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Existence and uniqueness of fixed points for mixed monotone multivalued operators in Banach spaces

By

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Abstract

In this paper, the existence and approximation of fixed points for two classes of systems of mixed monotone (downward and upward) multivalued operators are discussed. We present some new fixed point theorems of mixed monotone(downward and upward)operators which need not be continuous and compact. We also indicate the condition to ensure the uniqueness of the fixed point. At last we get some applications of our theorems.

1. Introduction

Mixed monotone operators are a class of important operators which are extensively used in nonlinear differential and integral equations (see $[5-6]$), also frequently used tools for studying dynamic programming (see [4]). The theory is based on the iterative technique which is brought forward by S.Ishikawa(see [1]). The theory of fixed points for mixed monotone single-valued operators in ordered Banach spaces has been widely investigated (see [8–13]). Recently, some fixed point theorems for multivalued monotone operators have been considered (see [2–5]). For instant, in [5] author has obtained the following result (see Theorem 1 of [5]).

Theorem 1.1. *Let* $A, B : [u_0, v_0] \times [u_0, v_0] \to 2^E$ *be a mixed monotone upward operator, respectively, and satisfy the following conditions:*

(i) *There exists a constant* $\beta \in (0,1)$ *such that* $A(v,u) - B(u,v) =: \{y - x : y \in A(v,u), x \in B(u,v)\} \leq \beta(v-u), \quad \forall u_0 \leq u \leq v \leq v_0.$

(ii) $B(u,v) \leq A(v,u), \quad \forall u_0 \leq u \leq v \leq v_0.$

(iii) $u_0 \leq B(u_0,v_0), A(v_0,u_0) \leq v_0.$

(iv) $A(v,u)$ *has a maximum element for* $\forall u_0 \leq u \leq v \leq v_0.$

Then A *and* B *have a common fixed point* $x^* \in [u_0, v_0]$.

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The theorem just discuss a kind of mixed monotone operator, and though a amount of work has been done in this area, none of them pick out the condition of the fixed point uniqueness about mixed monotone multivalued operators. In this paper, we concern with two kinds of monotone operators, and get the result of uniqueness. Namely, we will discuss the existence and uniqueness of fixed points to mixed monotone multivalued operators under some sufficient conditions which are different from ones of the above cited works.

Let $(E, |\cdot|)$ be a real Banach space and P a normal cone of E. A partial ordering "$\leq$" is induced by the cone P, namely, $x, y \in E, x \leq y$ iff $y - x \in P$. For $u_0, v_0 \in E$ with $u_0 < v_0$ we denote by $[u_0, v_0]$ the order interval $\{u \in E : u_0 \leq u \leq v_0\}$.

For the sake of convenience, we first recall some definitions.

A subset $D \subset E$ is said to have a maximum element, if there exists $x \in D$ such that $y \leq x$ for all $y \in D$. By $\max D$ we denote the maximum element of D. Similarly, we can define a minimum element of D. Obviously, the maximum (minimum) element of D is unique if D has maximum (minimum)element.

Definition 1.1. For two subset X, Y of E, we mark $X \leq Y$, if $\forall x \in X, \exists y \in Y$ such that $x \leq y$.

Definition 1.2. Given a nonempty subset D of X we say that $A : D \to 2^X \backslash \emptyset$ is increasing (decreasing) upward if $u, v \in D$, $u \leq v$ and $\forall x \in A(u)$ imply that there exists $y \in A(v)$ such that $x \leq y$ $(x \geq y)$. A is increasing (decreasing)downward if $u, v \in D, u \leq v$ and $y \in A(v)$ imply an existence of $x \in A(u)$ such that $x \leq y (x \geq y)$. If A is increasing (decreasing) upward and downward we say that A is increasing (decreasing).

Definition 1.3. Assume $D \subset X$. A multivalued operator $A : D \times D \to 2^X$ is said to be mixed monotone upward operator if $A(x, y)$ is increasing upward in x, and decreasing downward in y. A multivalued operator $B : D \times D \to 2^X$ is said to be a mixed monotone downward operator if $B(x, y)$ is decreasing downward in x, and increasing upward in y, i.e.

