a concrete semimartingale.

Proof. We set

$$T_R := \inf\{t > s; |Y_{s,t}| \geq R\}$$

with convention that $\inf \emptyset = 1$. By the condition (7.7), we have $\lim_{R\to\infty} \mathbf{P}(T_R < 1) = 0$. Put $Y^R_{s,t} = Y_{s,t\wedge T_R}$. By the monotone convergence theorem, we get

$$\lim_{R\to\infty} \mathbf{E}[\sup_{s<r\leq 1} |Y^R_{s,r}|^p] = \mathbf{E}[\lim_{R\to\infty} \sup_{s<r\leq T_R\wedge 1} |Y_{s,r}|^p] = \mathbf{E}[\sup_{s<r\leq 1} |Y_{s,r}|^p].$$

Hence, if (7.9) holds for Y^R with a constant C_p which does not depend on R, then we complete the proof. We have

(7.10)
$$|Y^R_{s,t}(\epsilon)|^p \leq C^1_p \left(|Y_{s_0,s_0}(\epsilon)|^p + \left| \int_{s_0}^{t\wedge T_R} g_r(\epsilon)\, dr \right|^p + \left| \int_{s_0}^{t\wedge T_R} \int h_r(\epsilon, y) \tilde{N}(dr \times dy) \right|^p \right).$$

By Schwarz's inequality and (7.8), we have

$$\left| \int_{s_0}^{t\wedge T_R} g_r(\epsilon)\, dr \right|^p \leq C \left(\int_{s_0}^{t} |\eta_r(\epsilon)|^p\, dr + \int_{s_0}^{t} \sup_{s_0<u\leq r} |Y^R_{s_0,u}(\epsilon)|^p\, dr \right). \tag{7.11}$$

Put $M_{s_0,t} = \int_{s_0}^t \int h_r(\epsilon, y)\tilde{N}(dr \times dy)$, and $M^R_{s_0,t} = \int_{s_0}^{t\wedge T_R} \int h_r(\epsilon, y)\tilde{N}(dr \times dy)$. By the Itô formula, we have

$$|M^R_{s_0,t}|^p = \text{ a martingale with mean zero} + \int_{s_0}^{t\wedge T_R} \int \left(|M_{s_0,r} + h_{s_0,r}(y)|^p - |M_{s_0,r-}|^p - p h_{s_0,r}(y) |M_{s_0,r}|^{p-1} \right) \nu(dy) ds$$

The mean value theorem yields that

$$\begin{aligned} &\big| |M_{s_0,r} + h_{s,r} g(y)|^p - |M_{s_0,r-}|^p - p\, h_{s,r} g(y) |M_{s_0,r}|^{p-1} \big| \\ &\leq \frac{1}{2} p(p-1) |h_{s_0,r} g(y)|^2 |M_{s_0,r} + \theta h_{s_0,r} g(y)|^{p-2} \\ &\leq C^4_p (|h_{s_0,r}|^2 |M_{s_0,r}|^{p-2} + |h_{s_0,r}|^p), \end{aligned}$$

Using Doob's inequality, we have

$$\begin{aligned} &\mathbf{E}[\sup_{s_0<r\leq t} |M^R_{s_0,t}|^p] \\ &\leq \mathbf{E}\left[\int_{s_0}^{t\wedge T_R} \int \left(|h_r(\epsilon, y)|^2 |M_{s_0,r}|^{p-2} + |h_r(\epsilon, y)|^p \right) \nu(dy) ds \right]. \end{aligned} \tag{7.12}$$

In the case $p = 2$, one can deduce the assertion from (7.10), (7.11), (7.12), (7.8), and Gronwall's lemma. In the case $p > 2$, it follows from (7.8) that, for $r \le t \wedge T_R$

$$
\begin{aligned}
|M_{s_0,r}|^{p-2} &\le \left(|Y_{s_0,s_0}(\epsilon)| + \left| \int_s^r g_{r_1}(\epsilon) dr_1 \right| + \sup_{s_0 < r_1 \le r} |Y^R_{s_0,r_1}(\epsilon)| \right)^{p-2} \\
&\le C_p \left(|Y_{s_0,s_0}(\epsilon)| + \int_{s_0}^t |\eta_{s_0,r_1}(\epsilon)| dr_1 + \sup_{s_0 < r_1 \le r} |Y^R_{s_0,r_1}(\epsilon)| \right)^{p-2},
\end{aligned}
$$

and that

$$\int |h_r(\epsilon, y)|^2 \nu(dy) \le C_2 \left(|\eta_r(\epsilon)| + \sup_{s_0 < r_1 \le r} |Y^R_{s_0,r_1}(\epsilon)| \right)^2.$$

Hence, Hölder's inequality yields

$$
\begin{aligned}
&\int_0^{t \wedge T_R} \int |h_r(\epsilon, y)|^2 |M_{s_0,r}|^{p-2} \nu(dy) dr \\
&\le C_p \left[\int_{s_0}^t \left(|\eta_r(\epsilon)| + \sup_{s_0 < u \le r} |Y^R_{s_0,u}(\epsilon)| \right)^p dr \right]^{\frac{2}{p}} \\
&\quad \cdot \left[\int_{s_0}^t \left(|Y_{s_0,s_0}(\epsilon)| + \int_{s_0}^r |\eta_{r_1}(\epsilon)| dr_1 + \sup_{s_0 < r_1 \le r} |Y^R_{s_0,r_1}(\epsilon)| \right)^p dr \right]^{\frac{p-2}{p}} \\
&\le C_p^5 \left[\int_{s_0}^t \left(|Y_{s_0,s_0}(\epsilon)|^p + |\eta_r(\epsilon)|^p + \sup_{s < u \le r} |Y^R_{s_0,u}(\epsilon)|^p \right) dr \right].
\end{aligned}
$$

Therefore, the right hand side of (7.12) is bounded by

$$C_p \mathbf{E} \left[|Y_{s_0,s_0}(\epsilon)|^p + \int_{s_0}^t \left(|\eta_r(\epsilon)|^p + \sup_{s_0 < r_1 \le r} |Y^R_{s_0,r_1}|^p \right) dr \right]$$

