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Geometric variants of the Hofer norm

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This note discusses some geometrically defined seminorms on the group Ham (M,ω) of Hamiltonian diffeomorphisms of a closed symplectic manifold (M,ω), giving conditions under which they are nondegenerate and explaining their relation to the Hofer norm. As a consequence we show that if an element in Ham (M,ω) is sufficiently close to the identity in the C^2-topology then it may be joined to the identity by a path whose Hofer length is minimal among all paths, not just among paths in the same homotopy class relative to endpoints. Thus, true geodesics always exist for the Hofer norm. The main step in the proof is to show that a "weighted" version of the nonsqueezing theorem holds for all fibrations over S^2 generated by sufficiently short loops. Further, an example is given showing that the Hofer norm may differ from the sum of the one sided seminorms.

1. Introduction.

This paper considers some foundational questions about seminorms on the Hamiltonian group that were raised in Polterovich's lovely book [20]. The interest of these seminorms lies in their geometric interpretation in terms of the minimal area, or equivalently curvature, of associated fibered spaces over the 2-disc D and 2-sphere S^2. This allows one to use geometric methods to find lower bounds for these seminorms and hence also for the usual Hofer norm. The main question, to which we give only a partial answer, is whether they are norms.

Our approach gives just one way of measuring the size $\rho^+(\phi)$ of "one side" of a Hamiltonian symplectomorphism ϕ but there are three associated two sided seminorms, the largest of which is the Hofer norm ρ. The middle one, which we shall call ρ_f, is more natural from a geometric point of view, and we shall see that it is always nondegenerate. However we can prove the nondegeneracy of the smallest one, which is the sum of the one sided

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seminorms $\rho^+(\phi)+\rho^+(\phi^{-1})$, only in special cases, for example if $M=\mathbb{C}P^n$ or is weakly exact.

Lalonde–McDuff [6] established the nondegeneracy of the Hofer norm for arbitrary (even open) (M,ω) by using the nonsqueezing theorem for the product $(M\times S^2, \omega+dx\wedge dy)$. In order to extend this to the more general seminorms considered here, we need to understand which nontrivial symplectic fibrations $M\to P\to S^2$ still have the nonsqueezing property. As explained in §4 we tackle this question by looking at the modified Seidel representation:

$$\Psi:\pi_1(\mathrm{Ham}\,(M))\to (QH_{\mathrm{ev}}\,(M))^\times,$$

where $(QH_{\mathrm{ev}}\,(M))^\times$ denotes the commutative group of multiplicative units in the even part of the quantum homology ring of M.

In this introduction we first define the seminorms and their geodesics, then give their geometric interpretation, and finally discuss short loops and nonsqueezing. A different version of some of these results is being developed by Oh (see [15]) using the action functional of Floer homology. It would be very interesting to work out the relation between his new norm and the ones discussed here. We will assume throughout that (M,ω) is a closed connected symplectic manifold. Further we write $\mathrm{Ham}=\mathrm{Ham}\,(M,\omega)$ for the group of Hamiltonian symplectomorphisms and $\widetilde{\mathrm{Ham}}=\widetilde{\mathrm{Ham}}\,(M,\omega)$ for its universal cover.

1.1. The family of seminorms.

Each Hamiltonian $H_t: M\to\mathbb{R}, 0\le t\le 1$, defines a family of vector fields X_t (often called the **symplectic gradient**) given by[2]

$$\omega(X_t,\cdot)=-dH_t$$

that integrates to a path $\phi_t^H, 0\le t\le 1$, in Ham starting at the identity id. Let

$$c_t=\frac{1}{n!}\int_M H_t\,\omega^n$$

[2]Note the sign. Since signs are crucial when studying one sided norms, we have chosen to make the signs consistent with those in Polterovich's book [20] even though this is different from our usual conventions.

be the corresponding set of mean values. We define the positive (resp. negative) length of H_t to be:

$$\begin{aligned}\mathcal{L}^+(\phi_t^H) &= \mathcal{L}^+(H_t) &= \textstyle\int_0^1 (\max_{x\in M} H_t(x) - c_t)\, dt,\\ \mathcal{L}^-(\phi_t^H) &= \mathcal{L}^-(H_t) &= \textstyle\int_0^1 (c_t - \min_{x\in M} H_t(x))\, dt.\end{aligned}$$

These measurements of the length of paths give rise to seminorms on the groups $\widetilde{\mathrm{Ham}}$ and Ham as follows. Recall that an element $\widetilde{\phi}$ of $\widetilde{\mathrm{Ham}}$ is a homotopy class of paths from id to some element $\phi \in \mathrm{Ham}$.

Definition 1.1. We define $\widetilde{\rho}^+(\widetilde{\phi})$ (resp. $\widetilde{\rho}^-(\widetilde{\phi})$) to be the infimum of $\mathcal{L}^+(\phi_t^H)$ (resp. $\mathcal{L}^-(\phi_t^H)$) taken over all paths in the homotopy class $\widetilde{\phi}$. The corresponding seminorms $\rho^\pm(\phi)$ on Ham are defined by taking the infimum of $\mathcal{L}^\pm$ over **all** paths in Ham from id to ϕ.[3]

The seminorms $\nu = \widetilde{\rho}^\pm, \rho^\pm$ have the property that for all elements g, h of the appropriate group $\mathrm{G} = \widetilde{\mathrm{Ham}}$ or Ham

$$\nu(gh) \le \nu(g) + \nu(h), \qquad \nu(g) = \nu(hgh^{-1}).$$

Hence the corresponding "distance function"

$$d_\nu(g,h) = d_\nu(id, hg^{-1}) = \nu(hg^{-1})$$

is invariant under multiplication from both the left and the right. However, the seminorms ν under present consideration are not symmetric (i.e., invariant under taking the inverse), though clearly

$$\rho^+(\phi) = \rho^-(\phi^{-1}), \qquad \widetilde{\rho}^+(\widetilde{\phi}) = \widetilde{\rho}^-(\widetilde{\phi}^{-1}).$$

Correspondingly $d_\nu(g,h)$ need not equal $d_\nu(h,g)$.

These seminorms ν are one sided in the sense that they only measure the size of "half" of H_t, either the part $H_t - c_t$ above the mean, or the part below. We now discuss several different ways to obtain two sided norms. By definition these are symmetric, i.e., invariant under taking the inverse, so that the corresponding pseudometrics will be biinvariant.

The usual Hofer norm ρ on Ham is given by minimizing[4] the sum

$$\mathcal{L}(H_t) := \mathcal{L}^+(H_t) + \mathcal{L}^-(H_t);$$

[3] By using suitable reparametrizations, it is not hard to see that one gets the same seminorms if one defines $\mathcal{L}^+(H_t)$ (resp. $\mathcal{L}^-(H_t)$) to be the maximum of $H_t(x) - c_t$ (resp. $c_t - H_t(x)$) over all t, x: see Polterovich [19].

[4] By slight abuse of language we often use the word "minimizing" in situations when the minimum (or more properly infimum) is not attained.

we write $\widetilde{\rho}$ for the associated seminorm on $\widetilde{\mathrm{Ham}}$. Thus $\widetilde{\rho}(\widetilde{\phi})$ is the infimum of $\mathcal{L}^+(H_t)+\mathcal{L}^-(H_t)$ over all paths in the homotopy class $\widetilde{\phi}$, while $\rho(\phi)$ is its infimum over all paths from id to ϕ. Clearly,

$$\widetilde{\rho}(\widetilde{\phi}) = \widetilde{\rho}(\widetilde{\phi}^{-1}), \qquad \rho(\phi) = \rho(\phi^{-1}).$$

Although ρ is known to be a norm for all M (cf [6]), it is not clear whether $\widetilde{\rho}$ is always a norm. Its null set

$$\mathrm{null}\,(\widetilde{\rho}) = \{\widetilde{\phi} : \widetilde{\rho}(\widetilde{\phi}) = 0\}$$

is a normal subgroup, and so must lie in the kernel $\pi_1(\mathrm{Ham}\,)$ of the covering map $\widetilde{\mathrm{Ham}} \to \mathrm{Ham}$.[5] It is conceivable that $\mathrm{null}\,(\widetilde{\rho})$ is nonempty for some M. For example, there might be nontrivial elements of $\pi_1(\mathrm{Ham}\,(M,\omega))$ that are supported in a Darboux chart. Such elements lie in $\mathrm{null}\,(\widetilde{\rho})$ since the size of their generating Hamiltonian can be made arbitrarily small by a suitable conformally symplectic conjugation. They do not exist when when $\dim(M) \le 4$ since the group of compactly supported symplectomorphisms of Euclidean space is contractible in these dimensions. However, they might exist in higher dimensions. In any case, $\widetilde{\rho}$ descends to a norm on the quotient group $\widetilde{\mathrm{Ham}}\,/\mathrm{null}\,(\widetilde{\rho})$.