(a_1) for each $y \in D$ and any $x_1, x_2 \in D$ with $x_1 \leq x_2$, if $u_1 \in A(x_1, y)$ then there exists a $u_2 \in A(x_2, y)$ such that $u_1 \leq u_2$;

(a_2) for each $x \in D$ and any $y_1, y_2 \in D$ with $y_1 \leq y_2$, if $v_2 \in A(x, y_2)$ then there exist a $v_1 \in A(x, y_1)$ such that $v_1 \geq v_2$.

(b_1) for each $y \in D$ and any $x_1, x_2 \in D$ with $x_1 \leq x_2$, if $u_2 \in B(x_2, y)$ then there exists a $u_1 \in B(x_1, y)$ such that $u_1 \geq u_2$;

(b_2) for each $x \in D$ and any $y_1, y_2 \in D$ with $y_1 \leq y_2$, if $v_1 \in B(x, y_1)$ then there exist a $v_2 \in B(x, y_2)$ such that $v_1 \leq v_2$.

Remark 1. If A is a single-valued operator, then mixed monotone upward and mixed monotone are identical.

Definition 1.4. $x^* \in D$ is called a fixed point of multivalued operator A if $x^* \in A(x^*, x^*)$.

Definition 1.5. A multivalued operator K is said to have nonempty closed values if $K(x, y)$ is a nonempty closed subset of E for each $(x, y) \in E \times E$.

Throughout this paper we always assume that all multivalued operators have nonempty closed values.

2. Main Results

Theorem 2.1. *Let $A, B : [u_0, v_0] \times [u_0, v_0] \to 2^E$ be a mixed monotone upward operator and a mixed monotone downward operator, respectively, and satisfy the following conditions*:

(i) *There exists a constant $\lambda \in (0,1)$ such that $B(u,v) - A(u,v) =: \{y - x : y \in B(u,v), x \in A(u,v)\} \le \lambda(v-u), \quad \forall u_0 \le u \le v \le v_0$.*

(ii) *$A(u,v) \le B(u,v), \quad \forall u_0 \le u \le v \le v_0$.*

(iii) *$u_0 \le A(u_0,v_0), B(u_0,v_0) \le v_0$.*

(iv) *$B(u,v)$ has a maximum element for $\forall u_0 \le u \le v \le v_0$.*

Then A, B have a common fixed point $x^* \in [u_0, v_0]$. Further, for any real sequence $\{t_n\}$ satisfying $0 < \tau \le t_n \le 1, n = 1, 2, \cdots$, there exist mixed iterate sequences $\{u_n\}, \{v_n\}$ such that both of them converge to x^*, and we have the estimate of differences as follows

$$\begin{cases} |x^* - u_n| \le N[1-\tau(1-\lambda)]^n |u_0 - v_0|, \\ |x^* - v_n| \le N[1-\tau(1-\lambda)]^n |u_0 - v_0|. \end{cases}$$

Here N is the normal constant of P. In addition, if the following condition

(v) *For any $z \in A(x_1,y_1) - A(x_2,y_2) =: \{y - x : y \in A(x_1,y_1), x \in A(x_2,y_2)\}$ with $u_0 \le x_2 \le x_1 \le y_1 \le y_2 \le v_0$, we have $z \ge \theta$*

is satisfied, then A and B have the unique common fixed point.

Proof. We divide this proof into two steps.

Step 1. We prove the existence of fixed points. To end this, we first construct two sequences $\{u_n\}$ and $\{v_n\}$ such that

$$\begin{cases} u_{n+1} \in (1-t_n)u_n + t_n A(u_n, v_n), \\ v_{n+1} \in (1-t_n)v_n + t_n B(u_n, v_n), \quad (n = 0, 1, 2, \cdots). \end{cases} \tag{2.1}$$

From the condition (iii) it follows that there exists $x_0 \in A(u_0, v_0)$ such that $x_0 \ge u_0$, and from condition (iv) $B(u_0, v_0)$ has a maximum element y_0. Moreover, condition (iii) shows $y_0 \le v_0$. Let

$$\begin{cases} u_1 = (1-t_0)u_0 + t_0 x_0, \\ v_1 = (1-t_0)v_0 + t_0 y_0, \end{cases}$$

from the choice of x_0 and y_0, together with the condition (ii), we infer $u_0 \le x_0 \le y_0 \le v_0$. Set

$$x_0 = (1-t_0)x_0 + t_0 x_0 \ge u_1 = (1-t_0)u_0 + t_0 x_0 \ge (1-t_0)u_0 + t_0 u_0 = u_0,$$
$$y_0 = (1-t_0)y_0 + t_0 y_0 \le v_1 = (1-t_0)v_0 + t_0 y_0 \le (1-t_0)v_0 + t_0 v_0 = v_0.$$

