JOURNAL OF
SYMPLECTIC GEOMETRY
Volume 6, Number 2, 189–237, 2008

REDUCED LAGRANGIAN AND HAMILTONIAN FORMULATIONS OF EULER–YANG–MILLS FLUIDS

François Gay-Balmaz and Tudor S. Ratiu

The Lagrangian and Hamiltonian structures for an ideal gauge-charged fluid are determined. Using a Kaluza–Klein point of view, the equations of motion are obtained by Lagrangian and Poisson reductions associated to the automorphism group of a principal bundle. As a consequence of the Lagrangian approach, a Kelvin–Noether theorem is obtained. The Hamiltonian formulation determines a non-canonical Poisson bracket associated to these equations.

1. Introduction

The equations of motion of an ideal incompressible fluid on an oriented Riemannian manifold (M,g) are given by the Euler equations

$$\frac{\partial v}{\partial t} + \nabla_v v = -\operatorname{grad} p, \tag{1.1}$$

where the Eulerian velocity v is a divergence free vector field, p is the pressure, and ∇ is the Levi–Civita covariant derivative associated to g. Arnold [**1**] has shown that equations (1.1) are formally the spatial representation of the geodesic spray on the volume-preserving diffeomorphism group $\mathcal{D}_\mu(M)$ of M with respect to the L^2 Riemannian metric, where μ is the Riemannian volume form on M. See also [**2**, Section 5.5.8], for a quick exposition of this fact. Ebin and Marsden [**3**] give the analytic formulation and many rigorous results concerning the Euler and Navier–Stokes equations derived from this geometric point of view. From the Hamiltonian perspective, equations (1.1) are the Lie–Poisson equations on the Lie algebra $\mathfrak{X}_{\mathrm{div}}(M)$ of $\mathcal{D}_\mu(M)$, consisting of divergence free vector fields (see [**4**]). Here, $\mathfrak{X}_{\mathrm{div}}(M)$ is identified with its dual by the weak L^2-pairing.

In [**5**, **6**], this approach is generalized to the case of the motion of an ideal compressible adiabatic fluid

$$\begin{cases} \dfrac{\partial v}{\partial t} + \nabla_v v = \dfrac{1}{\rho} \operatorname{grad} p, \\ \dfrac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) = 0, \\ \dfrac{\partial s}{\partial t} + \mathbf{d}s(v) = 0, \end{cases} \tag{1.2}$$

where ρ is the mass density, s is the specific entropy, and p is the pressure. In this case, the configuration space is the full diffeomorphism group $\mathcal{D}(M)$ and equations (1.2) are obtained via Lie–Poisson reduction for semidirect products. The Euler-Poincaré approach is given in [**7**].

In this work, we generalize the two previous procedures to the case of a classical charged ideal fluid. More precisely, using a Kaluza–Klein point of view, we obtain the equations of motion by Lagrange–Poincaré and Poisson reduction by a symmetry group (see, e.g., [**8**] and [**9**, Section 10.5]). We consider on M a G-principal bundle $P \to M$ and enlarge the configuration space from the group of diffeomorphisms of M to the product of the group of automorphisms of P with the field variables.

If $G = S^1$ we recover the Euler–Maxwell equations describing the motion of an electrically charged fluid

$$\begin{cases} \dfrac{\partial v}{\partial t} + \nabla_v v = \dfrac{q}{m}(\mathbf{E} + v \times \mathbf{B}) - \dfrac{1}{\rho} \operatorname{grad} p, \\ \dfrac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) = 0, \\ \dfrac{\partial s}{\partial t} + \mathbf{d}s(v) = 0, \\ \dfrac{\partial \mathbf{E}}{\partial t} = \operatorname{curl} \mathbf{B} - \dfrac{q}{m}\rho v, \quad \dfrac{\partial \mathbf{B}}{\partial t} = -\operatorname{curl} \mathbf{E}, \\ \operatorname{div} \mathbf{E} = \dfrac{q}{m}\rho, \quad \operatorname{div} \mathbf{B} = 0, \end{cases} \tag{1.3}$$

where v is the Eulerian velocity, $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, m is the mass of the charged fluid particles, and the constant q is the electric charge of the particles. The Hamiltonian structure of the incompressible Euler–Maxwell equations is already presented in [**5**].

Returning to the general case of a G-principal bundle, we will show that the Lagrange–Poincaré and Poisson reduction methods lead to the equations for an ideal compressible adiabatic fluid carrying a gauge-charge, as given in [**10**]. We call these equations the Euler–Yang–Mills equations. The physically relevant examples are obtained for $G = \mathrm{SU}(2)$ or $G = \mathrm{SU}(3)$ in which case the associated fluid motion goes also under the name of chromohydrodynamics. For a Lagrangian description of the Euler–Yang–Mills equations and the associated variational principle formulated in local coordinates both in the non-relativistic and relativistic versions, see [**11**, **12**]. In these papers, the variations are constrained according to the general Euler–Poincaré variational principle for field theories presented in [**13**].

The physical interpretation of the equations obtained by the methods given in this paper is the following. The evolution of the fluid particles as well as of the gauge-charge density of the fluid is given by a curve ψ_t in the automorphism group of a principal bundle $P \to M$. In fact, ψ_t is the flow of a time-dependent vector field U_t on P. This vector field induces a time-dependent vector field v_t on M, which represents the Eulerian velocity of the fluid. Given the evolution of the Yang–Mills fields potential and of the mass density, the vector field U_t also induces a Lie algebra valued and time-dependent function which represents the gauge-charge density of the fluid. Note the analogy with the classical Kaluza–Klein construction appearing in the formulation of the equations of motion for a charged particle in a Yang–Mills field. See also [**14**] which generalizes the Eulerian fluid velocity vector to include a non-Abelian, or gauge, index.

The paper is organized as follows. To fix notations and conventions, we summarize in Section 2 some basic facts about principal bundles, connections, automorphisms, and gauge groups. The Hamiltonian and Lagrangian formulations of the Maxwell equations are recalled in Section 3 and generalized to the case of the Yang–Mills fields equations. The Lagrangian formulation of the motion of a charged classical particle in a Yang–Mills field, that is, the Wong equations, are presented in Section 4. In Section 5 it is shown that the compressible and incompressible Euler–Yang–Mills equations consist of coupled equations. These are the Euler–Poincaré equations of a semidirect product (associated to the automorphism group of a principal bundle) for the fluid and charge variables and the Yang–Mills equations for the vector potential (that is, the connection) and the "electric part" of the Yang–Mills field. The Hamiltonian counterpart of this result is presented in Section 6: one obtains coupled equations consisting of Lie–Poisson equations on the same dual for the fluid and charge variables together with the Yang–Mills equations. Formally the Gauss equation relating the gauge-charge and the "electric part" of the Yang–Mills field is missing from this system. However, it is obtained by conservation of the momentum map associated to the invariance under gauge transformations. We naturally obtain the non-canonical Poisson bracket associated to the Hamiltonian formulation of the Euler–Yang–Mills equations. By applying the general process of reduction by stages, we recover some already known results about the Euler–Maxwell equations. We also show that the two different Poisson brackets derived in [**10**] and in [**5**] are in fact obtained by Poisson reduction, at different stages, of the same canonical Poisson structure. Finally, in Section 7, we present a Kelvin–Noether theorem for the Euler–Yang–Mills equations.

2. Connections, automorphisms, and gauge transformations

In this section, we recall basic notions related to principal bundles. We also introduce notations and conventions that will be used throughout the paper.

2.1. Principal and adjoint bundles. Consider a smooth free and proper right action

$$\Phi : G \times P \longrightarrow P, \quad (g,p) \longmapsto \Phi_g(p)$$

of a Lie group G on a manifold P. Thus we get the principal bundle

$$\pi : P \longrightarrow M := \frac{P}{G},$$

where M is endowed with the unique manifold structure for which π is a submersion.

To any element ξ in the Lie algebra $\mathfrak{g}$ of G, there corresponds a vector field ξ_P on P, called the infinitesimal generator, defined by

$$\xi_P(p) := \left.\frac{d}{dt}\right|_{t=0} \Phi_{\exp(t\xi)}(p).$$

At any $p \in P$, these vector fields generate the vertical subspace

$$V_pP := \{\xi_P(p) \mid \xi \in \mathfrak{g}\} = \ker(T_p\pi).$$

Recall that the adjoint vector bundle is

$$\operatorname{Ad} P := P \times_G \mathfrak{g} \longrightarrow M,$$

where the quotient is taken relative to the right action $(g,(p,\xi)) \mapsto (\Phi_g(p), \operatorname{Ad}_{g^{-1}}(\xi))$. The elements of $\operatorname{Ad} P$ are denoted by $[p,\xi]_G$, for $(p,\xi) \in P \times \mathfrak{g}$. There is a Lie bracket operation $[\cdot,\cdot]_x$ on each fiber $(\operatorname{Ad} P)_x$ depending smoothly on $x \in M$; it is defined by

$$[[p,\xi]_G, [p,\eta]_G]_x := [p,[\xi,\eta]]_G$$

for $[p,\xi]_G, [p,\eta]_G \in \operatorname{Ad} P$, $\pi(p) = x$.

2.2. Exterior forms on adjoint bundles. Consider the space $\Omega^k(P,\mathfrak{g})$ of $\mathfrak{g}$-valued k-forms on P and let $\overline{\Omega^k}(P,\mathfrak{g})$ be the subspace of $\Omega^k(P,\mathfrak{g})$ consisting of $\mathfrak{g}$-valued k-forms ω such that:

(1) $\Phi_g^*\omega = \operatorname{Ad}_{g^{-1}} \circ \omega$,
(2) if one of $u_1,\dots,u_k \in T_pP$ is vertical, then $\omega(u_1,\dots,u_k) = 0$.

The real vector space $\overline{\Omega^k}(P,\mathfrak{g})$ is naturally isomorphic to $\Omega^k(M, \operatorname{Ad} P)$, the space of $\operatorname{Ad} P$-valued k-forms on M. Indeed, to each $\omega \in \overline{\Omega^k}(P,\mathfrak{g})$

corresponds a k-form $\widetilde{\omega} \in \Omega^k(M, \operatorname{Ad} P)$ whose value on $v_1, \ldots, v_k \in T_xM$ is given by

$$\widetilde{\omega}(x)(v_1, \ldots, v_k) := [p, \omega(p)(u_1, \ldots, u_k)]_G, \tag{2.1}$$

where $p \in P$ is such that $\pi(p) = x$ and $u_i \in T_pP$ are such that $T_p\pi(u_i) = v_i$.

To define the inverse of the map $\widetilde{\ }: \overline{\Omega^k}(P, \mathfrak{g}) \to \Omega^k(M, \operatorname{Ad} P)$, we introduce first for every $p \in P$ the $\mathbb{R}$-linear map $i_p : (\operatorname{Ad} P)_x \to \mathfrak{g}$, $x := \pi(p) \in M$, by

$$i_p\left([q, \eta]_G\right) := \xi, \quad \text{where } \xi \text{ is such that } [p, \xi]_G = [q, \eta]_G;$$

in this formula $p, q \in P$ and $\xi, \eta \in \mathfrak{g}$. Equivalently, this definition can be restated as

$$i_p\left([q, \eta]_G\right) := \operatorname{Ad}_g \eta, \text{ where } g \in G \text{ is uniquely determined by } q = \Phi_g(p).$$

Then the definition of the Lie bracket on each fiber $(\operatorname{Ad} P)_x$ of the adjoint bundle $\operatorname{Ad} P$ immediately implies that $i_p : (\operatorname{Ad} P)_x \to \mathfrak{g}$ is a Lie algebra isomorphism. In addition, $i_{\Phi_g(p)} = \operatorname{Ad}_{g^{-1}} \circ i_p$ for every $p \in P$ and $g \in G$. Using the maps i_p for every $p \in P$, define the inverse of $\widetilde{\ }: \overline{\Omega^k}(P, \mathfrak{g}) \to \Omega^k(M, \operatorname{Ad} P)$ by

$$\omega(p)(u_1, \ldots, u_k) := i_p\Big(\widetilde{\omega}(\pi(p))\big(T_p\pi(u_1), \ldots, T_p\pi(u_1)\big)\Big), \tag{2.2}$$

for any $p \in P$ and $u_1, \ldots, u_k \in T_pP$. The identity $i_{\Phi_g(p)} = \operatorname{Ad}_{g^{-1}} \circ i_p$ ensures that $\omega \in \overline{\Omega^k}(P, \mathfrak{g})$.

Since $\overline{\Omega^0}(P, \mathfrak{g}) = \mathcal{F}_G(P, \mathfrak{g}) := \{f : P \to \mathfrak{g} \mid f \circ \Phi_g = \operatorname{Ad}_{g^{-1}} \circ f\}$ and $\Omega^0(M, \operatorname{Ad} P) = \Gamma(\operatorname{Ad} P)$, the space of sections of $\operatorname{Ad} P$, we shall use the notations $\overline{\Omega^0}(P, \mathfrak{g}) = \mathcal{F}_G(P, \mathfrak{g})$ and $\Omega^0(M, \operatorname{Ad} P) = \Gamma(\operatorname{Ad} P)$ interchangeably. We have hence $\mathcal{F}_G(P, \mathfrak{g}) \cong \Gamma(\operatorname{Ad} P)$ as Lie algebras, the isomorphism $f \in \mathcal{F}_G(P, \mathfrak{g}) \mapsto \widetilde{f} \in \Gamma(\operatorname{Ad} P)$ being given by (2.1), that is, $\widetilde{f}(\pi(p)) = [p, f(p)]_G$.

2.3. Connections and covariant differentials. A principal connection on P is a $\mathfrak{g}$-valued 1-form $\mathcal{A} \in \Omega^1(P, \mathfrak{g})$ such that

$$\Phi_g^* \mathcal{A} = \operatorname{Ad}_{g^{-1}} \circ \mathcal{A} \quad \text{and} \quad \mathcal{A}(\xi_P) = \xi.$$

The set of all connections will be denoted by $\mathcal{C}onn(P)$. It is an affine space with underlying vector space $\overline{\Omega^1}(P, \mathfrak{g})$. Recall that a connection induces a splitting $T_pP = V_pP \oplus H_pP$ of the tangent space into the vertical and horizontal subspace defined by

$$H_pP := \ker(\mathcal{A}(p)).$$

The covariant exterior differential associated to $\mathcal{A}$ is the map $\mathbf{d}^{\mathcal{A}} : \Omega^k(P, \mathfrak{g}) \to \Omega^{k+1}(P, \mathfrak{g})$ defined by

$$\mathbf{d}^{\mathcal{A}}\omega(p)(u_1, \ldots, u_k) := \mathbf{d}\,\omega(p)\left(\operatorname{hor}_p(u_1), \ldots, \operatorname{hor}_p(u_k)\right),$$

where $\mathrm{hor}_p(u_i)$ is the horizontal part of $u_i \in T_pP$, $i = 1, \ldots, k$. Note that for $\omega \in \overline{\Omega^k}(P, \mathfrak{g})$ we have $\mathbf{d}^{\mathcal{A}}\omega \in \overline{\Omega^{k+1}}(P, \mathfrak{g})$. For $f \in \mathcal{F}_G(P, \mathfrak{g})$ and $\omega \in \overline{\Omega^1}(P, \mathfrak{g})$, we have the formulas

$$\mathbf{d}^{\mathcal{A}} f(p)(u) = \mathbf{d}f(p)(u) + [\mathcal{A}(p)(u), f(p)]$$

and

$$\mathbf{d}^{\mathcal{A}}\omega(p)(u,v) = \mathbf{d}\omega(p)(u,v) + [\mathcal{A}(p)(u), \omega(p)(v)] - [\mathcal{A}(p)(v), \omega(p)(u)]$$

for any $u, v \in T_pP$.

The curvature of the connection $\mathcal{A}$ is, by definition, the 2-form

$$\mathcal{B} := \mathbf{d}^{\mathcal{A}}\mathcal{A} \in \overline{\Omega^2}(P, \mathfrak{g}).$$

The curvature $\mathcal{B}$ verifies the Cartan structure equations and the Bianchi identity given respectively by

$$\mathcal{B}(u,v) = \mathbf{d}\,\mathcal{A}(u,v) + [\mathcal{A}(u), \mathcal{A}(v)] \quad \text{and} \quad \mathbf{d}^{\mathcal{A}}\mathcal{B} = 0.$$

The following lemma will be useful for future computations.

Lemma 2.1. *Let $\mathcal{A} \in \mathcal{C}onn(P)$, $\mathcal{B}$ its curvature, and $f \in \mathcal{F}_G(P, \mathfrak{g})$. Then*

$$\mathbf{d}^{\mathcal{A}}\mathbf{d}^{\mathcal{A}} f(u,v) = [\mathcal{B}(u,v), f].$$

Proof. For any $U, V \in \mathfrak{X}(P)$, we have

$$\begin{aligned}
\mathbf{d}^{\mathcal{A}}(\mathbf{d}^{\mathcal{A}} f)(U,V) &= \mathbf{d}(\mathbf{d}^{\mathcal{A}} f)(U,V) + [\mathcal{A}(U), \mathbf{d}^{\mathcal{A}} f(V)] - [\mathcal{A}(V), \mathbf{d}^{\mathcal{A}} f(U)] \\
&= \mathbf{d}\left(\mathbf{d}f + [\mathcal{A}, f]\right)(U,V) + [\mathcal{A}(U), \mathbf{d}f(V) + [\mathcal{A}(V), f]] \\
&\quad - [\mathcal{A}(V), \mathbf{d}f(U) + [\mathcal{A}(U), f]] \\
&= 0 + \mathbf{d}([\mathcal{A}(V), f])(U) - \mathbf{d}([\mathcal{A}(U), f])(V) - [\mathcal{A}([U,V]), f] \\
&\quad + [\mathcal{A}(U), \mathbf{d}f(V)] + [\mathcal{A}(U), [\mathcal{A}(V), f]] - [\mathcal{A}(V), \mathbf{d}f(U)] \\
&\quad - [\mathcal{A}(V), [\mathcal{A}(U), f]] \\
&= [\mathbf{d}(\mathcal{A}(V))(U), f] - [\mathbf{d}(\mathcal{A}(U))(V), f] - [\mathcal{A}([U,V]), f] \\
&\quad + [\mathcal{A}(U), [\mathcal{A}(V), f]] - [\mathcal{A}(V), [\mathcal{A}(U), f]] \\
&= [\mathbf{d}\mathcal{A}(U,V), f] + [[\mathcal{A}(U), \mathcal{A}(V)], f] \\
&= [\mathcal{B}(U,V), f].
\end{aligned}$$

□

Recall also that a principal connection $\mathcal{A}$ on P induces an affine connection and a covariant derivative, denoted respectively by $\nabla^{\mathcal{A}}$ and $\frac{D^{\mathcal{A}}}{dt}$, on the vector bundles $\mathrm{Ad}\,P \to M$ and $(\mathrm{Ad}\,P)^* \to M$ (see e.g., [**8**, **15**]).

Given a Riemannian metric g on M and a connection $\mathcal{A}$ on P, we can define the covariant codifferential

$$\delta^{\mathcal{A}} : \overline{\Omega^k}(P, \mathfrak{g}) \longrightarrow \overline{\Omega^{k-1}}(P, \mathfrak{g});$$

see, e.g., [**16**, Definition 4.2.8].

2.4. Bundle metrics. We assume throughout this paper that the Lie algebra $\mathfrak{g}$ has a distinguished inner product γ satisfying

$$\gamma(\mathrm{Ad}_g \xi, \mathrm{Ad}_g \eta) = \gamma(\xi, \eta), \quad \text{for all } g \in G \text{ and all } \xi, \eta \in \mathfrak{g}.$$

Such an inner product is said to be Ad-invariant and satisfies the relation

$$\gamma([\zeta, \xi], \eta) + \gamma(\xi, [\zeta, \eta]) = 0 \quad \text{for all } \xi, \eta, \zeta \in \mathfrak{g}. \tag{2.3}$$

For example, if G is compact, such an inner product always exists. If G is reductive, one can always find such a non-degenerate γ but it may be indefinite.

Given a Riemannian metric g on M and an Ad-invariant inner product γ on $\mathfrak{g}$, we can define a Riemannian metric $g\gamma$ on the vector bundles $\Lambda^k(M, \mathrm{Ad}\, P) \to M$ of $\mathrm{Ad}\, P$-valued exterior k-forms on M. Indeed, the inner product γ induces a Riemannian metric on the vector bundle $\mathrm{Ad}\, P \to M$ whose value on $[p, \xi]_G, [p, \eta]_G \in (\mathrm{Ad}\, P)_x$, $x := \pi(p)$, $p \in P$, is given by

$$\gamma_x\left([p, \xi]_G, [p, \eta]_G\right) := \gamma(\xi, \eta).$$

Denote, by abuse of notation, by the same letter γ the smooth vector bundle metric on $\mathrm{Ad}\, P$ defined by $\gamma|_{(\mathrm{Ad}\, P)_x} := \gamma_x$. Let $\overline{g}$ denote the Riemannian metric induced by g on the vector bundles $\Lambda^k M \to M$ of exterior k-forms on M. The Riemannian metric $g\gamma$ on the vector bundle $\Lambda^k(M, \mathrm{Ad}\, P) \to M$ is then constructed in the following manner. If $\alpha_x, \beta_x \in \Lambda^k(M, \mathrm{Ad}\, P)_x$, write $\alpha_x = \alpha^a f_a$ and $\beta_x = \beta^a f_a$, where $\{f_a\}$ is a basis of the fiber $(\mathrm{Ad}\, P)_x$, and $\alpha^a, \beta^a \in (\Lambda^k M)_x$. Then define

$$(g\gamma)_x(\alpha_x, \beta_x) := \gamma_{ab}\, \overline{g}(\alpha^a, \beta^b),$$

where $\gamma_{ab} := \gamma_x(f_a, f_b)$. It is easy to verify that this construction is independent on the choice of the basis in each fiber $(\mathrm{Ad}\, P)_x$.

Let M be a compact oriented boundaryless manifold. If $\alpha \in \Omega^k(M, \mathrm{Ad}\, P)$ and $\beta \in \Omega^{k+1}(M, \mathrm{Ad}\, P)$, we have (see, e.g., [**16**, Theorem 4.2.9]):

$$\int_M (g\gamma)(\mathbf{d}^{\mathcal{A}}\alpha, \beta)\mu = \int_M (g\gamma)(\alpha, \delta^{\mathcal{A}}\beta)\mu, \tag{2.4}$$

where μ denotes the volume form associated to the Riemannian metric g.

Given a connection $\mathcal{A}$, a Riemannian metric g on M, and an Ad-invariant inner product γ on $\mathfrak{g}$, we can define the Kaluza–Klein metric $K_{\mathcal{A}}$ on TP by

$$K_{\mathcal{A}}(u_p, v_p) := g(T_p\pi(u_p), T_p\pi(v_p)) + \gamma(\mathcal{A}(u_p), \mathcal{A}(v_p)). \tag{2.5}$$

The Kaluza–Klein metric is G-invariant, that is, $\Phi_g^* K_{\mathcal{A}} = K_{\mathcal{A}}$ for any $g \in G$.