We will also consider the two sided seminorm on $\widetilde{\mathrm{Ham}}$ given by the sum $\widetilde{\rho}^{\,+} + \widetilde{\rho}^{\,-}$. It is easy to see that

$$\widetilde{\rho}(\widetilde{\phi}) \ge \widetilde{\rho}^{\,+}(\widetilde{\phi}) + \widetilde{\rho}^{\,-}(\widetilde{\phi}).$$

In principle, there could be strict inequality here since the one sided seminorms $\widetilde{\rho}^{\,+}(\widetilde{\phi})$ and $\widetilde{\rho}^{\,-}(\widetilde{\phi})$ could well be realized by different minimizing sequences of Hamiltonians. However we have so far found no example to illustrate this possibility.[6] Sometimes we will write $\widetilde{\rho}_f$ instead of $\widetilde{\rho}^+ + \widetilde{\rho}^-$.

There are two possible ways to obtain a related seminorm on Ham; one may either consider the sum $\rho^+ + \rho^-$ or consider the seminorm ρ_f on Ham induced by $\widetilde{\rho}^{\,+} + \widetilde{\rho}^{\,-}$, viz:

$$\rho_f = \inf\{\widetilde{\rho}^{\,+}(\widetilde{\phi}) + \widetilde{\rho}^{\,-}(\widetilde{\phi}) : \widetilde{\phi} \text{ lifts } \phi\}.$$

[5] Recall that $\mathrm{Ham}\,(M)$ is a simple group when M is closed. Thus a symmetric, conjugation invariant seminorm on $\mathrm{Ham}\,(M)$ must be either nondegenerate or everywhere zero.

[6] In [4] Entov considers the seminorm $\max(\widetilde{\rho}^{\,+}, \widetilde{\rho}^{\,-})$. This is clearly equivalent to $\widetilde{\rho}^{\,+} + \widetilde{\rho}^{\,-}$, and it is easy to check that it has the same geodesics.

Both $\rho^+ + \rho^-$ and ρ_f are symmetric. Further

$$\rho^+(\phi) + \rho^-(\phi) \le \rho_f(\phi) \le \rho(\phi).$$

Polterovich asked in [20] whether the seminorm $\rho^+ + \rho^-$ must always be a norm, and indeed whether it always equals ρ. The proof given in [7] that ρ is a norm adapts almost immediately to show that ρ_f is always a norm. However, the question for $\rho^+ + \rho^-$ is much more difficult and revolves around properties of $\pi_1(\mathrm{Ham})$ and the quantum homology of M that are discussed in more detail in §1.4. We have only succeeded in showing that it is a norm in special cases, some of which are mentioned in the next result. Recall that (M, ω) is said to be **weakly exact** if ω vanishes on the image $H_2^S(M, \mathbb{Z})$ of $\pi_2(M)$ in $H_2(M, \mathbb{Z})$.

Theorem 1.2.

(i) *ρ_f is a norm for all closed M.*

(ii) *$\rho^+ + \rho^-$ is a norm if M is weakly exact or $M = \mathbb{C}P^n$.*

(iii) *There is a symplectic form ω on the one point blow up M_* of $\mathbb{C}P^2$ such that $\rho^+ + \rho^-$ is a norm on* $\mathrm{Ham}\,(M_*, \omega)$ *that is distinct from ρ_f and hence also from ρ.*

We also show that $\rho^+ + \rho^-$ is a norm whenever there are no asymmetric short loops in the sense of Definition 1.14 below. Other conditions under which $\rho^+ + \rho^-$ is a norm are given later. For example, it would suffice that the nonsqueezing theorem hold for all $[\lambda] \in \pi_1(\mathrm{Ham}\,(M, \omega))$: see Corollary 1.7 and Lemma 1.16. The example in (iii) is discussed in more detail in §1.5.

The question of whether one sided seminorms such as ρ^+ are nondegenerate seems intractable. Since the corresponding null set is a conjugation invariant semigroup rather than a normal subgroup, the simplicity of the group $\mathrm{Ham}\,(M, \omega)$ is now of no help: to prove nondegeneracy one must find a lower bound for $\rho^+(\phi)$ for *every* element $\phi \in \mathrm{Ham}\,(M, \omega)$, a task beyond the reach of the techniques used here.

Remark 1.3 (Noncompact manifolds). The above definitions are generalizations of notions first introduced for Euclidean space $(\mathbb{R}^{2n}, \omega_0)$. More generally, given any noncompact manifold (M, ω) (without boundary) let $\mathrm{Ham}^c(M, \omega)$ be the group of compactly supported Hamiltonian symplectomorphisms and consider Hamiltonians H with compact support. Then set

$$\mathcal{L}^+(H_t) := \int \max(H_t)\, dt, \qquad \mathcal{L}^-(H_t) := -\int \min(H_t)\, dt,$$

and use these length measurements to define seminorms just as before. Clearly, in this case there are elements $\phi \neq id$ such that $\rho^+(\phi) = 0$, for example the time 1 map of a nonzero function H_t with $H_t \leq 0$. One can recover a situation more like the closed case by restricting attention to the subgroup $\mathrm{Ham}\,_0^c(M, \omega)$ generated by compactly supported Hamiltonians of zero mean. This subgroup, the kernel of the Calabi homomorphism, is simple just as in the closed case, and even in the case $M = \mathbb{R}^{2n}$ it is not yet known whether the seminorms $\rho^\pm$ are nondegenerate on it.

On the other hand a few of the questions considered here are more tractable when $M = \mathbb{R}^{2n}$. For example, the sum $\rho^+ + \rho^-$ is always nondegenerate. This was first proved by Viterbo [24] who used generating functions to construct a section $\phi \mapsto c(\phi)$ of the action spectrum bundle with the property that

$$0 < c(\phi) \leq \rho^+(\phi) + \rho^-(\phi), \quad \text{when} \quad \phi \neq id.$$

It also follows from our arguments since the fact that the elements of $\pi_1(\mathrm{Ham}\,^c(\mathbb{R}^{2n}, \omega_0))$ have representatives with arbitrarily short Hofer length implies that $\rho^+ + \rho^- = \rho_f$: see §1.4. However, it is still unknown whether the two norms $\rho^+ + \rho^-$ and ρ must always agree, even when $M = \mathbb{R}^2$ or $\mathbb{R}^4$. Note also that when M is Euclidean space there are several other possible one sided seminorms arising from various selectors and it is not clear that $\rho^\pm$ are the most interesting ones to consider. For further discussion see Polterovich [20] and Schwarz [21].

1.2. Results on Geodesics.

We now describe our results on geodesics. It is possible to define geodesics as critical points of a suitable length functional: see [20, 7]. However the lack of smoothness of this functional causes some problems. We will take a different approach that nevertheless gives rise to the same geodesics. We write G to denote one of the groups $\widetilde{\mathrm{Ham}}$, Ham.

Observe that the seminorms under consideration are of two kinds. If ν is one of $\widetilde{\rho}^\pm, \widetilde{\rho}, \rho^\pm$ and ρ then $\nu(g)$ is the infimum of an appropriate length functional $\mathcal{L}_\nu$ over all paths in G from id to g, while the value of the other seminorms $\widetilde{\rho}_f, \rho_f$ and $\rho^+ + \rho^-$ at an element g depend on minimizing the functionals $\mathcal{L}^\pm$ on two possibly different paths. If ν is one of the three latter seminorms we set $\mathcal{L}_\nu = \mathcal{L}$.