It follows that $u_0 \le u_1 \le x_0 \le y_0 \le v_1 \le v_0$. It is easy to see $x_0 \in A(u_0, v_0)$ and $B(u_0, v_0) \le y_0$. This yields $u_1 \le A(u_0, v_0)$,$B(u_0, v_0) \le v_1$. So from the condition (ii) we have $u_0 \le u_1 \le A(u_0, v_0) \le A(u_1, v_1) \le B(u_1, v_1) \le B(u_0, v_0) \le v_1 \le v_0$.

Inductively, suppose that there exist u_n and u_{n-1} satisfying $u_n = (1 - t_{n-1})u_{n-1} + t_{n-1}x_{n-1}$ with $x_{n-1} \in A(u_{n-1}, v_{n-1})$ and $x_{n-1} \ge u_{n-1}$, also, v_n and v_{n-1} satisfying $v_n = (1 - t_{n-1})v_{n-1} + t_{n-1}y_{n-1}$ with $y_{n-1} = \max B(u_{n-1}, v_{n-1})$. This implies that u_n and v_n satisfy (2.1). Moreover, suppose

$$u_0 \le u_{n-1} \le u_n \le x_{n-1} \le y_{n-1} \le v_n \le v_{n-1} \le v_0.$$

Then we have

$$\begin{aligned} u_{n-1} \le u_n \le A(u_{n-1}, v_{n-1}) &\le A(u_n, v_n) \\ &\le B(u_n, v_n) \le B(u_{n-1}, v_{n-1}) \le v_n \le v_{n-1}. \end{aligned}$$

Now let us consider the natural number $n + 1$. By the preceding inequality, there exists $x_n \in A(u_n, v_n), x_n \ge u_n$. Let y_n be a maximum element of $B(u_n, v_n)$ and

$$\begin{cases} u_{n+1} = (1 - t_n)u_n + t_n x_n, \\ v_{n+1} = (1 - t_n)v_n + t_n y_n. \end{cases} \tag{2.2}$$

So we have

$$\begin{aligned} u_n \le u_{n+1} \le A(u_n, v_n) &\le A(u_{n+1}, v_{n+1}) \\ &\le B(u_{n+1}, v_{n+1}) \le B(u_n, v_n) \le v_{n+1} \le v_n. \end{aligned} \tag{2.3}$$

and also have

$$u_0 \le u_1 \le u_2 \le \cdots \le u_n \le \cdots \le v_n \le v_{n-1} \le \cdots \le v_1 \le v_0. \tag{2.4}$$

Next, we prove the both of sequences $\{u_n\}$ and $\{v_n\}$ converge to the common fixed point of A and B. From condition(i) and (2.2) we have

$$\begin{aligned} \theta &\le v_{n+1} - u_{n+1} = (v_n - u_n) + t_n(u_n - v_n + y_n - x_n) \\ &\le (v_n - u_n) + \tau(u_n - v_n + y_n - x_n) \\ &\le (1 - \tau(1 - \lambda))(v_n - u_n) \le \cdots \\ &\le [1 - \tau(1 - \lambda)]^{n+1}(v_0 - u_0). \end{aligned} \tag{2.5}$$

For any natural number m, from (2.4) and the above inequality it follows that

$$\begin{aligned} \theta &\le u_{n+m} - u_n \le v_n - u_n \le [1 - \tau(1 - \lambda)]^n(v_0 - u_0), \\ \theta &\le v_n - v_{n+m} \le v_n - u_n \le [1 - \tau(1 - \lambda)]^n(v_0 - u_0). \end{aligned}$$

Note that P is a normal cone, we have

$$|v_n - u_n| \le N[1 - \tau(1 - \lambda)]^n|v_0 - u_0|. \tag{2.6}$$

$$|u_{n+m} - u_n| \le N[1 - \tau(1 - \lambda)]^n|v_0 - u_0|. \tag{2.7}$$

$$|v_{n+m} - v_n| \le N[1 - \tau(1 - \lambda)]^n|v_0 - u_0|. \tag{2.8}$$

This implies that u_n and v_n are Cauchy sequence in E, By virtue of the completeness of E, there exist $x, y \in E$ such that

$$\lim_{n\to\infty} u_n = x,$$
$$\lim_{n\to\infty} v_n = y.$$

This, together with (2.6), implies that $x = y$. Denote $x = y = x^*$. Then we have

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = x^*. \tag{2.9}$$

By means of (2.3) and the monotonicity of A, B, we get, for $n = 0, 1, 2, \cdots$,

$$u_n \leq A(u_n, v_n) \leq A(x^*, x^*) \leq B(x^*, x^*) \leq B(u_n, v_n) \leq v_n.$$

Therefore, for each $n = 0, 1, 2, \cdots$, there exists $z_n \in A(x^*, x^*)$ such that $u_n \leq z_n \leq v_n$. Let n goes to infinity, in view of (2.9) and the normality of P, we obtain $\lim_{n\to\infty} z_n = x^*$. In the light of the fact that A has closed values, we have $x^* \in A(x^*, x^*)$. Similarly, we can verify $x^* \in B(x^*, x^*)$, that is, x^* is a common fixed point of A and B.

Step 2. We prove the uniqueness of the fixed point. In this part, the condition (v) will be used.

Let us suppose that, besides x^* of Step 1, $y^* \in [u_0, v_0]$ is also a common fixed point A and B. From $u_1 = (1 - t_0)u_0 + t_0 x_0$ for $x_0 \in A(u_0, v_0)$ and the upward mixed monotonicity of A it follows that $A(u_0, v_0) \leq A(y^*, y^*)$, and from the condition (v) it yields $y^* \geq A(u_0, v_0)$. So $y^* \geq u_1$. $v_1 = (1 - t_0)v_0 + y_0$ and the downward mixed monotonicity of B guarantee that $B(u_0, v_0) \geq B(y^*, y^*)$. Since y_0 is the maximum element of $B(u_0, v_0)$, we have $y_0 \geq B(y^*, y^*)$, and $y_0 \geq y^*$. Consequently we have $u_1 \leq y^* \leq v_1$.

In the same way we get $u_n \leq y^* \leq v_n$ for $n = 2, 3, 4, \cdots$. Since $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = x^*$, by the normality of P, we get $x^* = y^*$. Uniqueness is proved. This completes the proof of Theorem 2.1. □

Corollary 2.1. *Let $A, B : [u_0, v_0] \times [u_0, v_0] \to E$ be mixed monotone single-valued operators and satisfy the conditions* (i)–(iii) *of Theorem 1, then A and B have an unique common fixed point in* $[u_0, v_0]$.

According to Theorem 2.1, we easily obtain the following result:

Theorem 2.2. *Let $A, B : [u_0, v_0] \to 2^E$ be decreasing downward multi-valued operators satisfying the following conditions*:

(i) *There exists a constant $\lambda \in (0, 1)$ such that $B(u) - A(v) =: \{y - x : y \in B(u), x \in A(v)\} \leq \lambda(v - u), \quad \forall u_0 \leq u \leq v \leq v_0$.*

(ii) $A(v) \leq B(u), \quad \forall u_0 \leq u \leq v \leq v_0$.

(iii) $u_0 \leq A(v_0), B(u_0) \leq v_0$.

(iv) *$B(u)$ has a maximum element for all $\forall u \in [u_0, v_0]$.*
Then A, B have a common fixed point $x^* \in [u_0, v_0]$. Further, for any real sequence $\{t_n\}$ satisfying $0 < \tau \leq t_n \leq 1, n = 1, 2, \cdots$, there exist mixed iterate sequences $\{u_n\}, \{v_n\}$, defined by

$$\begin{cases} u_{n+1} \in (1-t_n)u_n + t_n A(v_n), \\ v_{n+1} \in (1-t_n)v_n + t_n B(u_n), \quad (n = 0, 1, 2, \cdots) \end{cases}$$

such that both of them converge to x^*,and we have the estimate of differences as follows

$$\begin{cases} |x^* - u_n| \leq N[1 - \tau(1-\lambda)]^n |u_0 - v_0|, \\ |x^* - v_n| \leq N[1 - \tau(1-\lambda)]^n |u_0 - v_0|. \end{cases}$$

Here N is the normal constant of P. If A also satisfies the following condition:
(v) *For any $z \in A(x_1) - A(x_2) =: \{y - x : y \in A(x_1), x \in A(x_2)\}$ with $u_0 \leq x_1 \leq x_2 \leq v_0$ we have $z \geq \theta$.*
Then the common fixed point is unique.

Corollary 2.2. *Let $A, B : [u_0, v_0] \to E$ be decreasing single-valued operators satisfying the following conditions*:
(i) *There exists a constant $\lambda \in (0, 1)$ such that $B(u) - A(v) \leq \lambda(v - u), \quad \forall u_0 \leq u \leq v \leq v_0$.*
(ii) *$A(v) \leq B(u), \quad \forall u_0 \leq u \leq v \leq v_0$.*
(iii) *$u_0 \leq A(v_0), B(u_0) \leq v_0$.*
Then A, B have a unique common fixed point $x^* \in [u_0, v_0]$.

Theorem 2.3. *Let $A, B : [u_0, v_0] \to 2^E$ be increasing upward multivalued operators satisfying the following conditions*:
(i) *There exists a constant $\lambda \in (0, 1)$ such that $B(v) - A(u) =: \{y - x : y \in B(v), x \in A(u)\} \leq \lambda(v - u), \quad \forall u_0 \leq u \leq v \leq v_0$.*
(ii) *$A(u) \leq B(v), \quad \forall u_0 \leq u \leq v \leq v_0$.*
(iii) *$u_0 \leq A(u_0), B(v_0) \leq v_0$.*
(iv) *$B(v)$ has a maximum element for all $\forall v \in [u_0, v_0]$.*
Then A, B have a common fixed point $x^* \in [u_0, v_0]$. Further, for any real sequence $\{t_n\}$ satisfying $0 < \tau \leq t_n \leq 1, n = 1, 2, \cdots$, there exist mixed iterate sequences $\{u_n\}, \{v_n\}$, defined by

$$\begin{cases} u_{n+1} \in (1-t_n)u_n + t_n A(u_n), \\ v_{n+1} \in (1-t_n)v_n + t_n B(v_n), \quad (n = 0, 1, 2, \cdots) \end{cases}$$

such that both of them converge to x^*, and we have the estimate of differences as follows

$$\begin{cases} |x^* - u_n| \leq N[1 - \tau(1-\lambda)]^n |u_0 - v_0|, \\ |x^* - v_n| \leq N[1 - \tau(1-\lambda)]^n |u_0 - v_0|. \end{cases}$$

Here N is the normal constant of P. If A also satisfies the following condition:
(v) *For any $z \in A(x_1) - A(x_2) =: \{y - x : y \in A(x_1), x \in A(x_2)\}$ with $u_0 \leq x_2 \leq x_1 \leq v_0$,we have $z \geq \theta$.*

Then the common fixed point is unique.

Corollary 2.3. *Let $A, B : [u_0, v_0] \to E$ be increasing single-valued operators satisfying the following conditions*:

(i) *There exists a constant $\lambda \in (0, 1)$ such that $B(v) - A(u) \le \lambda(v - u)$, $\forall u_0 \le u \le v \le v_0$.*

(ii) $A(u) \le B(v)$, $\forall u_0 \le u \le v \le v_0$.

(iii) $u_0 \le A(u_0), B(v_0) \le v_0$.

Then A, B have a unique common fixed point $x^* \in [u_0, v_0]$.

3. Applications

In this section, we shall provide a result about the following integral inclusion systems

$$\begin{cases} x(t) \in \int_0^t k(t,s)[F_1(x(s),x'(s)) - G_1(y(s),y'(s))]ds, \\ x(t) \in \int_0^t k(t,s)[F_2(x(s),x'(s)) - G_2(y(s),y'(s))]ds \end{cases} \tag{3.1}$$

which is based upon what is obtained from the previous sections. Here, $t \in [0, 1]$, k is continuous function from $\Omega = \{(s, t) : 0 \le s \le t \le 1\}$ into $\mathbb{R}^+$ with $\mathbb{R}^+ = (0, +\infty)$, F_i, G_i $(i = 1, 2)$ are multivalued functions from $\mathbb{R} \times \mathbb{R}$ into $2^{\mathbb{R}^+}$.

Let $E = C^1[0, 1]$ with the norm $\| \cdot \| = \sup\limits_{t\in[0,1]} | \cdot (t)|$ and $E_0 = \{x \in E : x(0) = 0\}$. Obviously, E_0 is a Banach space. We induce the order in E_0 by $x \le y$ if and only if $x(t) \le y(t)$ and $x'(t) \le y'(t)$ for all $t \in [0, 1]$. Denote $C^+ = \{z \in E_0 : z \ge 0\}$, then C^+ is normal cone with the normal constant $N = 1$.

Theorem 3.1. *Assume that the following conditions are satisfied*

(H1) *F_i, G_i $(i = 1, 2)$ have closed values and F_1 is increasing upward, G_1 is increasing downward, F_2 is decreasing downward and G_2 is decreasing upward.*

(H2) $0 \le F_1(x(t), x'(t)) - G_1(y(t).y'(t)) \le F_2(x(t), x'(t)) - G_2(y(t), y'(t)) \le u(t)$ *for each $x, y \in C^+$ and $t \in [0, 1]$, where $u \in L^1([0, 1], \mathbb{R}^+)$.*

(H3) $0 \le F_2(x(t), x'(t)) - G_2(y(t), y'(t)) - [F_1(x(t), x'(t)) - G_1(y(t), y'(t))] \le v(t)(y'(t) - x'(t))$ *with $0 \le x \le y$ and $v \in L^1([0, 1], \mathbb{R}^+)$ satisfies* $\sup\limits_{(t,s)\in\Omega} [k(t, s)v(t)] < 1$.

(H4) *For each $x \in C^+$, $F_2(x(t), x'(t))$ and $G_2(x(t), x'(t))$ have a maximum element and minimum element, respectively.*

Then system (3.1) *has at least a solution $x \in C^+$. Moreover, if the following hypothesis holds, then* (3.1) *has a unique solution.*

(H5) *For any $z \in F_1(x_1(s), x_1'(s)) - F_1(x_2(s), x_2'(s)) =: \{y - x : y \in F_1(x_1(s), x_1'(s)), x \in F_1(x_2(s), x_2'(s))\}$ with $x_2 \le x_1$, we have $z \ge \theta$;For any $z \in G_1(y_2(s), y_2'(s)) - G_1(y_1(s), y_1'(s)) =: \{y - x : y \in G_1(y_2(s), y_2'(s)), x \in G_1(y_1(s), y_1'(s))\}$ with $y_1 \le y_2$, we have $z \ge \theta$.*

Proof. Let us define multivalued mappings A and B by

$$\begin{cases} A(x,y)(t) = \int_0^t k(t,s)[F_1(x(s),x'(s)) - G_1(y(s),y'(s))]ds, \\ B(x,y)(t) = \int_0^t k(t,s)[F_2(x(s),x'(s)) - G_2(y(s),y'(s))]ds. \end{cases}$$

Evidently A is mixed monotone upward operator and B is mixed monotone downward operator. It is sufficient to prove that A and B have a common fixed point in C^+. To this end, we check that A and B satisfy all conditions of Theorem 2.1.

For any $\xi \in A(x,y), \eta \in B(x,y)$, there exist $z_i \in F_i(x,x'), w_i \in G(y,y')$ $(i=1,2)$ such that

$$\xi(t) = \int_0^t k(t,s)[z_1(s) - w_1(s)]ds, \quad \eta(t) = \int_0^t k(t,s)[z_2(s) - w_2(s)]ds.$$

By (H3), we have

$$\begin{aligned}\eta(t) - \xi(t) &= \int_0^t k(t,s)[(z_2(s) - w_2(s)) - (z_1(s) - w_1(s))]ds \\ &\le \int_0^t k(t,s)v(s)[y'(s) - x'(s)]ds \\ &\le \sup_{(t,s)\in\Omega} k(t,s)v(s) \int_0^t [y'(s) - x'(s)]ds \\ &= \lambda[y(t) - x(t)],\end{aligned}$$

where $\lambda =: \sup_{(t,s)\in\Omega} k(t,s)v(s) < 1$. This implies that the condition (i) holds.

It is easy to see that condition (ii) is satisfied by condition (H2).

The condition (iii) trivially holds. If we set $u_0 = 0$ and $v_0 = \int_0^t k(t,s)u(s)ds$.

(H4) guarantees that $B(x,y)$ has a maximum element with $0 \le x \le y \le v_0$, namely, the condition (iv) is hold. By (H5) we can see that the condition (v) is true. Consequently all of conditions of Theorem 2.1 are satisfied. □

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