Hence, (7.10), (7.11), and Gronwall's lemma also show (7.9) for $p > 2$. □

Lemma 7.2. *For any $p \ge 2$, there exists a constant $C_p > 0$ such that the following inequalities hold*

$$\mathbf{E}[\sup_{0 \le s \le 1} |X_s(\epsilon) - X_s(\epsilon_1)|^p] \le C_p |\epsilon - \epsilon_1|^p \tag{7.13}$$

$$\mathbf{E} \left[\sup_{0 \le s \le 1} \left| \frac{X_s(\epsilon) - X_s(\epsilon_1)}{\epsilon - \epsilon_1} - \frac{X_s(\epsilon) - X_s(\epsilon_2)}{\epsilon - \epsilon_2} \right|^p \right] \le C_p |\epsilon_1 - \epsilon_2|^p. \tag{7.14}$$

Remark 21. From (7.14), one can check that

$$\mathbf{E} \left[\sup_{0 \le s \le 1} \left| \frac{X_s(\epsilon) - X_s(\epsilon_1)}{\epsilon - \epsilon_1} - \frac{X_s(\epsilon') - X_s(\epsilon_2)}{\epsilon' - \epsilon_2} \right|^p \right] \le C_p (|\epsilon - \epsilon'|^p + |\epsilon_1 - \epsilon_2|^p).$$

By applying Lemma 1.1 in Fujiwara-Kunita [4], we know that there is a version $X'_t(\epsilon)$ of $X_t(\epsilon)$ such that $\epsilon \to X'_t(\epsilon)$ is continuously differentiable.

Proof. We shall prove (7.13). Define $Y_t(\epsilon, \epsilon_1) = X_t(\epsilon) - X_t(\epsilon_1)$. We have

$$\begin{aligned} Y_t(\epsilon, \epsilon_1) = &\int_0^t (\, b(\, X_{s-}(\epsilon)\,) - b(\, X_{s-}(\epsilon)\,))\; ds \\ &+ \int_{0+}^t (\, \epsilon a(X_{s-}(\epsilon), y) - \epsilon_1 a(X_{s-}(\epsilon), y)\,)\; \tilde{N}(ds \times dy) \end{aligned}$$

Note that

$$|b(\, X_{s-}(\epsilon)\,) - b(\, X_{s-}(\epsilon_1)| \le \sup_x |b^{(1)}(x)|\, |\, Y_{s-}(\epsilon, \epsilon_1)|,$$

and

$$\begin{aligned} &\int |\epsilon a(X_{s-}(\epsilon), y) - \epsilon_1 a(X_{s-}(\epsilon), y)|^p \nu(dy) \\ &\le C_p \left[|\epsilon - \epsilon_1|^p \int |a(X_{s-}(\epsilon), y)|^p \nu(dy) \right. \\ &\quad \left. + \int |a(X_{s-}(\epsilon), y) - a(X_{s-}(\epsilon_1), y)|^p \nu(dy) \right] \\ &\le C_p \left(\; |\epsilon - \epsilon_1|\, (1 + |X_{s-}|)^p + |Y_{s-}(\epsilon, \epsilon_1)|^p \; \right), \end{aligned}$$

where we used (7.5). By applying Lemma 7.1 with $s_0 = 0$ and $\eta_{s_0,s}(y) = |\epsilon - \epsilon_1|\, (\, 1 + |X_{s-}(\epsilon)|\,)$, we have (7.13).

Next, by using Lemma 7.1, we shall also show (7.14). Note that

$$\begin{aligned} &\frac{Y_t(\epsilon, \epsilon_1)}{\epsilon - \epsilon_1} - \frac{Y_t(\epsilon, \epsilon_2)}{\epsilon - \epsilon_2} \\ &\quad = \int_0^t \left(\frac{b(\, X_{s-}(\epsilon)\,) - b(\, X_{s-}(\epsilon)}{\epsilon - \epsilon_1} - \frac{b(\, X_{s-}(\epsilon)\,) - b(\, X_{s-}(\epsilon)\,)}{\epsilon - \epsilon_2} \right) ds \\ &\quad + \int_{0+}^t \int \Xi(\epsilon, \epsilon_1, s, y)\, \tilde{N}(ds \times dy), \end{aligned}$$

where $\Xi(\epsilon, \epsilon_1, s, y)$ is given by

$$\frac{\epsilon a(X_{s-}(\epsilon), y) - \epsilon_1 a(X_{s-}(\epsilon), y)}{\epsilon - \epsilon_1} - \frac{\epsilon a(X_{s-}(\epsilon), y) - \epsilon_1 a(X_{s-}(\epsilon), y)}{\epsilon - \epsilon_1}$$

We define $\eta_s = \left(1 + \left|\frac{Y_{s-}(\epsilon, \epsilon_2)}{\epsilon - \epsilon_2}\right|\right) |Y_{s-}(\epsilon_1, \epsilon_2)|$. Then, the inequality (7.13) shows that $\mathbf{E}[\sup_{0 \le s \le 1} |\eta_s|^p] \le |\epsilon_1 - \epsilon_2|^p$.One can write

$$\frac{b(\, X_{s-}(\epsilon)\,) - b(\, X_{s-}(\epsilon_1))}{\epsilon - \epsilon_1} = \frac{Y_{s-}(\epsilon, \epsilon_1)}{\epsilon - \epsilon_1} \int_0^1 b^{(1)}(X_{s-}(\epsilon_1) + \theta Y_{s-}(\epsilon, \epsilon_1))\, d\theta.$$

Because b has bounded derivatives, we have

$$
\begin{aligned}
&(7.15)\\
&\left|\frac{b(X_{s-}(\epsilon))-b(X_{s-}(\epsilon_1))}{\epsilon-\epsilon_1}-\frac{b(X_{s-}(\epsilon))-b(X_{s-}(\epsilon_2))}{\epsilon-\epsilon_2}\right|\\
&\leq\left|\frac{Y_{s-}(\epsilon,\epsilon_1)}{\epsilon-\epsilon_1}-\frac{Y_{s-}(\epsilon,\epsilon_2)}{\epsilon-\epsilon_2}\right|\int_0^1|b^{(1)}(X_{s-}(\epsilon_1)+\theta Y_{s-}(\epsilon,\epsilon_1))|\,d\theta\\
&+\left|\frac{Y_{s-}(\epsilon,\epsilon_2)}{\epsilon-\epsilon_2}\right|\int_0^1\left|b^{(1)}(X_{s-}(\epsilon_1)+\theta Y_{s-}(\epsilon,\epsilon_1))-b^{(1)}(X_{s-}(\epsilon_2)+\theta Y_{s-}(\epsilon,\epsilon_2))\right|\,d\theta\\
&\leq C\left(\left|\frac{Y_{s-}(\epsilon,\epsilon_1)}{\epsilon-\epsilon_1}-\frac{Y_{s-}(\epsilon,\epsilon_2)}{\epsilon-\epsilon_2}\right|+\eta_s\right).
\end{aligned}
$$