Definition 1.4. Let $(\nu, \mathcal{L}_\nu)$ be one of the pairs defined above. A path $\{g_t\}_{t\in[a,b]}$ in G is said to be **ν-minimizing** if it achieves the minimum of

$\mathcal{L}_\nu$, i.e., if

$$\nu(g_b g_a^{-1}) = d_\nu(g_a, g_b) = \mathcal{L}_\nu(\{g_t\}).$$

When defining geodesics, it is convenient to restrict attention to paths $\{g_t\}$ that are **regular** in the sense that their generating vector field $\dot{g}_t$ is never zero. We then say that a regular path $\{g_t\}_{t\in[a,b]}$ is a **ν-geodesic** from g_a to g_b if there is $\varepsilon > 0$ such that for all $t_0 \in [a,b]$ each path

$$[a,b] \cap [t_0, t_0 + \varepsilon] \to \mathrm{G} : \quad t \mapsto g_t g_{t_0}^{-1}$$

is ν-minimizing.

It follows immediately that any ρ_f or $(\rho^+ + \rho^-)$-geodesic is also a ρ-geodesic. Observe also this definition is somewhat stronger than is usual in this context (cf. [7, 20]) since it requires that short pieces of ν-geodesics minimize $\mathcal{L}_\nu$ among *all* paths with the given endpoints rather than simply among the homotopic paths. Though natural, it was not used previously for the group $\mathrm{G} = \mathrm{Ham}\,(M)$ because it was not known whether there always are paths satisfying this stronger condition. This is the problem of "short loops" that is discussed in §1.4. The main new step in the proof of Theorem 1.6 below is to get around this difficulty.

Before stating it, we recall some ideas from Bialy–Polterovich [2] and Lalonde–McDuff [7]. A path $\gamma = \{\phi_t\}_{t\in[a,b]}$ in Ham that is generated by the Hamiltonian H_t is said to have a **fixed maximum (minimum)** if there is a point $x_0 \in M$ such that

$$H_t(x) \quad \le (\ge) \quad H_t(x_0), \quad \text{for all } \ x \in M, \ t \in [a,b].$$

It is said to have a fixed maximum (minimum) **at each moment** if there is $\varepsilon > 0$ such that each subpath $\{\phi_t \phi_{t_0}^{-1}\}_{t\in[a,b]\cap[t_0,t_0+\varepsilon]}$ has a fixed maximum (minimum). It was shown in [2] that a path γ in $\mathrm{Ham}^c(\mathbb{R}^{2n})$ is a ρ-geodesic if and only if it has both a fixed maximum and a fixed minimum at each moment. The result proved in [7] for general M can be stated in our current language as follows:[7] a path γ in $\widetilde{\mathrm{Ham}}\,(M,\omega)$ is a $\widetilde{\rho}$-geodesic if and only if

[7]The papers Lalonde–McDuff [7] were written before it was understood how to define Gromov–Witten invariants on arbitrary symplectic manifolds. Therefore, many of the results in part II have unnecessary restrictions. In particular, in Theorems 1.3 (i) and 1.4 and in Propositions 1.14 and 1.19 (i) one can remove the hypothesis that M has dimension ≤ 4 or is semi-monotone since the proofs are based on the fact that quasicylinders $Q = (M \times D^2, \Omega)$ have the nonsqueezing property. The result we now quote from [7] incorporates these improvements. All the details of its proof (besides the construction of general Gromov–Witten invariants) occur in [7]: see §3 below. Oh gives a new proof in [15].

it has both a fixed maximum and a fixed minimum at each moment. This generalizes to the one sided seminorms on $\widetilde{\mathrm{Ham}}$ as follows.

Proposition 1.5. *Let $\gamma = \{\phi_t\}_{t\in[a,b]}$ be a path in* $\mathrm{Ham}\,(M)$.

(i) *A lift of γ to $\widetilde{\mathrm{Ham}}$ is a geodesic with respect to $\widetilde{\rho}^{\,+}$ (resp. $\widetilde{\rho}^{\,-}$) if and only if γ has a fixed maximum (resp. minimum) at each moment;*

(ii) *A lift of γ to $\widetilde{\mathrm{Ham}}$ is a geodesic with respect to $\widetilde{\rho}^{\,+} + \widetilde{\rho}^{\,-}$ if and only if γ has both a fixed maximum and a fixed minimum at each moment.*

It is much harder to get results on $\mathrm{Ham}\,(M)$. Here is our main result.

Theorem 1.6. *Let $\gamma = \{\phi_t\}_{t\in[a,b]}$ be a path in* $\mathrm{Ham}\,(M)$.

(i) *If ν is one of the norms ρ or ρ_f on $\mathrm{Ham}\,(M)$, then γ is a ν-geodesic if and only if it has both a fixed maximum and a fixed minimum at each moment.*

(ii) *If $M = \mathbb{C}P^n$ or is weakly exact, the same result holds with $\nu = \rho^+ + \rho^-$.*

Corollary 1.7. *ρ_f is nondegenerate for all M, while $\rho^+ + \rho^-$ is nondegenerate when M satisfies the conditions in (ii) above.*

Proof. Because $\mathrm{Ham}\,(M)$ is a simple group, a symmetric conjugation invariant seminorm ν on $\mathrm{Ham}\,(M)$ is either identically zero or is nondegenerate. But in the former case every path in Ham would be a geodesic. Hence, the existence of nongeodesic paths implies that ν is nondegenerate. □

The proof of the above theorem finds lower bounds for $\nu(\phi)$ when ϕ is close to the identity by using the geometric characterization of the seminorms given in §1.3 and suitable extensions of the nonsqueezing theorem. Further, we generalize [2, 7] by proving the following local flatness result for a neighborhood of id in $\mathrm{Ham}\,(M,\omega)$. Recall that $\mathrm{Ham}\,(M,\omega)$ is the kernel of a surjective homomorphism

$$\mathrm{Flux}: \quad \mathrm{Symp}_0(M,\omega) \to H^1(M,\mathbb{R})/\Gamma_\omega$$

where the flux group Γ_ω is finitely generated but is not known to be discrete in all cases. Hence the most we can say in general is that $\mathrm{Ham}\,(M,\omega)$ sits inside the identity component Symp_0 as the leaf of a foliation. Therefore, we do not use the topology on Ham induced from Symp_0 but instead use the topology on Ham induced from the C^2-topology on the Lie algebra of Hamiltonian functions with zero mean. Thus a neighborhood of the identity consists of all time 1-maps of Hamiltonian flows generated by Hamiltonians H_t that are sufficiently small in the C^2 topology.

Proposition 1.8. *There is a path connected neighborhood $\mathcal{N} \subset \mathrm{Ham}(M)$ of the identity in the C^2-topology such that any element $\phi \in \mathcal{N}$ can be joined to the identity by a path that minimizes both ρ and ρ_f. In particular, these two norms agree on $\mathcal{N}$. Moreover, $(\mathcal{N}, \rho)$ is isometric to a neighborhood of $\{0\}$ in a normed vector space.*

One can also look for longer length minimizing paths starting from *id*. Note that one cannot reach an arbitrary $\phi \in \mathrm{Ham}(M)$ by such a path: an example is given in [7] II of an element in $\mathrm{Ham}(S^2)$ that cannot be reached by any ρ-minimizing path. Moreover, since $(\rho^+ + \rho^-)$-geodesics are the same as ρ-geodesics (when they exist), any path that minimizes $\rho^+ + \rho^-$ also minimizes ρ. Hence elements τ with $0 \neq \rho^+(\tau) + \rho^-(\tau) < \rho(\tau)$ as in Theorem 1.2(iii) cannot be reached by $(\rho^+ + \rho^-)$-minimizing paths.[8]

The following result is a mild extension of work by Entov [4] and McDuff–Slimowitz [13]. (See also Oh [15] where the result is generalized to some time dependent paths.) We will say that a time independent Hamiltonian H is **slow** if neither its flow nor the linearized flows at its critical points have nonconstant contractible periodic orbits of period < 1.