2.5. Expressions in a local trivialization. Consider a local trivialization $P_U := \pi^{-1}(U) \to U \times G$, $p \mapsto (x, g)$. This induces a local trivialization of the vector bundle $\operatorname{Ad} P \to M$, given by

$$\operatorname{Ad} P \supset P_U \times_G \mathfrak{g} \longrightarrow U \times \mathfrak{g}, \quad [(x, g), \xi]_G \longmapsto (x, \operatorname{Ad}_g \xi). \tag{2.6}$$

It is useful to note that for $\omega \in \overline{\Omega^k}(P, \mathfrak{g})$, we can locally write

$$\omega(x, g)((v_1, a_1), \ldots, (v_k, a_k)) = \operatorname{Ad}_{g^{-1}}(\overline{\omega}(x)(v_1, \ldots, v_k)),$$

where $\overline{\omega}(x)(v_1, \ldots, v_k) := \omega(x, e)((v_1, 0), \ldots, (v_k, 0)) \in \mathfrak{g}$, $v_i \in T_xU$, $a_i \in T_gG$, $i = 1, \ldots, k$. Moreover, in the local trivialization (2.6), the $\operatorname{Ad} P$-valued k-form $\widetilde{\omega} \in \Omega^k(M, \operatorname{Ad} P)$ defined in (2.1) is given by

$$\tilde{\omega}(x)(v_1, \ldots, v_k) = (x, \overline{\omega}(x)(v_1, \ldots, v_k)), \quad x \in U, \ v_i \in T_xU.$$

Recall also that, in a local trivialization, a connection $\mathcal{A}$ can be written as

$$\mathcal{A}(x, g)(v_x, \xi_g) = \operatorname{Ad}_{g^{-1}}(\overline{\mathcal{A}}(x)(v_x) + TR_{g^{-1}}(\xi_g)), \tag{2.7}$$

where $v_x \in T_xU$, $\xi_g \in T_gG$, and $\overline{\mathcal{A}}$ is a 1-form on $U \subset M$ with values in $\mathfrak{g}$. Locally, for $f \in \mathcal{F}_G(P, \mathfrak{g})$ and the curvature $\mathcal{B}$, we can write

$$\begin{aligned} \overline{\mathbf{d}^{\mathcal{A}} f}(v) &= \mathbf{d}\overline{f}(v) + [\overline{\mathcal{A}}(v), \overline{f}], \\ \overline{\mathcal{B}}(v, w) &= \mathbf{d}\overline{\mathcal{A}}(v, w) + [\overline{\mathcal{A}}(v), \overline{\mathcal{A}}(w)]. \end{aligned}$$

If the principal bundle is trivial, the previous formulas hold globally and the adjoint bundle is also trivial $\operatorname{Ad} P \cong M \times \mathfrak{g}$, so we have $\Omega^k(M, \operatorname{Ad} P) = \Omega^k(M, \mathfrak{g})$ and for $\omega \in \overline{\Omega^k}(P, \mathfrak{g})$ we get $\widetilde{\omega} = \overline{\omega}$.

2.6. Automorphisms and gauge transformations. We say that a diffeomorphism φ of P is an automorphism if it is equivariant, that is, $\Phi_g \circ \varphi = \varphi \circ \Phi_g$, for all $g \in G$. The Fréchet Lie group of all automorphisms is denoted by $\mathcal{A}ut(P)$. See [**17**] for an account of Fréchet Lie groups in the framework of manifold of maps from the point of view of the "convenient calculus." An automorphism φ of P induces a unique diffeomorphism $\overline{\varphi}$ of M defined by the condition $\pi \circ \varphi = \overline{\varphi} \circ \pi$. The Lie algebra $\mathfrak{aut}(P)$ consists of G-invariant vector fields on P. Its (left) Lie bracket is denoted by $[U, V]_{\mathrm{L}}$ and is the negative of the usual Jacobi–Lie bracket $[U, V]_{\mathrm{JL}}$. For $U \in \mathfrak{aut}(P)$, we denote by $[U] \in \mathfrak{X}(M)$ the unique vector field on M defined by the condition

$$T\pi \circ U = [U] \circ \pi.$$

The subgroup $\mathcal{A}ut_\mu(P)$ consists, by definition, of automorphisms φ of P such that the induced diffeomorphism $\overline{\varphi}$ preserves the volume form μ on M. For $U \in \mathfrak{aut}_\mu(P)$, we have $[U] \in \mathfrak{X}_{\mathrm{div}}(M)$, the space of all divergence free vector fields on M.

The normal subgroup $\mathcal{G}au(P)$ of gauge transformations contains, by definition, all automorphisms φ on P with $\overline{\varphi} = \mathrm{id}_M$. Note that we can identify the gauge group $\mathcal{G}au(P)$ with the group

$$\mathcal{F}_G(P,G) := \{\tau \in \mathcal{F}(P,G) \mid \tau\circ\Phi_g = \mathrm{AD}_{g^{-1}} \circ \tau\}, \quad \text{where } \mathrm{AD}_g(h) := ghg^{-1}.$$

The identification is given by the group isomorphism $\widehat{\ }: \mathcal{G}au(P) \to \mathcal{F}_G(P,G)$, which associates to $\varphi \in \mathcal{G}au(P)$, the map $\widehat{\varphi} \in \mathcal{F}_G(P,G)$ defined by the condition

$$\varphi(p) = \Phi_{\widehat{\varphi}(p)}(p).$$

The Lie algebra $\mathfrak{gau}(P)$ consists of G-invariant vertical vector fields on P. Therefore, when $U \in \mathfrak{gau}(P)$, we have $[U] = 0$. Note the identifications

$$\mathfrak{gau}(P) \cong \mathcal{F}_G(P,\mathfrak{g}) \cong \Gamma(\mathrm{Ad}\, P).$$

Indeed, to $f \in \mathcal{F}_G(P,\mathfrak{g})$ we can associate the G-invariant vertical vector field $\sigma(f) \in \mathfrak{gau}(P)$ given by

$$\sigma(f)(p) := f(p)_P(p). \tag{2.8}$$

The second isomorphism is given by the map (2.1). A direct computation shows that $\sigma : \mathcal{F}_G(P,\mathfrak{g}) \to \mathfrak{gau}(P)$ is a Lie algebra isomorphism, that is,

$$\sigma([f,g]) = [\sigma(f),\sigma(g)]_{\mathrm{L}}.$$

The transformation law of a connection $\mathcal{A}$ under $\varphi \in \mathcal{G}au(P)$ is given by

$$\varphi^*\mathcal{A} = \mathrm{Ad}_{\widehat{\varphi}^{-1}} \circ\mathcal{A} + TL_{\widehat{\varphi}^{-1}} \circ T\widehat{\varphi}. \tag{2.9}$$

If the principal bundle $P \to M$ is trivial, the automorphism group is the semidirect product of $\mathcal{D}(M)$ with $\mathcal{F}(M,G)$. To see this, note first that each $\varphi \in \mathcal{A}ut(P)$ is in this case of the form

$$\varphi(x,g) = (\overline{\varphi}(x), \overline{\overline{\varphi}}(x)g),$$

where $\overline{\varphi} \in \mathcal{D}(M)$, the diffeomorphism group of M, and $\overline{\overline{\varphi}} \in \mathcal{F}(M,G)$, the smooth G-valued functions on M. Thus the map $\varphi \in \mathcal{A}ut(P) \mapsto (\overline{\varphi},\overline{\overline{\varphi}}) \in \mathcal{D}(M) \times \mathcal{F}(M,G)$ is bijective. Second, the pair $\left(\overline{\varphi_1\circ\varphi_2}, \overline{\overline{\varphi_1\circ\varphi_2}}\right)$ corresponding to the product $\varphi_1 \circ \varphi_2$ is uniquely determined by the right hand side of the identity

$$(\varphi_1\circ\varphi_2)(x,g) = \left((\overline{\varphi_1}\circ\overline{\varphi_2})(x), (\overline{\overline{\varphi_1}}\circ\overline{\varphi_2})(x)\overline{\overline{\varphi_2}}(x)g\right).$$

This shows that the map $\varphi \in \mathcal{A}ut(P) \mapsto (\overline{\varphi},\overline{\overline{\varphi}}) \in \mathcal{D}(M) \circledS \mathcal{F}(M,G)$ is a group isomorphism, where the semidirect product is defined by the right action of $\mathcal{D}(M)$ by group automorphisms on $\mathcal{F}(M,G)$ given by $(\chi,\lambda) \in \mathcal{F}(M,G) \times \mathcal{D}(M) \mapsto \chi\circ\lambda \in \mathcal{F}(M,G)$. In particular, if $\psi \in \mathcal{G}au(P)$, then $\overline{\psi} = \mathrm{id}_M$ and we have $\psi(x,g) = (x, \overline{\overline{\psi}}(x)g)$, which shows that the map $\psi \in \mathcal{G}au(P) \mapsto \overline{\overline{\psi}} \in \mathcal{F}(M,G)$ is a group isomorphism.

The same considerations hold for the volume preserving case. We have shown hence that if the principal G-bundle $\pi : P \to M$ is trivial, then we have the group isomorphisms

$$\mathcal{A}ut(P) \simeq \mathcal{D}(M) \circledS \mathcal{G}au(P) \quad \text{and} \quad \mathcal{A}ut_\mu(P) \simeq \mathcal{D}_\mu(M) \circledS \mathcal{G}au(P)$$

and the corresponding Lie algebra isomorphisms

$$\mathfrak{aut}(P) \simeq \mathfrak{X}(M) \circledS \mathcal{F}(M, \mathfrak{g}) \quad \text{and} \quad \mathfrak{aut}_\mu(P) \simeq \mathfrak{X}_{\rm div}(M) \circledS \mathcal{F}(M, \mathfrak{g}).$$

Using the general formula for the Lie bracket associated the Lie algebra of a semidirect product of two groups (see formula (6.4.2) in [**18**], for example), we find that the (left) Lie bracket on $\mathfrak{aut}(P)$ and $\mathfrak{aut}_\mu(P)$ is

$$[(v, \theta), (w, \omega)]_{\rm L} = \left([v, w]_{\rm L}, \mathbf{d}\theta(w) - \mathbf{d}\omega(v) + [\theta, \omega]\right). \tag{2.10}$$

If the principal bundle $P \to M$ is not trivial, the situation is more involved. First, the sequence

$$0 \longrightarrow \mathfrak{gau}(P) \longrightarrow \mathfrak{aut}(P) \longrightarrow \mathfrak{X}(M) \longrightarrow 0$$

is exact. The second arrow is the inclusion and the third is the Lie algebra homomorphism given by $U \in \mathfrak{aut}(M) \mapsto [U] \in \mathfrak{X}(M)$ which is surjective because any $X \in \mathfrak{X}(M)$ is covered by its horizontal lift relative to some connection. Note, however, that the horizontal lift of vector fields relative to a connection is, in general, not a Lie algebra homomorphism since the bracket of two horizontally lifted vector fields has a vertical part. This is an indication that if $P \to M$ is non-trivial, then $\mathfrak{aut}(P)$ is, in general, not the semidirect product of $\mathfrak{X}(M)$ with $\mathfrak{gau}(P)$.

Second, at group level, the map $\mathcal{A}ut(P) \to \mathcal{D}(M)$ is not surjective, in general. For example, let $P = S^3 \subset \mathbb{R}^4$ be the unit sphere, thought of as the unit quaternions, and let $S^1 := \{x + y\mathbf{k} \mid x, y \in \mathbb{R}, x^2 + y^2 = 1\}$ act on P by $q \mapsto q(x + y\mathbf{k})$. The Hopf fibration map $\pi : q \in S^3 \mapsto q\mathbf{k}\overline{q} \in S^2$ defines a principal S^1-bundle over $M := S^2$. We shall prove that in this case the map $\mathcal{A}ut(P) \to \mathcal{D}(M)$ is not surjective.[1] Let $\eta \in \mathcal{D}(S^2)$ be the antipodal map whose degree is -1 and is hence not homotopic to the identity. If there were some $\varphi \in \mathcal{A}ut(S^3)$ descending to η, then φ would not have any fixed points and hence its degree would be one. By the Hopf Degree Theorem φ would then be homotopic to the identity which would imply that η was homotopic to the identity, a contradiction.

2.7. Duality. In this paper, we will identify the cotangent space $T^*_\varphi \mathcal{A}ut(P)$ with the space of G-invariant 1-forms on P along $\varphi \in \mathcal{A}ut(P)$. The duality pairing is

$$\langle \mathbf{M}_\varphi, U_\varphi \rangle := \int_M \mathbf{M}_\varphi(U_\varphi)\mu,$$

[1] We thank Marco Castrillón-López for this example.

where $\mathbf{M}_\varphi \in T^*_\varphi \mathcal{A}ut(P)$ and $U_\varphi \in T_\varphi \mathcal{A}ut(P)$. Note that in this formula we used the fact that $\mathbf{M}_\varphi(U_\varphi)$ is a smooth function on P that does not depend on the fiber variables and hence induces a unique smooth function on M which is then integrated using the volume form μ on M. In particular, we have $\mathfrak{aut}(P)^* = \Omega^1_G(P)$, the space of right-invariant 1-forms on P.

We identify the cotangent space $T^*_\varphi \mathcal{G}au(P)$ with the tangent space $T_\varphi \mathcal{G}au(P)$ via the duality

$$\langle U_\varphi, V_\varphi \rangle := \int_M \gamma \left(\widetilde{\mathcal{A}(U_\varphi)}, \widetilde{\mathcal{A}(V_\varphi)} \right) \mu, \tag{2.11}$$

for any principal connection $\mathcal{A}$ on P. Note that, since U_φ and V_φ are vertical, the pairing (2.11) does not depend on $\mathcal{A}$ since $\mathcal{A}(U_\varphi) = \sigma^{-1}(U_\varphi \circ \varphi^{-1}) \circ \varphi$ for any connection $\mathcal{A}$.

3. Equations for the fields

In this section, we give the Lagrangian and Hamiltonian formulations for the Yang–Mills fields in the vacuum. We will see that it is not possible to pass from one to the other by a simple Legendre transformation.

3.1. Lagrangian formulation of the Maxwell equations. On the Lagrangian side, the variables are the magnetic potential $A \in \Omega^1(M)$ and the electric potential $A_0 \in \mathcal{F}(M)$, where M is a three dimensional compact manifold without boundary. The Lagrangian is defined on the tangent bundle $T(\mathcal{F}(M) \times \Omega^1(M))$ and is given by

$$L(A_0, \dot{A}_0, A, \dot{A}) = \frac{1}{2} \int_M \|E\|^2 \mu - \frac{1}{2} \int_M \|B\|^2 \mu,$$

where $E := -\dot{A} + \mathbf{d}A_0$, $B := \mathbf{d}A$, and $\| \cdot \|$ is the norm associated to the Riemannian metric induced by g on the vector bundle $\Lambda^k M \to M$, for $k = 1, 2$. The Euler–Lagrange equations associated to L are

$$\delta E = 0 \quad \text{and} \quad \frac{\partial E}{\partial t} = \delta B. \tag{3.1}$$

The relations $E = -\dot{A} + \mathbf{d}A_0$ and $B = \mathbf{d}A$ give

$$\frac{\partial B}{\partial t} = -\mathbf{d}E \quad \text{and} \quad \mathbf{d}B = 0. \tag{3.2}$$

Using the vector field variables $\mathbf{E} := E^\sharp$ and $\mathbf{B} := (\star B)^\sharp$, where $\star : \Omega^k(M) \to \Omega^{3-k}(M)$, is the Hodge-star operator associated to the Riemannian metric g on M, we obtain the Maxwell equations in the vacuum

$$\begin{cases} \dfrac{\partial \mathbf{E}}{\partial t} = \operatorname{curl} \mathbf{B}, & \operatorname{div} \mathbf{E} = 0, \\ \dfrac{\partial \mathbf{B}}{\partial t} = -\operatorname{curl} \mathbf{E}, & \operatorname{div} \mathbf{B} = 0, \end{cases} \tag{3.3}$$

where curl : $\mathfrak{X}(M) \to \mathfrak{X}(M)$ is the operator $\operatorname{curl}(\mathbf{X}) := [\star(\mathbf{d}\mathbf{X}^\flat)]^\sharp$, for any $\mathbf{X} \in \mathfrak{X}(M)$.

Let us recall the classical argument that we can choose $A_0 = 0$. Assume that A_0' and A' satisfy Maxwell's equations (3.1) and (3.2). We search a function $\varphi \in \mathcal{F}(M)$ such that $A := A' + \mathbf{d}\varphi$ leaves equations (3.1) and (3.2) unchanged and $A_0 = 0$. Since these equations are second order, we have $\dot{A}'_0 = \partial A_0'/\partial t$ and $\dot{A}' = \partial A'/\partial t$. Let $E' := -\dot{A}' + \mathbf{d}A_0'$, $B' := \mathbf{d}A'$. The requirement is that $E' = E$ and $B' = B$. Therefore,

$$-\dot{A}' + \mathbf{d}A_0' = E' = E = -\dot{A} + \mathbf{d}A_0 = -\dot{A}' - \mathbf{d}\dot{\varphi} + \mathbf{d}A_0,$$

which is equivalent to $\mathbf{d}\dot{\varphi} = \mathbf{d}(A_0 - A_0')$ and hence it is sufficient to choose $A_0 := A_0' + \dot{\varphi}$. This shows that one can choose $A_0 = 0$ provided $\dot{\varphi} = -A_0'$. Note that equations (3.1) and (3.2) are unchanged under this transformation, as required.

We now recall the four dimensional formulation of the Maxwell equations. Consider the Lorentzian manifold (X, γ) given by $X = M \times \mathbb{R}$ and $\gamma := \tau_1^* g - \tau_2^* dt^2$, where $\tau_1 : X \to M$ and $\tau_2 : X \to \mathbb{R}$ are the natural projections and g is a Riemannian metric on M. Consider the 1-form G on X defined by $G := \tau_1^* A_t + \tau_1^* A_0 \wedge \tau_2^* dt$. We have

$$\mathbf{d}G = \tau_1^* \mathbf{d}A_t - \tau_1^* \dot{A}_t \wedge \tau_2^* dt + \mathbf{d}A_0 \wedge \tau_2^* dt = \tau_1^* B + \tau_1^* E \wedge \tau_2^* dt =: F,$$

and the Maxwell equations (3.3) can be simply written as (see, e.g., [**19**, Section 22.4])

$$\mathbf{d}F = 0 \quad \text{and} \quad \delta F = 0.$$

In a general slicing of space-time, not just $M \times \mathbb{R}$, the derivation of these equations and much more information can be found, for example, in [**20**, **21**].

The Legendre transformation associated to the Maxwell Lagrangian L is not bijective. Thus, it is not possible to pass in the usual way from the Lagrangian to the Hamiltonian formulation by the Legendre transformation. This degeneracy is typical of relativistic field theories and is resolved by the Dirac theory of constraints; see, for example, [**20**, **21**] and references therein. In the next subsection, we directly generate the Hamiltonian formulation for the Maxwell equations.

3.2. Hamiltonian formulation of the Maxwell equations. On the Hamiltonian side (see [**5**]), the configuration space variable is the magnetic potential $A \in \Omega^1(M)$. The Hamiltonian is defined on the cotangent bundle $T^*\Omega^1(M) \simeq \Omega^1(M) \times \Omega^1(M)$, where the cotangent space at any point A is identified with $\Omega^1(M)$ using the natural L^2-pairing, and is given by

$$H(A, Y) = \frac{1}{2}\int_M \|E\|^2 \mu + \frac{1}{2}\int_M \|B\|^2 \mu$$

for $E := -Y$ and $B := \mathbf{d}A$. Hamilton's equations are

$$\frac{\partial B}{\partial t} = -\mathbf{d}E \quad \text{and} \quad \frac{\partial E}{\partial t} = \delta B,$$

and the relation $B = \mathbf{d}A$ gives

$$\mathbf{d}B = 0.$$

To obtain the last equation $\delta E = 0$, we use the invariance of the Hamiltonian under gauge transformations. The action of the gauge group $\mathcal{F}(M)$ on $\Omega^1(M)$ is given by

$$\mathcal{F}(M) \times \Omega^1(M) \longrightarrow \Omega^1(M), \quad (\varphi, A) \longmapsto A + \mathbf{d}\varphi, \tag{3.4}$$

and is Hamiltonian. The associated momentum map is

$$\mathbf{J} : T^*\Omega^1(M) \longrightarrow \mathcal{F}(M)^* \simeq \mathcal{F}(M), \quad \mathbf{J}(A, Y) = \delta Y,$$

where $\mathcal{F}(M)^*$ is identified with $\mathcal{F}(M)$ using the natural L^2-pairing. So the condition $\mathbf{J}(A, Y) = 0$ gives the fourth Maxwell equation $\delta E = 0$.

Note that in the Hamiltonian formulation we have used only the configuration variable A, whereas in the Lagrangian formulation the configuration space consisted of pairs (A_0, A). As we have seen, the variable A_0 can be set equal to zero without any effect on Maxwell's equations. Note also that the Euler–Lagrange equation $\delta E = 0$ was obtained from the variation of the Lagrangian relative to A_0, whereas in the Hamiltonian set-up this equation appears as a conservation law for the gauge group action (3.4).

3.3. Generalization to any principal bundle. We now generalize the previous formulations to the case of a G-principal bundle $P \to M$ over an arbitrary compact boundaryless manifold M. We will show that if M is three dimensional, $G = S^1$, and the bundle is trivial, then we recover the Maxwell equations.

Lagrangian formulation. The Lagrangian $L : T(\mathcal{F}_G(P, \mathfrak{g}) \times \mathcal{C}onn(P)) \to \mathbb{R}$ is defined by

$$L(\mathcal{A}_0, \dot{\mathcal{A}}_0, \mathcal{A}, \dot{\mathcal{A}}) = \frac{1}{2}\int_M \|E\|^2 \mu - \frac{1}{2}\int_M \|B\|^2 \mu, \tag{3.5}$$

where

(1) $E := \widetilde{\mathcal{E}} \in \Omega^1(M, \operatorname{Ad} P)$ is the $\operatorname{Ad} P$-valued 1-form associated, through the map (2.1), to the "electric part" $\mathcal{E} \in \overline{\Omega^1}(P, \mathfrak{g})$ of the Yang–Mills field, given by

$$\mathcal{E} := -\dot{\mathcal{A}} + \mathbf{d}^{\mathcal{A}}\mathcal{A}_0 \in \overline{\Omega^1}(P, \mathfrak{g});$$

(2) $B := \widetilde{\mathcal{B}} \in \Omega^2(M, \operatorname{Ad} P)$ is the $\operatorname{Ad} P$-valued 2-form associated, through the map (2.1), to the "magnetic part" $\mathcal{B} \in \overline{\Omega^2}(P, \mathfrak{g})$ of the Yang–Mills field, given by the curvature

$$\mathcal{B} := \mathbf{d}^{\mathcal{A}}\mathcal{A} \in \overline{\Omega^2}(P, \mathfrak{g});$$

(3) $\|\cdot\|$ is the norm associated to the metric $g\gamma$ on the vector bundles $\Lambda^k(M, \operatorname{Ad} P) \to M$, for $k = 1, 2$.