In a similar way, we have

$$
\begin{aligned}
&\int\left|\frac{a(X_{s-}(\epsilon),y)-a(X_{s-}(\epsilon_1),y)}{\epsilon-\epsilon_1}-\frac{a(X_{s-}(\epsilon),y)-a(X_{s-}(\epsilon_2),y)}{\epsilon-\epsilon_2}\right|^p\nu(dy)\\
&\quad\leq C_p\left(\left|\frac{Y_{s-}(\epsilon,\epsilon_1)}{\epsilon-\epsilon_1}-\frac{Y_{s-}(\epsilon,\epsilon_2)}{\epsilon-\epsilon_2}\right|^p+|\eta_{s-}|^p\right).
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&\int|\Xi(\epsilon,\epsilon_1,s,y)|^p\,\nu(dy)\leq C_p\int|\,a(X_{s-}(\epsilon_1),y)-a(X_{s-}(\epsilon_2),y)|^p\,\nu(dy)\\
&+C_p\int\left|\frac{a(X_{s-}(\epsilon),y)-a(X_{s-}(\epsilon_1),y)}{\epsilon-\epsilon_1}-\frac{a(X_{s-}(\epsilon),y)-a(X_{s-}(\epsilon_2),y)}{\epsilon-\epsilon_2}\right|^p\nu(dy)\\
&\leq C_p\left(\left|\frac{Y_{s-}(\epsilon,\epsilon_1)}{\epsilon-\epsilon_1}-\frac{Y_{s-}(\epsilon,\epsilon_2)}{\epsilon-\epsilon_2}\right|^p+|\eta_{s-}|^p\right),
\end{aligned}
$$

where, in the last inequality, we have also used

$$
\begin{aligned}
&\int|\,a(X_{s-}(\epsilon_1),y)-a(X_{s-}(\epsilon_2),y)|^p\,\nu(dy)\\
&\quad\leq\left(\sup_x\int|a^{(1)}(x,y)|^p\nu(dy)\right)\,|Y_{s-}(\epsilon_1,\epsilon_2)|^p\leq C_p\eta_s.
\end{aligned}
$$

By applying Lemma 7.1, we get the result. □

As we mentioned in Remark 21, we can take a modification of $X(\epsilon)$ such that $\epsilon\to X(\epsilon)$ is continuously differentiable. We denote by the same symbol $X(\epsilon)$ the continuously differential version. We denote also $X_t^{(0)}(\epsilon)=X_t(\epsilon)$ and $X^{(1)}(\epsilon)=\frac{dX_t(\epsilon)}{d\epsilon}$. Then, one can check that $X^{(1)}(\epsilon)$ satisfies

$$
\begin{aligned}
X_t^{(1)}(\epsilon)=&\int_0^t b^{(1)}(X_{s-}(\epsilon))X_{s-}^{(1)}ds+\int_0^t a(X_{s-}(\epsilon),y)\tilde N(ds\times dy)\\
&+\epsilon\int_0^t\int a^{(1)}(X_{s-}(\epsilon),y)\,X_{s-}^{(1)}(\epsilon)\tilde N(ds\times dy).
\end{aligned}
$$

Further, applying the same argument to the process $X^{(1)}(\epsilon)$, we see that $\epsilon \to X^{(1)}(\epsilon)$ has continuously differential version. Repeating the same argument inductively, we have:

Theorem 7.1. *There exists a version of $X(\epsilon)$ such that $\epsilon \to X_t(\epsilon)$ is infinitely differentiable. Moreover, for $n = 2, 3 \ldots$, $X^{(n)}(\epsilon) = \frac{d^n}{d\epsilon^n} X(\epsilon)$ is given by formal $n-$times derivative of $X(\epsilon)$ with respect to ϵ:*

(7.16)
$$X_t^{(n)}(\epsilon) = \int_0^t \frac{d^n}{d\epsilon^n} b(X_{s-}(\epsilon))ds + \int_0^t \int \frac{d^n}{d\epsilon^n} \left(\epsilon a(X_{s-}(\epsilon), y) \right) \tilde{N}(ds \times dy).$$

Remark 22. The stochastic differential equation is defined by induction. More precisely, the stochastic differential equation (7.16) has the unique solution if the processes $X_t^{(0)}, \ldots, X_t^{(n-1)}$ are given.

Lemma 7.3. *For any n, any ϵ, and any t, $X_t^{(n)}(\epsilon)$ is in $\mathbf{D}_\infty$. Further, we have*

$$\sup_{0<t\leq 1} \sup_{\epsilon \in (-1,1)} \left| X_t^{(n)}(\epsilon) \right|_{k,p} < \infty. \tag{7.17}$$

Proof. To prove this, we shall use induction with respect to n. In the case $n = 0$, the claim is true because of (7.6). Let $n \geq 1$ be fixed. Suppose that for each $l \leq n$, and for each non-negative integer k and for $p \geq 2$, (7.17) holds. Under this assumption, we shall prove

$$\sup_{\epsilon \in (-1,1)} \left| X_t^{(n+1)}(\epsilon) \right|_{k,p} < \infty. \tag{7.18}$$

We set

$$F_s^1(\epsilon) = \frac{d^{n+1}}{d\epsilon^{n+1}} b(X_s(\epsilon)) - b^{(1)}(X_{s-}(\epsilon)) X^{(n+1)}(\epsilon)$$
$$F_s^2(\epsilon, y) = \frac{d^{n+1}}{d\epsilon^{n+1}} (\epsilon a(X_s(\epsilon), y)) - \epsilon a^{(1)}(X_{s-}(\epsilon), y) X^{(n+1)}(\epsilon).$$