Proposition 1.9. *Let H be a slow Hamiltonian. Then the path $\phi_t^H, 0 \le t \le 1$ minimizes both $\widetilde{\rho}^-$ and $\widetilde{\rho}^+$ and hence also minimizes $\widetilde{\rho}$ on $\widetilde{\mathrm{Ham}}(M)$. If in addition (M, ω) is weakly exact, this path minimizes all the norms $\rho^+ + \rho^-$, ρ_f and ρ on $\mathrm{Ham}(M)$.*

1.3. Geometric interpretations of the seminorms.

Consider a smooth fibration $\pi : P \to B$ with fiber M, where B is either S^2 or the 2-disc D. Here we consider S^2 to be the union $D_+ \cup D_-$ of two copies of D, with the same orientation as D_+. We denote the equator $D_+ \cap D_-$ by ∂, oriented as the boundary of D_+, and choose some point $*$ on ∂ as the base point of S^2. Similarly, $B = D$ is provided with a basepoint $*$ lying

[8] In fact, such an element τ cannot be reached even by a $(\rho^+ + \rho^-)$-minimizing sequence γ_i of paths, i.e. such that $\rho^+(\tau) + \rho^-(\tau)$ is the limit of $\mathcal{L}^+(\gamma_i) + \mathcal{L}^-(\gamma_i)$. Similarly, if there were an element ϕ such that $\rho_f(\phi) \neq \rho(\phi)$ then it could not be reached by a ρ_f-minimizing sequence of paths.

on $\partial = \partial D$. In both cases, we assume that the fiber over $*$ has a chosen identification with M.

In this paper we will be considering triples (P, π, Ω) where $\pi : P \to B$ is a fibration as above and Ω is a normalized π-compatible symplectic form on P. π-compatibility means that the restriction ω_b of Ω to the fiber $M_b = \pi^{-1}(b)$ is nondegenerate for each $b \in B$, and the normalization condition is that $\omega_* = \omega$. For short we will write (P, Ω) instead of the triple (P, π, Ω), and will use the words "ω-compatible" instead of "normalized π-compatible."

In §2.2 we will describe in more detail the geometric structure that is induced by Ω on the fibration $\pi : P \to B$. The most important point is that Ω defines a connection on π whose horizontal distribution is Ω-orthogonal to the fibers. If α is any path in B then $\pi^{-1}(\alpha)$ is a hypersurface in P whose characteristic foliation consists of the horizontal lifts of α, and it is not hard to check that the resulting holonomy is Hamiltonian round every contractible loop. Because B is simply connected, it follows that the structural group of π can be reduced to $\mathrm{Ham}\,(M, \omega)$. Further, π can be symplectically trivialized over each disc D by parallel translation along a suitable set of rays. This means that there is a fiber preserving mapping

$$\Phi : \pi^{-1}(D) \to M \times D, \quad \Phi|_{M_*} = id_M$$

such that the pushforward $\Phi_* \Omega$ restricts to the same form ω on each fiber $M \times pt$. The fibration $(P, \Omega) \to S^2$ is said to be **symplectically trivial**[9] if such a map exists from P to $M \times S^2$.

Definition 1.10. The **monodromy** $\phi = \phi(P) \in \mathrm{Ham}\,(M)$ of a fibration $(P, \Omega) \to B$ is defined to be the monodromy of the connection determined by Ω around the based loop $(\partial, *)$. Using the trivialization of P over ∂ provided by B itself if $B = D$ or by D_+ if $B = S^2$, one gets a well defined lift $\widetilde{\phi}$ of ϕ to $\widetilde{\mathrm{Ham}}$. Sometimes we will write $P_{\widetilde{\phi}}$ (resp. P_ϕ) for a fibration $(P, \Omega) \to B$ with monodromy $\widetilde{\phi}$ (resp. ϕ).

The next definition describes various different area measurements.

Definition 1.11. The **area** of a fibration $(P, \Omega) \to B$ is defined to be:

$$\mathrm{area}\,(P, \Omega) = \frac{\mathrm{vol}\,(P, \Omega)}{\mathrm{vol}\,(M, \omega)} = \frac{\int_P \Omega^{n+1}}{(n+1) \int_M \omega^n}.$$

Further:

[9] The words "symplectically trivial" mean "trivial as a symplectic fibration." In the present context, this implies triviality as a Hamiltonian fibration.

(i) $\widetilde{a}^+(\widetilde{\phi})$ (resp. $a^+(\phi)$) is the infimum of area (P,Ω) taken over all ω-compatible symplectic forms Ω on the fibration $P \to D$ with monodromy $\widetilde{\phi}$ (resp. ϕ).

(ii) $\widetilde{a}(\widetilde{\phi})$ (resp. $a_f(\phi)$) is the infimum of area (P,Ω) taken over all symplectically trivial fibrations $(P,\Omega) \to S^2$ with monodromy $\widetilde{\phi}$ (resp. ϕ).

(iii) $a(\phi)$ is the infimum of area (P,Ω) taken over *all* fibrations $(P,\Omega) \to S^2$ with monodromy ϕ.

(iv) $\widetilde{a}^-(\widetilde{\phi}) = \widetilde{a}^+(\widetilde{\phi}^{-1})$ and $a^-(\phi) = a^+(\phi^{-1})$

It is easy to see that $a^+(\phi)$ (resp. $a_f(\phi)$) is the infimum of $\widetilde{a}^+(\widetilde{\phi})$ (resp. $\widetilde{a}(\widetilde{\phi})$) over all lifts $\widetilde{\phi}$ of ϕ to $\widetilde{\mathrm{Ham}}$. The following lemma amplifies Polterovich's results in [18].

Proposition 1.12.

(i) $\widetilde{\rho}^+(\widetilde{\phi}) = \widetilde{a}^+(\widetilde{\phi})$*;*

(ii) $\rho^+(\phi) + \rho^-(\phi) = a(\phi)$*;*

(iii) $\rho_f(\phi) = a_f(\phi)$.

This is proved in §2.2. We have here interpreted our seminorms in terms of area since this is what our methods estimate. However, as is clear from the proof of the above Proposition, these area measurements are equivalent to suitable measurements of curvature: see Polterovich's remarks about K-area in [16, 18] and also Entov [4].

The Hofer norm also has a geometric interpretation: $\rho(\phi)$ is the infimum of area(P_ϕ,Ω) taken over all fibrations (P_ϕ,Ω) for which there is a symplectomorphism

$$\Phi : (P_\phi,\Omega) \to (M \times S^2, \omega + \alpha)$$

that is the identity on the fiber over the base point and takes the hypersurface $\pi^{-1}(\delta)$ lying over the equator to a specially situated hypersurface Γ_H in the product. (The precise condition on Γ_H is described at the end of §2.2. Note that Φ need not preserve the given fibered structure on P_ϕ.) Even if we ignore the condition on Γ_H, we may be minimizing over a smaller set than in the definition of a_f, since it is not clear whether every ω-compatible form on the trivial fibration $M \times S^2 \to S^2$ is symplectomorphic to a product form. ¿From a geometric point of view the minimizing sets used to define $a(\phi)$ and $a_f(\phi)$ are much more natural than that used to define the Hofer norm. Thus $\rho^+ + \rho^-$ and ρ_f are the seminorms with the most geometric meaning. We consider ρ_f to be the geometric analog of the Hofer norm and hence have called it ρ_f where f denotes "fibered." Correspondingly we will sometimes write $\widetilde{\rho}_f$ instead of $\widetilde{\rho}^+ + \widetilde{\rho}^-$ for its lift to $\widetilde{\mathrm{Ham}}$.

Finally, let us consider the case when $\widetilde{\phi}$ is a lift of id; in other words, $\widetilde{\phi}$ is a homotopy class $[\lambda]$ of loops in Ham. Above we have measured $\widetilde{\rho}^+([\lambda])$ by means of the area of a fibration over D with boundary monodromy $[\lambda]$. On the other hand, the natural geometric object associated to a loop is a fibration over S^2 constructed by using the loop λ as a clutching function:

$$P_\lambda = \left(D_+^2 \times M\right) \cup_\lambda \left(D_-^2 \times M\right),$$

where[10]

$$\lambda : (2\pi t, x)_- \mapsto (2\pi t, \lambda_t(x))_+.$$

By fixing an identification of the fiber of P_λ at the basepoint $* \in \partial D_+$ with M, we can normalize the loop $\lambda = \{\lambda_t\}$ by requiring that $\lambda_* = id$. It is not hard to see that the symplectic form on the fibers has a closed extension to P_λ precisely when λ is homotopic to a loop in $\mathrm{Ham}\,(M,\omega)$. Thus there is a bijective correspondence between classes $[\lambda] \in \pi_1(\mathrm{Ham}\,(M,\omega))$ and Hamiltonian fibrations P_λ with one fiber identified with M. Hence, the most obvious geometric way to measure the size of $[\lambda]$ is to take the minimum area of ω-compatible forms Ω on this fibration $P_\lambda \to S^2$. The next lemma says that this gives nothing new.