The Euler–Lagrange equations associated to L are

$$\delta^{\mathcal{A}}\mathcal{E} = 0 \quad \text{and} \quad \frac{\partial \mathcal{E}}{\partial t} + [\mathcal{A}_0, \mathcal{E}] = \delta^{\mathcal{A}}\mathcal{B}.$$

Indeed, using the L^2 pairing

$$\langle \alpha, \beta \rangle = \int_M (g\gamma)(\alpha, \beta)\mu, \quad \alpha, \beta \in \Omega^k(M, \operatorname{Ad} P), \tag{3.6}$$

we can identify the cotangent bundles of $\mathcal{F}_G(P, \mathfrak{g})$ and $\mathcal{C}onn(P)$ with their tangent bundles. Using formulas (2.3), (2.4), and the identity

$$\left.\frac{d}{dt}\right|_{t=0} \mathbf{d}^{\mathcal{A}+t\mathcal{C}}(\mathcal{A} + t\mathcal{C}) = \mathbf{d}^{\mathcal{A}}\mathcal{C}, \tag{3.7}$$

where $\mathcal{A} \in \mathcal{C}onn(P)$, $\mathcal{C} \in T_{\mathcal{A}}\mathcal{C}onn(P) = \overline{\Omega^1}(P, \mathfrak{g})$, we get

$$\frac{\partial L}{\partial \mathcal{A}_0} = \delta^{\mathcal{A}}\mathcal{E}, \quad \frac{\partial L}{\partial \dot{\mathcal{A}}_0} = 0, \quad \frac{\partial L}{\partial \mathcal{A}} = -\delta^{\mathcal{A}}\mathcal{B} + [\mathcal{A}_0, \mathcal{E}], \quad \frac{\partial L}{\partial \dot{\mathcal{A}}} = -\mathcal{E}.$$

Thus, the Euler–Lagrange equations

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{\mathcal{A}}_0} - \frac{\partial L}{\partial \mathcal{A}_0} = 0 \quad \text{and} \quad \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{\mathcal{A}}} - \frac{\partial L}{\partial \mathcal{A}} = 0$$

become

$$\delta^{\mathcal{A}}\mathcal{E} = 0 \quad \text{and} \quad \frac{\partial \mathcal{E}}{\partial t} + [\mathcal{A}_0, \mathcal{E}] = \delta^{\mathcal{A}}\mathcal{B},$$

as stated above.

The relations $\mathcal{E} := -\dot{\mathcal{A}} + \mathbf{d}^{\mathcal{A}}\mathcal{A}_0$ and $\mathcal{B} := \mathbf{d}^{\mathcal{A}}\mathcal{A}$ give the equations

$$\frac{\partial \mathcal{B}}{\partial t} + [\mathcal{A}_0, \mathcal{B}] = -\mathbf{d}^{\mathcal{A}}\mathcal{E} \quad \text{and} \quad \mathbf{d}^{\mathcal{A}}\mathcal{B} = 0.$$

Indeed, for the first equality we have, using (3.7) and Lemma 2.1,

$$\dot{\mathcal{B}} = \mathbf{d}^{\mathcal{A}}\dot{\mathcal{A}} = -\mathbf{d}^{\mathcal{A}}\mathcal{E} + \mathbf{d}^{\mathcal{A}}\mathbf{d}^{\mathcal{A}}\mathcal{A}_0 = -\mathbf{d}^{\mathcal{A}}\mathcal{E} + [\mathcal{B}, \mathcal{A}_0].$$

The second equality is the Bianchi identity. Summarizing, we get the system

$$\begin{cases} \dfrac{\partial \mathcal{E}}{\partial t} + [\mathcal{A}_0, \mathcal{E}] = \delta^{\mathcal{A}}\mathcal{B}, & \delta^{\mathcal{A}}\mathcal{E} = 0, \\[2ex] \dfrac{\partial \mathcal{B}}{\partial t} + [\mathcal{A}_0, \mathcal{B}] = -\mathbf{d}^{\mathcal{A}}\mathcal{E}, & \mathbf{d}^{\mathcal{A}}\mathcal{B} = 0. \end{cases} \tag{3.8}$$

To recover Maxwell's equations, we take a trivial S^1-principal bundle $P = M \times S^1$. Then $\operatorname{Ad} P = M \times \mathbb{R}$ and $\Omega^k(M, \operatorname{Ad} P) = \Omega^k(M)$. Since the structure group S^1 of the principal bundle P is Abelian, the covariant differential does not depend on the connection, that is, $\mathbf{d}^{\mathcal{A}} = \mathbf{d}$. We obtain the following identifications.

(1) From the equality $\mathcal{E} = -\dot{\mathcal{A}} + \mathbf{d}^{\mathcal{A}}\mathcal{A}_0$, we obtain that the electric field $E := \widetilde{\mathcal{E}} \in \Omega^1(M)$ is given by

$$E = -\dot{A} + \mathbf{d}A_0,$$

where $A_0 := \widetilde{\mathcal{A}_0} \in \mathcal{F}(M)$ and $\dot{A} := \widetilde{\dot{\mathcal{A}}} \in \Omega^1(M)$.

(2) From the equality $\mathcal{B} = \mathbf{d}^{\mathcal{A}}\mathcal{A}$, we obtain that the magnetic field $B := \widetilde{\mathcal{B}} \in \Omega^2(M)$ is given by

$$B = \mathbf{d}A,$$

where $A \in \Omega^1(M)$ is given by $A := \overline{\mathcal{A}}$ (see equation (2.7)).

Returning to the general case, let us show, as in the case of Maxwell's equations, that we can choose $\mathcal{A}_0 = 0$. Assume that $\mathcal{A}_0'$ and $\mathcal{A}'$ satisfy equations (3.8). We search a $\varphi \in \mathcal{G}au(P)$ such that $\mathcal{A} := \varphi^*\mathcal{A}'$ leaves the equations (3.8) unchanged and $\mathcal{A}_0 = 0$. Since these equations are second order, we have $\dot{\mathcal{A}}_0' = \partial\mathcal{A}_0'/\partial t$ and $\dot{\mathcal{A}}' = \partial\mathcal{A}'/\partial t$. Let $\mathcal{E}' := -\dot{\mathcal{A}}' + \mathbf{d}^{\mathcal{A}'}\mathcal{A}_0'$, $\mathcal{B}' := \mathbf{d}^{\mathcal{A}}\mathcal{A}'$. Since $\mathcal{B} = \varphi^*\mathcal{B}'$, the requirement is that $\mathcal{E} = \varphi^*\mathcal{E}'$. Therefore, emphasizing the time-dependence, we have the equivalences

$$\begin{aligned} & -\dot{\mathcal{A}}_t + \mathbf{d}^{\mathcal{A}_t}\mathcal{A}_{0t} = \varphi_t^*\left(-\dot{\mathcal{A}}_t' + \mathbf{d}^{\mathcal{A}_t'}\mathcal{A}_{0t}'\right) \\ \iff\quad & -\frac{\partial}{\partial t}(\varphi_t^*\mathcal{A}_t') + \mathbf{d}^{\mathcal{A}_t}\mathcal{A}_{0t} = -\varphi_t^*\frac{\partial}{\partial t}\mathcal{A}_t' + \mathbf{d}^{\mathcal{A}_t}\varphi_t^*\mathcal{A}_{0t}' \\ \iff\quad & \mathbf{d}^{\mathcal{A}_t}\mathcal{A}_{0t} = \left.\frac{\partial}{\partial t}\right|_{t=s}\varphi_s^*\mathcal{A}_t' + \mathbf{d}^{\mathcal{A}_t}\varphi_t^*\mathcal{A}_{0t}'. \end{aligned} \tag{3.9}$$

Taking the time derivative of (2.9), we get

$$\left.\frac{d}{dt}\right|_{t=0}\psi_t^*\mathcal{A} = \mathbf{d}^{\mathcal{A}}\left(\sigma^{-1}(\dot{\psi}_0)\right), \tag{3.10}$$

for any smooth curve $\psi_t \in \mathcal{G}au(P)$ such that $\psi_0 = \mathrm{id}$. Therefore we conclude that

$$\begin{aligned} \left.\frac{\partial}{\partial s}\right|_{s=t}\varphi_s^*\mathcal{A}_t' = \left.\frac{\partial}{\partial s}\right|_{s=t}\varphi_t^*(\varphi_s\circ\varphi_t^{-1})^*\mathcal{A}_t' &= \varphi_t^*\mathbf{d}^{\mathcal{A}_t'}\left(\sigma^{-1}(\dot{\varphi}_t\circ\varphi_t^{-1})\right) \\ &= \mathbf{d}^{\mathcal{A}_t}\varphi_t^*\left(\sigma^{-1}(\dot{\varphi}_t\circ\varphi_t^{-1})\right). \end{aligned}$$

So (3.9) is equivalent to $\mathbf{d}^{\mathcal{A}_t}\mathcal{A}_{0t} = \mathbf{d}^{\mathcal{A}_t}\varphi_t^*\left(\sigma^{-1}(\dot{\varphi}_t\circ\varphi_t^{-1}) + \mathcal{A}_{0t}'\right)$ and hence it is sufficient to choose $\mathcal{A}_{0t} := \varphi_t^*\left(\sigma^{-1}(\dot{\varphi}_t\circ\varphi_t^{-1}) + \mathcal{A}_{0t}'\right)$ in order to get $\mathcal{E} = \varphi^*\mathcal{E}'$. Thus one can choose $\mathcal{A}_{0t} = 0$ provided that $\dot{\varphi}_t\circ\varphi_t^{-1} = -\sigma(\mathcal{A}_{0t}')$.

A direct computation shows that equations (3.8) are unchanged under this transformation, as required.

Let $(X = M \times \mathbb{R}, \gamma)$ be the Lorentzian manifold given in Section 3.1. Let $\overline{P} := P \times \mathbb{R}$ and define the free G-action $\overline{\Phi}_g(p,t) := (\Phi_g(p), t)$ on $\overline{P}$. We get the principal G-bundle $\overline{\pi} : \overline{P} := P \times \mathbb{R} \to X$. From $\mathcal{A}_t \in \mathcal{C}onn(P)$ and $\mathcal{A}_0 \in \mathcal{F}_G(P, \mathfrak{g})$, we can construct the 1-form $\mathcal{C} \in \Omega^1(\overline{P}, \mathfrak{g})$

$$\mathcal{C} := \tau^*\mathcal{A}_t + \tau^*\mathcal{A}_0 \wedge (\tau_2 \circ \overline{\pi})^* dt,$$

where $\tau : \overline{P} \to P$ is the natural projection and $\tau_2 : X \to \mathbb{R}$ is the projection on the second factor. One can check that $\mathcal{C}$ is a connection on $\overline{P}$ since $\tau^*\mathcal{A} \in \mathcal{C}onn(\overline{P})$ and $\tau^*\mathcal{A}_0 \wedge (\tau_2 \circ \overline{\pi})^* dt \in \overline{\Omega^1}(\overline{P}, \mathfrak{g})$.

Finally we obtain

$$\mathbf{d}^{\mathcal{C}}\mathcal{C} = \tau^*\mathcal{B} + \tau^*\mathcal{E} \wedge (\tau_2 \circ \overline{\pi})^* dt =: \mathcal{F},$$

and equations (3.8) are equivalent to the Yang–Mills equations together with the Bianchi identity (see, e.g., [**22**–**24**])

$$\delta^{\mathcal{C}}\mathcal{F} = 0 \quad \text{and} \quad \mathbf{d}^{\mathcal{C}}\mathcal{F} = 0.$$

Hamiltonian formulation. As in the electromagnetic case, the configuration space variable is the magnetic potential $\mathcal{A} \in \mathcal{C}onn(P)$ and the Hamiltonian is defined on the cotangent bundle $T^*\mathcal{C}onn(P)$ by

$$H(\mathcal{A}, \mathcal{Y}) = \frac{1}{2}\int_M \|E\|^2 \mu + \frac{1}{2}\int_M \|B\|^2 \mu,$$

where:

(1) $E := \widetilde{\mathcal{E}} \in \Omega^1(M, \operatorname{Ad} P)$ is the $\operatorname{Ad} P$-valued 1-form associated, through the map (2.1), to the "electric part" $\mathcal{E} \in \overline{\Omega^1}(P, \mathfrak{g})$ of the Yang–Mills field, given by
$$\mathcal{E} := -\mathcal{Y} \in \overline{\Omega^1}(P, \mathfrak{g}),$$

(2) $B := \widetilde{\mathcal{B}} \in \Omega^2(M, \operatorname{Ad} P)$ is the $\operatorname{Ad} P$-valued 2-form associated, through the map (2.1), to the "magnetic part" $\mathcal{B} \in \overline{\Omega^2}(P, \mathfrak{g})$ of the Yang–Mills field, given by the curvature
$$\mathcal{B} := \mathbf{d}^{\mathcal{A}}\mathcal{A} \in \overline{\Omega^2}(P, \mathfrak{g}).$$

As before, we identify the cotangent bundle of $\mathcal{C}onn(P)$ with the tangent bundle, using the L^2 pairing (3.6).

Hamilton's equations associated to H are

$$\frac{\partial \mathcal{B}}{\partial t} = -\mathbf{d}^{\mathcal{A}}\mathcal{E} \quad \text{and} \quad \frac{\partial \mathcal{E}}{\partial t} = \delta^{\mathcal{A}}\mathcal{B},$$

and the Bianchi identity gives

$$\mathbf{d}^{\mathcal{A}}\mathcal{B} = 0.$$

To obtain the last equation, we use the invariance of the Hamiltonian under gauge transformations. The action of $\varphi \in \mathcal{G}au(P)$ on $\mathcal{A} \in \mathcal{C}onn(P)$ is $\varphi^*\mathcal{A}$ and the cotangent-lift of this action is $(\varphi^*\mathcal{A}, \varphi^*\mathcal{Y})$. Under this action, $\mathcal{E}$ and $\mathcal{B}$ are transformed into $\varphi^*\mathcal{E}$ and $\varphi^*\mathcal{B}$, so H is gauge-invariant. The momentum mapping associated to this Hamiltonian action is

$$\mathbf{J} : T^*\mathcal{C}onn(P) \longrightarrow \mathfrak{gau}(P)^* \simeq \mathfrak{gau}(P), \quad \mathbf{J}(\mathcal{A}, \mathcal{Y}) = \sigma(\delta^{\mathcal{A}}\mathcal{Y}),$$

so the conservation law $\mathbf{J}(\mathcal{A}, \mathcal{Y}) = 0$ gives the last equation

$$\delta^{\mathcal{A}}\mathcal{E} = 0.$$

Note that we identify $\mathfrak{gau}(P)^*$ with $\mathfrak{gau}(P)$ via the L^2 pairing (3.6).

4. Equations for the particles

We consider the evolution of a non-relativistic Yang–Mills charged particle of mass m in a given Yang–Mills field.

Fix a connection $\mathcal{A} \in \mathcal{C}onn(P)$ and an equivariant function $\mathcal{A}_0 \in \mathcal{F}_G(P, \mathfrak{g})$. The Yang–Mills field is given by its electric part $\mathcal{E} = \mathbf{d}^{\mathcal{A}}\mathcal{A}_0$ and its magnetic part $\mathcal{B} = \mathbf{d}^{\mathcal{A}}\mathcal{A}$.

Consider the right-invariant Lagrangian $L : TP \to \mathbb{R}$, given by

$$L(u_p) = \frac{m}{2} g_{[p]}(T\pi(u_p), T\pi(u_p)) + \frac{1}{2}\gamma(\mathcal{A}(u_p) + \mathcal{A}_0(p), \mathcal{A}(u_p) + \mathcal{A}_0(p)).$$

Since L is G-invariant, it induces a Lagrangian on $(TP)/G$. We use the identification of $(TP)/G$ with $TM \oplus \operatorname{Ad} P$ through the connection dependent vector bundle isomorphism (see [**8**])

$$\Psi_{\mathcal{A}} : \frac{TP}{G} \longrightarrow TM \oplus \operatorname{Ad} P, \quad \Psi_{\mathcal{A}}([u_p]) := (T_p\pi(u_p), [p, \mathcal{A}(u_p)]_G).$$

The reduced Lagrangian on $TM \oplus \operatorname{Ad} P$ is given by

$$l(v_x, \xi_x) = \frac{m}{2} g_x(v_x, v_x) + \frac{1}{2}\gamma_x(\xi_x + A_0(x), \xi_x + A_0(x)),$$

where $A_0 \in \Gamma(\operatorname{Ad} P)$ is associated to $\mathcal{A}_0$ via the map (2.1). By Lagrangian reduction (see [**8**]), $p(t) \in P$ is a solution of the Euler–Lagrange equations for L if and only if $x(t) := \pi(p(t)) \in M$ and $\xi(t) := [p(t), \mathcal{A}(\dot{p}(t))]_G \in \operatorname{Ad} P_{x(t)}$ are solutions of the Lagrange-Poincaré equations

$$\begin{cases} \dfrac{\partial l}{\partial x}(\dot{x}, \xi) - \dfrac{D^g}{dt}\dfrac{\partial l}{\partial v}(\dot{x}, \xi) = \left\langle \dfrac{\partial l}{\partial \xi}(\dot{x}, \xi), B(\dot{x}, .) \right\rangle \\ \dfrac{D^{\mathcal{A}}}{dt}\dfrac{\partial l}{\partial \xi}(\dot{x}, \xi) = -\operatorname{ad}^*_{\xi}\dfrac{\partial l}{\partial \xi}(\dot{x}, \xi), \end{cases}$$

where D^g/dt and $D^{\mathcal{A}}/dt$ denote the covariant derivatives induced by g on T^*M and by $\mathcal{A}$ on $(\operatorname{Ad} P)^*$, respectively,

$$\frac{\partial l}{\partial v}(v_x, \xi_x) \in T_x^*M \quad \text{and} \quad \frac{\partial l}{\partial \xi}(v_x, \xi_x) \in (\operatorname{Ad} P)_x^*$$

are usual fiber derivatives of l in the vector bundles TM and Ad P, and

$$\frac{\partial l}{\partial x}(v_x, \xi_x) \in T_x^*M$$

is the partial covariant derivative of l relative to the Levi–Civita connection on M and the principal connection $\mathcal{A}$ on P. See [**8**] for details regarding the Lagrange–Poincaré equations.

In terms of the functional derivatives

$$\frac{\delta l}{\delta x}(v_x, \xi_x), \quad \frac{\delta l}{\delta v}(v_x, \xi_x) \in T_xM, \quad \text{and} \quad \frac{\delta l}{\delta \xi}(v_x, \xi_x) \in (\operatorname{Ad} P)_x,$$

defined similarly, the Lagrange–Poincaré equations become

$$\begin{cases} \dfrac{\delta l}{\delta x}(\dot{x}, \xi) - \dfrac{D^g}{dt}\dfrac{\delta l}{\delta v}(\dot{x}, \xi) = \gamma_x\left(\dfrac{\delta l}{\delta \xi}(\dot{x}, \xi), B(\dot{x}, \cdot)\right)^{\sharp} \\ \dfrac{D^{\mathcal{A}}}{dt}\dfrac{\delta l}{\delta \xi}(\dot{x}, \xi) = \left[\xi, \dfrac{\delta l}{\delta \xi}(\dot{x}, \xi)\right]_x, \end{cases}$$

where $[\cdot,\cdot]_x$ is the bracket of elements in $(\operatorname{Ad} P)_x$ and $\frac{D^g}{dt}$ and $\frac{D^{\mathcal{A}}}{dt}$ denote the covariant derivatives on TM and $\operatorname{Ad} P$, respectively. Using that

$$\begin{aligned} &\frac{\delta l}{\delta x}(v_x, \xi_x) = \frac{\delta \bar{l}}{\delta x}(v_x, \xi_x) + \gamma_x(\xi_x + A_0(x), \mathbf{d}^{\mathcal{A}} A_0(\,.\,))^{\sharp}, \\ &\frac{\delta l}{\delta v}(v_x, \xi_x) = \frac{\delta \bar{l}}{\delta v}(v_x, \xi_x), \\ &\frac{\delta l}{\delta \xi}(v_x, \xi_x) = \xi_x + A_0(x), \\ &\frac{\delta \bar{l}}{\delta x}(\dot{x}, \xi) - \frac{D^g}{dt}\frac{\delta \bar{l}}{\delta v}(\dot{x}, \xi) = -\frac{D^g}{dt}\dot{x}(t), \end{aligned}$$

where

$$\bar{l}(v_x, \xi_x) = \frac{m}{2} g_x(v_x, v_x),$$

we obtain that the trajectory $x(t) := \pi(p(t)) \in M$ and the charge

$$q(t) = [p(t), \mathcal{A}(\dot{p}(t)) + \mathcal{A}_0(p(t))]_G = \frac{\delta l}{\delta \xi}(\dot{x}(t), \xi(t)) \in (\operatorname{Ad} P)_{x(t)},$$

are solutions of

$$\begin{cases} m\dfrac{D^g}{dt}\dot{x}(t) = \gamma_{x(t)}(q(t), E(\cdot) + B(\cdot, \dot{x}(t)))^\sharp, \\ \dfrac{D^{\mathcal{A}}}{dt}q(t) + [A_0(x(t)), q(t)]_{x(t)} = 0. \end{cases}$$

The first line is the non-Abelian Lorentz equation and the second line represents the covariant conservation of charge. These equations are the non-relativistic Wong equations [**8**, **25**, **26**].

In the case of the trivial S^1-bundle $P = M \times S^1$, recall that $\operatorname{Ad} P = M \times \mathbb{R}$. The Lagrangian is

$$L(v_x, \theta, \dot{\theta}) = \frac{m}{2} g_x(v_x, v_x) + \frac{1}{2}(A(x)(v_x) + \dot{\theta} + A_0(x))^2,$$

where $A := \overline{\mathcal{A}} \in \Omega^1(M)$ and $A_0 := \widetilde{\mathcal{A}_0}$. We have $(TP)/S^1 = TM \oplus \operatorname{Ad} P = TM \times \mathbb{R}$, and $\Psi_{\mathcal{A}}(v_x, \dot{\theta}) = (v_x, A(x)(v_x) + \dot{\theta})$. So the reduced Lagrangian is

$$l(v_x, \xi) = \frac{m}{2} g_x(v_x, v_x) + \frac{1}{2}(\xi + A_0(x))^2.$$

By the Lagrange–Poincaré reduction [**8**], we obtain that $p(t) = (x(t), \theta(t))$ is a solution of the Euler–Lagrange equations if and only if

$$\begin{cases} m\dfrac{D^g}{dt}\dot{x}(t) = q(t)(E(\cdot) + B(\cdot, \dot{x}(t)))^\sharp, \\ \dfrac{d}{dt}q(t) = 0, \end{cases}$$

where $q(t) := A(\dot{x}(t)) + \dot{\theta}(t) + A_0(x(t))$ is the charge. If $\dim M = 3$, in terms of the vector fields $\mathbf{E} := E^\sharp$ and $\mathbf{B} = (\star B)^\sharp$ and using that the charge $q(t) = q$ is conserved, the previous system becomes simply the Lorentz force law

$$m\frac{D^g}{dt}\dot{x}(t) = q(\mathbf{E} + \dot{x}(t) \times \mathbf{B}),$$

describing the motion of a charged particle of mass m in a fixed electromagnetic field.