The variable $F_{s-}^2(\epsilon, y)$ can be written by a linear sum of random variables such as

$$\begin{aligned} &a^{(l)}(X_s(\epsilon), y)\, (X_s^{(1)})^{l_1} \cdots (X_s^{(n)})^{l_n} \quad (l = 0, \ldots, n) \quad \text{or} \\ &\epsilon a^{(l')}(X_s(\epsilon), y)\, (X_s^{(1)})^{l_1} \cdots (X_s^{(n)})^{l_n} \quad (l' = 2, \ldots, n+1), \end{aligned} \tag{7.19}$$

where $0 \leq l_1, \ldots, l_n \leq n$. Hence, by the assumption of the induction, we have

$$\sup_{\epsilon} \sup_{0<s\leq 1} \operatorname{ess\,sup}_{\sigma \in A(1)^k} \mathbf{E} \left[\int \left| \frac{D_\sigma^k F_s^2(\epsilon, y)}{\gamma(\sigma)} \right|^p \nu(dy) \right] < \infty. \tag{7.20}$$

In a similar way, one can check that

$$\sup_{\epsilon} \sup_{0<s\leq 1} |F^1(s, \epsilon)|_{k,p} < \infty. \tag{7.21}$$

Pick $u=(s_1,y_1)\in A(1)$. Note that $D_uX_t^{(n+1)}=0$, if $s_1>t$. Hence, to prove (7.18) for $k=1$, we have to see that

$$\sup_{\epsilon}\sup_{0<t\le 1}\operatorname*{ess\,sup}_{(s_1,y_1)\in A(1)\cap(0,t]\times\mathbf{R}}\left\|\frac{D_{s_1,y_1}X_t(\epsilon)}{|y_1|}\right\|_p<\infty.$$

We set $\tilde X_{s_1,t}(\epsilon,y_1)=D_uX_t(\epsilon)$. Then one can check that $\tilde X_{s_1,t}(\epsilon,y_1)$ satisfies

$$\begin{aligned}\tilde X_{s_1,t}(\epsilon,y_1)&=F^2_{s_1}(\epsilon,y_1)+a^{(1)}(X^{(0)}_{s_1}(\epsilon),y_1)X^{(n+1)}_{s_1}(\epsilon)\\&\quad+\tilde Y_{s_1,t}+\int_{s_1}^t b^{(1)}(X^{(0)}_{s-}(\epsilon)+\tilde X_{s_1,s-}(\epsilon,y_1))\tilde X_{s_1,s-}(\epsilon,y_1)\,ds\\&\quad+\epsilon\int_{s_1}^t\int a^{(1)}(X^{(0)}_{s-}(\epsilon)+\tilde X_{s_1,s-}(\epsilon,y_1),y)\tilde X_{s_1,s-}(\epsilon,y_1)\,\tilde N(ds\times dy).\end{aligned}$$

where

$$\begin{aligned}\tilde Y_{s_1,t}&=\int_{s_1}^t(D_u(F^1_{s-}(\epsilon)+b^{(1)}(X^0_{s-}(\epsilon))))X^{(n+1)}_{s-}(\epsilon)\,ds\\&\quad+\int_{s_1}^t\int(\,D_u(F^2_{s-}(\epsilon,y)+a^{(1)}(X^{(0)}_{s-}(\epsilon),y)\,)\,)X^{(n+1)}_{s-}(\epsilon)\,\tilde N(ds\times dy).\end{aligned}$$

Recall that $F^2_{s-}(\epsilon,y)$ can be written by a linear sum of random variables such as (7.19). We use the convention that $D^0_uG=G$. Then, using Lemma 7.7 with $s_0=s$ and

$$\eta_t=\left(|D_uX_t^{(0)}|+\sum_{\substack{k_j\in\{0,1\}\\1\le k_0+\cdots k_n\le n}}|D_u^{k_0}X_t^{(0)}(\epsilon)\cdots D_u^{k_n}X_t^{(n)}(\epsilon)|\right)|X_t^{(n+1)}(\epsilon)|,$$

we have

$$\sup_{0<s_1<t\le 1}\|Y_{s_1,t}\|_p\le C\,\mathbf{E}\left[\int_{s_0}^1|\eta_r|^p\,dr\right]^{\frac1p}\le C|y_1|,$$

where, in the last inequality, we used the assumption of the induction. By Assumption 1, we have

$$\begin{aligned}&|b^{(1)}(X^{(0)}_{s-}(\epsilon)+D_uX^{(0)}_{s-})\tilde X_{s_1,s-}(\epsilon,y_1)|\le C|\tilde X_{s_1,s-}(\epsilon,y_1)|,\\&\int|a^{(1)}(X^{(0)}_{s-}(\epsilon)+\tilde X_{s_1,s-}(\epsilon,y_1),y)\tilde X_{s_1,s-}(\epsilon,y_1)|^p\nu(dy)\le C|\tilde X_{s_1,s-}(\epsilon,y_1)|^p.\end{aligned}$$

It follows from (7.3), (7.4), and (7.20) that

$$\sup_{s_1}\|F^2_{s_1}(\epsilon,y_1)+a^{(1)}(X^{(0)}_{s_1}(\epsilon),y_1)X^{(n+1)}_{s_1}(\epsilon)\|_p\le C|y_1|. \tag{7.22}$$

Hence, using Lemma 7.1 with $s_0 = s_1$, $\eta_t = \tilde{Y}_{s_1,t}$ and $Y_{s_0,s_0} = F^2_{s_1}(\epsilon, y_1) + a^{(1)}(X^{(0)}_{s_1}(\epsilon), y_1)X^{(n+1)}_{s_1}(\epsilon)$, we have

$$\begin{aligned}\sup_{\epsilon} \mathbf{E}[\sup_{s_1<t\le 1} |\tilde{X}_{s_1,t}(y_1,\epsilon)|^p] &\le C_p\|F^2_{s_1}(\epsilon,y_1) + a^{(1)}(X^{(0)}_{s_1}(\epsilon),y_1)X^{(n+1)}_{s_1}(\epsilon)\|^p_p| + \mathbf{E}\left[\int_{s_1}^t |Y_{s_1,1}|^p\,dr\right] \\ &\le C_p|y_1|^p.\end{aligned}$$

This implies that $\sup_\epsilon \sup_t |X^{(n+1)}_t(\epsilon)|_{1,p} < \infty$. Repeating this argument, one can get the result. □

Theorem 7.2. *The Taylor expansion*

$$X_1(\epsilon) \sim \sum_{n=0}^{\infty} \epsilon^n \frac{X^{(n)}_1(0)}{n!}$$

is the asymptotic expansion.