Lemma 1.13. *For each $[\lambda] \in \pi_1(\mathrm{Ham})$, $\widetilde{\rho}^+([\lambda])$ is the infimum of* $\mathrm{area}(P_\lambda, \Omega)$*, where Ω ranges over all ω-compatible forms on the fibration $P_\lambda \to S^2$ constructed above.*

Observe here that even though $P_\lambda \to S^2$ is a fibration over the closed manifold S^2 the corresponding area measure is the *one sided* seminorm $\widetilde{\rho}^+([\lambda])$. One way to understand this is to consider P_λ as made by gluing together two fibrations, the first $(P_+, \Omega) \to D_+$ with monodromy λ and area $\geq \widetilde{\rho}^+([\lambda])$ and the second $(P_-, \Omega) \to D_-$ with trivial monodromy and hence arbitrarily small area.

[10] Note the direction of this attaching map: this is a different convention from [10, 11] though the same as [20].

1.4. Short loops and the Nonsqueezing theorem.

Though one can find lower bounds for the seminorms $\widetilde{\rho}, \widetilde{\rho}^{\pm}$ on $\widetilde{\mathrm{Ham}}$ by the methods of [7, 4, 13] it is usually more difficult to find lower bounds on Ham. There is one lower bound for $\rho(\phi)$ that is independent of the path from id to ϕ, namely the energy–capacity inequality:

> *Let $B = \phi(B^{2n}(r))$ be a symplectically embedded ball in (M, ω) of radius r. If $\phi(B) \cap B = \emptyset$, then $\rho(\phi) \geq \pi r^2/2$.*

This was proved for $\mathbb{R}^{2n}$ by Hofer [5] (without the constant $1/2$), and for any symplectic manifold in [6]. It follows immediately that ρ is a norm. However, this inequality does not hold for $\rho^{\pm}$: one can find counterexamples by adapting the construction in Eliashberg–Polterovich [3] that shows the degeneracy of the L^p-metrics for $p < \infty$.

The paper [7] II proposes another way to get lower bounds for $\widetilde{\rho}(\widetilde{\phi})$ in the case when $\widetilde{\phi}$ is C^2-close to id. The idea is to define the "graph" Γ_H of a Hamiltonian isotopy and to embed symplectic balls $B^{2n+2}(\varepsilon)$ of radius ε both "under" and "over" this graph. If ϕ_t is sufficiently C^2-close to the id and has fixed extrema, one can construct such embeddings with

$$\pi\varepsilon^2 = \mathcal{L}(\phi_t).$$

A simple argument using the nonsqueezing theorem then implies that the path $\{\phi_t\}_{t\in[0,1]}$ in $\widetilde{\mathrm{Ham}}(M)$ minimizes $\widetilde{\rho}$. (This argument is explained in more detail in §3 below.)

As remarked in [7], in order to go from here to estimates on the group Ham we need to understand the **short loops**. More precisely, given a seminorm $\widetilde{\nu}$ on $\widetilde{\mathrm{Ham}}(M)$ define

$$\ell_{\widetilde{\nu}} : \pi_1(\mathrm{Ham}\,(M, \omega)) \to [0, \infty)$$

to be the restriction of $\widetilde{\nu}$ to the subgroup $\pi_1(\mathrm{Ham}\,) \subset \widetilde{\mathrm{Ham}}$. To simplify the notation, we will denote its value on the homotopy class $[\lambda]$ of a loop λ in Ham by $\ell_{\widetilde{\nu}}(\lambda)$. Further, we define $r_{\widetilde{\nu}}(M)$ to be the minimum of the *positive* values of $\ell_{\widetilde{\nu}}$. If $\ell_{\widetilde{\nu}}(\lambda) = 0$ for all $[\lambda]$ then we set $r_{\widetilde{\nu}}(M) = \infty$. For short, we will write

$$\ell^{\pm}, r^{\pm} \quad \text{for} \quad \ell_{\widetilde{\nu}}, r_{\widetilde{\nu}} \quad \text{when} \quad \widetilde{\nu} = \widetilde{\rho}^{\pm}.$$

Further, we define $r_a(M)$ to be the supremum of $\delta \geq 0$ such that

$$\big(\ell^-(\lambda) < \delta \text{ or } \ell^+(\lambda) < \delta\big) \implies \big(\ell^-(\lambda) = 0 = \ell^+(\lambda).\big)$$

Thus $r_a(M) = 0$ if there is a sequence of loops λ_i in $\mathrm{Ham}\,(M)$ such that $\ell^-(\lambda_i) \to 0$ while $\inf_i \ell^+(\lambda_i) > 0$. In particular, $r_a(M) = 0$ if there is a loop λ such that $\ell^-(\lambda) = 0$ while $\ell^+(\lambda) > 0$.

Definition 1.14. The manifold M is said to have **$\widetilde{\nu}$-short loops** if 0 is not an isolated point in the image of $\ell_{\widetilde{\nu}}$, or equivalently if $r_{\widetilde{\nu}}(M) = 0$. It is said to have **asymmetric short loops** if $r_a(M) = 0$.

The next lemma shows that $\widetilde{\nu}$-geodesics in $\widetilde{\mathrm{Ham}}$ descend to ν-geodesics in Ham when there are no (asymmetric) short loops.

Lemma 1.15.

(i) *Let* $(\widetilde{\nu}, \nu)$ *be one of the pairs* $(\widetilde{\rho}, \rho), (\widetilde{\rho}_f, \rho_f)$. *Suppose that the path* γ *has* $\widetilde{\nu}$*-length* $\le r_{\widetilde{\nu}}(M)/2$ *and minimizes* $\widetilde{\nu}$ *in* $\widetilde{\mathrm{Ham}}\,(M)$. *Then it is* ν*-minimizing in* $\mathrm{Ham}\,(M)$.

(ii) *If* $(\widetilde{\nu}, \nu) = (\widetilde{\rho}_f, \rho^+ + \rho^-)$ *the same statement holds with* $r_{\widetilde{\nu}}(M)$ *replaced by* $r_a(M)$.

Proof. (i) We know that γ minimizes $\mathcal{L}$ among all homotopic paths and need to see that it minimizes $\mathcal{L}$ among all paths with the same endpoints. If not, there is another shorter path γ' with the same endpoints. Then $\lambda = (-\gamma') * \gamma$ is a loop with length

$$\mathcal{L}(\lambda) \le \mathcal{L}(\gamma) + \mathcal{L}(-\gamma') < r_{\widetilde{\nu}}(M).$$

Hence it has zero length. This means that we can compose γ' with an arbitrarily ν-short loop homotopic to λ to obtain a path that is homotopic to γ but shorter than it. This contradiction proves the lemma.

In case (ii) essentially the same argument works. Now we may only assume that γ' reduces one of the seminorms $\widetilde{\rho}^{\pm}$, say $\widetilde{\rho}^-$. Then

$$\widetilde{\rho}^+(\lambda) = \widetilde{\rho}^-(-\lambda) = \widetilde{\rho}^-((-\gamma) * \gamma') \le \mathcal{L}^+(\gamma) + \mathcal{L}^-(\gamma') < r_a(M).$$

Hence both $\widetilde{\rho}^+(\lambda)$ and $\widetilde{\rho}^-(\lambda)$ are 0 and the argument proceeds as before. □

In the general case, when there are (asymmetric) short loops, we establish the existence of geodesics in $\mathrm{Ham}\,(M)$ by generalizing the nonsqueezing theorem to nontrivial fibrations.

As in §1.3 consider a fibration $(P, \Omega) \to B$ where $B = D$ or S^2. We will say that **the nonsqueezing theorem holds for** (P, Ω) if area(P, Ω) constrains the radius of any embedded symplectic ball $B^{2n+2}(r)$ in (P, Ω) by the inequality

$$\pi r^2 \leq \text{area}\,(P, \Omega).$$

For example, if (P, Ω) is the product $(M \times D, \omega + \alpha)$, where α is an area form on D, then (P, Ω) has area equal to $\int_D \alpha$, so that this reduces to the usual nonsqueezing theorem. Similarly, we say that the nonsqueezing theorem holds for the loop λ if it holds for the corresponding fibration $(P_\lambda, \Omega) \to S^2$ where Ω is *any* ω-compatible symplectic form on P_λ.

The following result is proved in §3.

Lemma 1.16.