We remark that when the variable $\mathcal{A}_0$ is absent, the Lagrangian is given by the Kaluza–Klein metric,

$$L(u_p) = \frac{1}{2}K_{\mathcal{A}}(p)(u_p, u_p) = \frac{m}{2} g_{[p]}(T_p\pi(u_p), T\pi_p(u_p)) + \gamma(\mathcal{A}(u_p), \mathcal{A}(u_p)).$$

In this case, the Legendre transformation is invertible and the corresponding Hamiltonian on T^*P is

$$H(\alpha_p) = \frac{1}{2}K^*_{\mathcal{A}}(p)(\alpha_p, \alpha_p),$$

where $K^*_{\mathcal{A}}$ is the dual metric on T^*P, defined by

$$K^*_{\mathcal{A}}(p)\left(K_{\mathcal{A}}(p)(u_p, \cdot), K_{\mathcal{A}}(p)(v_p, \cdot)\right) := K_{\mathcal{A}}(p)(u_p, v_p).$$

5. Lagrangian formulation of Euler–Yang–Mills

We begin by recalling some facts about the Euler–Poincaré reduction for semidirect products [**7**, **8**, **27**]. Let $\rho : G \to \mathcal{A}ut(V)$ denote a *right* Lie group representation of G in the vector space V. As a set, the semidirect product $S = G \circledS V$ is the Cartesian product $S = G \times V$ whose group multiplication is given by

$$(g_1, v_1)(g_2, v_2) = (g_1 g_2, v_2 + \rho_{g_2}(v_1)).$$

The Lie algebra of S is the semidirect product Lie algebra, $\mathfrak{s} = \mathfrak{g} \circledS V$, whose bracket has the expression

$$\mathrm{ad}_{(\xi_1, v_1)}(\xi_2, v_2) = [(\xi_1, v_1), (\xi_2, v_2)] = ([\xi_1, \xi_2], v_1\xi_2 - v_2\xi_1),$$

where $v\xi$ denotes the induced action of $\mathfrak{g}$ on V, that is,

$$v\xi := \left.\frac{d}{dt}\right|_{t=0} \rho_{\exp(t\xi)}(v) \in V.$$

From the expression for the Lie bracket, it follows that for $(\xi, v) \in \mathfrak{s}$ and $(\mu, a) \in \mathfrak{s}^*$, we have

$$\mathrm{ad}^*_{(\xi,v)}(\mu, a) = (\mathrm{ad}^*_\xi \mu + v \diamond a, a\xi),$$

where $a\xi \in V^*$ and $v \diamond a \in \mathfrak{g}^*$ are given, respectively, by

$$a\xi := \left.\frac{d}{dt}\right|_{t=0} \rho^*_{\exp(-t\xi)}(a) \quad \text{and} \quad \langle v \diamond a, \xi\rangle_{\mathfrak{g}} := -\langle a\xi, v\rangle_V,$$

where $\langle\cdot,\cdot\rangle_{\mathfrak{g}} : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ and $\langle\cdot,\cdot\rangle_V : V^* \times V \to \mathbb{R}$ are the duality parings.

Lagrangian semidirect product theory with parameter.

- Let Q be a manifold on which G acts trivially and assume that we have a function $L : TG \times TQ \times V^* \to \mathbb{R}$ which is right G-invariant.
- In particular, if $a_0 \in V^*$, define the Lagrangian $L_{a_0} : TG \times TQ \to \mathbb{R}$ by $L_{a_0}(v_g, u_q) := L(v_g, u_q, a_0)$. Then L_{a_0} is right invariant under the lift to $TG \times TQ$ of the right action of G_{a_0} on $G \times Q$, where G_{a_0} is the isotropy group of a_0.
- Right G-invariance of L permits us to define $l : \mathfrak{g} \times TQ \times V^* \to \mathbb{R}$ by
$$l(T_gR_{g^{-1}}(v_g), u_q, \rho^*_g(a_0)) = L(v_g, u_q, a_0).$$
- For a curve $g(t) \in G$, let $\xi(t) := TR_{g(t)^{-1}}(\dot{g}(t))$ and define the curve $a(t)$ as the unique solution of the linear differential equation with time-dependent coefficients $\dot{a}(t) = -a(t)\xi(t)$ with initial condition $a(0) = a_0$. Its solution can be written as $a(t) = \rho^*_{g(t)}(a_0)$.

Theorem 5.1. *The following are equivalent:*

i. *Hamilton's variational principle holds:*

$$\delta \int_{t_1}^{t_2} L_{a_0}(g(t), \dot{g}(t), q(t), \dot{q}(t))dt = 0,$$

for variations of g and q with fixed endpoints.

ii. $(g(t), q(t))$ *satisfies the Euler–Lagrange equations for L_{a_0} on $G \times Q$.*

iii. *The constrained variational principle*

$$\delta \int_{t_1}^{t_2} l(\xi(t), q(t), \dot{q}(t), a(t))dt = 0,$$

holds on $\mathfrak{g} \times Q$, upon using variations of the form

$$\delta\xi = \frac{\partial \eta}{\partial t} - [\xi, \eta], \quad \delta a = -a\eta,$$

where $\eta(t) \in \mathfrak{g}$ vanishes at the endpoints and $\delta q(t)$ is unrestricted except for vanishing at the endpoints.

iv. *The following system of Euler–Poincaré equations (with a parameter) coupled with Euler–Lagrange equations holds on $\mathfrak{g} \times TQ \times V^*$:*

$$\frac{\partial}{\partial t}\frac{\delta l}{\delta \xi} = -\operatorname{ad}^*_\xi \frac{\delta l}{\delta \xi} + \frac{\delta l}{\delta a} \diamond a, \tag{5.1}$$

and

$$\frac{\partial}{\partial t}\frac{\partial l}{\partial \dot{q}} - \frac{\partial l}{\partial q} = 0.$$

Note that the Euler–Poincaré equation (5.1) can be written, in weak form, as

$$\frac{d}{dt}\mathbf{D}l(\xi)(\eta) = -\mathbf{D}l(\xi)([\xi, \eta]) + \left\langle \frac{\delta l}{\delta a} \diamond a, \eta \right\rangle_{\mathfrak{g}}, \quad \text{for all } \eta \in \mathfrak{g}, \tag{5.2}$$

where $\mathbf{D}$ denotes the Fréchet derivative. This formulation will be useful below.

Ideal compressible adiabatic fluid. Before treating the Yang–Mills fluid, we apply the preceding theory to the case of the compressible adiabatic fluid. For this particular case we choose $G = \mathcal{D}(M)$ and $V = \mathcal{F}(M) \times \mathcal{F}(M)$ (in this case Q is absent). We identify the dual $\mathcal{F}(M)^*$ with $\mathcal{F}(M)$ via the natural L^2 pairing. The action of $\eta \in \mathcal{D}(M)$ on $(\rho, s) \in V^*$ is

$$(\rho, s) \longmapsto ((J\eta)(\rho \circ \eta), s \circ \eta),$$

where $J\eta$ is the Jacobian determinant of η, ρ is the density of the fluid, and s is its specific entropy. As usual, we treat the mass density ρ as a density on M and the entropy s as a function on M; this is why in the previous formula the action of the diffeomorphism group is different on the two components.

The Lagrangian is given by

$$L_{(\rho,s)}(u_\eta)=\frac{1}{2}\int_M \rho g(u_\eta,u_\eta)\mu-\int_M \rho e(\rho(J\eta)^{-1},s)\mu, \tag{5.3}$$

where e is the fluid's specific internal energy. Application of part **iv** in Theorem 5.1 gives the equations of motion

$$\begin{cases}\dfrac{\partial v}{\partial t}+\nabla_v v=-\dfrac{1}{\rho}\operatorname{grad}p,\\ \dfrac{\partial \rho}{\partial t}+\operatorname{div}(\rho v)=0,\\ \dfrac{\partial s}{\partial t}+\mathbf{d}s(v)=0,\end{cases} \tag{5.4}$$

where the pressure is given by $p=\rho^2\frac{\partial e}{\partial \rho}(\rho,s)$.

5.1. Yang–Mills ideal fluid. In the case of the Yang-Mills fluid, we choose $G=\mathcal{A}ut(P)$, $Q=\mathcal{F}_G(P,\mathfrak{g})\times\mathcal{C}onn(P)$, and $V=\mathcal{F}(M)\times\mathcal{F}(M)$. As before, we use the notations $\varphi\in\mathcal{A}ut(P)$, $(\mathcal{A}_0,\mathcal{A})\in\mathcal{F}_G(P,\mathfrak{g})\times\mathcal{C}onn(P)$, and $(\rho,s)\in V^*$. The action of φ on (ρ,s) is given by

$$(\rho,s)\longmapsto((J\overline{\varphi})(\rho\circ\overline{\varphi}),s\circ\overline{\varphi}),$$

where $\overline{\varphi}\in\mathcal{D}(M)$ is the map induced on the base M by φ. From the expressions of the Lagrangian (5.3) and of the Lagrangians for the fields and particles given in Sections 3 and 4, it follows that the Lagrangian for the Yang–Mills ideal fluid is defined on the tangent bundle $T(\mathcal{A}ut(P)\times\mathcal{F}_G(P,\mathfrak{g})\times\mathcal{C}onn(P))$ by

$$\begin{aligned}L_{(\rho,s)}(U_\psi,\mathcal{A}_0,\dot{\mathcal{A}}_0,\mathcal{A},\dot{\mathcal{A}})=&\frac{1}{2}\int_M\rho g([U_\psi],[U_\psi])\mu\\ &+\frac{1}{2}\int_M\rho\|\,(\mathcal{A}(U_\psi)+\mathcal{A}_0\circ\psi)\tilde{}\,\|^2\mu\\ &-\int_M\rho e(\rho(J\overline{\psi})^{-1},s)\mu\\ &+\frac{1}{2}\int_M\|E\|^2\mu-\frac{1}{2}\int_M\|B\|^2\mu,\end{aligned} \tag{5.5}$$

where $[U_\psi]\in T_{\overline{\psi}}\mathcal{D}(M)$ is such that $T\pi\circ U_\psi=[U_\psi]\circ\pi$. Note that $\mathcal{A}(U_\psi)+\mathcal{A}_0\circ\psi\in\mathcal{F}_G(P,\mathfrak{g})$, so we can consider the section $(\mathcal{A}(U_\psi)+\mathcal{A}_0\circ\psi)\tilde{}\in\Gamma(\operatorname{Ad}P)$ and its L^2 norm $\|\,(\mathcal{A}(U_\psi)+\mathcal{A}_0\circ\psi)\tilde{}\,\|$ relative to the Riemannian metric γ_x.

The two last terms of (5.5) are given as in the Lagrangian (3.5). Roughly speaking, this Lagrangian has the following structure

$$\begin{Bmatrix}\text{Integration of the Lagrangian}\\ \text{for the particles}\end{Bmatrix} - \{\text{Internal energy}\} + \begin{Bmatrix}\text{Lagrangian for the}\\ \text{Yang–Mills fields}\end{Bmatrix}.$$

Note that L verifies the invariance property needed for an application of Theorem 5.1, that is, L is invariant under the right action of $\varphi \in \mathcal{A}ut(P)$

$$(U_\psi, \rho, s) \longmapsto (U_\psi \circ \varphi, (J\overline{\varphi})(\rho \circ \overline{\varphi}), s \circ \overline{\varphi}).$$

Indeed, we have $[U_\psi \circ \varphi] = [U_\psi] \circ \overline{\varphi}$, so the invariance of the first term follows by a change of variable in the integral. The invariance of the second integral follows from the fact that $\mathcal{A}(U_\psi) + \mathcal{A}_0 \circ \psi \in \mathcal{F}_G(P, \mathfrak{g})$ and that for $f, g \in \mathcal{F}_G(P, \mathfrak{g})$ and $\varphi \in \mathcal{A}ut(P)$ we have

$$\gamma(\widetilde{f \circ \varphi}, \widetilde{g \circ \varphi}) = \gamma(\widetilde{f}, \widetilde{g}) \circ \overline{\varphi},$$

as functions on M.

The reduced Lagrangian l on $\mathfrak{aut}(P) \times T(\mathcal{F}_G(P, \mathfrak{g}) \times \mathcal{C}onn(P)) \times (\mathcal{F}(M)^* \times \mathcal{F}(M)^*)$ has the expression

$$\begin{aligned}(5.6)\quad l(U, \mathcal{A}_0, \dot{\mathcal{A}}_0, \mathcal{A}, \dot{\mathcal{A}}, \rho, s) = {} & \frac{1}{2}\int_M \rho g([U], [U])\mu \\ & + \frac{1}{2}\int_M \rho \| (\mathcal{A}(U) + \mathcal{A}_0)^{\sim} \|^2 \mu - \int_M \rho e(\rho, s)\mu \\ & + \frac{1}{2}\int_M \|E\|^2 \mu - \frac{1}{2}\int_M \|B\|^2 \mu\end{aligned}$$

and the Euler–Poincaré equations in weak form are

$$(5.7)\qquad \frac{\partial}{\partial t}\mathbf{D}l(U)(V) = -\mathbf{D}l(U)([U, V]_{\mathrm{L}}) + \left\langle \frac{\delta l}{\delta(\rho, s)} \diamond (\rho, s), V \right\rangle,$$

for all $V \in \mathfrak{aut}(P)$. We now compute these equations.

Recall that the (left) Lie bracket on the Lie algebra $\mathfrak{aut}(P)$ is

$$[U, V]_{\mathrm{L}} = \mathrm{ad}_U V = -[U, V]_{\mathrm{JL}},$$

where $[\,,\,]_{\mathrm{JL}}$ denotes the usual Jacobi–Lie bracket of vector fields. The following lemma gives the decomposition of $[U, V]_{\mathrm{L}}$ into the horizontal and vertical parts.

Lemma 5.2. *Let $\mathcal{A}$ be a connection on the principal bundle P and let $U, V \in \mathfrak{aut}(P)$. Then we have*

$$\begin{aligned}(5.8)\qquad [U, V]_{\mathrm{L}} = \sigma\Big(& [\mathcal{A}(U), \mathcal{A}(V)] + \mathbf{d}^{\mathcal{A}}(\mathcal{A}(U))(V) \\ & - \mathbf{d}^{\mathcal{A}}(\mathcal{A}(V))(U) + \mathcal{B}(U, V)\Big) + \mathrm{hor}([U, V]_{\mathrm{L}}),\end{aligned}$$

where hor *denotes the horizontal part relative to the connection* $\mathcal{A}$. *In particular we have the equality*

$$\mathcal{A}\left([U,V]_{\mathrm{L}}\right)=[\mathcal{A}(U),\mathcal{A}(V)]+\mathbf{d}^{\mathcal{A}}(\mathcal{A}(U))(V)-\mathbf{d}^{\mathcal{A}}(\mathcal{A}(V))(U)+\mathcal{B}(U,V).$$

Proof. First note that using the Cartan structure equations and the fact that $\mathcal{B}\in\overline{\Omega^2}(P,\mathfrak{g})$, we have

$$\mathbf{d}\mathcal{A}(\operatorname{hor}U,\sigma(\mathcal{A}(V)))=\mathcal{B}(\operatorname{hor}U,\sigma(\mathcal{A}(V)))-[\mathcal{A}(\operatorname{hor}U),\mathcal{A}(\sigma(\mathcal{A}(V)))]=0.$$

We also have

$$\begin{aligned}\mathbf{d}\mathcal{A}(\operatorname{hor}U,\sigma(\mathcal{A}(V)))&=\mathbf{d}(\mathcal{A}(\sigma(\mathcal{A}(V))))(\operatorname{hor}U)-\mathbf{d}(\mathcal{A}(\operatorname{hor}U))(\sigma(\mathcal{A}(V)))\\&\qquad-\mathcal{A}([\operatorname{hor}U,\sigma(\mathcal{A}(V))]_{\mathrm{JL}})\\&=\mathbf{d}(\mathcal{A}(V))(\operatorname{hor}U)+\mathcal{A}([\operatorname{hor}U,\sigma(\mathcal{A}(V))]_{\mathrm{L}}).\end{aligned}$$

These formulas prove that

$$\mathbf{d}(\mathcal{A}(V))(\operatorname{hor}U)=-\mathcal{A}([\operatorname{hor}U,\sigma(\mathcal{A}(V))]_{\mathrm{L}}).\tag{5.9}$$

We now compute the Lie bracket $[U,V]_{\mathrm{L}}$. By decomposing U and V into their vertical and horizontal parts, that is, we write $U=\sigma(\mathcal{A}(U))+\operatorname{hor}U$ and $V=\sigma(\mathcal{A}(V))+\operatorname{hor}V$, we obtain four terms. The first term is

$$\begin{aligned}[\operatorname{hor}U,\operatorname{hor}V]_{\mathrm{L}}&=\sigma(\mathcal{A}([\operatorname{hor}U,\operatorname{hor}V]_{\mathrm{L}}))+\operatorname{hor}[\operatorname{hor}U,\operatorname{hor}V]_{\mathrm{L}}\\&=\sigma(\mathcal{B}(U,V))+\operatorname{hor}[U,V]_{\mathrm{L}},\end{aligned}$$

where we used the equalities

$$\mathcal{B}(U,V)=-\mathcal{A}([\operatorname{hor}U,\operatorname{hor}V]_{\mathrm{JL}})$$

and

$$T\pi\circ[U,V]_{\mathrm{L}}=[[U],[V]]_{\mathrm{L}}\circ\pi.$$

Since $[\sigma(\mathcal{A}(U)),\operatorname{hor}V]_{\mathrm{L}}$ is vertical (apply the formula above), the second term is

$$\begin{aligned}[\sigma(\mathcal{A}(U)),\operatorname{hor}V]_{\mathrm{L}}&=\sigma(\mathcal{A}([\sigma(\mathcal{A}(U)),\operatorname{hor}V]_{\mathrm{L}}))=\mathbf{d}(\mathcal{A}(U))(\operatorname{hor}V)\\&=\mathbf{d}^{\mathcal{A}}(\mathcal{A}(U))(V),\end{aligned}$$

by formula (5.9). There is an analogous formula for the third term $[\operatorname{hor}U,\sigma(\mathcal{A}(V))]_L$. Using the Lie algebra isomorphism $\sigma:\mathcal{F}_G(P,\mathfrak{g})\to\mathfrak{gau}(P)$ defined in (2.8), the fourth term is

$$[\sigma(\mathcal{A}(U)),\sigma(\mathcal{A}(V))]=\sigma([\mathcal{A}(U),\mathcal{A}(V)]_{\mathrm{L}}).$$

Summing these four terms we obtain the desired formula (5.8). □

Inspired by the Kaluza-Klein metric (2.5), we define on $\mathfrak{aut}(P)$ a nondegenerate bilinear form given by

$$\langle U,V\rangle_{\mathcal{A}}:=\int_M g([U],[V])\mu+\int_M\gamma\left(\widetilde{\mathcal{A}(U)},\widetilde{\mathcal{A}(V)}\right)\mu.$$

Therefore, we have

(5.10)

$$
\begin{aligned}
\langle W, [U,V]_{\mathrm{L}}\rangle_{\mathcal{A}} &= \int_M g([W],[[U,V]_{\mathrm{L}}])\mu + \int_M \gamma\left(\widetilde{\mathcal{A}(W)}, \widetilde{\mathcal{A}([U,V]_{\mathrm{L}})}\right)\mu \\
&= \int_M g([W],[[U],[V]]_{\mathrm{L}})\mu + \int_M \gamma\left(\widetilde{\mathcal{A}(W)}, [\widetilde{\mathcal{A}(U),\mathcal{A}(V)}]\right)\mu \\
&\quad + \int_M \gamma\left(\widetilde{\mathcal{A}(W)}, \widetilde{\mathbf{d}^{\mathcal{A}}(\mathcal{A}(U))[V]}\right)\mu \\
&\quad - \int_M \gamma\left(\widetilde{\mathcal{A}(W)}, \widetilde{\mathbf{d}^{\mathcal{A}}(\mathcal{A}(V))[U]}\right)\mu \\
&\quad + \int_M \gamma\left(\widetilde{\mathcal{A}(W)}, \widetilde{\mathcal{B}}([U],[V])\right)\mu \\
&= \int_M g\left(\mathrm{ad}^{\dagger}_{[U]}[W] + \gamma\left(\widetilde{\mathcal{A}(W)}, \widetilde{\mathbf{d}^{\mathcal{A}}(\mathcal{A}(U))(\cdot)} + \widetilde{\mathcal{B}}([U],\cdot)\right)^{\sharp}, [V]\right)\mu \\
&\quad + \int_M \gamma\left([\widetilde{\mathcal{A}(W),\mathcal{A}(U)}] + \widetilde{\mathbf{d}^{\mathcal{A}}(\mathcal{A}(W))[U]}\right. \\
&\quad \left. + \operatorname{div}([U])\widetilde{\mathcal{A}(W)},\ \widetilde{\mathcal{A}(V)}\right)\mu,
\end{aligned}
$$

where in the last equality, $\mathrm{ad}^{\dagger}$ denotes the L^2 adjoint of ad relative to the metric g, and $\sharp$ is the index raising operator associated to g. Note that for $u, w \in \mathfrak{X}(M)$, $\mathrm{ad}^{\dagger}$ is given by

$$
\mathrm{ad}^{\dagger}_u w = \nabla_u w + \nabla u^{\mathrm{T}} \cdot w + w \operatorname{div} u. \tag{5.11}
$$

In the second summand of the last equality in (5.10) we used the following lemma.