Proof. Taylor's formula yields that

$$X_1(\epsilon) - \sum_{n=0}^{m} \epsilon^n \frac{X^{(n)}_1(0)}{n!} = \frac{\epsilon^{m+1}}{m!} \int_0^1 (1-\theta)^m X^{(m+1)}_1(\theta\epsilon)\,d\theta$$

Therefore, it follows from Lemma 7.3 that

$$\left| X_1(\epsilon) - \sum_{n=0}^{m} \epsilon^n \frac{X^{(n)}_1(0)}{n!} \right|_{k,p} \le C_m \sup_{\epsilon'} |X^{m+1}_1(\epsilon')|_{k,p}\epsilon^{m+1}.$$

□

We define $F(\epsilon) := \frac{X_1(\epsilon)-X_1(0)}{\epsilon}$, then $F(\epsilon)$ has the asymptotic expansion $\sum_{n=0}^{\infty} \epsilon^n f_n$ with $f_n = \frac{X^{(n+1)}_1(0)}{(n+1)!}$. We shall give a sufficient condition that $F(\epsilon)$ satisfies the uniformly non-degenerate condition. Let Z^s_t $(s<t)$ be a solution to the following linear stochastic differential equation;

$$Z_t(\epsilon) = 1 + \int_s^t b^{(1)}(X_{r-}(\epsilon))\,Z_{r-}(\epsilon)\,dr + \epsilon\int_s^t\int a^{(1)}(X_{r-}(\epsilon),y)\,Z_{r-}(\epsilon)\,\tilde{N}(ds\times dy).$$

This equation is given by the derivative of the stochastic differential equation :

$$X_{s,t}(\epsilon) = x + \int_s^t b(X_{s,r-}(\epsilon))\,dr + \epsilon\int_s^t\int a(X_{s,r-}(\epsilon),y)(\epsilon)\,\tilde{N}(ds\times dy).$$

with respect to the initial value x. Put $\psi_s(\epsilon) = Z_{s,1}(\epsilon)\bar{a}(X_s(\epsilon))$. Then, as in Picard [12], one can check that for any $p \ge 2$ there is a constant $C > 0$ such that for any $u = (s,x) \in A(1)$

$$\sup_{\epsilon} \|D_{(s,x)}F(\epsilon) - G_t(\epsilon)x\|_p \le C\gamma(u)^{r_0} \tag{7.23}$$

where r_0 is given in condition (7.4).

Theorem 7.3. *Suppose that* $\limsup_{\epsilon\to 0}\sup_t \mathbf{E}[|X_t(\epsilon)|^{-p}] < \infty$ *for any* $p \geq 2$, *and that there is a positive constant* $c > 0$ *such that*

$$|\bar{a}(x)| \geq c|x|, \qquad \liminf_{\epsilon\to 0}\inf_{x,y}(1+\epsilon a^{(1)}(x,y)) \geq c$$

then $F(\epsilon) := \frac{X_1(\epsilon)-X_1(0)}{\epsilon}$ *satisfies the uniformly non-degenerate condition.*

Proof. We use Theorem 5.2 with $\psi_t(\epsilon) = Z_{s,1}(\epsilon)\bar{a}(X_s(\epsilon))$ defined above. Because of (7.23), all we have to do is to show that the condition (5.2) holds. Thanks to Jensen's inequality, we have

$$\left\| \left(\int_0^1 (\psi_t(\epsilon))^2 dt \right)^{-1} \right\|_p \leq \left\| \int_0^1 |\,(\psi_t(\epsilon))^2\,|^{-1} dt \right\|_p \leq \sup_t \|\psi_t^{-1}(\epsilon)\|_{2p}.$$

We define

$$W_{s,t}(\epsilon) = -K_{s,t}(\epsilon) + \int_s^t \int \frac{(\epsilon a(^{(1)}(X_{r-}(\epsilon),y))^2}{1+\epsilon a^{(1)}(X_{r-}(\epsilon),y)} N(dr\times dy),$$

where $K_{s,t}(\epsilon) = \int_s^t b^{(1)}(X_{r-})dr + \epsilon\int_s^t\int a^{(1)}(X_{r-}(\epsilon),y)\tilde{N}(dr\times dy)$. Then, by the assumption, $W_{s,t}(\epsilon)$ is well-defined for each $\epsilon > 0$ small enough. It follows from Theorem 63 in [14] that $Z_{s,t}^{-1}(\epsilon)$ $(s < t)$ satisfies

$$Z_{s,t}^{-1}(\epsilon) = 1 + \int_s^t Z_{s,r-}^{-1}(\epsilon) dW_{s,r}(\epsilon).$$

Hence, one can check that $\limsup_{\epsilon\to 0}\sup_s \|Z_{s,1}^{-1}(\epsilon)\|_p < \infty$. From the condition $\limsup_{\epsilon\to 0}\sup_t \mathbf{E}[|X_t(\epsilon)|^{-p}] < \infty$ and $|\bar{a}(X_s(\epsilon)| \geq c|X_s(\epsilon)|$, the assertion follows. □

If $\bar{a}$ is uniformly non-degenerate, then we need not to assume the condition $\limsup_{\epsilon\to 0}\sup_t \mathbf{E}[|X_t(\epsilon)|^{-p}] < \infty$

Example 7.1. Suppose that the Lévy measure $\nu(du)$ is given by

$$\nu(dx) = 1_{\{x\,;\,|x|\leq R\}} C\left(1_{(-\infty,0)}(x)e^{Gx} + 1_{(0,\infty)}(x)e^{-Mx}\right)|x|^{-(1+Y)}dx,$$

where C, G, M are positive constants and $Y < 2$. This is known as CGMY-model in mathematical finance if $R = \infty$. We suppose that $0 < Y < 2$, then Condition 1 holds. Put $g_\epsilon(y) = \log(1+\epsilon(e^y - 1))$. Let $1 < R < \infty$ be fixed. Then, one can check that, for any $y \in [-R, R]$,

$$|g_\epsilon(y)| \leq Re^R \epsilon(1 \wedge |y|). \tag{7.24}$$

We define $b(\epsilon) = \int_{-R}^R (\,e^{g_\epsilon(y)} - 1 - g_\epsilon(y)\,)\nu(dy)$, and consider the following process;

$$\tilde{L}_t(\epsilon) = (b - b(\epsilon))t + \int_0^t \int g_\epsilon(y)\tilde{N}(dr\times dy).$$