(i) *Suppose that there is $\varepsilon > 0$ such that the nonsqueezing theorem holds for all loops $\lambda \in \pi_1(\mathrm{Ham}\,(M, \omega))$ with $\widetilde{\rho}_f([\lambda]) \leq \varepsilon$, and let $(\widetilde{\nu}, \nu)$ be one of the pairs $(\widetilde{\rho}, \rho), (\widetilde{\rho}_f, \rho_f)$. Then there is a C^2-neighborhood $\mathcal{N}$ of id in $\mathrm{Ham}\,(M, \omega)$ such that every $\widetilde{\nu}$-minimizing path in $\mathcal{N}$ also minimizes ν.*

(ii) *The same statement holds for the pair $(\widetilde{\nu}, \nu) = (\widetilde{\rho}_f, \rho^+ + \rho^-)$ provided that the nonsqueezing theorem holds for all loops λ with either $\ell^-(\lambda)$ or $\ell^+(\lambda) < \varepsilon$.*

Using ideas from [10, 11] and Seidel [23] we show in §4.3 that the hypothesis in (i) above holds for all spherically rational symplectic manifolds (M, ω). For such a manifold, the **index of rationality** $q(M)$ is the smallest positive number q such that $[\omega](A) \in q\mathbb{Z}$ for all $A \in H_2^S(M, \mathbb{Z})$. In the weakly exact case, we set $q(M) = \infty$.

Proposition 1.17. *If (M, ω) is a spherically rational symplectic manifold with index of rationality $q(M)$, the nonsqueezing theorem holds for all loops λ in $\mathrm{Ham}\,(M, \omega)$ with $\widetilde{\rho}_f(\lambda) < q(M)/2$.*

For general manifolds (M, ω) we establish the existence of ρ-geodesics by using a suitable modification of Lemma 1.16. Here the relevant quantity is the minimum size of a class B for which there is a nontrivial Gromov–Witten invariant $n_M(a, b, c; B)$. Here nontrivial means that $B \neq 0$, and a, b, c can be any elements in $H_*(M)$. Thus we set

$$\hbar = \hbar(M) = \min\{\omega(B) > 0 : \text{ some } \; n_M(a, b, c; B) \neq 0\},$$

so that $\hbar = \infty$ if all nontrivial Gromov–Witten invariants vanish. Note that $\hbar > 0$ for all (M, ω): standard compactness results imply that for each κ there are only finitely many classes B with $\omega(B) \le \kappa$ that can be represented by a J-holomorphic curve for generic J, and it is only such classes that give rise to nonzero invariants.

Proposition 1.18. *Suppose that λ is a loop in $\pi_1(\mathrm{Ham}\,(M), \omega))$ such that $\ell^{\pm}(\lambda) < \hbar(M)/2$. Then there is $\kappa \in \mathbb{R}$ with $|\kappa| \le \max(\ell^-(\lambda), \ell^+(\lambda))$ such that the radii of all symplectically embedded balls in $(P_{\pm\lambda}, \Omega)$ are constrained by the inequalities*

$$\pi r^2 \le \mathrm{area}\,(P_\lambda, \Omega) + \kappa, \quad \pi r^2 \le \mathrm{area}\,(P_{-\lambda}, \Omega) - \kappa.$$

In particular, if $\ell^{\pm}(\lambda) = 0$ then the nonsqueezing theorem holds for $\pm\lambda$.

We call the above property **weighted nonsqueezing**. For the proof see §4.4.

Corollary 1.19. *Let $(\widetilde{\nu}, \nu)$ be one of the pairs $(\widetilde{\rho}, \rho), (\widetilde{\rho}_f, \rho_f)$. Suppose that the path γ has $\widetilde{\nu}$-length $< \hbar(M)/4$ and minimizes $\widetilde{\nu}$ in $\widetilde{\mathrm{Ham}}\,(M)$. Then it is ν-minimizing in $\mathrm{Ham}\,(M)$.*

The hypothesis that just one of $\ell^+(\lambda), \ell^-(\lambda)$ is small does not seem to give useful information towards proving the nonsqueezing theorem. Therefore, to understand the $(\rho^+ + \rho^-)$-geodesics we use the following result. Here we have written c_1 for the first Chern class of (TM, J), where J is any ω-tame almost complex structure on M.

Proposition 1.20.

(i) *If M is weakly exact the nonsqueezing theorem holds for all loops λ in $\mathrm{Ham}\,(M)$.*

(ii) *The same statement holds if c_1 vanishes on $\pi_2(M)$ and all 3-point Gromov–Witten invariants $n_M(a, b, c; B)$ vanish.*

(iii) *If $M = \mathbb{C}P^n$ the nonsqueezing theorem holds for the $(n+1)$st multiple $(n+1)[\lambda]$ of each loop λ.*

These are sample results; our methods could doubtless be used to find other manifolds for which the nonsqueezing theorem holds for all loops. However, as is clear from Lemma 1.16 above, this is more than is needed to show that $(\rho^+ + \rho^-)$-geodesics exist. There certainly are fibrations (such as the nontrivial S^2-bundle over S^2 with suitable Ω) for which the nonsqueezing theorem does not hold. Also the nonsqueezing theorem may well fail for general fibrations $(P,\Omega) \to D$ with nontrivial boundary monodromy. (See Remark 4.14 for further discussion on this point.)

1.5. Calculating the seminorm $\rho^+ + \rho^-$.

This section describes the example mentioned in Theorem 1.2 (iii) with $\rho(\phi) \neq \rho^+(\phi) + \rho^-(\phi)$. In the case considered here, the path that gives the minimum of ρ^- is not homotopic to the one that minimizes ρ^+. Therefore, the example does not show that the norms $\widetilde{\rho}$ and $\widetilde{\rho}^+ + \widetilde{\rho}^-$ on $\widetilde{\mathrm{Ham}}\,(M)$ are different. (The latter question appears much more delicate: see §2.2.)

Proposition 1.21. *Suppose that $\{\phi_t\}_{t\in[0,1]}$ is a loop λ in $\mathrm{Ham}\,(M)$ generated by a function $H_t : M \to \mathbb{R}$ with $\mathcal{L}^+(H_t) \neq \mathcal{L}^-(H_t)$. Suppose further that*

$$\mathcal{L}(\lambda) = r_{\widetilde{\rho}}(M) = \inf\{\widetilde{\rho}([\lambda]) : \widetilde{\rho}([\lambda]) > 0,\ [\lambda] \in \pi_1(\mathrm{Ham}\,)\}.$$

Then, if $\tau = \phi_T^H$ is the halfway point of this loop, that is if $\mathcal{L}(\{H_t\}_{t\in[0,T]}) = \mathcal{L}(\{H_t\}_{t\in[T,1]})$,

$$\rho(\tau) > \rho^+(\tau) + \rho^-(\tau).$$

Proof. There are two natural paths to τ, namely β^- given by $\{\phi_t\}_{t\in[0,T]}$ and β^+ given by $\{\phi_{1-t}\}_{t\in[0,1-T]}$. Without loss of generality we may suppose that $q = \mathcal{L}^-(H_t) < \mathcal{L}^+(H_t) = p$. Hence

$$q = \mathcal{L}^-(H_t) = \mathcal{L}^-(\beta^-) + \mathcal{L}^+(\beta^+) < \mathcal{L}^+(\beta^-) + \mathcal{L}^-(\beta^+) = p.$$

(This holds because the direction of β^+ is the opposite of ϕ_t.) Thus

$$\rho^-(\tau) + \rho^+(\tau) \leq \mathcal{L}^-(\beta^-) + \mathcal{L}^+(\beta^+) = q.$$

On the other hand, λ, by hypothesis, is an $\mathcal{L}$-minimizing representative of its homotopy class. This implies that both paths β^+ and β^- minimize $\mathcal{L}$ in their homotopy class. Further, since these paths have length precisely $\mathcal{L}(\lambda)/2 = (p+q)/2$, any shorter path β' from id to τ would create a nonconstant loop $\lambda' = (-\beta^+) * \beta'$ in $\mathrm{Ham}\,(M)$ with length $< \mathcal{L}(\lambda)$. If $\ell_{\widetilde{\rho}}([\lambda']) = 0$, then we could alter β' by an arbitrarily short path to be homotopic to β^+ which contradicts the minimality of $\mathcal{L}(\beta^+)$. On the other hand, if $\ell_{\widetilde{\rho}}([\lambda']) > 0$ then we must have $\ell_{\widetilde{\rho}}([\lambda']) \geq r_{\widetilde{\rho}}(M) = \mathcal{L}(\lambda)$. Hence the paths $\beta^{\pm}$ must achieve the minimum of ρ, and

$$\rho(\tau) = (p+q)/2 > q \geq \rho^-(\tau) + \rho^+(\tau).$$

$\square$

In order to apply this argument to the norm ρ_f instead of ρ we need to start with a loop λ whose length minimizes $\widetilde{\rho}_f$ over $\pi_1(\mathrm{Ham})$ and that reaches the halfway mark with respect to both $\mathcal{L}^-$ and $\mathcal{L}^+$ at the same time. The latter condition is most easily achieved if H_t is time independent, i.e., if λ is given by a circle action. Therefore, we have the following corollary.