Lemma 5.3. *Consider an* Ad*-invariant inner product* γ *on* $\mathfrak{g}$ *and the induced vector bundle metric on* $\operatorname{Ad}P$*, also denoted by* γ*. Then for* $v \in \mathfrak{X}(M)$ *and* $f, g \in \mathcal{F}_G(P,\mathfrak{g})$*, we have*

$$
\int_M \gamma\left(\widetilde{\mathbf{d}^{\mathcal{A}} f}(v), \widetilde{g}\right)\mu = -\int_M \gamma\left(\widetilde{f}, \widetilde{\mathbf{d}^{\mathcal{A}} g}(v)\right)\mu - \int_M \gamma(\widetilde{f},\widetilde{g})(\operatorname{div} v)\mu.
$$

Proof. One verifies that for any $v \in \mathfrak{X}(M)$, we have

$$
\mathbf{d}\left(\gamma(\widetilde{f},\widetilde{g})\right)(v) = \gamma\left(\widetilde{\mathbf{d}^{\mathcal{A}} f}(v), \widetilde{g}\right) + \gamma\left(\widetilde{f}, \widetilde{\mathbf{d}^{\mathcal{A}} g}(v)\right).
$$

Integrating over M gives the result. Indeed, denoting by h the real valued function $\gamma(\widetilde{f},\widetilde{g})$, we obtain

$$
\int_M \mathbf{d}h(v)\mu = \int_M \operatorname{div}(hv)\mu - \int_M h(\operatorname{div} v)\mu = -\int_M h(\operatorname{div} v)\mu,
$$

by the divergence theorem. $\square$

Using (5.10) and the formula

$$\begin{aligned}\mathbf{D}l(U)(V) &= \frac{1}{2}\int_M g(\rho[U],[V])\mu + \frac{1}{2}\int_M \gamma\left(\rho\left(\mathcal{A}(U)+\mathcal{A}_0\right)\widetilde{\ }, \widetilde{\mathcal{A}(V)}\right)\mu\\ &= \langle \rho U + \sigma(\mathcal{A}_0), V\rangle_{\mathcal{A}},\end{aligned}$$

we obtain

(5.12)

$$\begin{aligned}\mathbf{D}l(U)([U,V]_{\mathrm{L}}) &= \langle \rho U + \sigma(\mathcal{A}_0), [U,V]_{\mathrm{L}}\rangle_{\mathcal{A}}\\ &= \int_M g\left(\operatorname{ad}^\dagger_{[U]}\rho[U] + \gamma\left(\widetilde{\mathcal{Q}}, \widetilde{\mathbf{d}^{\mathcal{A}}(\mathcal{A}(U))}(.) + \widetilde{\mathcal{B}}([U],.)\right)^\sharp, [V]\right)\mu\\ &\quad + \int_M \gamma\left(\widetilde{[\mathcal{Q},\mathcal{A}(U)]} + \widetilde{\mathbf{d}^{\mathcal{A}}\mathcal{Q}(U)} + \operatorname{div}([U])\widetilde{\mathcal{Q}}, \widetilde{\mathcal{A}(V)}\right)\mu,\end{aligned}$$

where

$$\mathcal{Q} := \rho(\mathcal{A}(U)+\mathcal{A}_0) \in \mathcal{F}_G(P,\mathfrak{g}) \tag{5.13}$$

is the charge density.

On the other hand we have, using the notations $A_0 := \widetilde{\mathcal{A}_0} \in \Gamma(\operatorname{Ad}P)$, $\dot{A} := \widetilde{\dot{\mathcal{A}}} \in \Omega^1(M,\operatorname{Ad}P)$, and $Q := \widetilde{\mathcal{Q}} \in \Gamma(\operatorname{Ad}P)$,

$$\begin{aligned}\frac{\partial}{\partial t}\mathbf{D}l(U)(V) &= \frac{d}{dt}\left[\int_M g\left(\rho[U],[V]\right)\mu + \int_M \gamma\left(Q, \widetilde{\mathcal{A}(V)}\right)\mu\right]\\ &= \int_M g\left(\frac{\partial}{\partial t}\rho[U],[V]\right)\mu + \int_M \gamma\left(\dot{Q}, \widetilde{\mathcal{A}(V)}\right)\mu\\ &\quad + \int_M \gamma\left(Q, \dot{A}[V]\right)\mu\\ &= \int_M g\left(\rho\frac{\partial}{\partial t}[U] - \operatorname{div}(\rho[U])[U] + \gamma\left(Q,\dot{A}(.)\right)^\sharp, [V]\right)\mu\\ &\quad + \int_M \gamma\left(\dot{Q}, \widetilde{\mathcal{A}(V)}\right)\mu,\end{aligned}$$

where we used the equation $\dot{\rho} = -\operatorname{div}(\rho[U])$. Using the equalities

$$\begin{aligned}\frac{\delta l}{\delta\rho} &= \frac{1}{2}g([U],[U]) + \frac{1}{2}\gamma\left(\frac{Q}{\rho},\frac{Q}{\rho}\right) - e - \rho\frac{\partial e}{\partial\rho},\\ \frac{\delta l}{\delta s} &= -\rho\frac{\partial e}{\partial s},\\ \frac{\delta l}{\delta(\rho,s)}\diamond(\rho,s) &= \rho\operatorname{grad}\frac{\delta l}{\delta\rho} - \frac{\delta l}{\delta s}\operatorname{grad}s,\\ &= \rho\nabla[U]^{\mathrm{T}}\cdot[U] + \frac{1}{2}\rho\operatorname{grad}\gamma\left(\frac{Q}{\rho},\frac{Q}{\rho}\right) - \operatorname{grad}\left(\rho^2\frac{\partial e}{\partial\rho}\right),\end{aligned}$$

equation (5.7) yields the system

$$
\begin{cases}
\rho\dfrac{\partial}{\partial t}[U]+\rho\nabla_{[U]}[U]+\gamma\left(Q,\dot{A}(.)+\widetilde{\mathbf{d}^{\mathcal{A}}(\mathcal{A}(U))}(\cdot)+B([U],\cdot)\right)^{\sharp}\\
\quad=\dfrac{1}{2}\rho\operatorname{grad}\gamma\left(\dfrac{Q}{\rho},\dfrac{Q}{\rho}\right)-\operatorname{grad}p\\
\dfrac{\partial}{\partial t}\mathcal{Q}+[\mathcal{Q},\mathcal{A}(U)]+\mathbf{d}^{\mathcal{A}}\mathcal{Q}(U)+\operatorname{div}([U])\mathcal{Q}=0.
\end{cases}
\tag{5.14}
$$

Denoting $\mathcal{S}:=\frac{\mathcal{Q}}{\rho}=\mathcal{A}(U)+\mathcal{A}_0$, several applications of Lemma 5.3, give

$$
\int_M\gamma\left(S,\widetilde{\mathbf{d}^{\mathcal{A}}(\mathcal{A}(U))}(v)\right)\mu=\frac{1}{2}\int_M\mathbf{d}(\gamma(S,S))(v)\mu-\int_M\gamma\left(S,\widetilde{\mathbf{d}^{\mathcal{A}}\mathcal{A}_0}(v)\right)\mu,
$$

for all $v\in\mathfrak{X}(M)$. So we get

$$
\gamma\left(Q,\widetilde{\mathbf{d}^{\mathcal{A}}(\mathcal{A}(U))}(\cdot)\right)^{\sharp}=\frac{1}{2}\rho\operatorname{grad}\gamma\left(\frac{Q}{\rho},\frac{Q}{\rho}\right)-\gamma\left(Q,\widetilde{\mathbf{d}^{\mathcal{A}}\mathcal{A}_0}(\cdot)\right)^{\sharp}.
$$

With this formula and the equality

$$
[\mathcal{Q},\mathcal{A}(U)]=[\mathcal{A}_0,\mathcal{Q}],
$$

the system (5.14) is equivalent to

$$
\begin{cases}
\dfrac{\partial}{\partial t}[U]+\nabla_{[U]}[U]=\dfrac{1}{\rho}\gamma\left(Q,-\dot{A}(\cdot)+\widetilde{\mathbf{d}^{\mathcal{A}}\mathcal{A}_0}(\cdot)+B(\cdot,[U])\right)^{\sharp}\\
-\dfrac{1}{\rho}\operatorname{grad}p\\
\dfrac{\partial}{\partial t}\mathcal{Q}+[\mathcal{A}_0,\mathcal{Q}]+\mathbf{d}^{\mathcal{A}}\mathcal{Q}(U)+\operatorname{div}([U])\mathcal{Q}=0,
\end{cases}
\tag{5.15}
$$

which is the same as

$$
\begin{cases}
\dfrac{\partial v}{\partial t}+\nabla_v v=\dfrac{1}{\rho}\gamma\left(Q,E(\cdot)+B(\cdot,v)\right)^{\sharp}-\dfrac{1}{\rho}\operatorname{grad}p,\\
\dfrac{\partial Q}{\partial t}+[A_0,Q]+\nabla_v^{\mathcal{A}}Q+Q\operatorname{div}v=0,
\end{cases}
\tag{5.16}
$$

where $v:=[U]\in\mathfrak{X}(M)$ is the Eulerian velocity.

We compute now the Euler–Lagrange equations relative to the Lagrangian (5.6) and the variables $(\mathcal{A}_0,\mathcal{A})$. The computations are similar to those done in Paragraph 3.3. We find

$$
\frac{\partial l}{\partial\mathcal{A}_0}=\delta^{\mathcal{A}}\mathcal{E}+\mathcal{Q},\quad\frac{\partial l}{\partial\dot{\mathcal{A}}_0}=0,\quad\frac{\partial l}{\partial\mathcal{A}}=-\delta^{\mathcal{A}}\mathcal{B}+[\mathcal{A}_0,\mathcal{E}]+\mathcal{Q}\otimes\pi^*v^\flat,\quad\frac{\partial l}{\partial\dot{\mathcal{A}}}=-\mathcal{E},
$$

where $\mathcal{Q}\otimes\pi^*v^\flat\in\overline{\Omega^1}(P,\mathfrak{g})$ is given by

$$\left(\mathcal{Q}\otimes\pi^*v^\flat\right)(u_p):=\mathcal{Q}(p)\left(\pi^*v^\flat(u_p)\right)=\mathcal{Q}(p)g_x(v(x),T_p\pi(u_p)),\quad x=\pi(p).$$

Note that we have $\widetilde{\mathcal{Q}\otimes\pi^*v^\flat}=Q\otimes v^\flat$, the 1-form on M, with values in $\operatorname{Ad}P$, given by

$$\left(Q\otimes v^\flat\right)(u_x)=Q(x)v^\flat(u_x)=Q(x)g_x(v(x),u_x)\in(\operatorname{Ad}P)_x.$$

For the computation of the partial derivative $\frac{\partial l}{\partial\mathcal{A}}$, we use the identity

$$\gamma(Q,\dot{A}(v))=(g\gamma)(Q\otimes v^\flat,\dot{A}).$$

The resulting Euler–Lagrange equations are

$$\delta^{\mathcal{A}}\mathcal{E}=-\mathcal{Q}\quad\text{and}\quad\frac{\partial\mathcal{E}}{\partial t}+[\mathcal{A}_0,\mathcal{E}]=\delta^{\mathcal{A}}\mathcal{B}-\mathcal{Q}\otimes\pi^*v^\flat.$$

As in Paragraph 3.3, the relations $\mathcal{E}=-\dot{\mathcal{A}}+\mathbf{d}^{\mathcal{A}}\mathcal{A}_0$ and $\mathcal{B}=\mathbf{d}^{\mathcal{A}}\mathcal{A}$ give the equations

$$\frac{\partial\mathcal{B}}{\partial t}+[\mathcal{A}_0,\mathcal{B}]=-\mathbf{d}^{\mathcal{A}}\mathcal{E}\quad\text{and}\quad\mathbf{d}^{\mathcal{A}}\mathcal{B}=0.$$

Summarizing, we have proved the following theorem, which is one of the main results of this paper.

Theorem 5.4. *Let $(\psi,\mathcal{A}_0,\mathcal{A})$ be a curve in $\mathcal{A}ut(P)\times\mathcal{F}_G(P,\mathfrak{g})\times\mathcal{C}onn(P)$ and consider the curve $(U,\mathcal{A}_0,\mathcal{A}):=(\dot{\psi}\circ\psi^{-1},\mathcal{A}_0,\mathcal{A})$ in $\mathfrak{aut}(P)\times\mathcal{F}_G(P,\mathfrak{g})\times\mathcal{C}onn(P)$. Then $(\psi,\mathcal{A}_0,\mathcal{A})$ is a solution of the Euler–Lagrange equations associated to the Lagrangian $L_{(\rho_0,s_0)}$ given in* (5.5) *if and only if $(U,\mathcal{A}_0,\mathcal{A})$ is a solution of the Euler–Yang–Mills equations:*

$$\text{(5.17)}\quad\begin{cases}\dfrac{\partial v}{\partial t}+\nabla_v v=\dfrac{1}{\rho}\gamma\left(Q,E(\cdot)+B(\cdot,v)\right)^\sharp-\dfrac{1}{\rho}\operatorname{grad}p,\\[2mm]\dfrac{\partial\rho}{\partial t}+\operatorname{div}(\rho v)=0,\quad\rho(0)=\rho_0,\quad\dfrac{\partial s}{\partial t}+\mathbf{d}s(v)=0,\quad s(0)=s_0,\\[2mm]\dfrac{\partial Q}{\partial t}+[A_0,Q]+\nabla^{\mathcal{A}}_v Q+Q\operatorname{div}v=0,\\[2mm]\dfrac{\partial\mathcal{E}}{\partial t}+[\mathcal{A}_0,\mathcal{E}]=\delta^{\mathcal{A}}\mathcal{B}-\mathcal{Q}\otimes\pi^*v^\flat,\quad\delta^{\mathcal{A}}\mathcal{E}=-\mathcal{Q},\\[2mm]\dfrac{\partial\mathcal{B}}{\partial t}+[\mathcal{A}_0,\mathcal{B}]=-\mathbf{d}^{\mathcal{A}}\mathcal{E},\quad\mathbf{d}^{\mathcal{A}}\mathcal{B}=0,\end{cases}$$

where

$$\begin{aligned}
&p := \rho^2 \frac{\partial e}{\partial \rho}(\rho, s), \quad v := [U] \in \mathfrak{X}(M),\\
&\mathcal{E} := -\dot{\mathcal{A}} + \mathbf{d}^{\mathcal{A}} \mathcal{A}_0 \in \overline{\Omega^1}(P, \mathfrak{g}) \quad \text{and} \quad E := \widetilde{\mathcal{E}} \in \Omega^1(M, \operatorname{Ad} P),\\
&\mathcal{B} := \mathbf{d}^{\mathcal{A}} \mathcal{A} \quad \text{and} \quad B := \widetilde{\mathcal{B}},\\
&\mathcal{Q} := \rho(\mathcal{A}(U) + \mathcal{A}_0) \quad \text{and} \quad Q = \widetilde{\mathcal{Q}}.
\end{aligned}$$

Corollary 5.5. *In the case of the trivial bundle $P = M \times S^1$ and assuming that the fluid is composed of particles of mass m and charge q, we obtain the Euler–Maxwell equations*

$$(5.18) \quad \begin{cases}
\dfrac{\partial v}{\partial t} + \nabla_v v = \dfrac{q}{m}(\mathbf{E} + v \times \mathbf{B}) - \dfrac{1}{\rho} \operatorname{grad} p,\\
\dfrac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) = 0, \quad \rho(0) = \rho_0, \quad \dfrac{\partial s}{\partial t} + \mathbf{d}s(v) = 0, \quad s(0) = s_0,\\
\dfrac{\partial \mathbf{E}}{\partial t} = \operatorname{curl} \mathbf{B} - \dfrac{q}{m}\rho v, \quad \dfrac{\partial \mathbf{B}}{\partial t} = -\operatorname{curl} \mathbf{E},\\
\operatorname{div} \mathbf{E} = \dfrac{q}{m}\rho, \quad \operatorname{div} \mathbf{B} = 0,
\end{cases}$$

where

$$\mathbf{E} := E^\sharp \quad \text{and} \quad \mathbf{B} := (\star B)^\sharp.$$

Proof. If we define $Q_t = \rho_t \frac{q_t}{m}$, the equation for Q in (5.17) becomes

$$0 = \frac{\partial q_t}{\partial t} + \mathbf{d}q_t(v) = \frac{d}{dt} q_t(x(t)),$$

where $x(t)$ is the trajectory of the particle starting at $x(0)$. Since all particles have the same charge $q \in \mathbb{R}$ by hypothesis, we conclude that $q(t, x)$ is a constant. Therefore, the equation for Q in (5.17) disappears. It is easily seen that the other equations become the ones in (5.18). □

We end this section by examining more carefully the case of a trivial principal bundle $P = M \times G$. We use the fact, already pointed out in the introduction, that in this case the automorphism group is a semidirect product of two groups.

In the trivial bundle case, we have a connection independent L^2 pairing on $\mathfrak{aut}(P)$, given by

$$\langle (m, \nu), (v, \theta) \rangle = \int_M g(m, v)\mu + \int_M \gamma(\nu, \theta)\mu.$$

Using this pairing, the expression (2.10) for the Lie bracket on the semidirect product Lie algebra, the expression (5.11), and integration by parts, we obtain the following expression for $\mathrm{ad}^\dagger$:

$$\begin{aligned}\mathrm{ad}^\dagger_{(v,\theta)}(m,\nu) &= \Big(\mathrm{ad}^\dagger_v m + \gamma(\nu, \mathbf{d}\theta(\cdot))^\sharp, \nu \operatorname{div} v + \mathbf{d}\nu(v) + [\nu,\theta]\Big)\\ &= \Big(\nabla_v m + \nabla v^{\mathrm{T}} \cdot m + m \operatorname{div} v + \gamma(\nu, \mathbf{d}\theta(\cdot))^\sharp, \nu \operatorname{div} v\\ &\qquad + \mathbf{d}\nu(v) + [\nu,\theta]\big). \end{aligned} \tag{5.19}$$

The reduced Lagrangian

$$l : \mathfrak{aut}(P) \times T(\mathcal{F}_G(P,\mathfrak{g}) \times \mathcal{C}onn(P)) \times (\mathcal{F}(M)^* \times \mathcal{F}(M)^*) \longrightarrow \mathbb{R}$$

is

$$\begin{aligned} l(v,\theta,\mathcal{A}_0,\dot{\mathcal{A}}_0,\mathcal{A},\dot{\mathcal{A}},\rho,s) &= \frac{1}{2}\int_M \rho g(v,v)\mu + \frac{1}{2}\int_M \rho\|\overline{\mathcal{A}}(v)+\theta+A_0\|^2\mu\\ &\quad - \int_M \rho e(\rho,s)\mu + \frac{1}{2}\int_M \|E\|^2\mu - \frac{1}{2}\int_M \|B\|^2\mu. \end{aligned} \tag{5.20}$$

and we have

$$\frac{\delta l}{\delta v} = \rho(v + \gamma(\overline{\mathcal{A}}(v)+\theta+A_0, \overline{\mathcal{A}}(\cdot))^\sharp) \quad \text{and} \quad \frac{\delta l}{\delta \theta} = \rho\left(\overline{\mathcal{A}}(v)+\theta+A_0\right).$$

The Euler–Poincaré equations are

$$\frac{\partial}{\partial t}\left(\frac{\delta l}{\delta v}, \frac{\delta l}{\delta \theta}\right) = -\mathrm{ad}^\dagger_{(v,\theta)}\left(\frac{\delta l}{\delta v}, \frac{\delta l}{\delta \theta}\right) + \frac{\delta l}{\delta(\rho,s)} \diamond (\rho,s),$$

and a long direct computation gives, as expected, the system

$$\begin{cases} \dfrac{\partial v}{\partial t} + \nabla_v v = \dfrac{1}{\rho}\gamma\left(Q, E(\cdot) + B(\cdot,v)\right)^\sharp - \dfrac{1}{\rho}\operatorname{grad} p,\\ \dfrac{\partial Q}{\partial t} + [\overline{\mathcal{A}}(v) + A_0, Q] + \mathbf{d}Q(v) + Q\operatorname{div} v = 0, \end{cases} \tag{5.21}$$

where

$$Q := \frac{\delta l}{\delta \theta} = \rho\left(\overline{\mathcal{A}}(v)+\theta+A_0\right) \in \mathcal{F}(M,\mathfrak{g}), \quad E \in \Omega^1(M,\mathfrak{g}), \quad B \in \Omega^2(M,\mathfrak{g}).$$

5.2. The incompressible and homogeneous case. In the incompressible case, we choose $G = \mathcal{A}ut_\mu(P)$, the Lie group of all automorphisms $\varphi \in \mathcal{A}ut(P)$ such that $\overline{\varphi} \in \mathcal{D}_\mu(M)$. Since the fluid is homogeneous, the advected variables ρ and s are absent. Therefore, we can use the standard Euler–Poincaré reduction with parameters $(\mathcal{A}_0, \mathcal{A}) \in \mathcal{F}_G(P,\mathfrak{g}) \times \mathcal{C}onn(P)$ (take $V = 0$ in the semidirect theory). The Lagrangian for the incompressible homogeneous Yang–Mills ideal fluid is defined on the tangent bundle $T(\mathcal{A}ut_\mu(P) \times \mathcal{F}_G(P,\mathfrak{g}) \times \mathcal{C}onn(P))$ and is given by

$$L(U_\psi, \mathcal{A}_0, \dot{\mathcal{A}}_0 \mathcal{A}, \dot{\mathcal{A}}) = \frac{1}{2}\int_M g([U_\psi],[U_\psi])\mu + \frac{1}{2}\int_M \|\,(\mathcal{A}(U_\psi) + \mathcal{A}_0 \circ \psi)^{\sim}\|^2\mu + \frac{1}{2}\int_M \|E\|^2\mu - \frac{1}{2}\int_M \|B\|^2\mu. \tag{5.22}$$

The computations of the Euler–Poincaré equations are similar to those done in the compressible case, except that we have $\operatorname{div}([U]) = 0$ and we must replace formula (5.12) by formula

$$\begin{aligned}\mathbf{D}l(U)([U,V]_L) &= \int_M g\left(\mathrm{P}_e\left(\operatorname{ad}^\dagger_{[U]}[U] + \gamma\left(\widetilde{\mathcal{Q}}, \mathbf{d}^{\mathcal{A}}\widetilde{(\mathcal{A}(U))}(.) + \widetilde{\mathcal{B}}([U],\cdot)\right)^\sharp\right), [V]\right)\mu \\ &\quad + \int_M \gamma\left([\widetilde{\mathcal{Q}, \mathcal{A}(U)}] + \widetilde{\mathbf{d}^{\mathcal{A}}\mathcal{Q}(U)}, \widetilde{\mathcal{A}(V)}\right)\mu,\end{aligned}$$

where $\mathrm{P}_e : \mathfrak{X}(M) \to \mathfrak{X}_{\mathrm{div}}(M)$ is the projector associated to the L^2 orthogonal Hodge decomposition

$$\mathfrak{X}(M) = \mathfrak{X}_{\mathrm{div}}(M) \oplus \operatorname{grad}(\mathcal{F}(M)).$$

We finally get the following result.