Then, by the Itô formula, we have

$$X_t(\epsilon) := e^{\tilde{L}_t(\epsilon)} = 1 + b\int_0^t X_{r-}dr + \epsilon\int_0^t\int_{-R}^R X_{r-}(e^y-1)\tilde{N}(dr\times du). \tag{7.25}$$

On the other hand, for $\epsilon > 0$, the process $\tilde{L}_t$ can be considered as a Lévy process with characteristic exponent $\Phi(\boldsymbol{\xi})$

$$\Phi(\boldsymbol{\xi}) = \log \mathbf{E}[e^{i\,\boldsymbol{\xi}\,\tilde{L}_1(\epsilon)}] = b - b(\epsilon) + \int_{\log(1+\epsilon(e^{-R}-1))}^{\log(1+\epsilon(e^R-1))} (e^{i\,\boldsymbol{\xi}\,y} - 1 - i\,\boldsymbol{\xi}\,y)\nu_\epsilon(dy),$$

where $\nu_\epsilon(dy) = \frac{e^y}{\epsilon+e^y-1}\left(\frac{\nu(dx)}{dx}\Big|_{x=\log(1+\epsilon^{-1}(e^y-1))}\right)dy$. By Lemma 25.6 in Sato [16], we see that $\mathbf{E}[|X_t|^p] = \mathbf{E}[e^{p\tilde{L}_t(\epsilon)}]$ exists for any $p \in \mathbf{R}$, and by Theorem 25.17 in Sato [16], we get

$$\begin{aligned}\limsup_{\epsilon\to 0}\sup_{0<t\le 1}\mathbf{E}[|X_t|^p] &= \limsup_{\epsilon\to 0}\sup_{0<t\le 1}\mathbf{E}[e^{p\tilde{L}_t(\epsilon)}]\\ &\le \limsup_{\epsilon\to 0}\exp\left\{\left|p(b-b(\epsilon)) + \int (e^{py}-1-py)\nu_\epsilon(dy)\right|\right\} < \infty\end{aligned}$$

where, in the last inequality, we used (7.24) after changing variable $y = \log(1+\epsilon^{-1}(e^x-1))$. Hence, the stochastic differential equation (7.25) satisfies the condition in Theorem 7.3. We cannot choose $R=\infty$; $X_t(\epsilon)$ does not belong to $\mathbf{L}^p$ for large p if $R=\infty$. However, it seems important to study the asymptotic expansion of (7.25), if one want to know the property of geometric CGMY process.

8. Appendix: Proof of Theorem 3.3

Proof of Theorem 3.3. We divide the proof of Theorem 3.3 into several steps.

Step 1. For simplicity, we denote $Z_u^{(j)} = Z_u^{(j)}(\boldsymbol{\xi})$ and $D_{\mathbf{Z}}^{*(n)}(G) = D_{\mathbf{Z}(\boldsymbol{\xi})}^{*(n)}(G)$. Put $Z_u^{(j),l} := Z_u^{(j)}1_{A(\frac{1}{l})^c}(u)$. We also define $D_{\mathbf{Z}^l}^{*(n)}(G)$ as in Lemma 3.2. By Lemma 3.2, all we have to do is to show that for any sufficiently large l

$$\|D_{\mathbf{Z}^l}^{*(n)}(G)\|_2 \le C(1+|\,\boldsymbol{\xi}\,|^2)^{-(1-\frac{\alpha}{2\beta})\frac{n}{2}}|G|_{n,q},$$

where C does not depend on l. The assumption of Theorem 3.3 yields that for any k, j and for any $p \ge 2$

$$|Z^{(j),l}|_{k,p,|\,\boldsymbol{\xi}\,|^{-\frac{1}{\beta}}} \le |Z^{(j)}|_{k,p,|\,\boldsymbol{\xi}\,|^{-\frac{1}{\beta}}} \le C_{k,p,j}(\,|\,\boldsymbol{\xi}\,|\Gamma(|\,\boldsymbol{\xi}\,|^{-\frac{1}{\beta}})\,)^{-1} \tag{8.1}$$

where $C_{k,p,j}$ is a constant which depends on p, k, j.

Lemma 8.1. *Under the notation above, we have*

$$D^{*(n)}_{\mathbf{Z}^l}(G) = \int_{S_n} G \circ \varepsilon^-_{\bar{u}_n} \left(\prod_{j=1}^n Z^{(j),l}_{u_j} \circ \varepsilon^-_{\bar{u}_j} \right) \tilde{N}(du_1)\tilde{N}(du_2)\dots\tilde{N}(du_n),$$

where $\bar{u}_j = (u_1, \dots, u_j)$ *and* $S_n = \{(u_1, \dots, u_n) \in E^n\,;\, u_i \neq u_j \text{ if } i \neq j\}$.

Remark 23. See proof of Lemma 2.5 in Picard [13].

From Lemma 8.1, we can write

$$\left(D^{*(n)}_{\mathbf{Z}^l}(G)\right)^2 = \int_{S_n \times S_n} G \circ \varepsilon^-_{\bar{u}_n}\, G \circ \varepsilon^-_{\bar{v}_n}\, \hat{Z}^l(\tau)\, \tilde{N}(d\tau),$$

where,

$$\hat{Z}^l(\tau) = \left(\prod_{j=1}^n Z^{(j),l}_{u_j} \circ \varepsilon^-_{\bar{u}_j} \right) \left(\prod_{j=1}^n Z^{(j),l}_{v_j} \circ \varepsilon^-_{\bar{v}_j} \right)$$

with $\tau = (u_1, \dots, u_n, v_1, \dots, v_n)$. We divide the region $S_n \times S_n$ as follows; for $J_1 = \{j^1_1, \dots, j^1_m\}, J_2 = \{j^2_1, \dots, j^2_m\} \subset \{1, 2, \dots, n\}$, we define

$$\Delta_{J_1,J_2} = \left\{ (u_1, \dots, u_n, v_1, \dots, v_n)\,; \quad \begin{array}{c} u_{i_1} = v_{i_2} \text{ if and only if } i_1 = j^1_q \\ \text{and } i_2 = j^2_q \text{ for some } 1 \leq q \leq m \end{array} \right\}.$$

Then we can write

$$D^{*(n)}_{\mathbf{Z}^l}(G)^2 = \sum_{m=0}^n \sum_{\substack{J_1,J_2 \\ |J_1|=|J_2|=m}} \int_{\Delta_{J_1,J_2}} \hat{Z}^l(\tau)\tilde{N}(d\tau),$$

where $|J|$ is the cardinal number of the set J. Let $J_1 = \{j^1_1, \dots, j^1_m\}, J_2 = \{j^2_1, \dots, j^2_m\}$ be fixed. Let us estimate the expectation of $\int_{\Delta_{J_1,J_2}} \hat{Z}^l(\tau)\tilde{N}(d\tau)$.

Step 2. Put $m = |J_1|$ and $k = n - m$. Define $\{i^1_1, \dots, i^k_1\} = \{1, \dots, n\} \cap J^c_1$ and $\{i^2_1, \dots, i^2_k\} = \{1, \dots, n\} \cap J^c_2$. We denote $\sigma_1 = (u_{j^1_1}, \dots, u_{j^1_m})$ and $\sigma_2 = (u_{i^1_1}, \dots, u_{i^1_k}, v_{i^2_1}, \dots, v_{i^2_k})$. Then, on the set Δ_{J_1,J_2}, we can regard $\hat{Z}^l(\tau)$ as the process parametrized by $(\sigma_1, \sigma_2) \in S_{m+2k}$. We set

$$\bar{Z}^l(\sigma_1, \sigma_2) = \prod_{q=1}^m \frac{Z^{(j^1_q),l}_{u_{j^1_q}} \circ \varepsilon^-_{\bar{u}_{j^1_q}}}{\gamma(u_{j^1_q})} \frac{Z^{(j^2_q),l}_{u_{j^1_q}} \circ \varepsilon^-_{\bar{v}_{j^2_q}}}{\gamma(u_{j^1_q})} \cdot \prod_{q=1}^k \frac{Z^{(i^1_m),l}_{u_{i^1_m}} \circ \varepsilon^-_{\bar{u}_{i^1_m}}}{\gamma(u_{i^1_m})} \frac{Z^{(i^2_m),l}_{u_{i^2_m}} \circ \varepsilon^-_{\bar{v}_{i^2_m}}}{\gamma(u_{i^2_m})}$$

Then, we can write

$$\frac{\hat{Z}^l(\tau)}{\gamma^2(\sigma_1)\gamma(\sigma_2)} = \bar{Z}^l(\sigma_1, \sigma_2) \tag{8.2}$$

Note that $\tilde{N}(du) \times \tilde{N}(du) = N(du)$. Recall that the measure $\lambda_0(du)$ is given by (3.2). By applying Lemma 2.4 in Picard [12], we have

$$\begin{aligned}
&\mathbf{E}\left[\int_{\Delta_{J_1,J_2}} G\circ\varepsilon^-_{\bar{u}_n}\, G\circ\varepsilon^-_{\bar{v}_n}\hat{Z}^l(\tau)\,\tilde{N}(d\tau)\right] \\
&= \mathbf{E}\left[\int_{\Delta_{J_1,J_2}} G\circ\varepsilon^-_{\bar{u}_n}\, G\circ\varepsilon^-_{\bar{v}_n}\hat{Z}^l(\tau)\,\tilde{N}(d\sigma_2)\,N(d\sigma_1)\right] \\
&= \mathbf{E}\left[\int_{S_{m+2k}} D_{\sigma_2}\left(G\circ\varepsilon^-_{\bar{u}_n}\, G\circ\varepsilon^-_{\bar{v}_n}\hat{Z}^l(\sigma_1,\sigma_2)\right)\circ\varepsilon^+_{\sigma_1}\ \lambda(d\sigma_1)\lambda(d\sigma_2)\right] \\
&= c^{m+2k}\,\mathbf{E}\left[\int_{S_{m+2k}} \frac{D_{\sigma_2}\left(G\circ\varepsilon^-_{\bar{u}_n}\, G\circ\varepsilon^-_{\bar{v}_n}\bar{Z}^l(\sigma_1,\sigma_2)\right)}{\gamma(\sigma_2)}\circ\varepsilon^+_{\sigma_1}\ \lambda_0(d\sigma_1)\lambda_0(d\sigma_2)\right] \\
&= c^{m+2k}\sum\mathbf{E}\left[\int_{S_{m+2k}} \frac{D_{\sigma_1'}D_{\sigma_2}\left(G\circ\varepsilon^-_{\bar{u}_n}\, G\circ\varepsilon^-_{\bar{v}_n}\bar{Z}^l(\sigma_1,\sigma_2)\right)}{\gamma(\sigma_2)}\ \lambda_0(d\sigma_1)\lambda_0(d\sigma_2)\right],
\end{aligned}$$

where $c = \int_{A(1)}\gamma^2(u)\lambda(du)$, σ_1' is extracted from σ_1, and the sum is obtained by the formulas (2.10) and (2.8). The formula (2.10) and the fact that $\gamma(u)\le 1$ on $A(|\boldsymbol{\xi}|^{\frac{-1}{\beta}})$ yields

$$\begin{aligned}
&\mathbf{E}\left[\int_{S_{m+2k}}\left|\frac{D_{\sigma_1'}D_{\sigma_2}\left(G\circ\varepsilon^-_{\bar{u}_n}\, G\circ\varepsilon^-_{\bar{v}_n}\bar{Z}^l(\sigma_1,\sigma_2)\right)}{\gamma(\sigma_2)}\right|\lambda_0(\sigma_1)\lambda_0(d\sigma_2)\right] \\
&\quad\le\sum\mathbf{E}\left[\int_{S_{m+2k}}\left|\frac{D_{\nu_1}G\circ\varepsilon^-_{\bar{u}_n}}{\gamma(\nu_1)}\frac{D_{\nu_2}G\circ\varepsilon^-_{\bar{v}_n}}{\gamma(\nu_2)}\right|\left|\frac{D_\nu\bar{Z}^l(\sigma_1,\sigma_2)}{\gamma(\nu)}\right|\lambda_0(\sigma_1)\lambda_0(d\sigma_2)\right]
\end{aligned}$$

where ν_1,ν_2, and ν are extracted from (σ_1,σ_2) and the sum is given by the formula (2.10).

Step 3. We shall estimate

$$\mathbf{E}\left[\int_{S_{m+2k}}\left|\frac{D_{\nu_1}G\circ\varepsilon^-_{\bar{u}_n}}{\gamma(\nu_1)}\frac{D_{\nu_2}G\circ\varepsilon^-_{\bar{v}_n}}{\gamma(\nu_2)}\right|\left|\frac{D_\nu\bar{Z}^l(\sigma_1,\sigma_2)}{\gamma(\nu)}\right|\lambda_0(\sigma_1)\lambda_0(d\sigma_2)\right] \tag{8.3}$$

For any $r > 2$, we take $r' > 1$ such that $\frac{2}{r}+\frac{1}{r'} = 1$. By applying Hölder's inequality, (8.3) is bounded by

(8.4)
$$\begin{aligned}
&\int_{S_{m+2k}}\left\|\frac{D_{\nu_1}G\circ\varepsilon^-_{\bar{u}_n}}{\gamma(\nu_1)}\frac{D_{\nu_2}G\circ\varepsilon^-_{\bar{v}_n}}{\gamma(\nu_2)}\right\|_{\frac{r}{2}}\left\|\frac{D_\nu\bar{Z}^l(\sigma_1,\sigma_2)}{\gamma(\nu)}\right\|_{r'}\lambda_0(\sigma_1)\lambda_0(d\sigma_2) \\
&\quad\le\int_{S_{m+2k}}\left\|\frac{D_{\nu_1}G\circ\varepsilon^-_{\bar{u}_n}}{\gamma(\nu_1)}\right\|_r\left\|\frac{D_{\nu_2}G\circ\varepsilon^-_{\bar{v}_n}}{\gamma(\nu_2)}\right\|_r\left\|\frac{D_\nu\bar{Z}^l(\sigma_1,\sigma_2)}{\gamma(\nu)}\right\|_{r'}\lambda_0(\sigma_1)\lambda_0(d\sigma_2).
\end{aligned}$$

It follows from (2.1) that for any random process X_u

$$X_u\circ\varepsilon^-_\tau = X_u\quad\text{a.e.-}\lambda_0(du)\lambda(d\tau).$$

Hence, it follows from (8.1) and (2.10) that,

$$\sup_l \operatorname*{ess\,sup}_{\sigma_1,\sigma_2\in A^{m+2k}(|\boldsymbol{\xi}|^{\frac{-1}{\beta}})} \left\| \frac{D_\nu \bar{Z}^l(\sigma_1,\sigma_2)}{\gamma(\nu)} \right\|_{r'} \le C\left(|\boldsymbol{\xi}|\Gamma(|\boldsymbol{\xi}|^{-\frac{1}{\beta}})\right)^{-2n},$$

where $C>0$ does not depend on l. Note that the region of the integral in (8.4) is $A(|\boldsymbol{\xi}|^{\frac{1}{\beta}})^{2k+m}$, because $Z_u^{(j)}=0$ on $A(|\boldsymbol{\xi}|^{\frac{1}{\beta}})$. For $j=1,2$, applying Hölder's inequality, we have

$$\begin{aligned}
&\sup_{|G|_{n,r}=1} \int_{A(|\boldsymbol{\xi}|^{-\frac{1}{\beta}})} \left\| \frac{D_{\nu_j} G}{\gamma(\nu_j)} \right\|_r \lambda_0(d\nu_j) \\
&\le \sup_{|G|_{n,r}=1} \mathbf{E}\left[\int_{A(|\boldsymbol{\xi}|^{-\frac{1}{\beta}})} \left| \frac{D_{\nu_j} G}{\gamma(\nu_j)} \right|^r \lambda_0(d\nu_j) \right]^{\frac{1}{r}} \lambda_0\left(A(|\boldsymbol{\xi}|^{-\frac{1}{\beta}})\right)^{(1-\frac{1}{r})|\nu_j|} \\
&\le \Gamma(|\boldsymbol{\xi}|^{-\frac{1}{\beta}})^{(1-\frac{1}{r})|\nu_j|}.
\end{aligned}$$

where $|\nu_j|$ $(j=1,2)$ is the length of ν_j $(j=1,2)$, respectively. Remark that (2.9) and definition of σ_1,σ_2 imply that ν_1 and ν_2 have no same component, further ν_1 and ν_2 have to be extracted from σ_2. Hence, $|\nu_1|+|\nu_2|\le 2k$ holds. Recall that $r>2$. The right hand side of (8.4) is bounded by

$$\begin{aligned}
&\prod_{j=1,2} \int \left\| \frac{D_{\nu_j} G}{\gamma(\nu_j)} \right\|_r \lambda_0(d\nu_j) \cdot \operatorname*{ess\,sup}_{\sigma_1,\sigma_2} \left\| \frac{D_\nu \bar{Z}^l(\sigma_1,\sigma_2)}{\gamma(\nu)} \right\|_{r'} \Gamma(|\boldsymbol{\xi}|^{-\frac{1}{\beta}})^{m+2k-(|\nu_1|+|\nu_2|)} \\
&\le C|\boldsymbol{\xi}|^{-2n}\,\Gamma(|\boldsymbol{\xi}|^{-\frac{1}{\beta}})^{-2n+m+2k-(|\nu_1|+|\nu_2|)+(1-\frac{1}{r})(|\nu_1|+|\nu_2|)} \\
&\le C|\boldsymbol{\xi}|^{-2n}\,\Gamma(|\boldsymbol{\xi}|^{-\frac{1}{\beta}})^{-n+k-\frac{|\nu_1|+|\nu_2|}{2}} \\
&\le C|\boldsymbol{\xi}|^{-2n}\,\Gamma(|\boldsymbol{\xi}|^{-\frac{1}{\beta}})^{-n} \le C|\boldsymbol{\xi}|^{-2(1-\frac{\alpha}{2\beta})n},
\end{aligned}$$

where, in the last inequality, we have used Γ-2) in Condition 1 in Section 2. The constant C does not depend on $l>0$. This completes the proof. □

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