Corollary 1.22. *Suppose that a Hamiltonian H with $\mathcal{L}^-(H) \neq \mathcal{L}^+(H)$ generates a circle action $\lambda = \{\phi_t^H\}$ and let $\tau = \phi_{1/2}^H$. Then if $\mathcal{L}(\lambda)$ minimizes $\widetilde{\rho}_f$ over $\pi_1(\mathrm{Ham})$*

$$\rho(\tau) = \rho_f(\tau) > \rho^+(\tau) + \rho^-(\tau).$$

The above results are easy. Note that they imply that the norms ρ (or ρ_f) and $\rho^+ + \rho^-$ take different values on all elements sufficiently close to τ in the C^2-topology.

The next result is harder.

Proposition 1.23. *There is a symplectic manifold (M, ω) and a loop λ in* $\mathrm{Ham}(M, \omega)$ *that satisfies the conditions of Corollary* 1.22.

The main problem is to find a loop λ whose length minimizes $\widetilde{\rho}_f$. It is shown in McDuff–Slimowitz [13] that any semifree action of a circle achieves the minimum of $\widetilde{\rho}$ in its homotopy class. However, it is much more complicated to estimate the lengths of all other loops. For this we will apply a method due to Seidel [22, 23] that uses the representation of $\pi_1(\mathrm{Ham}(M))$ on the quantum homology of M.

There are few manifolds for which $\pi_1(\mathrm{Ham}(M))$ is known. The easiest manifold to try would be S^2. But this does not work because the Hamiltonian for the generating loop of $\pi_1(\mathrm{Ham}(S^2)) = \pi_1(\mathrm{SO}(3))$ is symmetric. Therefore, following Polterovich [17], we will work out an explicit example on the one point blowup M_* of $\mathbb{C}P^2$. We think of this as the region

$$\{(z_1, z_2) \in \mathbb{C}^2 : a^2 \leq |z_1|^2 + |z_2|^2 \leq 1\}$$

with boundaries collapsed along the Hopf flow. Abreu–McDuff [1] show that $\pi_1(\mathrm{Ham}(M_*)) = \mathbb{Z}$ for all $0 < a < 1$, with generator given by the rotation

$$\alpha : \quad (z_1, z_2) \mapsto (e^{-2\pi i t} z_1, z_2), \quad 0 \leq t \leq 1.$$

However the S^1-action that satisfies the hypothesis in Corollary 1.22 is

$$\lambda : \quad (z_1, z_2) \mapsto (e^{-2\pi it}z_1, e^{-2\pi it}z_2), \quad 0 \le t \le 1.$$

which is homotopic to 2α. Thus $\tau(z_1, z_2) = (-z_1, -z_2)$. We will show that the hypotheses are satisfied for all a. Details are in §5.

Organization of the paper. §2 proves Proposition 1.12 about the geometric interpretations the the seminorms. §3 discusses geodesics. The results here are based on the results in §4 about the nonsqueezing theorem. Finally, §5 contains the calculations on the one point blow up of $\mathbb{C}P^2$ needed to prove Theorem 1.2(iii).

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2. The geometric interpretation of $\rho^{\pm}$.

Our arguments are based on the geometric approach to estimating the Hofer norm proposed in [7] II. We will begin by reminding the reader of some definitions and notation from that paper. Throughout we will assume that ω is normalized so that $\int_M \omega^n = n!$

2.1. The regions over and under the graph.

In this section, unless explicit mention is made to the contrary we will assume that H_t is **positively normalized** in the following sense.

We want to arrange that the function

$$t \mapsto \min(t) = \min_{x \in M} H_t(x)$$

is nonnegative and has arbitrarily small integral. If the function $\min(t)$ is smooth, we simply replace H_t by $H_t - \min(t)$. This will be the case if H_t has a fixed minimum. In general, we replace H_t by $H_t - m(t)$ where $m(t)$ is a smooth function that is everywhere $\le \min(t)$ and is such that $\min(t) - m(t)$ has arbitrarily small integral.

Finally we reparametrize so that $H_t \equiv 0$ for t near $0, 1$. It is easy to see that this can be done without changing $\mathcal{L}^{\pm}(H_t)$. In particular, every time independent Hamiltonian H may be replaced by one of the form $\beta(t)H$ that satisfies the above condition and has the same length and time 1-map as before.

We denote the graph Γ_H of H_t by

$$\Gamma_H = \{(x, t, H_t(x))\} \subset M \times [0,1] \times \mathbb{R}.$$

For some small $\varepsilon > 0$ choose a smooth function $\mu'(t) : [0,1] \to [-2\varepsilon, 0]$ such that

$$\int_0^1 \big(\min(t) - m(t) - \mu'(t)\big)\, dt = \varepsilon.$$

(This is possible provided that $m(t)$ is properly chosen.) A **thickening of the region under** Γ_H is

$$R_H^-(\varepsilon) = \{(x, t, h) \,|\, \mu'(t) \le h \le H_t(x)\} \subset M \times [0,1] \times \mathbb{R}.$$

Note that if μ' is suitably chosen so that its graph is tangent to the lines $t = 0, 1$ we may arrange that $R_H^-(\varepsilon)$ is a manifold with corners. (Recall that $H_t \equiv 0$ for t near $0, 1$.)

Similarly, we can define $R_H^+(\varepsilon)$ to be a slight thickening of the region above Γ_H:

$$R_H^+(\varepsilon) = \{(x, t, h) \,|\, H_t(x) \le h \le \mu_H(t)\} \subset M \times [0,1] \times \mathbb{R}$$

where $\mu_H(t)$ is chosen so that

$$\mu_H(t) \ge \max_{x \in M} H_t(x) = \max(t), \qquad \int_0^1 (\mu_H(t) - \max(t)) dt = \varepsilon.$$

We define

$$R_H(2\varepsilon) \;=\; R_H^-(\varepsilon) \cup R_H^+(\varepsilon) \;\subset M \times [0,1] \times \mathbb{R},$$

and equip $R_H^-(\varepsilon)$, $R_H^+(\varepsilon)$, and $R_H(2\varepsilon)$ with the product symplectic form $\Omega_0 = \omega + \alpha$ where $\alpha = dt \wedge dh$. In particular, for any Hamiltonian H_t, $(R_H(2\varepsilon), \Omega_0)$ is symplectomorphic to the product $(M \times D(a), \Omega_0)$ where $D(a)$ denotes the 2-disc D^2 with area $a = \mathcal{L}(H) + 2\varepsilon$.

Clearly, there is a projection $\pi : R_H^{\pm}(\varepsilon) \to D$ with fibers of the form

$$\pi^{-1}(b) = \{(x, t_b, h_b(x)) : x \in M, b \in D\}.$$

Thus both spaces $(R_H^{\pm}(\varepsilon), \Omega_0)$ fiber over the 2-disc. Further, if X_t is the symplectic gradient of H_t,

$$\iota(X_t + \partial_t)(\omega + dt \wedge dh) = -dH_t + dh$$

vanishes on Γ_H. Therefore the monodromy along Γ_H in the direction of increasing t is simply given by the flow ϕ_t^H. We can smooth the corners of the regions $R_H^{\pm}(\varepsilon)$ so that the boundary monodromy is trivial except along Γ_H. Therefore, the boundary monodromy when taken in the direction corresponding to increasing t is the element $\widetilde{\phi}$ in $\widetilde{\mathrm{Ham}}$. We have chosen here to orient the ambient space $M \times [0,1] \times \mathbb{R}$ in such a way that t increases (resp. decreases) as one goes positively round the boundary of $R_H^+(\varepsilon)$ (resp. $R_H^-(\varepsilon)$). (Here the boundary is oriented so that if one puts the outwards normal vector in front of a positively oriented basis for the boundary one gets a positively oriented basis for the ambient space.) Hence the boundary monodromy of $(R_H^+(\varepsilon), \Omega_0)$ is $\widetilde{\phi}$ while that of $(R_H^-(\varepsilon), \Omega_0)$ is $\widetilde{\phi}^{-1}$.