Theorem 5.6. *Let $(\psi, \mathcal{A}_0, \mathcal{A})$ be a curve in $\mathcal{A}ut_\mu(P) \times \mathcal{F}_G(P,\mathfrak{g}) \times \mathcal{C}onn(P)$ and consider the curve $(U, \mathcal{A}_0, \mathcal{A}) := (\dot{\psi}\circ\psi^{-1}, \mathcal{A}_0, \mathcal{A})$ in $\mathfrak{aut}_\mu(P) \times \mathcal{F}_G(P,\mathfrak{g}) \times \mathcal{C}onn(P)$. Then $(\psi, \mathcal{A}_0, \mathcal{A})$ is a solution of the Euler–Lagrange equations associated to the Lagrangian* (5.22) *if and only if $(U, \mathcal{A}_0, \mathcal{A})$ is a solution of the incompressible homogeneous Euler–Yang–Mills equations:*

$$\begin{cases} \dfrac{\partial v}{\partial t} + \nabla_v v = \gamma\left(Q, E(\cdot) + B(\cdot, v)\right)^\sharp - \operatorname{grad} p, \\ \dfrac{\partial Q}{\partial t} + [A_0, Q] + \nabla^{\mathcal{A}}_v Q = 0, \\ \dfrac{\partial \mathcal{E}}{\partial t} + [\mathcal{A}_0, \mathcal{E}] = \delta^{\mathcal{A}}\mathcal{B} - \mathcal{Q} \otimes \pi^* v^\flat, \quad \delta^{\mathcal{A}}\mathcal{E} = -\mathcal{Q}, \\ \dfrac{\partial \mathcal{B}}{\partial t} + [\mathcal{A}_0, \mathcal{B}] = -\mathbf{d}^{\mathcal{A}}\mathcal{E}, \quad \mathbf{d}^{\mathcal{A}}\mathcal{B} = 0, \end{cases} \tag{5.23}$$

where

$$
\begin{aligned}
v &:= [U] \in \mathfrak{X}_{\mathrm{div}}(M),\\
\mathcal{E} &:= -\dot{\mathcal{A}} + \mathbf{d}^{\mathcal{A}}\mathcal{A}_0 \in \overline{\Omega^1}(P,\mathfrak{g}) \quad \textit{and} \quad E := \widetilde{\mathcal{E}} \in \Omega^1(M, \operatorname{Ad} P),\\
\mathcal{B} &:= \mathbf{d}^{\mathcal{A}}\mathcal{A} \quad \textit{and} \quad B := \widetilde{\mathcal{B}},\\
\mathcal{Q} &:= \mathcal{A}(U) + \mathcal{A}_0 \quad \textit{and} \quad Q = \widetilde{\mathcal{Q}}.
\end{aligned}
$$

Note that the pressure is in this case determined from v, E, and B through Green's function of the Laplacian on M. This is in contrast to (5.17) where the pressure was given by the internal energy.

If P is a trivial bundle, one gets the incompressible homogeneous version of the Euler–Yang–Mills equations (corresponding to the group $\mathcal{D}_\mu(M)\,Ⓢ\,\mathcal{F}(M,G)$) by replacing in formula (5.19) the vector fields by their projection onto their divergence free part, namely,

$$
\begin{aligned}
\operatorname{ad}^\dagger_{(v,\theta)}(m,\nu) &= \left(\operatorname{ad}^\dagger_v m + \mathrm{P}_e\left(\gamma(\nu, \mathbf{d}\theta(\cdot))^\sharp\right), \mathbf{d}\nu(v) + [\nu,\theta]\right)\\
&= \left(\mathrm{P}_e\left(\nabla_v m + \nabla v^{\mathrm{T}}\cdot m + \gamma(\nu, \mathbf{d}\theta(\cdot))^\sharp\right), \mathbf{d}\nu(v) + [\nu,\theta]\right). \qquad (5.24)
\end{aligned}
$$

One can also adapt our method to the case of the incompressible but non-homogeneous Yang–Mills fluid. It suffices to apply the semidirect product theory with $G = \mathcal{A}ut_\mu(P)$, $Q = \mathcal{F}_G(P,\mathfrak{g}) \times \mathcal{C}onn(P)$, and $V = \mathcal{F}(M)$, where the mass density ρ is an element of V^*. Note that in geophysical incompressible fluid dynamics, there is also a second scalar advected quantity, namely the buoyancy (for details, see [**7**, **28**]) which plays the role that entropy plays in a compressible fluid. In this case we would take $V = \mathcal{F}(M) \times \mathcal{F}(M)$, where the second factor is thought of as the space of densities on M, thereby making the buoyancy, an element of its dual, into a function.

6. Hamiltonian formulation of Euler–Yang–Mills

Once the Lagrangian formulation of a theory is known, one usually passes to the Hamiltonian formulation by a Legendre transformation, if the Lagrangian function is non-degenerate. Unfortunately, in our case, this is not possible because the Legendre transformation is not invertible, as we have already seen when studying the Maxwell equations. The trouble is that the Lagrangian function does not depend on $\dot{\mathcal{A}}_0$. To deal with this, we shall work with a new Lagrangian function obtained by eliminating $\mathcal{A}_0$ from (5.5). For this new Lagrangian function, the Legendre transformation is invertible and we can deduce the associated Hamiltonian formulation. However, in this process, an equation gets lost, namely, Gauss' law $\delta^{\mathcal{A}}\mathcal{E} = -\mathcal{Q}$ in (5.17). This equation will be recovered as a conservation law of the momentum map associated to the gauge transformation group. We begin by quickly recalling some facts about the Hamiltonian semidirect product reduction theory.

6.1. Hamiltonian semidirect product reduction with parameter. Let $S := G \circledS V$ be the semidirect product defined at the beginning of Section 5. The lift of right translation of S on T^*S induces a right action on $T^*G \times V^*$. Let Q be another manifold (without any G or V-action). Consider a Hamiltonian function $H : T^*G \times T^*Q \times V^* \to \mathbb{R}$ right invariant under the S-action on $T^*G \times T^*Q \times V^*$; recall that the S-action on T^*Q is trivial. In particular, the function $H_{a_0} := H|_{T^*G\times T^*Q\times\{a_0\}} : T^*G \times T^*Q \to \mathbb{R}$ is invariant under the induced action of the isotropy subgroup $G_{a_0} := \{g \in G \mid \rho_g^* a_0 = a_0\}$ for any $a_0 \in V^*$. The following theorem is an easy consequence of the semidirect product reduction theorem [**6**] and the reduction by stages method [**18**].

Theorem 6.1. *For $\alpha(t) \in T^*_{g(t)}G$ and $\mu(t) := T^*R_{g(t)}(\alpha(t)) \in \mathfrak{g}^*$, the following are equivalent*:

i. *$(\alpha(t), q(t), p(t))$ satisfies Hamilton's equations for H_{a_0} on $T^*(G \times Q)$.*
ii. *The following system of Lie–Poisson equations with parameter coupled with Hamilton's equations holds on $\mathfrak{s}^* \times T^*Q$:*

$$\begin{aligned}\frac{\partial}{\partial t}(\mu, a) &= -\operatorname{ad}^*_{(\delta h/\delta\mu, \delta h/\delta a)}(\mu, a)\\ &= -\left(\operatorname{ad}^*_{\delta h/\delta\mu}\mu + \frac{\delta h}{\delta a}\diamond a, a\frac{\delta h}{\delta\mu}\right), \quad a(0) = a_0\end{aligned}$$

and

$$\frac{dq^i}{dt} = \frac{\partial h}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial h}{\partial q^i},$$

where $\mathfrak{s}$ is the semidirect product Lie algebra $\mathfrak{s} = \mathfrak{g} \circledS V$. The associated Poisson bracket is the sum of the Lie–Poisson bracket on the Lie algebra $\mathfrak{s}^*$ and the canonical bracket on the cotangent bundle T^*Q, that is,

$$\begin{aligned}\{f, g\}(\mu, a, q, p) = \left\langle \mu, \left[\frac{\delta f}{\delta\mu}, \frac{\delta g}{\delta\mu}\right]\right\rangle + \left\langle a, \frac{\delta f}{\delta a}\frac{\delta g}{\delta\mu} - \frac{\delta g}{\delta a}\frac{\delta f}{\delta\mu}\right\rangle\\ + \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i}\frac{\partial f}{\partial p_i}.\end{aligned}$$

For example, one can start with a Lagrangian L_{a_0} as in the previous section, suppose that the Legendre transformation $\mathbb{F}L_{a_0}$ is invertible, and form the corresponding Hamiltonian $H_{a_0} = E_{a_0} \circ \mathbb{F}L_{a_0}^{-1}$, where E_{a_0} is the energy of L_{a_0}. Then the function $H : T^*G \times T^*Q \times V^* \to \mathbb{R}$ so defined is S-invariant and one can apply this theorem. This is the method we shall use below to find the Hamiltonian formulation of the Euler–Yang–Mills equations.

6.2. The Hamiltonian. Recall that we identify the cotangent space $T^*_\psi\mathcal{A}ut(P)$ with the space of G-invariant 1-forms on P along $\psi\in\mathcal{A}ut(P)$. The duality is given by

$$\langle \mathbf{M}_\psi, U_\psi\rangle := \int_M \mathbf{M}_\psi(U_\psi)\mu,$$

where $\mathbf{M}_\psi\in T^*_\psi\mathcal{A}ut(P)$ and $U_\psi\in T_\psi\mathcal{A}ut(P)$. Recall that the integrand defines a function on M (it is independent on the fiber variables in the bundle $\pi: P\to M$).

For $U_\psi, V_\psi\in T_\psi\mathcal{A}ut(P)$, the expression $K_\mathcal{A}(p)\left(U_\psi(p), V_\psi(p)\right)$ depends only on the class $x=\pi(p)$. Thus $K_\mathcal{A}(U_\psi, V_\psi)$, which is a smooth function on P, does not depend on the fibers and hence induces a smooth function on M. Therefore, the integral

$$\int_M K_\mathcal{A}(U_\psi, V_\psi)\mu$$

is well defined. Moreover, the definition of $K_\mathcal{A}$ immediately implies the equality

$$\int_M K_\mathcal{A}(U_\psi, V_\psi)\mu = \int_M g([U_\psi],[V_\psi])\mu + \int_M \gamma\left(\widetilde{\mathcal{A}(U_\psi)}, \widetilde{\mathcal{A}(V_\psi)}\right)\mu.$$

Similarly, for $\mathbf{M}_\psi, \mathbf{N}_\psi\in T^*_\psi\mathcal{A}ut(P)$, we can define the integral

$$\int_M K^*_\mathcal{A}(\mathbf{M}_\psi, \mathbf{N}_\psi)\mu,$$

where $K^*_\mathcal{A}$ denotes the dual metric induced on T^*P by the Kaluza–Klein metric.

The Hamiltonian for the Euler–Yang–Mills equations is defined on the cotangent bundle $T^*(\mathcal{A}ut(P)\times\mathcal{C}onn(P))$ and is given, for $(\rho,s)\in\mathcal{F}(M)\times\mathcal{F}(M)$, by

$$\begin{aligned}(6.1)\qquad H_{(\rho,s)}(\mathbf{M}_\psi,\mathcal{A},\mathcal{Y}) = \frac{1}{2}\int_M \frac{1}{\rho}K^*_\mathcal{A}(\mathbf{M}_\psi,\mathbf{M}_\psi)\mu + \int_M \rho e(\rho(J\overline{\psi})^{-1}, s)\mu \\ + \frac{1}{2}\int_M \|E\|^2\mu + \frac{1}{2}\int_M \|B\|^2\mu.\end{aligned}$$

This Hamiltonian is obtained by Legendre transforming the Lagrangian $L_{(\rho,s)}$ in the case the variable $\mathcal{A}_0$ is absent. Indeed, we have

$$\mathbf{M}_\psi(p) := \mathbb{F}L(U_\psi)(p) = \rho(x)K_\mathcal{A}(p)(U_\psi(p),\cdot),\quad x=\pi(p).$$

By Theorem 6.1, Hamilton's equations for $H_{(\rho,s)}$ are equivalent to the Lie–Poisson equations on the dual of the semidirect product Lie algebra $\mathfrak{aut}(P)\,\circledS\,(\mathcal{F}(M)\times\mathcal{F}(M))$, together with the standard Hamilton equations on $T^*\mathcal{C}onn(P)$, relative to the reduced Hamiltonian h given on $\left(\mathfrak{aut}(P)\circledS(\mathcal{F}(M)\times\mathcal{F}(M))\right)^*\times T^*\mathcal{C}onn(P)$ by

$$\begin{aligned}h(\mathbf{M},\rho,s,\mathcal{A},\mathcal{Y}) &= \frac{1}{2}\int_M \frac{1}{\rho}K^*_{\mathcal{A}}(\mathbf{M},\mathbf{M})\mu + \int_M \rho e(\rho,s)\mu\\ &\quad + \frac{1}{2}\int_M \|E\|^2\mu + \frac{1}{2}\int_M \|B\|^2\mu.\end{aligned}$$

By the Legendre transformation $U\mapsto \mathbf{M}=K_{\mathcal{A}}(\rho U,\cdot)$, these equations are equivalent to equations (5.17) with $\mathcal{A}_0=0$ but without the non-Abelian Gauss equation

$$\delta^{\mathcal{A}}\mathcal{E}=-\mathcal{Q}.$$

6.3. The momentum map of the gauge group. This last equation is obtained by invariance of the Hamiltonian under gauge transformations. Indeed, consider the action of the gauge group given for $\eta\in\mathcal{G}au(P)$, by

$$(\psi,\mathcal{A})\longmapsto(\eta^{-1}\circ\psi,\eta^*\mathcal{A}). \tag{6.2}$$

The cotangent-lift of this action leaves the Hamiltonian invariant. So, the associated momentum map, which is computed in the following lemma, is a conserved quantity.

Lemma 6.2. *The momentum map associated to the cotangent-lift of the gauge group action is*

$$\boldsymbol{J}(\mathbf{M}_\psi,\mathcal{A},\mathcal{Y})=\sigma\left(\delta^{\mathcal{A}}\mathcal{Y}-\mathcal{A}\left(\left(J\overline{\psi}^{-1}\right)V_\psi\circ\psi^{-1}\right)\right)\in\mathfrak{gau}(P)^*\simeq\mathfrak{gau}(P),$$

where $V_\psi\in T_\psi\mathcal{A}ut(P)$ is such that $\mathbf{M}_\psi=K_{\mathcal{A}}(V_\psi,\cdot)$ and $\sigma:\mathcal{F}_G(P,\mathfrak{g})\to\mathfrak{gau}(P)$ is defined in (2.8).

Proof. We will apply the formula $\mathbf{J}(\alpha_q)(\xi)=\langle\alpha_q,\xi_Q(q)\rangle$, which gives the momentum mapping associated to a cotangent-lifted action of a Lie group G on a cotangent bundle T^*Q. In our case we have $G=\mathcal{G}au(P)$, $Q=\mathcal{A}ut(P)\times\mathcal{C}onn(P)$ and for $\xi=\sigma(f)\in\mathfrak{gau}(P)$, the infinitesimal generator is given by (see (3.10))

$$\begin{aligned}\xi_Q(\psi,\mathcal{A}) &= \left.\frac{d}{dt}\right|_{t=0}\left(\exp(t\xi)^{-1}\circ\psi,\exp(t\xi)^*\mathcal{A}\right)\\ &= \left(-\xi\circ\psi,\mathbf{d}^{\mathcal{A}}f\right)\in T_{(\psi,\mathcal{A})}(\mathcal{A}ut_\mu(P)\times\mathcal{C}onn(P)).\end{aligned}$$

Thus, changing variables in the third equality below, using (2.4), we get

$$\begin{aligned}\langle\mathbf{J}(\mathbf{M}_\psi,\mathcal{A},\mathcal{Y}),\xi\rangle &= \langle(\mathbf{M}_\psi,\mathcal{A},\mathcal{Y}),(-\xi\circ\psi,\mathcal{A},\mathbf{d}^{\mathcal{A}}f)\rangle\\ &= -\int_M K_{\mathcal{A}}(V_\psi,\xi\circ\psi)\mu+\int_M(g\gamma)\left(\widetilde{\mathcal{Y}},\widetilde{\mathbf{d}^{\mathcal{A}}f}\right)\mu\\ &= -\int_M K_{\mathcal{A}}\left(\left(J\overline{\psi}^{-1}\right)V_\psi\circ\psi^{-1},\xi\right)\mu+\int_M\gamma\left(\widetilde{\delta^{\mathcal{A}}\mathcal{Y}},\widetilde{f}\right)\mu.\end{aligned}$$

Since $[\xi]=0$, the first term can be written as

$$-\int_M \gamma\left(\mathcal{A}\left(\widetilde{\left(J\overline{\psi}^{-1}\right) V_\psi} \circ \psi^{-1}\right), \widetilde{\mathcal{A}(\xi)}\right)\mu.$$

Thus, using the pairing (2.11) and the identity $\mathcal{A}\circ\sigma = \mathrm{id}_{\mathcal{F}_G(P,\mathfrak{g})}$, we get

$$\begin{aligned}\langle \mathbf{J}(\mathbf{M}_\psi,\mathcal{A},\mathcal{Y}),\xi\rangle &= -\int_M \gamma\left(\mathcal{A}\left(\widetilde{\left(J\overline{\psi}^{-1}\right) V_\psi} \circ \psi^{-1}\right), \widetilde{\mathcal{A}(\xi)}\right)\mu\\ &\quad + \int_M \gamma\left(\widetilde{\delta^{\mathcal{A}}\mathcal{Y}}, \widetilde{\mathcal{A}(\xi)}\right)\mu\\ &= \left\langle \sigma\left(\delta^{\mathcal{A}}\mathcal{Y} - \mathcal{A}\left(\left(J\overline{\psi}^{-1}\right) V_\psi\circ\psi^{-1}\right)\right), \xi\right\rangle.\end{aligned}$$

□

When $\mathbf{M}_\psi = \mathbb{F}L(U_\psi) = K_{\mathcal{A}}(\rho_0 U_\psi,\cdot)$, is a solution of Hamilton's equations associated to $H_{(\rho_0,s_0)}$, the conservation law $\mathbf{J}(\mathbf{M}_\psi,\mathcal{A},\mathcal{Y})=0$ gives

$$\mathcal{A}\left(\left(J\overline{\psi}^{-1}\right)(\rho_0\circ\overline{\psi}^{-1})U_\psi\circ\psi^{-1}\right) = \delta^{\mathcal{A}}\mathcal{Y}.$$

The definition of the charge density $\mathcal{Q}$ (see (5.13) without $\mathcal{A}_0$), the identities $U_\psi\circ\psi^{-1}=U$, $\left(J\overline{\psi}^{-1}\right)(\rho_0\circ\overline{\psi}^{-1})=\rho$, and the notation $\mathcal{E}=-\mathcal{Y}$, gives

$$\mathcal{Q} = -\delta^{\mathcal{A}}\mathcal{E}.$$

The following theorem summarizes the results of the present section.

Theorem 6.3. *Let* $(\mathbf{M}_\psi,\mathcal{A},\mathcal{Y})$ *be a curve in* $T^*(\mathcal{A}ut(P)\times\mathcal{C}onn(P))$ *and consider the induced curve* $(\mathbf{M},\mathcal{A},\mathcal{Y})\in\mathfrak{aut}(P)^*\times T^*\mathcal{C}onn(P)$ *given by* $\mathbf{M} := \left(J\overline{\psi}\right)\mathbf{M}_\psi\circ\psi^{-1}$. *Then* $(\mathbf{M}_\psi,\mathcal{A},\mathcal{Y})$ *is a solution of Hamilton's equations associated to the Hamiltonian* $H_{(\rho_0,s_0)}$ *given in* (6.1) *if and only if* $(\mathbf{M},\mathcal{A},\mathcal{Y})$ *is a solution of the system*

$$\left\{\begin{aligned}&\frac{\partial v}{\partial t}+\nabla_v v = \frac{1}{\rho}\gamma\left(Q, E(\cdot)+B(\cdot,v)\right)^\sharp - \frac{1}{\rho}\operatorname{grad}p,\\ &\frac{\partial\rho}{\partial t}+\operatorname{div}(\rho v)=0,\quad \rho(0)=\rho_0,\quad \frac{\partial s}{\partial t}+\mathbf{d}s(v)=0,\quad s(0)=s_0,\\ &\frac{\partial Q}{\partial t}+\nabla^{\mathcal{A}}_v Q + Q\operatorname{div}v = 0,\\ &\frac{\partial\mathcal{E}}{\partial t} = \delta^{\mathcal{A}}\mathcal{B} - \mathcal{Q}\otimes\pi^* v^\flat,\\ &\frac{\partial\mathcal{B}}{\partial t} = -\mathbf{d}^{\mathcal{A}}\mathcal{E},\quad \mathbf{d}^{\mathcal{A}}\mathcal{B}=0,\end{aligned}\right. \tag{6.3}$$

where we use the same notations as in Theorem 5.4 except that here v and $\mathcal{Q}$ are given in terms of $\mathbf{M}$ by

$$v = [U] \quad \textit{and} \quad \mathcal{Q} = \mathcal{A}(\rho U) \ \textit{where} \quad U = K_{\mathcal{A}}^{*}\left(\frac{\mathbf{M}}{\rho}, \cdot\right). \tag{6.4}$$

Conservation of the momentum map associated to the gauge transformations gives the equation

$$\delta^{\mathcal{A}}\mathcal{E} = -\mathcal{Q}.$$

One can adapt this theorem to the incompressible and homogeneous case.

6.4. The Poisson bracket. From Theorem 6.1, we know that the Euler–Yang–Mills equations (6.3) can be written as

$$\dot{f} = \{f, h\}$$

with respect to the Poisson bracket on $[\mathfrak{aut}(P) \circledS (\mathcal{F}(M) \times \mathcal{F}(M))]^* \times T^*\mathcal{C}onn(P)$

$$\begin{aligned}
\{f, g\}(\mathbf{M}, \rho, s, \mathcal{A}, \mathcal{Y}) = &\int_M \mathbf{M}\left(\left[\frac{\delta f}{\delta \mathbf{M}}, \frac{\delta g}{\delta \mathbf{M}}\right]_{\mathrm{L}}\right)\mu \\
&+ \int_M \rho\left(\mathbf{d}\left(\frac{\delta f}{\delta \rho}\right)\left[\frac{\delta g}{\delta \mathbf{M}}\right] - \mathbf{d}\left(\frac{\delta g}{\delta \rho}\right)\left[\frac{\delta f}{\delta \mathbf{M}}\right]\right)\mu \\
&+ \int_M s\left(\operatorname{div}\left(\frac{\delta f}{\delta s}\left[\frac{\delta g}{\delta \mathbf{M}}\right]\right) - \operatorname{div}\left(\frac{\delta g}{\delta s}\left[\frac{\delta f}{\delta \mathbf{M}}\right]\right)\right)\mu \\
&+ \int_M (g\gamma)\left(\frac{\delta f}{\delta \mathcal{A}}, \frac{\delta g}{\delta \mathcal{Y}}\right)\mu - \int_M (g\gamma)\left(\frac{\delta g}{\delta \mathcal{A}}, \frac{\delta f}{\delta \mathcal{Y}}\right)\mu.
\end{aligned} \tag{6.5}$$

We can obtain this bracket and the associated Hamilton equations (6.3) alternatively by a reduction by stages process [**18**]. The symplectic reduced spaces are of the form $\mathcal{O} \times T^*\mathcal{C}onn(P)$, where $\mathcal{O}$ is a coadjoint orbit of the semidirect product $S := \mathcal{A}ut(P) \circledS (\mathcal{F}(M) \times \mathcal{F}(M))$.

If the principal bundle is trivial, the automorphism group is the semidirect product $\mathcal{D}(M) \circledS \mathcal{F}(M, G)$ of the diffeomorphism group of M with the group of G-valued functions on M. In this case, the first term can be written more explicitly by taking advantage of the internal structure of $\mathcal{A}ut(P)$, and we recover (up to sign conventions) the Poisson bracket given in equation (5.14) of [**10**].

6.5. Summary. We comment now on the structure of the equations of motion (6.3) and the Poisson bracket (6.5). Note that in (6.3) there is an evolution equation for the gauge-charge Q but that the functions for which the Poisson bracket (6.5) is defined seem not to depend on Q. The explanation of this fact is given in Theorem 6.3; the discussion below summarizes briefly the key results and comments on the structure of both the equations and the Poisson bracket.

(1) The equations for v and Q are the "components" of a *single* equation: the Lie–Poisson equation on the dual of the Lie algebra of the automorphism group. The true variable is the fluid momentum $\mathbf{M}$ which defines both the Eulerian velocity v and the gauge-charge Q by using (6.4). Conversely, given ρ, v, and Q, the fluid momentum $\mathbf{M}$ is found by putting $\mathbf{M} := \rho K_{\mathcal{A}}(U, \cdot) = \rho g(v, T\pi(\cdot)) + \gamma(\mathcal{Q}, \mathcal{A}(\cdot))$, where $U := \mathrm{Hor}_{\mathcal{A}} \circ v + \frac{1}{\rho}\sigma(\mathcal{Q})$ (recall that for any $U \in \mathfrak{aut}(P)$ and $\mathcal{A} \in \mathcal{C}onn(P)$, we have the identity $U = \mathrm{Hor}_{\mathcal{A}} \circ [U] + \sigma(\mathcal{A}(U))$). In other words, the Lie–Poisson equation for $\mathbf{M}$ is equivalent to *two* equations: the equation for v and the equation for Q. This is the usual Kaluza–Klein point of view, namely, Q and v are constructed from $\mathbf{M}$ and vice-versa.
(2) The Q-equation looks like advection equation. To see this, recall that $Q \in \Gamma(\mathrm{Ad}\, P)$ and that $\delta^{\mathcal{A}} : \Omega^1(M, \mathrm{Ad}\, P) \to \Gamma(\mathrm{Ad}\, P)$ (see [**16**, Definition 4.2.8]). Defining
$$\mathrm{div}^{\mathcal{A}}(Qv) := -\delta^{\mathcal{A}}(Q \otimes v^\flat), \qquad v \in \mathfrak{X}(M),$$
where the 1-form $Q \otimes v^\flat \in \Omega^1(M, \mathrm{Ad}\, P)$ is given by $(Q \otimes v^\flat)(u_x) := Q(x) g_x(v(x), u_x)$, for any $u_x \in T_xM$, one easily deduces the formula
$$\mathrm{div}^{\mathcal{A}}(Qv) = \nabla^{\mathcal{A}}_v Q + Q \,\mathrm{div}\, v,$$
which allows us to write the Q-equation in the form
$$\dot{Q} + \mathrm{div}^{\mathcal{A}}(Qv) = 0.$$
However Q is not advected since its evolution is not given by the pull back of the flow of the velocity field v. Note that in this equation $\mathcal{A}$ is itself a variable that is time dependent.
(3) The equations for ρ and s are usual advection equations for a density and a function that appear in the Lie–Poisson equations of a semidirect product.
(4) The equations for $\mathcal{E}$ and $\mathcal{B}$ are Hamilton's equations for the conjugate variables $(\mathcal{A}, \mathcal{Y}) \in T^*\mathcal{C}onn(P)$.
(5) The equation $\delta^{\mathcal{A}}\mathcal{E} = -\mathcal{Q}$ comes from momentum conservation associated to gauge group symmetry and $\mathbf{d}^{\mathcal{A}}\mathcal{B} = 0$ is the Bianchi identity for the connection $\mathcal{A}$ and its curvature $\mathcal{B}$.
(6) The Poisson bracket (6.5) contains two types of terms: the first three are Lie–Poisson for a semidirect product and the fourth is the usual bracket on $T^*\mathcal{C}onn(P)$. However, note that the first summand in (6.5) gives rise to an evolution equation for $\mathbf{M}$ which, as we discussed above, is equivalent to *two* evolution equations, one for v and another one for Q. If the bundle is trivial, one can make formulas (6.4) more explicit,

as we shall see below when we carry out one more reduction. Note also that the Poisson bracket (6.5) is a product bracket; there is no coupling between the semidirect product fluid variables $(\mathbf{M}, \rho, s)$ and the Yang–Mills field variables $(\mathcal{A}, \mathcal{Y})$. The coupling in the equations is exclusively due to the Hamiltonian (6.1).

6.6. The second reduction. Note that right translation in the group $S = \mathcal{A}ut(P) \circledS (\mathcal{F}(M) \times \mathcal{F}(M))$ on itself and the action of $\mathcal{G}au(P)$ on $\mathcal{A}ut(P) \times \mathcal{C}onn(P)$ given by (6.2) commute if one views them as actions on $S \times \mathcal{C}onn(P)$. Therefore, by the general theory of commuting reduction by stages [**18**], since the momentum map associated to the gauge group action is $\mathcal{A}ut(P)$-invariant, it induces a momentum map $\mathbf{J}_{\mathfrak{s}^*}$ on $\mathfrak{s}^* \times T^*\mathcal{C}onn(P)$ which restricts to a momentum map $\mathbf{J}_{\mathcal{O}}$ on the reduced space $\mathcal{O} \times T^*\mathcal{C}onn(P)$. Here $\mathfrak{s} := \mathfrak{aut}(P) \circledS (\mathcal{F}(M) \times \mathcal{F}(M))$. A direct computation shows that the momentum map $\mathbf{J}_{\mathfrak{s}^*} : \mathfrak{s}^* \times T^*\mathcal{C}onn(P) \to \mathfrak{gau}(P)^*$ has the expression

$$\mathbf{J}_{\mathfrak{s}^*}(\mathbf{M}, \rho, s, \mathcal{A}, \mathcal{Y}) = \sigma\left(\delta^{\mathcal{A}}\mathcal{Y} - \mathcal{A}(V)\right), \tag{6.6}$$

where $V \in \mathfrak{aut}(P)$ is such that $\mathbf{M} = K_{\mathcal{A}}(V, \cdot)$, and σ denotes the map defined in (2.8). The gauge group action induced on $\mathfrak{s}^* \times T^*\mathcal{C}onn(P)$ and $\mathcal{O} \times T^*\mathcal{C}onn(P)$ is given by

$$(\mathbf{M}, \rho, s, \mathcal{A}, \mathcal{Y}) \longmapsto (\mathrm{Ad}^*_{\eta}\ \mathbf{M}, \rho, s, \eta^*\mathcal{A}, \eta^*\mathcal{Y}). \tag{6.7}$$

Using the notation $\mathbf{S} := (\mathbf{M}, \rho, s) \in \mathcal{O}$, it can be written as

$$(\mathbf{S}, \mathcal{A}, \mathcal{Y}) \longmapsto (\mathrm{Ad}^*_{(\eta,0,0)}\ \mathbf{S}, \eta^*\mathcal{A}, \eta^*\mathcal{Y}). \tag{6.8}$$

This action is simply the diagonal action given on the first factor by the coadjoint action of the subgroup $\mathcal{G}au(P)$ of $S = \mathcal{A}ut(P) \circledS (\mathcal{F}(M) \times \mathcal{F}(M))$, and on the second factor by gauge transformations. Note that when the center $Z(G)$ of the group G is trivial, then the transformation $\mathcal{A} \mapsto \eta^*\mathcal{A}$ is free. In this case, the reduced action (6.8) is also free and the second reduced symplectic spaces

$$\frac{\mathbf{J}_{\mathcal{O}}^{-1}(\mathbf{N})}{\mathcal{G}au(P)_{\mathbf{N}}}, \quad \mathbf{N} \in \mathfrak{gau}(P)^*,$$

have no singularities.

By the reduction by stages process, the reduced spaces $\mathbf{J}_{\mathcal{O}}^{-1}(\mathbf{N})/\mathcal{G}au(P)_{\mathbf{N}}$ are symplectically diffeomorphic to the reduced spaces obtained by a one step reduction from the cotangent bundle

$$T^*\big(S \times \mathcal{C}onn(P)\big)$$

with respect to the product of the two cotangent-lifted actions. Note that these reduced spaces are, up to connected components, the symplectic leaves in the Poisson manifold $\mathbf{J}_{\mathfrak{s}^*}^{-1}(\mathbf{N})/\mathcal{G}au(P)_{\mathbf{N}}$. This is a straightforward consequence of [**29**, Theorem 10.1.1(iv)], because the optimally reduced spaces are, up to connected components, precisely the symplectically reduced spaces for every leaf.

Note that the Euler–Yang–Mills equation (6.3) projects to the reduced space at zero momentum

$$\frac{\mathbf{J}_{\mathcal{O}}^{-1}(0)}{\mathcal{G}au(P)}. \tag{6.9}$$

The general case corresponds to the Yang–Mills fluid with an external charge $\mathbf{N}$.

In order to obtain the reduced Poisson structure concretely, we will identify the space $\mathfrak{s}^* \times T^*\mathcal{C}onn(P)$ with a space on which the gauge action is simpler. This identification is given in the following proposition.

Proposition 6.4. *Consider the group $K = \mathcal{D}(M) \circledS (\mathcal{F}(M) \times \mathcal{F}(M))$ and denote by $\mathfrak{k}^*$ the dual of its Lie algebra. There is a gauge-equivariant diffeomorphism*

$$i : \mathfrak{s}^* \times T^*\mathcal{C}onn(P) \longrightarrow \mathfrak{k}^* \times \mathcal{F}_G(P, \mathfrak{g}^*) \times T^*\mathcal{C}onn(P), \tag{6.10}$$

given by

$$i(\mathbf{M}, \rho, s, \mathcal{A}, \mathcal{Y}) := ((\mathrm{Hor}_{\mathcal{A}})^* \circ \mathbf{M}, \rho, s, \mathbb{J} \circ \mathbf{M}, \mathcal{A}, -\mathcal{Y}) =: (\mathbf{n}, \rho, s, \nu, \mathcal{A}, \mathcal{E}),$$

where the gauge group acts on $\mathfrak{s}^* \times T^*\mathcal{C}onn(P)$ by the action (6.8) and on $\mathfrak{k}^* \times T^*\mathcal{C}onn(P)$ only on the factor $\mathcal{F}_G(P, \mathfrak{g}^*) \times T^*\mathcal{C}onn(P)$ by the right action

$$(\nu, \mathcal{A}, \mathcal{E}) \longmapsto (\nu \circ \eta, \eta^*\mathcal{A}, \eta^*\mathcal{E}). \tag{6.11}$$

Moreover, the image of the level set $\mathbf{J}_{\mathfrak{s}^*}^{-1}(\mathbf{N})$ by the diffeomorphism i is

$$\{(\mathbf{n}, \rho, s, \nu, \mathcal{A}, \mathcal{E}) \mid \nu + \gamma(\delta^{\mathcal{A}}\mathcal{E} + f, \cdot) = 0\},$$

where $\mathbf{N} \in \mathfrak{gau}(P)$ and $f \in \mathcal{F}_G(P, \mathfrak{g})$ is such that $\sigma(f) = \mathbf{N}$. Thus $\mathbf{J}_{\mathfrak{s}^*}^{-1}(\mathbf{N})$ is diffeomorphic to $\mathfrak{k}^* \times T^*\mathcal{C}onn(P)$.

The map $\mathbb{J} : T^*P \to \mathfrak{g}^*$ denotes the momentum map $\mathbb{J}(\alpha_p)(\xi) := \langle \alpha_p, \xi_P(p) \rangle$, and $(\mathrm{Hor}_{\mathcal{A}})^*$ denotes the dual map of the horizontal-lift $\mathrm{Hor}_{\mathcal{A}} : TM \to TP$ with respect to $\mathcal{A}$.

Proof. We first prove that i is injective. Suppose that $(\mathbf{M}, \rho, s, \mathcal{A}, \mathcal{Y})$, $(\mathbf{M}', \rho', s', \mathcal{A}', \mathcal{Y}') \in \mathfrak{s}^* \times T^*\mathcal{C}onn(P)$ have the same image under i.

We clearly have $(\rho, s, \mathcal{A}, \mathcal{Y}) = (\rho', s', \mathcal{A}', \mathcal{Y}')$. Therefore, we have $(\mathrm{Hor}_{\mathcal{A}})^* \circ \mathbf{M} = (\mathrm{Hor}_{\mathcal{A}})^* \circ \mathbf{M}'$ and $\mathbb{J} \circ \mathbf{M} = \mathbb{J} \circ \mathbf{M}'$. This implies that $\mathbf{M} = \mathbf{M}'$. The map i is clearly surjective and hence invertible, its inverse being given by

$$i^{-1}(\mathbf{n}, \rho, s, \nu, \mathcal{A}, \mathcal{E}) = (\pi^*\mathbf{n} + \mathcal{A}^*\nu, \rho, s, \mathcal{A}, -\mathcal{E}),$$

where $\mathcal{A}^* : \mathfrak{g}^* \to T^*P$ denotes the dual map of $\mathcal{A}$. It follows that i is a diffeomorphism.

To prove gauge-equivariance, it suffices to show that for all $\eta \in \mathcal{G}au(P)$,

$$(\mathrm{Hor}_{\eta^*\mathcal{A}})^* \circ \eta^*\mathbf{M} = (\mathrm{Hor}_{\mathcal{A}})^* \circ \mathbf{M} \quad \text{and} \quad \mathbb{J} \circ \eta^*\mathbf{M} = (\mathbb{J} \circ \mathbf{M}) \circ \eta.$$

This is a direct computation using the formulas

$$\begin{gathered} T_p^*\pi(\mathrm{Hor}_{\mathcal{A}})_p^*(\mathbf{M}(p)) = \mathbf{M}(p) - \mathcal{A}(p)^*(\mathbb{J}(\mathbf{M}(p))) \\ \eta^*\mathcal{A} = \mathrm{Ad}_{\widehat{\eta}^{-1}} \circ \mathcal{A} + TL_{\widehat{\eta}^{-1}} \circ T\widehat{\eta} \\ \eta^*\mathbf{M}(p) = \mathbf{M}(p) + T_p^*\widehat{\eta}(T^*R_{\widehat{\eta}(p)^{-1}}(\mathbb{J}(\mathbf{M}(p)))), \end{gathered}$$

where $\widehat{\eta} \in \mathcal{F}_G(P, G)$ is such that $\eta(p) = \Phi_{\widehat{\eta}(p)}(p)$.

Recall that $\mathbf{J}_{\mathfrak{s}^*}(\mathbf{M}, \rho, s, \mathcal{A}, \mathcal{Y}) = \sigma(\delta^{\mathcal{A}}\mathcal{Y} - \mathcal{A}(V))$, where $V \in \mathfrak{aut}(P)$ is such that $\mathbf{M} = K_{\mathcal{A}}(V, \cdot)$. So for $\mathbf{N} = \sigma(f)$ the condition $\mathbf{J}_{\mathfrak{s}^*}(\mathbf{M}, \rho, s, \mathcal{A}, \mathcal{Y}) = \mathbf{N}$ reads $\delta^{\mathcal{A}}\mathcal{Y} - \mathcal{A}(V) = f$. Using that $\nu = \mathbb{J} \circ \mathbf{M} = \gamma(\mathcal{A}(V), \cdot)$ and $\mathcal{E} = -\mathcal{Y}$, we get the condition

$$\nu + \gamma(\delta^{\mathcal{A}}\mathcal{E} + f, \cdot) = 0.$$

□

This proposition shows that the reduced spaces $\mathbf{J}_{\mathfrak{s}^*}^{-1}(\mathbf{N})/\mathcal{G}au(P)_{\mathbf{N}}$ can be identified with the quotient $\mathfrak{k}^* \times [(\mathcal{F}_G(P, \mathfrak{g}^*) \times T^*\mathcal{C}onn(P))\,/\mathcal{G}au(P)_{\mathbf{N}}]$ via the diffeomorphism induced by i and given by

$$[(\mathbf{M}, \rho, s, \nu, \mathcal{A}, \mathcal{Y})] \longmapsto ((\mathrm{Hor}_{\mathcal{A}})^* \circ \mathbf{M}, \rho, s, [\nu, \mathcal{A}, \mathcal{E}]), \tag{6.12}$$

where $[\cdot]$ denote the corresponding equivalence classes.

We now compute the Poisson structure $\{\,,\}'$ induced by i on $\mathfrak{k}^* \times \mathcal{F}_G(P, \mathfrak{g}^*) \times T^*\mathcal{C}onn(P)$. For $f, g \in \mathcal{F}(\mathfrak{k}^* \times \mathcal{F}_G(P, \mathfrak{g}^*) \times T^*\mathcal{C}onn(P))$, we have the formulas

$$\frac{\delta(f \circ i)}{\delta \mathbf{M}} = \mathrm{Hor}_{\mathcal{A}} \circ \frac{\delta f}{\delta \mathbf{n}} + \mathbb{J}^* \circ \frac{\delta f}{\delta \nu}, \quad \left[\frac{\delta(f \circ i)}{\delta \mathbf{M}}\right] = \frac{\delta f}{\delta \mathbf{n}}, \quad \mathbf{n} := (\mathrm{Hor}_{\mathcal{A}})^* \circ \mathbf{M},$$

$$(g\gamma)\left(\frac{\delta(f \circ i)}{\delta \mathcal{A}}, \mathcal{C}\right) = (g\gamma)\left(\frac{\delta f}{\delta \mathcal{A}}, \mathcal{C}\right) - \nu\left(\mathcal{C}\left(\mathrm{Hor}_{\mathcal{A}}\left(\frac{\delta f}{\delta \mathbf{n}}\right)\right)\right), \quad \nu := \mathbf{J} \circ \mathbf{M}.$$

Using the equality (5.8), we obtain

(6.13)

$$\begin{aligned}
&\{f,g\}'(\mathbf{n},\rho,s,\nu,\mathcal{A},\mathcal{E}) := \{f\circ i, g\circ i\}(\mathbf{M},\rho,s,\mathcal{A},\mathcal{Y})\\
&\quad= \int_M \mathbf{n}\left(\left[\frac{\delta f}{\delta \mathbf{n}},\frac{\delta g}{\delta \mathbf{n}}\right]_L\right)\mu + \int_M \rho\left(\mathbf{d}\left(\frac{\delta f}{\delta \rho}\right)\frac{\delta g}{\delta \mathbf{n}} - \mathbf{d}\left(\frac{\delta g}{\delta \rho}\right)\frac{\delta f}{\delta \mathbf{n}}\right)\mu\\
&\qquad+\int_M s\left(\operatorname{div}\left(\frac{\delta f}{\delta s}\frac{\delta g}{\delta \mathbf{n}}\right) - \operatorname{div}\left(\frac{\delta g}{\delta s}\frac{\delta f}{\delta \mathbf{n}}\right)\right)\mu + \int_M (g\gamma)\left(\frac{\delta g}{\delta \mathcal{A}},\frac{\delta f}{\delta \mathcal{E}}\right)\mu\\
&\qquad-\int_M (g\gamma)\left(\frac{\delta f}{\delta \mathcal{A}},\frac{\delta g}{\delta \mathcal{E}}\right)\mu + \int_M \nu\left(\left[\frac{\delta f}{\delta \nu},\frac{\delta g}{\delta \nu}\right]\right)\mu + \int_M \nu\left(\frac{\delta g}{\delta \mathcal{E}}\left(\operatorname{Hor}_{\mathcal{A}}\circ\frac{\delta f}{\delta \mathbf{n}}\right)\right.\\
&\qquad-\frac{\delta f}{\delta \mathcal{E}}\left(\operatorname{Hor}_{\mathcal{A}}\circ\frac{\delta g}{\delta \mathbf{n}}\right) + \mathbf{d}^{\mathcal{A}}\left(\frac{\delta f}{\delta \nu}\right)\left(\operatorname{Hor}_{\mathcal{A}}\circ\frac{\delta g}{\delta \mathbf{n}}\right)\\
&\qquad\left.-\,\mathbf{d}^{\mathcal{A}}\left(\frac{\delta g}{\delta \nu}\right)\left(\operatorname{Hor}_{\mathcal{A}}\circ\frac{\delta f}{\delta \mathbf{n}}\right) + \mathcal{B}\left(\operatorname{Hor}_{\mathcal{A}}\circ\frac{\delta f}{\delta \mathbf{n}},\operatorname{Hor}_{\mathcal{A}}\circ\frac{\delta g}{\delta \mathbf{n}}\right)\right)\mu.
\end{aligned}$$

Note that the first three terms in (6.13) represent the Lie–Poisson bracket on $\mathfrak{k}^*$, the fourth and fifth terms represent the canonical bracket on $T^*\mathcal{C}onn(P)$, the sixth term is the Lie–Poisson bracket on $\mathcal{F}_G(P,\mathfrak{g}^*)$, and the last term provides the coupling of the fluid variables to the Yang–Mills fields.

By the general process of Poisson (point) reduction, the reduced spaces

$$\frac{\mathbf{J}_{\mathfrak{s}^*}^{-1}(\mathbf{N})}{\mathcal{G}au(P)_{\mathbf{N}}} \simeq \mathfrak{k}^* \times \left[\frac{(\mathcal{F}_G(P,\mathfrak{g}^*)\times T^*\mathcal{C}onn(P))}{\mathcal{G}au(P)_{\mathbf{N}}}\right]$$

inherit a Poisson bracket $\{\,,\}_{\mathbf{N}}$ given by

$$\{f_{\mathbf{N}},g_{\mathbf{N}}\}_{\mathbf{N}}(\mathbf{n},\rho,s,[\nu,\mathcal{A},\mathcal{E}]) := \{f,g\}'(\mathbf{n},\rho,s,\nu,\mathcal{A},\mathcal{Y}), \tag{6.14}$$

where f,g are any $\mathcal{G}au(P)$-invariant extensions of the functions $f_{\mathbf{N}}\circ\pi_{\mathbf{N}}$, $g_{\mathbf{N}}\circ\pi_{\mathbf{N}} : \mathbf{J}_{\mathfrak{s}^*}^{-1}(\mathbf{N}) \to \mathbb{R}$, relative to the projection $\pi_{\mathbf{N}} : \mathbf{J}_{\mathfrak{s}^*}^{-1}(\mathbf{N}) \to \mathbf{J}_{\mathfrak{s}^*}^{-1}(\mathbf{N})/\mathcal{G}au(P)_{\mathbf{N}}$.

There are no explicit formulas for the equations of motion on the Poisson point reduced space $\mathbf{J}_{\mathcal{O}}^{-1}(0)/\mathcal{G}au(P)$ because there is no concrete realization of this quotient, to our knowledge. However, there is an important particular case where this is possible, which we study next.

6.7. The case of a trivial bundle. We end this section by examining the case of a trivial principal bundle $P = M\times G$ and, more precisely, the case of the Euler–Maxwell equations which are obtained by taking by $G = S^1$. We then compare our results to those obtained for Euler–Maxwell in [**5**]. Recall that we have $\mathfrak{aut}(P) = \mathfrak{X}(M)\,\circledS\,\mathcal{F}(M,\mathfrak{g})$, so we obtain $\mathfrak{aut}(P)^* = \mathfrak{X}(M)^*\,\circledS\,\mathcal{F}(M,\mathfrak{g})^* = \Omega^1(M)\,\circledS\,\mathcal{F}(M,\mathfrak{g}^*)$. For $(\mathbf{m},\nu)\in\mathfrak{aut}(P)^*$, the gauge transformation (6.7) is given by

$$(\mathbf{m}, \nu, \rho, s, A, Y) \longmapsto (\mathbf{m} + T^*\widehat{\eta} \circ T^*R_{\widehat{\eta}^{-1}} \circ \nu, \mathrm{Ad}^*_{\widehat{\eta}} \circ \nu, \rho, s, \\ \mathrm{Ad}_{\widehat{\eta}^{-1}} \circ A + TL_{\widehat{\eta}^{-1}} \circ T\widehat{\eta}, \mathrm{Ad}_{\widehat{\eta}^{-1}} \circ Y),$$

where $A := \overline{\mathcal{A}}, Y := \overline{\mathcal{Y}} \in \Omega(M, \mathfrak{g})$. The equivariant diffeomorphism (6.10) is

$$i(\mathbf{m}, \nu, \rho, s, A, Y) = (\mathbf{m} - A^*\nu, \rho, s, \nu, A, -Y) =: (\mathbf{n}, \rho, s, \nu, A, E),$$

and the gauge transformation (6.11) is

(6.15)
$$(\mathbf{n}, \rho, s, \nu, A, E) \longmapsto (\mathbf{n}, \rho, s, \mathrm{Ad}^*_{\widehat{\eta}} \circ \nu, \mathrm{Ad}_{\widehat{\eta}^{-1}} \circ A + TL_{\widehat{\eta}^{-1}} \circ T\widehat{\eta}, \mathrm{Ad}_{\widehat{\eta}^{-1}} \circ E).$$

Recall that the relation between the charge density $\mathcal{Q}$ and the momentum $\mathbf{M}$ is

$$\mathcal{Q} = \mathcal{A}(V),$$

where $V \in \mathfrak{aut}(P)$ is such that $\mathbf{M} = K_{\mathcal{A}}(V, \cdot)$; see (6.4). When the bundle is trivial, this relation reads

$$\nu = \gamma(Q, \cdot). \tag{6.16}$$

In the case of Euler–Maxwell, since $G = S^1$, the gauge transformation is simply

$$(\mathbf{m}, \nu, \rho, s, A, Y) \longmapsto (\mathbf{m} + \nu\mathbf{d}\eta, \nu, \rho, s, A + \mathbf{d}\eta, Y),$$

and the relation (6.16) reads $\nu = Q$. Recall that we can write $Q = \rho\frac{q}{m}$, where $q \in \mathcal{F}(M)$ is the charge, see Corollary 5.5. This gauge transformation coincides with the one given in equation (36) of [**5**], where the notation $a := \frac{q}{m}$ is used. The zero level set of the momentum map is

$$\mathbf{J}_{\mathfrak{s}^*}^{-1}(0) = \left\{ \left(\mathbf{m}, \rho\frac{q}{m}, \rho, s, A, Y \right) \middle| \delta Y = \rho\frac{q}{m}, \right\}.$$

The bijection i reads

$$i\left(\mathbf{m}, \rho\frac{q}{m}, \rho, s, A, Y\right) = \left(\mathbf{m} - A\rho\frac{q}{m}, \rho, s, \rho\frac{q}{m}, A, -Y\right),$$

and the image of $\mathbf{J}_{\mathfrak{s}^*}^{-1}(0)$ is

$$\{(\mathbf{n}, \rho, s, \nu, A, E) \mid \operatorname{div} \mathbf{E} = \nu\},$$

where the notation $\mathbf{E} := E^\sharp \in \mathfrak{X}(M)$ is used. The gauge transformation (6.11) is

$$(\mathbf{n}, \rho, s, \nu, A, E) \longmapsto (\mathbf{n}, \rho, s, \nu, A + \mathbf{d}\eta, E).$$

Through the diffeomorphism i, the projection $\pi_0 : \mathbf{J}_{\mathfrak{s}^*}^{-1}(0) \to \mathbf{J}_{\mathfrak{s}^*}^{-1}(0)/\mathcal{G}au(P)$ is given by

$$(\mathbf{n}, \rho, s, \nu, A, E) \longmapsto (\mathbf{n}, \rho, s, [A], E),$$

where $[A]$ denotes the cohomology class of the 1-form A. Assuming that the first and second cohomology groups of M are trivial, $H^1(M) = H^2(M) = \{0\}$, we get the isomorphism

$$[A] \longmapsto B := \mathbf{d}A \in \Omega^2_{\mathrm{cl}}(M), \tag{6.17}$$

where $\Omega^2_{\mathrm{cl}}(M)$ denotes the space closed 2-forms. Thus i induces a diffeomorphism between $\mathbf{J}_{\mathfrak{s}^*}^{-1}(0)/\mathcal{G}au(P)$ and the space $\mathfrak{k}^* \times \Omega^2_{\mathrm{cl}}(M) \times \Omega^1(M)$ given by

$$[\mathbf{m}, \nu, \rho, s, A, Y] \longmapsto (\mathbf{m} - A\nu, \rho, s, \mathbf{d}A, -Y) =: (\mathbf{n}, \rho, s, B, E).$$

where $\Omega^2_{\mathrm{cl}}(M)$ denotes the closed 2-forms on M. This identification coincides with the one given in [**5**, Proposition 10.1].

Using the definition (6.14) and the bracket (6.13), the reduced Poisson bracket on $\mathfrak{k}^* \times \Omega^2_{\mathrm{cl}}(M) \times \Omega^1(M)$ is

$$\begin{aligned}\{f, g\}_0(\mathbf{n}, \rho, s, B, E) = &\int_M \mathbf{n}\left(\left[\frac{\delta f}{\delta \mathbf{n}}, \frac{\delta g}{\delta \mathbf{n}}\right]_{\mathrm{L}}\right)\mu + \int_M \rho\left(\mathbf{d}\left(\frac{\delta f}{\delta \rho}\right)\frac{\delta g}{\delta \mathbf{n}}\right.\\ &\left.- \mathbf{d}\left(\frac{\delta g}{\delta \rho}\right)\frac{\delta f}{\delta \mathbf{n}}\right)\mu + \int_M s\left(\operatorname{div}\left(\frac{\delta f}{\delta s}\frac{\delta g}{\delta \mathbf{n}}\right)\right.\\ &\left.- \operatorname{div}\left(\frac{\delta g}{\delta s}\frac{\delta f}{\delta \mathbf{n}}\right)\right)\mu + \int_M g\left(\delta\frac{\delta g}{\delta B}, \frac{\delta f}{\delta E}\right)\mu\\ &- \int_M g\left(\delta\frac{\delta f}{\delta B}, \frac{\delta g}{\delta E}\right)\mu + \int_M \rho\frac{q}{m}\left(\frac{\delta g}{\delta E}\left(\frac{\delta f}{\delta \mathbf{n}}\right)\right.\\ &\left.- \frac{\delta f}{\delta E}\left(\frac{\delta g}{\delta \mathbf{n}}\right) + B\left(\frac{\delta f}{\delta \mathbf{n}}, \frac{\delta g}{\delta \mathbf{n}}\right)\right)\mu,\end{aligned} \tag{6.18}$$

and the Euler–Maxwell equations can be written as

$$\dot{f} = \{f, h\}_0,$$

relative to the induced Hamiltonian h given by

$$\begin{aligned}h(\mathbf{n}, \rho, s, B, E) = &\frac{1}{2}\int_M \left(\frac{1}{\rho}g(\mathbf{n}, \mathbf{n}) + \frac{1}{\rho}(\delta E)^2\right)\mu + \int_M \rho e(\rho, s)\mu\\ &+ \frac{1}{2}\int_M \left(\|E\|^2 + \|B\|^2\right)\mu.\end{aligned}$$

Note that the function

$$C(\mathbf{n}, \rho, s, B, E) = \frac{1}{2}\int_M \frac{1}{\rho}(\delta E)^2$$

is a Casimir function, so an equivalent Hamiltonian is

$$\overline{h}(\mathbf{n}, \rho, s, B, E) = \frac{1}{2}\int_M \frac{1}{\rho}g(\mathbf{n}, \mathbf{n})\mu + \int_M \rho e(\rho, s)\mu + \frac{1}{2}\int_M \left(\|E\|^2 + \|B\|^2\right)\mu.$$

When M is three dimensional, we can use the notations $\mathbf{B} := (\star B)^\sharp$ and $\mathbf{E} := E^\sharp$. Therefore, the two last terms can be written as

$$\int_M g\left(\operatorname{curl}\frac{\delta g}{\delta \mathbf{B}}, \frac{\delta f}{\delta \mathbf{E}}\right)\mu - \int_M g\left(\operatorname{curl}\frac{\delta f}{\delta \mathbf{B}}, \frac{\delta g}{\delta \mathbf{E}}\right)\mu + \int_M \rho\frac{q}{m}\left(g\left(\frac{\delta g}{\delta \mathbf{E}}, \frac{\delta f}{\delta \mathbf{n}}\right)\right.$$
$$\left. - g\left(\frac{\delta f}{\delta \mathbf{E}}, \frac{\delta g}{\delta \mathbf{n}}\right) + g\left(\mathbf{B}, \frac{\delta f}{\delta \mathbf{n}} \times \frac{\delta g}{\delta \mathbf{n}}\right)\right)\mu.$$

This bracket coincides with the one derived in [**5**] by a direct computation. Note that the first line in the formula above is the Pauli–Born–Infeld Poisson bracket for the Maxwell equations (see, e.g., [**9**, §1.6]). The Hamiltonian $\bar{h}$ is very simple: it is the sum of the total energy of the fluid plus the energy of the electromagnetic field.

Remark. In the Euler–Maxwell case, the correspondence (6.17) is a bijective map if $H^1(M) = H^2(M) = \{0\}$. Indeed, for B, B' such that $\mathbf{d}B = \mathbf{d}B' = 0$, we have $B = \mathbf{d}A$ and $B' = \mathbf{d}A'$; therefore if $B = B'$, we have $A = A' + \mathbf{d}\eta$, that is, $[A] = [A']$.

This fact does not generalize to the case of a non-Abelian principal bundle, trivial or not: there exist gauge inequivalent connections (even on $\mathbb{R}^3 \times G$) with the same curvatures and holonomy groups; see [**30–32**].

7. The Kelvin–Noether theorem

The Kelvin–Noether theorem is a version of the Noether theorem that holds for solutions of the Euler–Poincaré equations. An application of this theorem to the ideal compressible adiabatic fluid (see (1.2)) gives the Kelvin circulation theorem

$$\frac{d}{dt}\oint_{\gamma_t} v^\flat = \oint_{\gamma_t} T\mathbf{d}s,$$

where $\gamma_t \subset M$ is a closed curve which moves with the fluid velocity v, and $T = \partial e/\partial s$ is the temperature.

7.1. Kelvin–Noether theorem for semidirect products. In order to apply this theorem to the Yang–Mills fluid, we recall some facts about the Kelvin–Noether theorem for semidirect products (see [**7**] for details).

We start with a Lagrangian L_{a_0} depending on a parameter $a_0 \in V^*$, as at the beginning of Section 5. We introduce a manifold $\mathcal{C}$ on which G acts on the left and suppose we have an equivariant map $\mathcal{K} : \mathcal{C} \times V^* \to \mathfrak{g}^{**}$, that is, for all $g \in G, a \in V^*, c \in \mathcal{C}$, we have

$$\langle \mathcal{K}(gc, \rho_g^*(a)), \mu\rangle = \langle \mathcal{K}(c, a), \operatorname{Ad}_g^* \mu\rangle,$$

where gc denotes the action of G on $\mathcal{C}$.

Define the Kelvin–Noether quantity $I : \mathcal{C} \times \mathfrak{g} \times TQ \times V^* \to \mathbb{R}$ by

$$I(c, \xi, q, \dot{q}, a) := \left\langle \mathcal{K}(c, a), \frac{\delta l}{\delta \xi}(\xi, q, \dot{q}, a) \right\rangle .$$

Theorem 7.1 (Kelvin–Noether.). *Fixing $c_0 \in \mathcal{C}$, let $\xi(t), q(t), \dot{q}(t), a(t)$ satisfy the Euler-Poincaré equations and define $g(t)$ to be the solution of $\dot{g}(t) = TR_{g(t)}\xi(t)$ and, say, $g(0) = e$. Let $c(t) = g(t)c_0$ and $I(t) := I(c(t), \xi(t), q(t), \dot{q}(t), a(t))$. Then*

$$\frac{d}{dt} I(t) = \left\langle \mathcal{K}(c(t), a(t)), \frac{\delta l}{\delta a} \diamond a \right\rangle .$$

7.2. The Kelvin–Noether theorem for Yang–Mills fluids. In the case of the Yang–Mills fluid, we shall choose for the abstract Lie group G above, the automorphism group $\mathcal{A}ut(P)$ and we let $\mathcal{C} = \{c \in \mathcal{F}(S^1, P) \mid \pi \circ c \in \operatorname{Emb}(S^1, M)\}$, where $\operatorname{Emb}(S^1, M)$ denotes the manifold of all embeddings of the circle S^1 in M. The left action of $\mathcal{A}ut(P)$ on $\mathcal{C}$ is given by $c \mapsto \varphi \circ c$. The map $\mathcal{K}$ is defined by

$$\langle \mathcal{K}(c, (\rho, s)), \mathbf{M} \rangle := \oint_c \frac{1}{\rho \circ \pi} \mathbf{M}, \quad \mathbf{M} \in \Omega^1_G(P).$$

A change of variables in the integral shows that $\mathcal{K}$ is equivariant, that is,

$$\langle \mathcal{K}(\varphi \circ c, (J\overline{\varphi}^{-1}(\rho \circ \overline{\varphi}^{-1}), s \circ \overline{\varphi}^{-1}), \mathbf{M} \rangle = \langle \mathcal{K}(c, (\rho, s)), \operatorname{Ad}^*_\varphi \mathbf{M} \rangle.$$

Using the Lagrangian l given in (5.6), we have

$$\frac{\delta l}{\delta U}(U, \mathcal{A}_0, \dot{\mathcal{A}}_0, \mathcal{A}, \dot{\mathcal{A}}, (\rho, s)) = \pi^*(\rho g([U], \cdot)) + \gamma(\mathcal{Q}, \mathcal{A}(\cdot)).$$

Therefore, the Kelvin–Noether quantity is

$$I(c, U, \mathcal{A}_0, \dot{\mathcal{A}}_0, \mathcal{A}, \dot{\mathcal{A}}, (\rho, s)) = \oint_{\overline{c}} [U]^\flat + \oint_c \frac{1}{\rho \circ \pi} \gamma\left(\mathcal{Q}, \mathcal{A}(\cdot)\right),$$

where $\overline{c} := \pi \circ c \in \operatorname{Emb}(S^1, M)$.

On the other hand, using the equality

$$\frac{\delta l}{\delta(\rho, s)} \diamond (\rho, s) = T^*\pi \left(\rho \, \mathbf{d} \frac{\delta l}{\delta \rho} - \frac{\delta l}{\delta s} \mathbf{d} s \right),$$

we get

$$\left\langle \mathcal{K}(c, (\rho, s)), \frac{\delta l}{\delta(\rho, s)} \diamond (\rho, s) \right\rangle = \oint_{\overline{c}} \frac{\partial e}{\partial s} \mathbf{d} s.$$

Thus, by Theorem 7.1, the *Kelvin circulation theorem for the Yang–Mills fluid* is

$$\frac{d}{dt} \left[\oint_{\overline{\gamma}_t} v^\flat + \oint_{\gamma_t} \frac{1}{\rho \circ \pi} \gamma\left(\mathcal{Q}, \mathcal{A}(\cdot)\right) \right] = \oint_{\overline{\gamma}_t} T \, \mathbf{d} s, \tag{7.1}$$

where $\gamma_t := \varphi_t \circ c_0 \subset P$, $\overline{\gamma}_t = \overline{\varphi_t} \circ \overline{c}_0 \subset M$ is a closed curve which moves with the fluid velocity $v := [U]$, and $T := \partial e/\partial s$ is the temperature.

When the principal bundle is trivial, formula (7.1) reads

$$\frac{d}{dt}\left[\oint_{\overline{\gamma}_t}\left(v^\flat + \frac{1}{\rho}\gamma\left(Q, \overline{\mathcal{A}}(\cdot)\right)\right) + \oint_{\gamma_t}\frac{1}{\rho}\gamma(Q,\cdot)\right] = \oint_{\overline{\gamma}_t} T\ \mathbf{d}s.$$

For the Euler–Maxwell fluid consisting of particles of mass m and charge q, since $Q = \rho\frac{q}{m}$, the second integral vanishes and we get

$$\frac{d}{dt}\oint_{\overline{\gamma}_t}\left(v^\flat + \frac{q}{m}A\right) = \oint_{\overline{\gamma}_t} T\ \mathbf{d}s,$$

which coincides with formula (7.37) in [**7**].

Acknowledgments

We thank Jerry Marsden for drawing our attention to this problem and for many illuminating discussions. Our thanks go to Marco Castrillón-López, Darryl Holm, Juan-Pablo Ortega, and the anonymous referee for several remarks that improved our exposition.

References

[1] V.I. Arnold, *Sur la géométrie différentielle des groupes de Lie de dimenson infinie et ses applications à l'hydrodynamique des fluides parfaits*, Ann. Inst. Fourier, Grenoble **16** (1966), 319–361.

[2] R. Abraham and J.E. Marsden, *Foundations of mechanics*, Benjamin-Cummings Publ. Co., Updated 1985 version, reprinted by Perseus Publishing, second edition, 1978.

[3] D.G. Ebin and J.E. Marsden, *Groups of diffeomorphisms and the motion of an incompressible fluid*, Ann. Math. **92** (1970), 102–163.

[4] J.E. Marsden, and A. Weinstein, *Coadjoint orbits, vortices and Clebsch variables for incompressible fluids*, Physica D **7** (1983), 305–323.

[5] J.E. Marsden, A. Weinstein, T.S. Ratiu, R. Schmid and R.G. Spencer, *Hamiltonian system with symmetry, coadjoint orbits and Plasma physics*, Proc. IUTAM-IS1MM Symposium on Modern Developments in Analytical Mechanics (Torino, 1982), Atti della Acad. della Sc. di Torino, **117** (1982), 289–340.

[6] J.E. Marsden, T.S. Ratiu and A. Weinstein, *Semidirect product and reduction in mechanics*, Trans. Amer. Math. Soc. **281** (1984), 147–177.

[7] D.D. Holm J.E. Marsden and T.S. Ratiu, *The Euler–Poincaré equations and semidirect products with applications to continuum theories*, Adv. Math. **137** (1998), 1–81.

[8] H. Cendra, J.E. Marsden and T.S. Ratiu, *Lagrangian reduction by stages*, Mem. Amer. Math. Soc. **152** (722) (2001).

[9] J.E. Marsden and T.S. Ratiu, *Introduction to mechanics and symmetry*, Texts in Applied Math., **17**, Springer-Verlag, 1994; Second ed., 1999, second printing 2003.

[10] J. Gibbons, D.D. Holm and B. Kupershmidt, *The Hamiltonian structure of classical chromohydrodynamics*, Physica D **6** (1983), 179–194.

[11] B. Bistrovic, R. Jackiw, H. Li, V.P. Nair and S.-Y. Pi, *Non-abelian fluid dynamics in Lagrangian formulation*, Phys. Rev. D **67**(2) (2003), 025013.

[12] R. Jackiw, V.P. Nair, S.-Y. Pi and A.P. *Polychronakos, Perfect fluid theory and its extensions*, J. Phys. A **37** (42) (2004), R327–R432.

[13] M. Castrillón-López, T.S. Ratiu and S. Shkoller, *Reduction in principal fiber bundles: covariant Euler–Poincaré equations*, Proc. Amer. Math. Soc. **128** (2000), 2155–2164.

[14] B.A. Bambah, S.M. Mahajan and C. Mukku, *Yang–Mills magnetofluid unification*, Phys. Rev. Lett. **97** (2006), 072301.

[15] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Wiley, 1963.

[16] D. Bleecker, *Gauge theory and variational principles*, Addison-Wesley Publ. Co., 1981.

[17] A. Kriegl and P.W. Michor, *The convenient setting of global analysis*, Mathematical Surveys and Monographs, **53**, Amer. Math. Soc., Providence, RI, 1997.

[18] J.E. Marsden, G. Misiołek, J.-P. Ortega, M. Perlmutter and T.S. Ratiu, *Hamiltonian reduction by stages*, Springer Lecture Notes in Math., **1913**, Springer-Verlag, 2007.

[19] C.W. Misner, K.S. Thorne and J.A. Wheeler, *Gravitation*, W.H. Freeman and Co., New York, 1973.

[20] M. Gotay, J. Isenberg and J.E. Marsden, *Momentum maps and classical relativistic fields, Part I: Covariant field theory*, 1997, available at: www.arxiv.org: [2004] physics/9801019.

[21] M. Gotay, J. Isenberg and J.E. Marsden, *Momentum maps and classical relativistic fields, Part II: Canonical analysis of field theories*, 1999, available at: www.arxiv.org: [2004] math-ph/0411032.

[22] J. Arms, *Linearization stability of gravitational and gauge fields*, J. Math. Phys. **20** (1979), 443–453.

[23] J.M. Arms, *The structure of the solution set for the Yang–Mills equations*, Math. Proc. Camb. Philos. Soc. **90** (1981), 361–372.

[24] J.M. Arms, J.E. Marsden and V. Moncrief, *The structure of the space solutions of Einstein's equations: II Several Killing fields and the Einstein–Yang–Mills equations*, Ann. Phys. **144** (1982), 81–106.

[25] S.K. Wong, *Field and particle equations for the classical Yang–Mills field and particles with isotopic spin*, Nuovo Cimento **65A** (1970), 689–693.

[26] R. Montgomery, *Canonical formulations of a classical particle in a Yang-Mills field and Wong's equations*, Lett. Math. Phys. **8** (1984), 59-67

[27] H. Cendra, D.D. Holm, M.J.W. Hoyle and J.E. Marsden, *The Maxwell–Vlasov equations in Euler–Poincaré form*, J. Math. Phys. **39** (1998), 3138–3157.

[28] D.D. Holm, J.E. Marsden and T.S. Ratiu, *The Euler-Poincaré equations in geophysical fluid dynamics*, in 'Large scale atmosphere-ocean dynamics II' (J. Norbury, I. Roulstone, eds.), Cambridge University Press, Cambridge, 2002, 251–300.

[29] J.-P. Ortega and T.S. Ratiu, *Momentum maps and Hamiltonian reduction*, Progress in Mathematics, 222, Birkhäuser, Boston, 2004.

[30] R. Montgomery, *The bundle picture in mechanics*, Ph.D. thesis, University of california, Berkeley, 1986.

[31] C.H. Gu and C.N. Yang, *Some problems on the gauge field theories. II*, Sci. Sinica **20** (1) (1977), 47–55.

[32] M.A. Mostow, *The field copy problem: to what extent do curvature (gauge field) and its covariant derivatives determine connection (gauge potential)*? Comm. Math. Phys. **78** (1) (1980), 137–150.

Section de Mathématiques and Bernoulli Center
École Polytechnique Fédérale de Lausanne
CH-1015 Lausanne, Switzerland
E-mail address: francois.gay-balmaz@epfl.ch
E-mail address: tudor.ratiu@epfl.ch

Received 4/2/2007, accepted 2/13/2008
Partially supported by Swiss NSF grant 200021-109111.