

**CLASSIFICATION OF MULTIPLICITY FREE
HAMILTONIAN ACTIONS OF ALGEBRAIC TORI
ON STEIN MANIFOLDS**

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A Hamiltonian action of an algebraic torus on a symplectic complex manifold is said to be *multiplicity free* if a general orbit is a lagrangian submanifold. To any multiplicity free Hamiltonian action of an algebraic torus $T \cong (\mathbb{C}^\times)^n$ on a Stein manifold X we assign a certain 5-tuple consisting of a Stein manifold Y , an étale map $Y \rightarrow \mathfrak{t}^*$, a set of divisors on Y and elements of $H^2(Y, \mathbb{Z})^{\oplus n}, H^2(Y, \mathbb{C})$. We show that X is uniquely determined by this invariants. Furthermore, we describe all 5-tuples arising in this way.

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1. Introduction

Let X be a smooth manifold with a symplectic form ω and T be a compact torus acting on X by symplectomorphisms. We recall that the action $T : X$ is called *Hamiltonian* if there are $n = \dim T$ functions $H_1, \dots, H_n \in C^\infty(X)$ such that

- (H1) $\{H_i, H_j\} = 0$, where $\{\cdot, \cdot\}$ denotes the Poisson bracket on X induced by ω .

(H2) The skew-gradient of H_i coincides with the velocity vector field of ξ_i , where ξ_1, \dots, ξ_n is a basis of the Lie algebra \mathfrak{t} of T .

It follows from (H1), (H2) that all functions H_i are T -invariant.

To a Hamiltonian action $T : X$ one assigns a map $\mu : X \rightarrow \mathfrak{t}^*$, $\langle \mu(x), \xi_i \rangle = H_i(x)$, called the *moment map*. This map is T -equivariant and satisfies the following identity:

$$\langle d_x \mu(v), \xi \rangle = \omega(\xi_* x, v), \forall x \in X, v \in T_x X, \xi \in \mathfrak{t},$$

where $\xi_* x$ denotes the velocity vector in x induced by ξ . Note that the moment map is defined uniquely up to the addition of a scalar. The manifold X (or, more precisely, the quadruple $(X, \omega, T : X, \mu)$) is called a *Hamiltonian T -manifold*.

We remark that, by (H1), (H2), any T -orbit in X is an isotropic submanifold. When a general orbit is lagrangian, the Hamiltonian T -manifold X is called *multiplicity free* or, shortly, *MF* (one also uses the term *symplectic toric manifold*).

The most crucial result concerning *compact* MF Hamiltonian T -manifolds is their classification due to Delzant [D]. If T_0 is the inefficiency kernel of the action $T : X$ (that is, the set of all elements of T acting trivially on X), then the action $T/T_0 : X$ is Hamiltonian with moment map $p \circ \mu$. Here μ is the moment map for the action $T : X$ and p is the natural projection $\mathfrak{t}^* \rightarrow (\mathfrak{t}/\mathfrak{t}_0)^*$. So it is enough to deal with effective actions. By the famous result of Atiyah [A], and Guillemin-Sternberg [GS] the image $\mu(X)$ is a convex polytope in \mathfrak{t}^* (provided X is compact). This image is called the *moment polytope* of X . Delzant proved that a compact MF Hamiltonian T -manifold is uniquely determined by its moment polytope. He also found a necessary and sufficient condition for a convex polytope in \mathfrak{t}^* to be a moment polytope of a compact MF Hamiltonian manifold.

The goal of the present paper is to solve a similar classificational problem for *algebraic* tori actions on complex symplectic manifolds (with holomorphic symplectic form). The definition of a (MF) Hamiltonian action is generalized to the complex case directly (the functions H_i are now taken from the algebra $\mathcal{O}(X)$ of holomorphic functions). Let T denote an algebraic torus $(\mathbb{C}^\times)^n$ and X a complex Hamiltonian T -manifold. Since the moment map μ is holomorphic, the manifold X is not compact unless the action $T : X$ is trivial. In this paper we concentrate on the case when X is a Stein manifold.

The data we use to classify MF Hamiltonian Stein T -manifolds are more complicated than in the Delzant case. Below X is a MF Stein Hamiltonian T -manifold.

By the results of [S], there is a categorical quotient Y for the action $T : X$, which is a Stein space (see Section 2 for details). Let $\pi : X \rightarrow Y$ be the quotient map. It turns out that in our case Y is even a Stein manifold (Lemma 2.7). Let ψ be a unique holomorphic map $Y \rightarrow \mathfrak{t}^*$ such

that $\mu = \psi \circ \pi$. One can show that ψ is étale (= a local isomorphism), Lemma 2.7. The manifold Y and the map ψ are the first two pieces of data we need.

For a one-dimensional subgroup $T_0 \subset T$ we denote by $\tilde{\mathcal{D}}(T_0)$ the set of all connected components X_0 of the fixed-point submanifold X^{T_0} such that there is $x \in X_0$ with $T_x = T_0$. Set $\mathcal{D}(T_0) := \{\pi(X_0), X_0 \in \tilde{\mathcal{D}}(T_0)\}$. We will see below (assertion 2 of Lemma 2.7) that any element of $\mathcal{D}(T_0)$ is a connected component of $\psi^{-1}(\alpha + \mathfrak{t}_0^\perp)$, where $\alpha \in \mathfrak{t}^*$ and \mathfrak{t}_0^\perp denotes the annihilator of \mathfrak{t}_0 in \mathfrak{t}^* . Set $\mathcal{D} := \cup_{T_0} \mathcal{D}(T_0)$. The set of divisors \mathcal{D} is the third piece of data we need.

The fourth piece of data is a system of certain elements of $H^2(Y, \mathbb{Z})$. To a character χ of T we assign the coherent sheaf \mathcal{O}_χ on Y defined by

$$\mathcal{O}_\chi(U) = \{f \in \mathcal{O}(\pi^{-1}(U)) \mid t.f = \chi(t)f, \forall t \in T\}.$$

We will see (assertion 3 of Lemma 2.7) that \mathcal{O}_χ is a line bundle for any χ . Let us fix a basis χ_1, \dots, χ_n in the character group $\mathfrak{X}(T)$ of T . Since Y is Stein, we see that the groups $\text{Pic}(Y)$ and $H^2(Y, \mathbb{Z})$ are naturally isomorphic. So we get n cohomology classes $c_i := [\mathcal{O}_{\chi_i}] \in H^2(Y, \mathbb{Z})$. We remark that $\chi \mapsto [\mathcal{O}_\chi]$ is not a group homomorphism despite for any characters χ_1, χ_2 there is a natural map $\mathcal{O}_{\chi_1} \otimes \mathcal{O}_{\chi_2} \rightarrow \mathcal{O}_{\chi_1 + \chi_2}$.

Finally, we will see (assertion 4 of Lemma 2.7) that there is a holomorphic 2-form ω_0 on Y such that $\omega = \pi^*(\omega_0) - d\alpha$, where α is a T -invariant holomorphic 1-form on X satisfying $\alpha(\xi_*) = H_\xi$ for any $\xi \in \mathfrak{t}$; here and below $H_\xi = \langle \mu, \xi \rangle$. Moreover, the class c_0 of ω_0 in the second DeRham cohomology group $H_{\text{DR}}^2(Y)$ is well-defined. Since Y is Stein, we know that $H_{\text{DR}}^2(Y)$ is naturally isomorphic to $H^2(Y, \mathbb{C})$, see [O], Theorem 4.16.

So to any MF Hamiltonian Stein T -manifold X we have assigned the 5-tuple $\mathcal{Y}_X := (Y, \psi, \mathcal{D}, (c_i)_{i=1}^n, c_0)$. Now let $\mathcal{Y} := (Y, \psi, \mathcal{D}, (c_i)_{i=1}^n, c_0)$, $\mathcal{Y}' := (Y', \psi', \mathcal{D}', (c'_i)_{i=1}^n, c'_0)$ be two 5-tuples of the indicated form. By a morphism between $\mathcal{Y}, \mathcal{Y}'$ we mean an étale holomorphic map $\varphi : Y \rightarrow Y'$ satisfying the following conditions:

- (a) $\psi = \psi' \circ \varphi$.
- (b) \mathcal{D} is the set of connected components of $\bigcup_{Y'_0 \in \mathcal{D}'} \varphi^{-1}(Y'_0)$.
- (c) $c_i := \varphi^*(c'_i), i = \overline{0, n}$.

Let X, X' be MF Hamiltonian Stein T -manifolds (recall that T is assumed to act effectively on both X, X') and $\omega, \omega', \mu, \mu'$ be the corresponding symplectic forms and the moment maps. By a *Hamiltonian morphism* between X and X' we mean a holomorphic T -equivariant map $f : X \rightarrow X'$ such that $f^*(\omega') = \omega, \mu = \mu' \circ f$. Since ψ, ψ' are both étale, we easily see that a Hamiltonian morphism $f : X \rightarrow X'$ gives rise to a unique morphism $\mathcal{Y}_f : \mathcal{Y}_X \rightarrow \mathcal{Y}_{X'}$ such that $\mathcal{Y}_f \circ \pi = \pi' \circ \mathcal{Y}_f$, where π, π' denote the quotient maps for X, X' .

Theorem 1.1. *Let X, X' be such as above. For any morphism $\varphi : \mathcal{Y}_X \rightarrow \mathcal{Y}_{X'}$ there is a Hamiltonian morphism $f : X \rightarrow X'$ with $\mathcal{Y}_f = \varphi$. In particular, X, X' are isomorphic if $\mathcal{Y}_X, \mathcal{Y}_{X'}$ are.*

We note that such a morphism f is not necessarily unique. For example, shifting f by an element of T , we again get a Hamiltonian morphism with desired properties.

Our next task is to give a characterization of 5-tuples \mathcal{Y} of the form \mathcal{Y}_X . We say that a 5-tuple \mathcal{Y} is *Delzant* if it enjoys the following property:

(Del) Let y be an arbitrary point of Y , Y_0^1, \dots, Y_0^k be all elements of \mathcal{D} containing y and T_0^1, \dots, T_0^k be the corresponding connected one-dimensional subgroups of T . Then there is a neighborhood U of y in Y such that any element of \mathcal{D} intersecting U is one of $Y_0^i, i = \overline{1, k}$. Furthermore, the primitive elements of $\mathfrak{X}(T_0^i)^* \hookrightarrow \mathfrak{X}(T)^*, i = \overline{1, k}$, (each of them is defined uniquely up to the multiplication by ± 1) constitute a part of a basis in the lattice $\mathfrak{X}(T)^*$.

Theorem 1.2. *Suppose a 5-tuple \mathcal{Y} is Delzant. Then there is a MF Hamiltonian Stein T -manifold X such that $\mathcal{Y} = \mathcal{Y}_X$.*

Let us describe the structure of this paper. Section 2 is devoted to some general results concerning MF Hamiltonian Stein T -manifolds. Firstly, we recall results of Snow concerning general reductive group actions on Stein manifolds. Then we establish the slice theorem 2.4, which asserts that locally a MF Hamiltonian Stein manifold looks like a model manifold introduced in Example 2.2. Using this theorem we prove some structure results on MF Hamiltonian Stein manifolds, Lemmas 2.6, 2.7. In Section 3 we establish a certain sheaf of groups $\mathcal{A}ut$ on the quotient Y of a MF Hamiltonian Stein manifold X . Its sections are Hamiltonian automorphisms that preserve the quotient map. This sheaf is important because it controls gluing of local pieces of MF Hamiltonian manifolds together. The main goal of the section is to determine the sheaf $\mathcal{A}ut$ up to an isomorphism (Lemmas 3.2, 3.3). In Section 4 we prove Theorem 1.1. The most crucial part of its proof is the special case of the identity morphism φ . To examine this case we consider a natural action of $H^1(Y, \mathcal{A}ut)$ on the set of all isomorphism classes of MF Hamiltonian Stein T -manifolds X such that $\mathcal{Y}_X = (Y, \psi, \mathcal{D}, \bullet, \bullet)$. Finally, in Section 5 we prove Theorem 1.2. The main idea of the proof is to consider certain local lagrangian sections of the quotient map.

2. Generalities on MF Hamiltonian Stein T -manifolds

Until a further notice T denotes an algebraic torus $(\mathbb{C}^\times)^n$ and X is a Stein T -manifold.

By [S], there is a Stein space Y and a surjective T -invariant holomorphic map $\pi : X \rightarrow Y$ satisfying the following conditions:

- (A) $U \subset Y$ is open iff $\pi^{-1}(U)$ is,
- (B) $\pi^*(\mathcal{O}(U)) = \mathcal{O}(\pi^{-1}(U))^T$ for any open subset $U \subset Y$.

Such Y, π have the following universal property: for any Stein space Z and a T -equivariant holomorphic map $\rho : X \rightarrow Z$ there is a unique T -equivariant holomorphic map $\bar{\rho} : Y \rightarrow Z$ such that $\rho = \bar{\rho} \circ \pi$. Therefore Y is called the *categorical quotient* for the action $T : X$ and π is called the quotient map. It is known, see [S], that any fiber of π contains a unique closed T -orbit.

In the sequel we will need the following notion.

Definition 2.1. A subset $X_0 \subset X$ is called *saturated* if it consists of fibers of the quotient map, that is $X_0 = \pi^{-1}(\pi(X_0))$.

If X_0 is open and saturated, then $\pi(X_0)$ is open.

Now we quote the Snow slice theorem. Let $x \in X$ be a point with closed T -orbit. Let T_0 denote the stabilizer T_x of x in T and V denote the slice T_0 -module $T_x X / \mathfrak{t}_* x$. Here and below $\mathfrak{t}_* x$ denotes the tangent space to the orbit Tx . Then there is an open saturated neighborhood U of x in X and an open T_0 -stable neighborhood U_0 of 0 in V such that U is T -equivariantly isomorphic to the homogeneous bundle $T *_{T_0} U_0$. By definition, $T *_{T_0} U_0$ is the orbit space for the action $T_0 : T \times U_0$ given by $t_0.(t, u) = (tt_0^{-1}, t_0 u), t \in T, u \in U, t_0 \in T_0$; it is easy to see that $T *_{T_0} U_0$ is a Stein manifold. Let π_0 denote the quotient map for the action $T_0 : V$. From property (B) above it follows that $\pi(U) \cong \pi_0(U_0)$ and the quotient $\pi : U \rightarrow \pi(U)$ is identified with $T *_{T_0} U_0 \rightarrow \pi_0(U_0)$. Since U is saturated, we see that U_0 is also saturated (w.r.t. T_0).

Now we would like to produce some examples of MF Hamiltonian Stein T -manifolds.

Example 2.2. Fix a connected subgroup $T_0 \subset T$, characters χ_1, \dots, χ_k of T_0 forming a basis of $\mathfrak{X}(T_0)$, and $\lambda \in \mathfrak{t}^*$. To these data we assign an MF Hamiltonian Stein manifold X as follows. Let V be a vector space with a basis v_1, \dots, v_k . Define the linear action $T_0 : V$ by $t.v_i = \chi_i(t)v_i, i = \overline{1, k}, t \in T_0$. Let X_0 be the homogeneous vector bundle $T *_{T_0} V$. There is a subtorus $T_1 \subset T$ such that $T_1 \times T_0 = T$. For any such T_1 we have $X_0 = T_1 \times V$. Choose a basis $\theta_1, \dots, \theta_l, l = n - k$, in $\mathfrak{X}(T_1)$, set $\beta_i := \frac{d\theta_i}{\theta_i} = d(\ln \theta_i)$. One can regard β_1, \dots, β_l as a basis in \mathfrak{t}_1^* . Let $\beta^i, i = \overline{1, l}$, be the basis in \mathfrak{t}_1 dual to β_i . Further, let v^1, \dots, v^k be the basis of V^* dual to v_1, \dots, v_k . Set $X := T^* X_0$. We can consider $\theta_i, \beta^i, v_j, v^j$ as functions on X and these functions form a coordinate system. The manifold X has the natural symplectic form $\omega = -d\alpha$, where $\alpha := \sum_{i=1}^l \beta^i d \ln \theta_i + \sum_{i=1}^k v_i dv^i$. The form α is T -invariant and the action $T : X$ is Hamiltonian with moment map $\mu(x) = \lambda + \sum_{i=1}^l \beta^i(x)\beta_i + \sum_{i=1}^k v_i(x)v^i(x)d(\ln \chi_i)$. One can easily check that α, μ do not depend on choices we have made (i.e., the choices of v_i, T_1, θ_j). We also note that the action $T : X$ is effective and $\alpha(\xi_*) = H_\xi$.

Let us check that the 5-tuple \mathcal{Y}_X is well-defined and compute it. We have $Y \cong \mathbb{C}^n$, $\beta^1, \dots, \beta^l, v_1 v^1, \dots, v_k v^k$ being coordinates. The map ψ is identical. We easily get $\mathcal{D} = \{D_i, i = \overline{1, k}\}$, where D_i is the affine hyperplane of the form $v_i v^i = \text{const.}$ containing λ .

Choose $\chi \in \mathfrak{X}(T)$, $\chi = \sum_{i=1}^l a_i \theta_i + \sum_{i=1}^k b_i \chi_i$. Then \mathcal{O}_χ is a trivial line bundle on Y generated by $(\prod_{i=1}^l \theta_i^{a_i})(\prod_{i=1}^k u_i^{|b_i|})$, where $u_i = v_i$ if $b_i \geq 0$ and v^i otherwise. So c_1, \dots, c_n are well-defined and equal to zero for any choice of a basis in $\mathfrak{X}(T)$.

Since ω is exact and $H^2(Y, \mathbb{C}) = 0$, we see that c_0 in this case is well-defined and equals 0.

We remark that the 5-tuple \mathcal{Y}_X is Delzant.

We call X a *model Hamiltonian T -manifold* associated with $T_0, \lambda, \chi_1, \dots, \chi_k$. By a *base point* in X we mean any point from a unique closed T -orbit in X_0 (i.e., any point $x \in X_0$ with $v^1(x) = \dots = v^k(x) = 0$).

Definition 2.3. Let X_1, X_2 be MF Hamiltonian Stein T -manifolds, $x_1 \in X_1, x_2 \in X_2$ be points with closed T -orbits. We say that the pairs $(X_1, x_1), (X_2, x_2)$ are *locally equivalent* if there are saturated open neighborhoods $U_i, i = 1, 2$, of x_i in X_i and a Hamiltonian isomorphism $\varphi : U_1 \rightarrow U_2$ such that $\varphi(x_1) = x_2$.

Till the end of the section X is an arbitrary MF Hamiltonian Stein T -manifold.

Theorem 2.4 (symplectic slice). *Let $x \in X$ be a point with closed orbit, $T_0 = T_x, \lambda = \mu(x)$. Then the following assertions hold:*

- (1) T_0 is connected, and the set of weights of T_0 in $T_x X$ has the form $\{0, \chi_1, \dots, \chi_k, -\chi_1, \dots, -\chi_k\}$, where χ_1, \dots, χ_k is a basis of the character group $\mathfrak{X}(T_0)$.
- (2) Let X' be a model Hamiltonian T -manifold associated with $T_0, \lambda, \chi_1, \dots, \chi_k$ and x' its base point. Then the pairs (X, x) and (X', x') are locally equivalent.

Proof. Note that $T_x X$ is a symplectic T_0 -module. There is a trivial isotropic T_0 -submodule $\mathfrak{t}_* x \subset T_x X$. Therefore $T_x X / (\mathfrak{t}_* x)^\perp$, where $^\perp$ denotes the skew-orthogonal complement, is also a trivial T_0 -module. Set $V := (\mathfrak{t}_* x)^\perp / \mathfrak{t}_* x$. This is a symplectic T_0 -module of dimension $\dim X - 2 \dim T_x = 2 \dim T_0$. By Snow's slice theorem, the action $T_0 : V$ is effective. It follows that the weights of T_0° in V have the form $\chi_1, \dots, \chi_k, -\chi_1, \dots, -\chi_k$, where χ_1, \dots, χ_k form a basis of $\mathfrak{X}(T_0^\circ)$. Therefore T_0° is a maximal torus in $\text{Sp}(V)$ whence $T_0 = T_0^\circ$. So assertion (1) is proved.

In view of the Snow slice theorem, the proof of assertion (2) is completely analogous to that of the symplectic slice theorem from [L] (where it was proved for Hamiltonian reductive group actions on affine algebraic varieties). \square

Remark 2.5. Now let X^1, X^2 be model Hamiltonian T -manifolds corresponding to the data $(T_0^1, \lambda^1, \chi_1^1, \dots, \chi_{k_1}^1), (T_0^2, \lambda^2, \chi_1^2, \dots, \chi_{k_2}^2)$ and x^1, x^2 be the corresponding base points. Then the pairs $(X^1, x^1), (X^2, x^2)$ are locally equivalent iff the following conditions are satisfied:

- (1) $T_0^1 = T_0^2$ whence $k_1 = k_2$.
- (2) $\lambda^1 = \lambda^2$.
- (3) After reordering $\chi_1^2, \dots, \chi_{k_2}^2$, we get $\chi_i^1 = \pm \chi_i^2, i = \overline{1, k_1}$.

Below we will often use a certain open submanifold of X associated with a “general enough” element $\zeta \in \mathfrak{X}(T) \otimes \mathbb{R}$.

We say that $\zeta \in \mathfrak{X}(T) \otimes \mathbb{R}$ is *general* for X if ζ is nonzero on the Lie algebra of T_x for any $x \in X$. Since the set of all T_x is countable, we see that such ζ does exist. By X^ζ we denote the subset of X consisting of all points x satisfying the following two conditions:

- (1) $T_x = \{1\}$.
- (2) Let $\tau : \mathbb{C}^\times \rightarrow T$ be a one-parameter subgroup. If $\lim_{t \rightarrow 0} \tau(t).x$ exists in X , then $\langle \zeta, \tau \rangle > 0$ (we note that the l.h.s of the previous inequality is always nonzero because $\text{im } \tau$ lies in the stabilizer of $\lim_{t \rightarrow 0} \tau(t).x$).

The following lemma describes some properties of the subset $X^\zeta \subset X$.

- Lemma 2.6.** (1) *If X_0 is an open saturated Stein subset of X , then $(X_0)^\zeta = X^\zeta \cap X_0$.*
- (2) *The set X^ζ is open in X .*
 - (3) *For any $y \in Y$ the intersection $X^\zeta \cap \pi^{-1}(y)$ is a single T -orbit. In particular, the map $\pi|_{X^\zeta} : X^\zeta \rightarrow Y$ is surjective.*
 - (4) *The map $\pi|_{X^\zeta} : X^\zeta \rightarrow Y$ is a locally trivial principal T -bundle whence the quotient map for the action $T : X^\zeta$.*

Proof. Since $\pi^{-1}(\pi(x)) \subset X$ is a closed subset, for a one parameter subgroup τ the following three conditions are equivalent:

- (1) $\lim_{t \rightarrow 0} \tau(t).x$ exists in X .
- (2) $\lim_{t \rightarrow 0} \tau(t).x$ exists in $\pi^{-1}(\pi(x))$.
- (3) $\lim_{t \rightarrow 0} \tau(t).x$ exists in X^0 .

Hence the first assertion.

Using the Snow slice theorem we reduce assertions (2), (3) to the case when X is a model Hamiltonian manifold (we do not need a symplectic structure here). Let $T_0, \chi_1, \dots, \chi_k, \lambda, \theta_i, \beta^i, v_j, v^j$ be such as in Example 2.2. Write the decomposition $\zeta = \sum_{i=1}^l a_i \theta_i + \sum_{i=1}^k b_i \chi_i$. The condition that ζ is general implies that all b_i are nonzero. Thanks to Remark 2.5, we may assume that all b_i are positive. In this case one checks directly that $X^\zeta = \{x \in X | v_i(x) \neq 0, i = \overline{1, k}\}$ and assertions (2), (3) follow.

Assertion (4) follows from the observation that there is a holomorphic slice S for the action $T : X^\zeta$ at any point of X^ζ and the restriction of π to S is an open embedding $S \hookrightarrow Y$. \square

Lemma 2.7. (1) Y is smooth and ψ is étale.

- (2) Let T_0 be a one-dimensional connected subgroup of T and $Y_0 \in \mathcal{D}(T_0)$. Then there is $\alpha \in \mathfrak{t}^*$ such that Y_0 is a connected component of $\psi^{-1}(\alpha + \mathfrak{t}_0^\perp)$.
- (3) For any $\chi \in \mathfrak{X}(T)$ the sheaf \mathcal{O}_χ is a line bundle.
- (4) There is a 2-form $\omega_0 \in \Omega^2(Y)$ and a T -invariant one-form $\alpha \in \Omega^1(X)$ such that $\omega = \pi^*(\omega_0) - d\alpha$, and $\alpha(\xi_*) = H_\xi$ for any $\xi \in \mathfrak{t}$. Moreover, the class of ω_0 in $H_{DR}^2(Y)$ does not depend on the choice ω_0, α .
- (5) The 5-tuple \mathcal{Y}_X is well-defined and Delzant.

In the proof below we will need the following elementary fact.

Lemma 2.8. Let $y \in Y$. Then the manifold $\pi^{-1}(U_0)$ is Stein for any open Stein neighborhood U_0 of y in Y .

Proof. Note that $\pi^{-1}(U_0)$ is the fiber product $U_0 \times_Y X$. Being a product of two Stein manifolds, $U_0 \times X$ is Stein. Since $\pi^{-1}(U_0) = U_0 \times_Y X$ is closed in $U_0 \times X$, we are done. \square

Proof of Lemma 2.7. Everything but assertion (4) follows directly from Theorem 2.4 and Example 2.2. Let us prove assertion (4).

Let us check that $[\omega_0] \in H_{DR}^2(Y)$ is unique if ω_0 exists. Let $\omega = \pi^*(\omega_0^1) - d\alpha^1 = \pi^*(\omega_0^2) - d\alpha^2$ be two representations of the required form. Then $\alpha^0 := \alpha^1 - \alpha^2$ is a T -invariant form such that $\alpha^0(\xi_*) = 0$. Choose a general for X element $\zeta \in \mathfrak{X}(T) \otimes_{\mathbb{Z}} \mathbb{R}$. By assertion 4 of Lemma 2.6, the map $\pi : X^\zeta \rightarrow Y$ is a principal T -bundle. It follows that there is $\alpha_0 \in \Omega^1(Y)$ such that $\alpha^0 = \pi^*(\alpha_0)$ on X^ζ and thence on the whole manifold X . So $\pi^*(\omega_0^1 - d\alpha_0) = \pi^*(\omega_0^2)$ whence $\omega_0^1 - d\alpha_0 = \omega_0^2$.

So it remains to prove that there exist ω_0, α with the required properties.

It follows from Example 2.2 and Lemma 2.8 that there exists an open covering $Y = \cup_{i \in I} Y_i$ satisfying the following conditions:

- (1) Both Y_i and $\pi^{-1}(Y_i)$ are Stein.
- (2) $\omega|_{\pi^{-1}(Y_i)} = -d\alpha_i$, where α_i is a holomorphic T -invariant 1-form on $\pi^{-1}(Y_i)$ such that $\alpha_i(\xi_*) = H_\xi$.

For a finite subset $J \subset I$ set $Y_J := \cap_{j \in J} Y_j$.

Set $\alpha_{ij} := \alpha_i - \alpha_j$. This is a T -invariant holomorphic 1-form on $\pi^{-1}(Y_{ij})$ with $\alpha_{ij}(\xi_*) = 0$. It follows that there is a (unique) holomorphic 1-form γ_{ij} on Y_{ij} such that $\alpha_{ij} = \pi^*(\gamma_{ij})$. The collection (γ_{ij}) is a 1-cocycle in Ω^1 . Since Y is Stein, we see that there are holomorphic 1-forms γ_i on Y_i such that $\gamma_{ij} = \gamma_i - \gamma_j$. Further, $0 = d\alpha_{ij} = \pi^*(d\gamma_{ij})$, whence $d\gamma_i = d\gamma_j$ on Y_{ij} .

Set $\omega_0 := d\gamma_i$ on Y_i . Then $\omega - \omega_0$ coincides with $d(\alpha_i - \pi^*(\gamma_i))$ on $\pi^{-1}(Y_i)$. Since $\alpha_i - \pi^*(\gamma_i) = \alpha_j - \pi^*(\gamma_j)$ on $\pi^{-1}(Y_j)$, we may set $\alpha = \alpha_i - \pi^*(\gamma_i)$. \square

Remark 2.9. Let $\mathcal{Y}_X = (Y, \psi, \mathcal{D}, (c_i)_{i=1}^n, c_0)$. Let us show how to recover the structure of X in a neighborhood of a point $x \in X$ with closed orbit. Let D_1, \dots, D_k be all elements of \mathcal{D} containing $\pi(x)$. For each i there is a unique affine hyperplane $\Gamma_i \subset \mathfrak{t}^*$ containing $\psi(D_i)$. Let $\eta_i, i = \overline{1, k}$, be a nonzero element from \mathfrak{t} lying in the annihilator of the (linear) hyperplane associated with Γ_i . Then the Lie algebra of T_0 is spanned by η_1, \dots, η_k , the character $\chi_i, i = \overline{1, k}$ is a primitive element of $\mathfrak{X}(T_0)$ annihilating all η_j with $j \neq i$ (a character χ_i is determined uniquely up to changing the sign), and $\lambda = \psi(x)$.

In the sequel we will need the following proposition that is a special case of Theorem in Section 1 of [HMP].

Proposition 2.10. *Let X be a complex T -manifold such that the action $T : X$ is effective, Y a Stein manifold, and $\pi : X \rightarrow Y$ a holomorphic T -invariant map. Suppose that there is an open covering $Y_i, i \in I$, of Y by Stein submanifolds such that $\pi^{-1}(Y_i)$ is a Stein T -manifold and $\pi : \pi^{-1}(Y_i) \rightarrow Y_i$ is the quotient map. Then X is Stein and $\pi : X \rightarrow Y$ is the quotient map.*

3. The sheaf $\mathcal{A}ut$

Let $T, X, \pi, \mathcal{Y}_X = (Y, \psi, \mathcal{D}, (c_i)_{i=1}^n, c_0)$ be such as above. The goal of this section is to study the sheaf of groups $\mathcal{A}ut^X$ on Y defined as follows: the group $\mathcal{A}ut^X(U)$ consists of all Hamiltonian morphisms of $\pi^{-1}(U)$ preserving π . This sheaf plays a crucial role in the subsequent development, compare with the proof of the uniqueness part of the Delzant theorem in [W].

At first, we construct a certain morphism of sheafs $\mathcal{O}_Y \rightarrow \mathcal{A}ut^X$.

To any function $f \in \mathcal{O}(U)$, where U is an open subset of X , we assign its skew-gradient $v(f)$ by

$$\omega_x(v(f), \eta) = \langle \eta, d_x f \rangle, x \in U, \eta \in T_x X.$$

Exponentiating the vector field $v(f)$ we get the local one-parameter subgroup $t \mapsto \exp(tv(f)), t \in \mathbb{C}$, of holomorphic automorphisms of X . Since $v(f) \cdot \omega = d\iota_{v(f)}\omega + \iota_{v(f)}d\omega$ ($\iota_{v(f)}$ denotes the contraction with $v(f)$), we see that $v(f)$ annihilates ω . So $\exp(tv(f))$ preserves ω .

Lemma 3.1. *Let U be an open subset of Y and $f \in \mathcal{O}(U) \cong \mathcal{O}(\pi^{-1}(U))^T$. Then the map $t \mapsto \exp(2\pi itv(f))$ is a well-defined homomorphism $\mathbb{C} \rightarrow \mathcal{A}ut^X(U)$ such that the corresponding action $\mathbb{C} : \pi^{-1}(U)$ is holomorphic and its velocity vector field coincides with $2\pi iv(f)$.*

Proof. Define two constant sheafs \mathfrak{t}_Y, T_Y on Y with fibers \mathfrak{t}, T , respectively. There is the natural epimorphism $\mathfrak{t}_Y \rightarrow T_Y, \xi \mapsto \exp(2\pi i\xi)$. Also there is the natural action $T_Y(U) : \pi^{-1}(U), \varphi \cdot x = \varphi(\pi(x))x$ by holomorphic T -equivariant automorphisms preserving π .

At first, suppose that the action $T : \pi^{-1}(U)$ is free. Note that the vector field $v(f)$ is T -invariant and tangent to all T -orbits (the latter stems from $\omega(v(f), \xi_*) = 0$ for any $\xi \in \mathfrak{t}$). So we may consider $v(f)$ as a section of \mathfrak{t}_Y . Applying the exponential map to $v(f)$, we get the one-parameter subgroup $\exp(2\pi i t v(f))$ of $T_Y(U)$. So we get the action $\mathbb{C} : \pi^{-1}(U)$, whose velocity vector field is $2\pi i v(f)$. Since $v(f)$ is a Hamiltonian vector field, this action preserves ω . Finally, by definition, the action preserves π and thus also μ .

Consider the general case. Choose general $\zeta \in \mathfrak{X}(T) \otimes_{\mathbb{Z}} \mathbb{R}$. Since the actions of $T_Y(U)$ and T commute, it follows from the definition of X^ζ that $\pi^{-1}(U) \cap X^\zeta$ is stable with respect to the action $T_Y(U) : \pi^{-1}(U)$. The action $\mathbb{C} : X^\zeta \cap \pi^{-1}(U)$ constructed above is factorized through a homomorphism $\mathbb{C} \rightarrow T_Y(U)$. Since $T_Y(U)$ acts on $\pi^{-1}(U)$ by holomorphic automorphisms, we see that the action $\mathbb{C} : \pi^{-1}(U) \cap X^\zeta$ can be extended to an action by holomorphic symplectomorphisms on the whole set $\pi^{-1}(U)$. \square

So we have the sheaf morphism $\mathcal{O}_Y \rightarrow \mathcal{A}ut^X$ given by

$$(3.1) \quad f \mapsto \exp(2\pi i v(f)).$$

Lemma 3.2. *The morphism of sheafs (3.1) is surjective.*

Proof. Again, choose an element $\zeta \in \mathfrak{X}(T) \otimes_{\mathbb{Z}} \mathbb{R}$ general for X and an open subset $U \subset Y$. We get a natural embedding $\mathcal{A}ut^X(U) \hookrightarrow \mathcal{A}ut^{X^\zeta}(U)$ induced by the restriction to $X^\zeta \cap \pi^{-1}(U)$. This embedding is compatible with morphisms $\mathcal{O}_Y \rightarrow \mathcal{A}ut^X, \mathcal{O}_Y \rightarrow \mathcal{A}ut^{X^\zeta}$. So we may replace X with X^ζ and assume that the action $T : X$ is free. Since the question is local, we may assume $X = T^*(T)$ and U is given by (in the notation of Example 2.2) $|\beta^i| < 1, i = \overline{1, n}$. We need to prove that the map $\mathcal{O}(U) \rightarrow \mathcal{A}ut^X(U)$ is surjective.

The group $\mathcal{A}ut^X(U) \hookrightarrow T_Y(U)$ consists of all holomorphic maps $\Phi = (\Phi_1, \dots, \Phi_n) : U \rightarrow T$ such that the map $(t, y) \mapsto (\Phi(y)t, y)$ preserves ω . Fix a basis $\theta_1, \dots, \theta_n$ of $\mathfrak{X}(T)$. Since U is simply connected, there is a holomorphic map $\varphi = (\varphi_1, \dots, \varphi_n) : U \rightarrow \mathfrak{t}$ such that $\Phi = \exp(2\pi i \varphi)$. A direct computation shows that the map $(t, y) \mapsto (\Phi(y)t, y)$ preserves ω iff $\sum_{i=1}^n \frac{d\Phi_i}{\Phi_i} \wedge dy_i = 0$. The latter is equivalent to the system of equations

$$\frac{\partial \varphi_i}{\partial \beta^j} = \frac{\partial \varphi_j}{\partial \beta^i}, \quad i, j = \overline{1, n}.$$

By the Dolbeaux lemma, there is $f \in \mathcal{O}(U)$ such that $\varphi_i = \frac{\partial f}{\partial \beta^i}$. It follows that $v(f) = \varphi$. The homomorphism $\mathcal{O}(U) \rightarrow \mathcal{A}ut^X(U)$ maps f to Φ . \square

Lemma 3.3. *The kernel of the epimorphism (3.1) is $\mathbb{C} \oplus \mathfrak{X}(T)^*$ (where \mathbb{C} denotes the sheaf of constant functions and $\mathfrak{X}(T)^*$ the sheaf of $H_\xi, \xi \in \mathfrak{X}(T)^*$).*

Proof. Again, it is enough to prove this lemma for $X = T^*(T)$, where it is checked directly. \square

Finally, $\mathcal{A}ut^X \cong \mathcal{O}_Y / (\mathbb{C} \oplus \mathfrak{X}(T)^*)$. In particular, $\mathcal{A}ut^X$ depends only on Y and $\psi : Y \rightarrow \mathfrak{t}^*$, so we write $\mathcal{A}ut$ instead of $\mathcal{A}ut^X$.

Corollary 3.4. $H^j(Y, \mathcal{A}ut) = H^{j+1}(Y, \mathbb{C} \oplus \mathfrak{X}(T)^*)$.

Proof. Since Y is Stein, we have $H^j(Y, \mathcal{O}_Y) = 0$ for $j > 0$. It remains to consider the long exact sequence associated with $0 \rightarrow \mathbb{C} \oplus \mathfrak{X}(T)^* \rightarrow \mathcal{O}_Y \rightarrow \mathcal{A}ut \rightarrow 0$. \square

4. Uniqueness

The goal of this section is to prove Theorem 1.1. Throughout this section X is an MF Hamiltonian Stein T -manifold and $\mathcal{Y}_X = (Y, \psi, \mathcal{D}, (c_i)_{i=1}^n, c_0)$ the corresponding 5-tuple.

Denote by \mathbb{X} the set of all isomorphism classes of multiplicity free Hamiltonian Stein T -manifolds with 5-tuples of the form $(Y, \psi, \mathcal{D}, \bullet, \bullet)$. Two such Hamiltonian T -manifolds X, X' are supposed to be isomorphic if there is a Hamiltonian isomorphism $\iota : X \rightarrow X'$ such that $\pi' \circ \iota = \pi$, where π, π' are the quotient maps for X, X' .

There is a natural action of $H^1(Y, \mathcal{A}ut)$ on \mathbb{X} , which we describe now.

Choose $c \in H^1(Y, \mathcal{A}ut)$. Let $Y_i, i \in I$, be an open covering and $\varphi_{ij} \in \mathcal{A}ut(Y_{ij})$, where $Y_{ij} := Y_i \cap Y_j$, be a 1-cocycle representing c . Set

$$X' := \coprod_{i \in I} \pi^{-1}(Y_i) / \sim,$$

where points $x_i \in \pi^{-1}(Y_i), x_j \in \pi^{-1}(Y_j)$ are equivalent if they both lie in $\pi^{-1}(Y_{ij})$ and $x_i = \varphi_{ij}(x_j)$ (the equality of points in X). Since (φ_{ij}) is a 1-cocycle, we see that \sim is a genuine equivalence relation. Clearly, X' has a unique structure of a Hamiltonian T -manifold such that the embedding $\pi^{-1}(Y_i) \rightarrow X'$ is a Hamiltonian morphism. It follows from Proposition 2.10 that X' is Stein.

It is clear that if φ'_{ij} is another 1-cycle and X'' is obtained by applying φ'_{ij} to X' , then X'' is obtained from X by applying $\varphi'_{ij}\varphi_{ij}$. Now suppose φ_{ij} is a 1-coboundary, that is, there is a 1-cochain $f_i \in \mathcal{A}ut(Y_i)$ with $\varphi_{ij} = f_i f_j^{-1}$. Then there is the isomorphism $X \rightarrow X'$ given by f_i on $\pi^{-1}(Y_i)$. So X' depends up to isomorphism only on c and we write X_c for X' . Also we note that $X_{c_1 c_2} = (X_{c_1})_{c_2}$, so we do have an action of $H^1(Y, \mathcal{A}ut)$ on \mathbb{X} .

The following proposition is the main property of this action.

Proposition 4.1. *Suppose \mathbb{X} is nonempty. Then the action of $H^1(Y, \mathcal{A}ut)$ on \mathbb{X} is free and transitive. Further, for any $(c'_i)_{i=0}^n \in H^2(\mathbb{C} \oplus \mathfrak{X}(T)^*)$ there is a unique element $X' \in \mathbb{X}$ with $\mathcal{Y}_X = (Y, \psi, \mathcal{D}, (c'_i)_{i=1}^n, c'_0)$.*

The proof of this proposition will be given at the end of the section after two auxiliary lemmas. Now we are going to derive Theorem 1.1 from Proposition 4.1.

Proof of Theorem 1.1. Let X' be an MF Hamiltonian Stein T -manifold and $\varphi : \mathcal{Y} = (Y, \psi, \mathcal{D}, (c_i)_{i=1}^n, c_0) \rightarrow \mathcal{Y}_{X'} = (Y', \psi', \mathcal{D}', (c'_i)_{i=1}^n, c'_0)$ be an arbitrary morphism of 5-tuples. Set $X = Y \times_{Y'} X'$ and let ψ denote the map $X \rightarrow X'$ arising from the Cartesian square. There is a unique structure of a Hamiltonian T -manifold on X such that $\tilde{\psi}$ is a Hamiltonian morphism. Being a closed submanifold in $Y \times X'$, the manifold X is Stein. The natural map $\pi : X \rightarrow Y$ is the quotient map. It is clear from construction that $\mathcal{Y}_X = \mathcal{Y}$.

The previous construction reduces the proof of the theorem to the case $\varphi = id$. This case stems from Proposition 4.1. \square

Let X' be constructed from X by applying $c \in H^1(Y, \mathcal{A}ut)$, π' denote the quotient map $X' \rightarrow Y$, and $\mathcal{Y}_{X'} = (Y, \psi', \mathcal{D}', (c'_i)_{i=1}^n, c'_0)$. It is clear from the construction of X' that $\psi' = \psi, \mathcal{D}' = \mathcal{D}$. Let us examine the behavior of c'_i . We recall that $H^1(Y, \mathcal{A}ut) \cong H^2(X, \mathbb{C} \oplus \mathfrak{X}(T)^*)$ (Corollary 3.4).

Recall also that there is a natural embedding of sheafs $\mathcal{A}ut \hookrightarrow T_Y$, see the proof of Lemma 3.1. So one can consider the homomorphism $\varphi \mapsto \langle \chi, \varphi \rangle$ from $\mathcal{A}ut$ to the constant sheaf with fiber \mathbb{C}^\times .

Lemma 4.2. *Choose $\chi \in \mathfrak{X}(T)$ and let c_χ, c'_χ be the classes of the line bundles $\mathcal{O}_\chi, \mathcal{O}'_\chi$ corresponding to X, X' in $H^2(Y, \mathbb{C})$. Then $c'_\chi = c_\chi + \langle \chi, c \rangle$.*

Proof. Let f_{ij} denote the transition functions for the sheaf \mathcal{O}_χ on Y . From the construction of the isomorphism $H^1(Y, \mathcal{A}ut) \cong H^2(Y, \mathbb{C} \oplus \mathfrak{X}(T)^*)$ it follows that we need to check that $f_{ij} \langle \chi, \varphi_{ij} \rangle$ is a system of transition functions for \mathcal{O}'_χ .

Indeed, let \mathcal{O}' be the line bundle on Y with transition functions $f_{ij} \langle \chi, \varphi_{ij} \rangle$. Let T act on sections of \mathcal{O}' by χ . We may assume that \mathcal{O}' is trivialized over each Y_i . Let σ be a global section of \mathcal{O}' . Our claim will follow if we check that σ can be regarded as a function on X' of weight χ . To verify this we need to check that $\sigma_i(x) = \sigma_j(\varphi_{ij}^{-1}x)$, where σ_i is the trivialization of σ . Here the subsets $\pi^{-1}(Y_{ij}) \subset \pi^{-1}(Y_i), \pi^{-1}(Y_j)$ are assumed to be identified as in X . But $\sigma_j(\varphi_{ij}^{-1}x) = \langle \chi, \varphi_{ij} \rangle^{-1} \sigma_j(x) = \sigma_i(x)$ (the factor f_{ij} does not appear because of the choice of the identification of two different $\pi^{-1}(Y_{ij})$). \square

Lemma 4.3. *Let $c \in H^2(Y, \mathbb{C}) \hookrightarrow H^2(Y, \mathbb{C} \oplus \mathfrak{X}(T)^*) \cong H^1(Y, \mathcal{A}ut)$. Then $2\pi ic = c_0 - c'_0$.*

Proof. As in the proof of Lemma 2.7, we may assume that the actions $T : X, T : X'$ are free. There is an $\mathcal{O}(Y)$ -valued 1-cochain (f_{ij}) such that $\varphi_{ij} = \exp(2\pi i v(f_{ij}))$. Let α_i be such as in the proof of Lemma 2.7. The 0-cocycle

in closed 1-forms constructed for ω' as in the proof of Lemma 2.7 equals $\alpha_j - \varphi_{ij}^{*-1}(\alpha_i) = (\alpha_j - \alpha_i) + \alpha_i - \varphi_{ij}^{*-1}(\alpha_i)$. Let us compute $\alpha_i - \varphi_{ij}^{*-1}\alpha_i$. We may assume that $\pi^{-1}(Y_i)$ is an open neighborhood in $T^*(T)$. Let θ_i, β^i be such as in Example 2.2. Then $\alpha_i = \sum_{i=1}^n \beta^i d(\ln \theta_i)$. The map φ_{ij} equals $(\theta_i, \beta^i) \mapsto (\theta_i \exp(2\pi i \frac{\partial f_{ij}}{\partial \beta^i}), \beta^i)$. Thus

$$\alpha_i - \varphi_{ij}^{*-1}\alpha_i = 2\pi i \sum_{k=1}^n \beta^k d\left(\frac{\partial f_{ij}}{\partial \beta^k}\right) = 2\pi i d\left(\sum_{k=1}^n \beta^k \frac{\partial f_{ij}}{\partial \beta^k} - f_{ij}\right).$$

So the 1-cocycle in $H^1(Y, \mathcal{O}_Y/\mathbb{C})$ corresponding to $c'_0 - c_0$ is given by $2\pi i(\mathcal{L}f_{ij} - f_{ij})$, where $\mathcal{L} = \sum_{k=1}^n H_i \frac{\partial}{\partial H_i}$ (by abusing notation, $\frac{\partial}{\partial H_i}$ stands for the lifting of the corresponding vector field on \mathfrak{t}^* to Y). Note that $\mathcal{L}(f_{ij} + f_{jk} + f_{ki}) = 0$, for $f_{ij} + f_{jk} + f_{ki} = \text{const}$. Therefore the classes of $c_0 - c'_0$ and $2\pi ic$ in $H^2(Y, \mathbb{C})$ coincide. \square

Proof of Proposition 4.1. Let us prove that the action of $H^1(Y, \mathcal{A}ut)$ on \mathbb{X} is transitive. Let $X, X' \in \mathbb{X}$ and π, π' be the corresponding quotient maps. By Remark 2.9, there is an open covering $Y_i, i \in I$, of Y such that there are Hamiltonian isomorphisms $\iota_i : \pi^{-1}(Y_i) \rightarrow \pi'^{-1}(Y_i), i \in I$, with $\pi' \circ \iota_i = \pi$. So $\iota_i \circ \iota_j^{-1}$ is a 1-cocycle in $\mathcal{A}ut$ and $X' = X_c$ for the corresponding cohomology class c . It follows that the action is transitive.

Choose $X \in \mathbb{X}$ and $(c'_i)_{i=0}^n \in H^2(Y, \mathbb{C} \oplus \mathfrak{X}(T)^*)$. Let $\mathcal{Y}_X = (Y, \psi, \mathcal{D}, (c_i)_{i=1}^n, c_0)$. By Lemma 4.2, there is $c^1 \in H^2(Y, \mathbb{C} \oplus \mathfrak{X}(T)^*)$ such that $\mathcal{Y}_{X_{c^1}} = (Y, \psi, \mathcal{D}, (c'_i)_{i=1}^n, c'_0)$ for some c'_0 and any two elements c^1 with this property differ by an element of $H^2(Y, \mathbb{C})$. Now, by Lemma 4.3, there is a unique element $c^2 \in H^2(Y, \mathbb{C})$ such that $\mathcal{Y}_{X_{c^1 c^2}} = (Y, \psi, \mathcal{D}, (c'_i)_{i=1}^n, c'_0)$. This completes the proof. \square

5. Existence

The goal of this section is to prove Theorem 1.2. Thanks to Proposition 4.1, it is enough to check that for any 5-tuple \mathcal{Y} of the form $(Y, \psi, \mathcal{D}, (c_i)_{i=1}^n, c_0)$ with some $c_i, i = \overline{0, n}$, there is X with $\mathcal{Y}_X = \mathcal{Y}$.

Choose an open covering $Y = \bigcup_i Y_i$ such that Y_i is an open disk and there is an open saturated subset X_i of some model Hamiltonian T -manifold such that $\mathcal{Y}_{X_i} = (Y_i, \psi|_{Y_i}, \mathcal{D}_i, 0, 0)$, where \mathcal{D}_i is the set of components of $D \cap Y_i, D \in \mathcal{D}$. Let $\pi_i : X_i \rightarrow Y_i$ be the quotient map. Choose some isomorphisms $\iota_{ij} : \pi_j^{-1}(Y_{ij}) \rightarrow \pi_i^{-1}(Y_{ij})$ of Hamiltonian manifolds such that $\pi_i \circ \iota_{ij} = \pi_j$. Such isomorphisms exist by Theorem 2.4. We may assume that $\iota_{ii} = id, \iota_{ji} = \iota_{ij}^{-1}$. Note, however, that, in general, $\iota_{ij} \iota_{jk} \iota_{ki} \neq id$. So our task is to modify these isomorphisms for the cocycle condition to hold.

There is $\zeta \in \mathfrak{X}(T) \otimes \mathbb{R}$ general for all X_i . Using Example 2.2, one easily gets that for any i there is a subvariety $\tilde{Y}_i \subset X_i^\zeta$ satisfying the following conditions:

- (1) The restriction of π_i to \tilde{Y}_i is an isomorphism $\tilde{Y}_i \rightarrow Y_i$.
- (2) \tilde{Y}_i is a lagrangian submanifold of X_i .

Proposition 5.1. *For any i, j there is a unique element $\varphi_{ij} \in \mathcal{A}ut(Y_{ij})$ such that*

$$(5.1) \quad \varphi_{ij}(\tilde{Y}_i \cap \pi_i^{-1}(Y_{ij})) = \iota_{ij}(\tilde{Y}_j \cap \pi_j^{-1}(Y_{ij})).$$

Proof. By Lemma 2.6, $\iota_{ij}(\pi_j^{-1}(Y_{ij})^\zeta) = \pi_i^{-1}(Y_{ij})^\zeta$. Since the group $\mathcal{A}ut(Y_{ij})$ depends only on Y_{ij}, ψ and does not depend on \mathcal{D} , we may assume that the actions $T : X_i, X_j$ are free. By (1), there is $\varphi_{ij} \in T_Y(Y_{ij})$ satisfying (5.1). Note that $\varphi_{ij*}\xi_* = \xi_*, \varphi_{ij}^*H_\xi = H_\xi$ for any ξ . Therefore for any $x \in \pi_i^{-1}(Y_{ij}), \eta \in T_x(\pi_i^{-1}(Y_{ij}))$ we have

$$\omega_{\varphi_{ij}(x)}(\xi_*, \varphi_{ij*}\eta) = \partial_{\varphi_{ij*}\eta}H_\xi(\varphi_{ij}x) = \partial_\eta H_\xi(x) = \omega_x(\xi_*, \eta).$$

Since both $\tilde{Y}_i \cap \pi_i^{-1}(Y_{ij}), \iota_{ij}(\tilde{Y}_j \cap \pi_j^{-1}(Y_{ij}))$ are lagrangian, we easily see that φ_{ij} is a symplectomorphism. Thus $\varphi_{ij} \in \mathcal{A}ut(Y_{ij})$. □

Set $\tilde{\iota}_{ij} := \varphi_{ij}^{-1}\iota_{ij}$. By the construction of φ_{ij} ,

$$\tilde{Y}_i \cap \pi_i^{-1}(Y_{ij}) = \tilde{\iota}_{ij}(\tilde{Y}_j \cap \pi_j^{-1}(Y_{ij})).$$

This equation and (1) imply the cocycle condition $\tilde{\iota}_{ij}\tilde{\iota}_{ji}\tilde{\iota}_{ki} = id$.

Set $X := \sqcup_i X_i / \sim$, where points $x_i \in X_i, x_j \in X_j$ are equivalent if $\pi_i(x_i) = \pi_j(x_j) \in Y_{ij}$ and $x_i = \tilde{\iota}_{ij}(x_j)$. The manifold X is Stein, see Proposition 2.10, it has the natural Hamiltonian structure and the action $T : X$ is MF. By construction, $\mathcal{Y}_X = (Y, \psi, \mathcal{D}, (c_i)_{i=1}^n, c_0)$ for some c_i .

6. An open problem

Here we would like to state an open problem communicated to us by F. Knop. Until a further notice T is a compact torus. Recall that any MF compact Hamiltonian T -manifold admits an invariant Kähler structure. This follows, for example, from the Delzant construction involving a Hamiltonian reduction.

For a moment, let X be an arbitrary smooth manifold. Recall that a *hyperkähler* structure on X is a quadruple (q, I, J, K) , where q is a Riemannian metric, and I, J, K are complex structures satisfying the quaternionic relations $IJ = K, JK = I, KI = J$ and such that the three 2-forms defined by $\omega_A(u, v) = q(Au, v), A = I, J, K$, are symplectic. The basic example here is $X = \mathbb{R}^{4n}$ considered as an n -dimensional quaternionic vector space \mathbb{H}^n . Note that $\omega_J + i\omega_K$ is a holomorphic symplectic form with respect to the complex structure I .

Now let X be a hyperkähler manifold equipped with an action of T preserving the hyperkähler structure. This action is called *hyperhamiltonian* if it is Hamiltonian for all symplectic forms ω_A , $A = I, J, K$. Let μ_I, μ_J, μ_K be the corresponding moment maps. Then one easily checks that $\mu_J + i\mu_K$ is a holomorphic map with respect to the complex structure I . If one can lift the action $T : X$ to a holomorphic action $T_{\mathbb{C}} : X$, where $T_{\mathbb{C}}$ denotes the complexification of T , then X becomes a Hamiltonian $T_{\mathbb{C}}$ -manifold.

We say that a hyperhamiltonian action $T : X$ is MF if $\dim X = 4(\dim T - \dim T_0)$, where T_0 denotes the inefficiency kernel for the action $T : X$.

Actually, there is a special class of MF hyperhamiltonian actions studied extensively in the last 10 years, so-called *hypertoric manifolds* (or toric hyperkähler manifolds), see, for example, [BD],[HS],[K],[P]. Let us give their definition.

There is a reduction procedure for hyperhamiltonian manifolds introduced in a more general setup in [HKLR]. Choose a connected subgroup $T_0 \subset T$. For $\alpha \in (\mathfrak{t}_0^*)^{\oplus 3}$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, we put $X//_{\alpha}T := (\mu_0^{-1}(\alpha))/T_0$, where $\mu = (p \circ \mu_I, p \circ \mu_J, p \circ \mu_K)$, $p : \mathfrak{t}^* \rightarrow \mathfrak{t}_0^*$ is the natural projection. If T_0 acts freely on $\mu^{-1}(\alpha)$, then $X//_{\alpha}T_0$ is a genuine manifold of dimension $\dim X - 4 \dim T_0$ possessing a natural hyperkähler structure. Moreover, X is a hyperhamiltonian T/T_0 -manifold. By a hypertoric manifold one means a reduction of \mathbb{H}^n by a subtorus in a maximal torus of $\mathrm{Sp}(n)$.

Problem 6.1. Let X be a MF Hamiltonian Stein $T_{\mathbb{C}}$ -manifold with symplectic form ω and moment map μ . Does there exist a hyperkähler structure (q, I, J, K) on X such that

- (1) I coincides with the initial complex structure on X .
- (2) $\omega_J = \mathrm{Re} \omega, \omega_K = \mathrm{Im} \omega$.
- (3) The action $T : X$ is hyperhamiltonian with $\mu_J = \mathrm{Re} \mu, \mu_K = \mathrm{Im} \mu$.

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