Finally, consider two Hamiltonians H_t and K_t on M such that $\phi_1^H = \phi_1^K$. There is a map $g : \Gamma_H \to \Gamma_K$ defined by

$$g(x,t,h) = \left(\phi_t^K \circ (\phi_t^H)^{-1}(x),\ t,\ h - H(x) + K(\phi_t^K \circ (\phi_t^H)^{-1}(x))\right).$$

The above formula defines a symplectomorphism of $(M \times [0,1] \times \mathbb{R}, \Omega_0)$ and so we use it to attach R_H^- to R_K^+ to form the space[11]

$$(R_{K,H}(2\varepsilon), \Omega_0) = (R_K^+(\varepsilon) \cup_g R_H^-(\varepsilon), \Omega_0).$$

We assume that the functions μ' and μ_H are chosen so that $R_{K,H}(2\varepsilon)$ is a smooth manifold with boundary. Note that Ω_0 has trivial monodromy round this boundary. Identifying this boundary to a point, we get a fibered space $(P_{K,H}(2\varepsilon), \Omega_0) \to S^2$.

In [7] II we only considered the case when the loop $\lambda = \phi_t^K \circ (\phi_t^H)^{-1}$ is contractible in $\mathrm{Ham}(M,\omega)$, in which case we showed that the fibration $(P_{K,H}(2\varepsilon), \Omega_0) \to S^2$ is symplectically trivial. (Such fibered spaces were called quasicylinders.) In general, we identify $P_{K,H}(2\varepsilon)$ with the space P_λ, where $\lambda = \{\phi_t^K \circ (\phi_t^H)^{-1}(x)\}$ and so think of Ω_0 as a form on the fibration $P_\lambda \to S^2$.

2.2. The area as norm.

We begin with a few words about fibrations of the form $(P, \Omega) \to B$. Even though we are only interested in the cases $B = D^2$ or S^2, it is worthwhile to understand the geometric structure imposed by the symplectic form Ω. As pointed out earlier, $P \to B$ inherits the structure of a Hamiltonian

[11]This definition is slightly different from that in [7] II to take account of the changed signs.

fibration, i.e. its structural group may be reduced to $\mathrm{Ham}(M)$. Conversely, it is shown in [8] and in Ch. 6 of [12] that if a smooth fibration $\pi : P \to B$ with closed fiber M has structural group $\mathrm{Ham}(M, \omega)$ then the family of symplectic forms ω_b on the fibers always has a closed extension. Moreover, if B itself is a symplectic manifold, we may assume that this extension is itself a symplectic form Ω. (The case $B = S^2$ is discussed in [22, 10].) We claim:

> *the choice of such a form Ω on a Hamiltonian fibration $\pi : P \to B$ with $B = D$ or S^2 is equivalent to the choice of a pair (Γ, α) where Γ is a connection on π with Hamiltonian holonomy and α is an area form on B.*

To see this, first suppose that $M \to P \to B$ is a fibration with structural group $\mathrm{Ham}(M, \omega)$. One can always provide it with a connection Γ with Hamiltonian holonomy, and then use the construction of Guillemin–Lerman–Sternberg to find a closed 2-form τ_Γ that extends the given forms on the fibers and induces the given connection Γ. This form τ_Γ is called the coupling form and is unique (at least for closed bases) if one requires that the integral over the fiber of the class $[\tau_\Gamma]^{n+1}$ is $0 \in H^2(B)$, where $2n = \dim M$. When $B = S^2$ this is equivalent to the condition

$$[\tau_\Gamma]^{n+1} = 0 \ \in H^{2n+2}(P). \tag{1}$$

(If $B = D$ uniqueness is given by a relative version of this condition. For more detail see the proof of Lemma 2.2 below.) Further, because M is compact one can obtain a symplectic extension Ω_Γ of the forms ω_b on the fibers by adding to τ_Γ the pullback of a suitable area form on B. Hence the data (Γ, α) does give rise to a symplectic extension $\Omega = \Omega_\Gamma$ of the forms ω_b. Note that Ω_Γ still induces the connection Γ on $P \to B$. Hence, if we start from Ω and let Γ be the corresponding connection, both Ω and the coupling form τ_Γ induce the same connection. Since the uniqueness condition for τ_Γ is cohomological, it is satisfied by some form of the type $\Omega - \pi^*(\alpha)$. Hence Ω may be written as

$$\Omega = \tau_\Gamma + \pi^*(\alpha)$$

where τ_Γ is the coupling form for the connection Γ defined by Ω and α is an area form on B. This proves the claim. (For more details, see for example [16, 12, 8].)

Lemma 2.1. *If $\Omega = \tau_\Gamma + \pi^*(\alpha)$ as above, then*

$$\mathrm{area}\,(P, \Omega) = \int_B \alpha.$$

Proof. This follows immediately from condition (1) above and the normalization condition $\int \omega^n = n!$ $\square$

We next explain Polterovich's ideas from [18, 20] that give geometric interpretations of the seminorms $\widetilde{\rho}^{\pm}$. Recall from Definition 1.11 that

$$\widetilde{a}^+(\widetilde{\phi}) = \inf \operatorname{area}(P, \Omega), \qquad \widetilde{a}^-(\widetilde{\phi}) = a^+(\widetilde{\phi}^{-1}),$$

where the infimum is taken over all ω-compatible fibered spaces $(P, \Omega) \to D$ with boundary monodromy equal to $\widetilde{\phi}$. A simple calculation shows that

$$\text{area } (R_H^+(\varepsilon), \Omega) = \mathcal{L}^+(H_t) + \varepsilon, \quad \text{area } (R_H^-(\varepsilon), \Omega) = \mathcal{L}^-(H_t) + \varepsilon.$$

Therefore,

$$\widetilde{a}^+(\widetilde{\phi}) \ \le \ \widetilde{\rho}^{\,+}(\widetilde{\phi}), \qquad \widetilde{a}^-(\widetilde{\phi}) \ \le \ \widetilde{\rho}^{\,-}(\widetilde{\phi}).$$

The following result combines Polterovich [16]§3.3 and [18]§3.3. We give a proof here partly because he considers loops rather than paths and partly because we simplify his argument by avoiding the use of K-area.

Lemma 2.2. $\widetilde{a}^+(\widetilde{\phi}) = \widetilde{\rho}^{\,+}(\widetilde{\phi})$ *and* $\widetilde{a}^-(\widetilde{\phi}) = \widetilde{\rho}^{\,-}(\widetilde{\phi})$.

Proof. By symmetry it suffices to prove the former result, which will follow if we show that

$$\widetilde{a}^+(\widetilde{\phi}) \ge \widetilde{\rho}^{\,+}(\widetilde{\phi}).$$

Suppose to the contrary that we are given a fibration $(P, \Omega) \to D$ with area $< \widetilde{\rho}^{\,+}(\widetilde{\phi})$ and monodromy $\widetilde{\phi}$. By Moser's theorem we may isotop Ω so that it is a product in some neighborhood $\pi^{-1}(\mathcal{N})$ of the base fiber M_*. Identify the base D with the unit square $K = \{0 \le x, y \le 1\}$ taking $\mathcal{N}$ to a neighborhood of $\partial' K = \partial K - \{1\} \times (0,1)$, and then identify P with $K \times M$ by parallel translating along the lines $\{(x,y) : x \in [0,1]\}$. In these coordinates, the form Ω may be written as

$$\Omega = \omega + d_M F' \wedge dy + L' dx \wedge dy$$

where F', L' are suitable functions on $K \times M$ and d_M denotes the fiberwise exterior derivative. Because Ω is a product near $\pi^{-1}(\partial' K)$, $d_M F' = 0$ there and L' reduces to a function of x, y only. By subtracting a suitable function $c(x,y)$ from F' we can arrange that $F = F' - c(x,y)$ has zero mean on each fiber $\pi^{-1}(x,y)$ and then write $L' + \partial_x c(x,y)$ as $-L + a(x,y)$ where L also has zero fiberwise means. