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Homotopy cofibres, higher coassociativity and homotopy coalgebras

By

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Abstract

In this note we show that, for a map of (weak or ordinary) k-fold homotopy coalgebras and under certain dimension and connectivity restrictions, the homotopy coalgebra structure is inherited by the homotopy cofibre of the given map.

Introduction

It is a classical and well know fact (its Eckmann-Hilton dual is also true and classical) that the cofibre C of a co-H-map $X \to Y$ between co-H-spaces inherits a co-H-structure in such a way that the map $Y \to C$ is also a co-H-map [6, 4.1]. Our objective in this note is to study whether one may include coassociativity and higher coassociativity, from the homotopy coalgebra structure approach, in this assertion, i.e., whether (weak and ordinary) k-fold homotopy coalgebra structures are preserved by cofibre sequences.

Recall that higher associativity of H-spaces was introduced in [10] through A_n-spaces, or equivalently, A_n-structures. However, the Eckmann-Hilton duals of these equivalent notions of higher associativity give rise to, in principle, different kinds of higher coassociativity. One is introduced through "co-A_n-spaces" [7], [9], while the other is based on the notion of a (weak and ordinary) homotopy coalgebra of higher order for the comonad $\Sigma^k\Omega^k$ (see the next section for precise definitions) [1], [3], [5], [9].

The latter is the approach we follow to prove:

Theorem 0.1. *Let $X \xrightarrow{f} Y \xrightarrow{q} C$ be a cofibration sequence in which X is a CW-complex with $\dim X < \operatorname{conn} C$, X and Y are weak k-fold homotopy coalgebras of order $j \geq 1$, $k \geq 1$, and f is a morphism of weak homotopy coalgebras of order j. Then, C is a weak k-fold homotopy coalgebra of order*

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j for which q is a morphism. Moreover, if $k+1 < \operatorname{conn} C$ and Y is a k-fold homotopy coalgebra, then C is also a k-fold homotopy coalgebra.

The assumptions of the theorem above are needed as shown in the following:

Example 0.1. Let p be an odd prime and consider an element $f\colon \mathbb{S}^{2p} \to \mathbb{S}^3$ of order p in $\pi_{2p}(\mathbb{S}^3)$. This is known to be a co-H-map [2, §3], i.e., a morphism of 1-fold homotopy coalgebras of order 1. However, for $p = 3$, the homotopy cofibre $\mathbb{S}^3 \cup_f e^7$ does not admit coassociative comultiplications (homotopy coalgebra structures of order 1) [3, Proposition 4.1]. Indeed, an $(n-1)$-connected co-H-space X of dimension at most $4n-5$ is coassociative if and only if it is homotopy equivalent to a suspension by a co-H-map. But, for $p = 3$, Berstein and Hilton have shown [2] that $\mathbb{S}^3 \cup_f e^7$ is not a suspension, and therefore, it is not a coassociative co-H-space.

Theorem 0.2. *Let $X \xrightarrow{f} Y \xrightarrow{q} C$ be a cofibration sequence in which X, Y and C are weak k-fold homotopy coalgebras of order $j \geq 1$ and q is a morphism of weak homotopy coalgebras of order j. If $k \leq \min\{\operatorname{conn} Y, \operatorname{conn} C\}$ and X is a CW-complex with $\dim X \leq 2\min\{\operatorname{conn} Y, \operatorname{conn} C\} - k$, then f is also a morphism.*

1. Homotopy coalgebra structures on homotopy cofibres

From now on we shall be working in the homotopy category $\mathrm{Ho}\mathcal{T}op^*$ of well based topological spaces. Given F, G adjoint functors in this category, denote by $\eta\colon 1_{\mathrm{Ho}\mathcal{T}op^*} \to GF$ and $\varepsilon\colon FG \to 1_{\mathrm{Ho}\mathcal{T}op^*}$ the unit and counit respectively.

Definition 1.1. ([1]) a *weak (FG)-homotopy coalgebra structure of order $j \geq 1$* on a space X consists of a sequence of spaces D_iX for $i = 0, \ldots, j$, together with maps $\gamma_i\colon X \to FD_iX$ (or γ_i^X) such that $D_0X = *$, D_iX is the homotopy pullback

$$\begin{array}{ccc} D_iX & \xrightarrow{\quad p_i \quad} & GX \\ {\scriptstyle q_i}\big\downarrow & & \big\downarrow{\scriptstyle G\gamma_{i-1}} \\ D_{i-1}X & \xrightarrow[\eta_{D_{i-1}}]{} & GFD_{i-1}X, \end{array}$$

and the composition $X \xrightarrow{\gamma_i} FD_iX \xrightarrow{F(p_i)} FGX \xrightarrow{\varepsilon_X} X$ is the identity on X for $i = 1, \ldots, j$. From now on, we denote $\varepsilon_i = \varepsilon_X \circ F(p_i)$ (or ε_i^X), so that $\varepsilon_i \circ \gamma_i = 1_X$. A space X is a weak (FG)-homotopy coalgebra structure of order j if it has a structure of such.

A weak (FG)-homotopy coalgebra of order $j \geq 1$ is an (ordinary) (FG)-

homotopy coalgebra of order j if, moreover, the square

$$\begin{array}{ccc} X & \xrightarrow{\gamma_i} & FD_iX \\ {\scriptstyle\gamma_1}\downarrow & & \downarrow{\scriptstyle F\eta_{D_iX}} \\ FGX & \xrightarrow[FG\gamma_i]{} & FGFD_iX. \end{array}$$

is commutative for $i = 1, \ldots, j$.

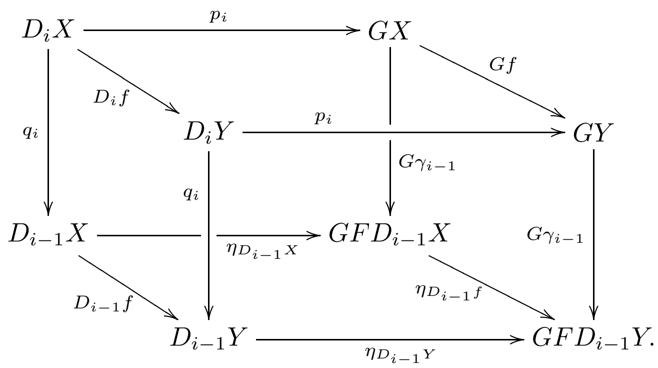
Note that a (weak or ordinary) homotopy coalgebra of order j (we shall omit (FG) if there is no ambiguity) has also homotopy coalgebra structures of order i for $1 \leq i \leq j$. Moreover, in [1, Theorem 1.6] it is proved that if X is a weak homotopy coalgebra of order j, then it is a homotopy coalgebra of order $j - 1$.

Definition 1.2. A map $f\colon X \to Y$ between weak homotopy coalgebras of order j is called a *morphism* if the following square

$$\begin{array}{ccc} X & \xrightarrow{\gamma_i} & FD_iX \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle FD_i(f)} \\ Y & \xrightarrow[\gamma_i]{} & FD_iY \end{array}$$

commutes for $i = 1, \ldots, j$:

Here, the maps $D_i(f)\colon D_iX \to D_iY$, $i \leq j$, are obtained inductively, as "whisker" maps, via the weak universal property of the homotopy pullback [8], to make the following diagram commutative, i.e., homotopy commutative and being the homotopies of each square compatible (the homotopies between the six maps from D_iX to $GFD_{i-1}Y$ are combined to form the boundary of a hexagon; compatibility means that the resulting map extends over the hexagon)



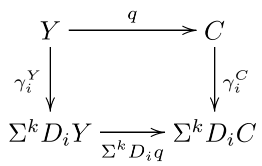
From now on, we shall be dealing with (weak) $\Sigma^k\Omega^k$-homotopy coalgebras ((weak) k-fold homotopy coalgebras or simply k-homotopy coalgebras henceforth). In this case $D_1X = \Omega^kX$.

Recall [3] that a co-H-structure on a space X is equivalent to the existence of a structure map $\gamma_1\colon X \to \Sigma\Omega X$ for which $\varepsilon_1 \circ \gamma_1 = 1_X$. Moreover, X is a coassociative co-H-space with an inverse if and only if it has a 1-fold homotopy coalgebra structure of order 1.

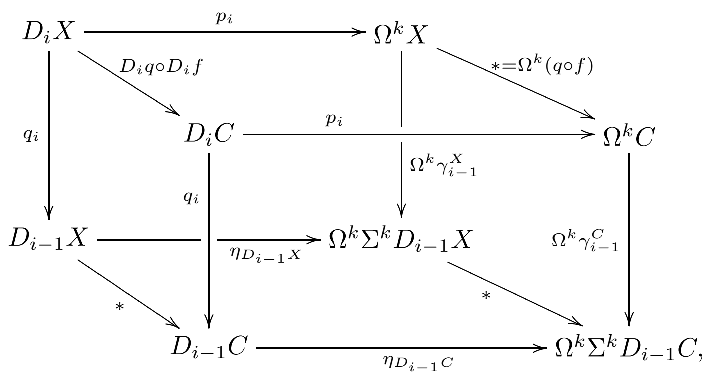
Next, we fix some notation on dimension and connectivity. Recall that a map $f\colon X \to Y$ is an n-equivalence or it has connectivity n, denoted conn $f = n$, if $\pi_i(f)$ is an isomorphism, $i < n$, and $\pi_n(f)$ is surjective. Observe that a space X is n-connected (denoted conn $X = n$) if and only if the constant map $X \to *$ has connectivity $n+1$. Note also that, for any map f, conn $F =$ conn $f - 1$, being F the homotopy fiber of f.

Proof of Theorem 0.1. For $0 \le i \le j$, we shall inductively construct a space D_iC and maps $\gamma_i^C\colon C \to \Sigma^k D_iC$ and $D_iq\colon D_iY \to D_iC$ satisfying:

1. D_iC fits into the homotopy pullback in Definition 1.1.
2. $D_iq \circ D_if = *$.
3. $\varepsilon_i^C \circ \gamma_i^C = 1_C$.
4. The following diagram is commutative:



For $i > 0$, define D_iC by the homotopy pullback in Definition 1.1 and D_if as the whisker map. Then, one has the following homotopy commutative diagram,



where the left and right "$*$" in the bottom square are $D_{i-1} \circ D_{i-1}f = *$ and $\Omega^k\Sigma^k(D_{i-1} \circ D_{i-1}f) = *$ respectively. Thus, by the weak universal property of the homotopy pullback, $D_iq \circ D_if = *$.

Therefore, by the universality of the homotopy pushout, there exists a map

$\gamma_i^C\colon C \to \Sigma^k D_i C$ such that the following diagram

$$\begin{array}{ccccc}
X & \xrightarrow{\;f\;} & Y & \xrightarrow{\;q\;} & C \\
\downarrow{\scriptstyle \gamma_i^X} & & \downarrow{\scriptstyle \gamma_i^Y} & & \downarrow{\scriptstyle \gamma_i^C} \\
\Sigma^k D_i X & \xrightarrow[\Sigma^k D_i f]{} & \Sigma^k D_i Y & \xrightarrow[\Sigma^k D_i q]{} & \Sigma^k D_i C
\end{array}$$

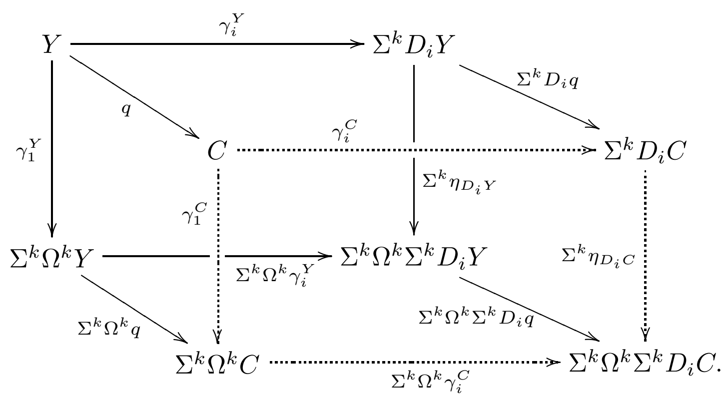
commutes.

To finish, we now show that $\varepsilon_i^C \circ \gamma_i^C = 1_C$. In view of the diagram above, and since $\varepsilon_i^Y \circ \gamma_i^Y = 1_Y$, it follows that $q^*(\varepsilon_i^C \circ \gamma_i^C) = q^*(1_C) = q$. Therefore, $\varepsilon_i^C \circ \gamma_i^C$ and 1_C live in the same orbit of the action of $[\Sigma X, C]$ on $[C, C]$. Thus, denoting by g^h the operation of $h \in [\Sigma X, C]$ on $g \in [C, C]$, there exists $\alpha \in [\Sigma X, C]$ such that $(\varepsilon_i^C \circ \gamma_i^C)^\alpha = 1_C$. However, since $\dim X < \text{conn } C$, $\alpha = *$ and $\varepsilon_i^C \circ \gamma_i^C = 1_C$. This completes the proof of the first assertion of the theorem.

To prove the second assertion, we show that, for $1 \leq i \leq j$, the following commutes:

$$\begin{array}{ccc}
C & \xrightarrow{\;\gamma_i^C\;} & \Sigma^k D_i C \\
\downarrow{\scriptstyle \gamma_1^C} & & \downarrow{\scriptstyle \Sigma^k \eta_{D_i C}} \\
\Sigma^k \Omega^k C & \xrightarrow{\;\Sigma^k \Omega^k \gamma_i^C\;} & \Sigma^k \Omega^k \Sigma^k D_i C.
\end{array}$$

First, observe that this diagram commutes after precomposing with q, due to the commutativity (except the dotted front square) of the following cube



In other words, $q^*(\Sigma^k \Omega^k \gamma_i^C \circ \gamma_1^C) = q^*(\Sigma^k \eta_{D_i X} \circ \gamma_i^C)$. Hence, there exists $\alpha \in [\Sigma X, \Sigma^k \Omega^k \Sigma^k D_i C]$ such that $(\Sigma^k \Omega^k \gamma_i^C \circ \gamma_1^C)^\alpha = \Sigma^k \eta_{D_i C} \circ \gamma_i^C$. But, as $k+1 < \text{conn } C$, we may apply [1, Proposition 2.3] to conclude that $\text{conn } D_i C \geq \text{conn } C - k$ for all i. It follows that $\text{conn } \Sigma^k \Omega^k \Sigma^k D_i C = \text{conn } D_i C + k \geq \text{conn } C$. Thus, since $\dim X < \text{conn } C$ it turns out that $\alpha = *$ and $\Sigma^k \Omega^k \gamma_i^C \circ \gamma_1^C = \Sigma^k \eta_{D_i X} \circ \gamma_i^C$. □

To prove Theorem 0.2 we need an immediate corollary of [1, Proposition 2.3]:

Lemma 1.1. *For any $(n-1)$-connected weak k-fold homotopy coalgebra X of order $j \geq 1$ with $k \leq n-1$,* $\operatorname{conn} \varepsilon_i^X \geq (i+1)n-(k+1)i+1$, *for $i \leq j+1$.*

Proof. Note first that, for any homotopy coalgebra X of order $j \geq 1$, ε_{j+1}^X is well defined. Moreover, for $i \leq j+1$, one has $\varepsilon_i^X = \varepsilon_X \circ F(p_i) = \varepsilon_X \circ \pi_{i-1,2}\Theta_{i-1}$ with $\pi_{i-1,2}, \Theta_{i-1}$ as in [1]. Now, from [1, Proposition 2.3(iii) and (iv)] it follows that , for $i \leq j+1$, $\operatorname{conn} \varepsilon_i^X = \operatorname{conn} \varepsilon_X \circ \pi_{i-1,2} \circ \Theta_{i-1} = \operatorname{conn} \varepsilon_X \circ \pi_{i-1,2} \geq (i+1)n-(k+1)i+1$. □

Proof of Theorem 0.2. For any $i \leq j$, consider the commutative diagram

$$\begin{CD}
F_iX @>{F_if}>> F_iY @>{F_iq}>> F_iC \\
@VVV @VVV @VVV \\
\Sigma^k D_iX @>>{\Sigma^k D_i f}> \Sigma^k D_iY @>>{\Sigma^k D_i q}> \Sigma^k D_iC \\
@V{\varepsilon_i^X}VV @V{\varepsilon_i^Y}VV @V{\varepsilon_i^C}VV \\
X @>{f}>> Y @>{q}>> C
\end{CD}$$

in which the vertical sequences are fibrations. Recall that the maps ε_i^X, ε_i^Y and ε_i^C have sections γ_i^X, γ_i^Y and γ_i^C, respectively which define the corresponding weak homotopy coalgebra structures. As q is a morphism, the square

$$\begin{CD}
Y @>{q}>> C \\
@V{\gamma_i^Y}VV @VV{\gamma_i^C}V \\
\Sigma^k D_iY @>>{\Sigma^k D_i q}> \Sigma^k D_iC
\end{CD}$$

also commutes while it remains to show that the same diagram commutes for f.

Observe that, for the fibration

$$F_iY \longrightarrow \Sigma^k D_iY \xrightarrow{\varepsilon_i^Y} Y,$$

the corresponding Puppe sequence of groups (recall that X is always a co-H-space) splits as

$$1 \to [X, F_iY] \longrightarrow [X, \Sigma^k D_iY] \xrightarrow{(\varepsilon_i^Y)_*} [X, Y] \to 1$$

via the section of $(\varepsilon_i^Y)_*$:

$$s\colon [X,Y] \to [X, \Sigma^k D_iY], \quad s(g) = \Sigma^k D_i(g) \circ \gamma_i^X.$$

Hence, we may consider the group morphism

$$\beta_Y \colon [X, \Sigma^k D_i Y] \to [X, F_i Y], \quad \beta_Y = 1_{[X,\Sigma^k D_i Y]} - s \circ (\varepsilon_i^Y)_*.$$

Indeed, the image of this morphism lies in $[X, F_i Y]$ as $(\varepsilon_i^Y)_* \circ \beta_Y = 0$.

Next, as q is a morphism, the following diagram commutes:

$$\begin{array}{ccccc} [X,Y] & \xrightarrow{(\gamma_i^Y)_*} & [X, \Sigma^k D_i Y] & \xrightarrow{\beta_Y} & [X, F_i Y] \\ {\scriptstyle q_*}\downarrow & & \downarrow{\scriptstyle (\Sigma^k D_i q)_*} & & \downarrow{\scriptstyle (F_i q)_*} \\ [X,C] & \xrightarrow[(\gamma_i^C)_*]{} & [X, \Sigma^k D_i C] & \xrightarrow[\beta_C]{} & [X, F_i C]. \end{array}$$

Then,

$$(F_i q)_* \circ \beta_Y \circ (\gamma_i^Y)_*(f) = \beta_C \circ (\gamma_i^C)_* \circ q_*(f) = 0.$$

Next, observe that

$$\text{conn } F_i q \geq \min\{\text{conn } F_i Y, \text{conn } F_i C\} = \min\{\text{conn } \varepsilon_i^Y, \varepsilon_i^C\} - 1.$$

Combining Lemma 1.1 with the inequality above it follows that, for $i \leq j$, conn $F_i q \geq i\alpha + \alpha - ki + 1 \geq 2\alpha - k + 1$, being $\alpha = \min\{\text{conn } Y, \text{conn } C\}$. By hypothesis, $\dim X \leq 2\alpha - k \leq \text{conn } F_i q - 1$. Hence, via classical obstruction theory $F_i(q)_*$ is injective and therefore $\beta_Y \circ (\gamma_i^Y)_*(f) = 0$. In other words, for $i \leq j$,

$$\gamma_i^Y \circ f = s(\varepsilon_i^Y \circ \gamma_i^Y \circ f) = s(f) = \Sigma^k D_i(f) \circ \gamma_i^X,$$

and f is a morphism. □

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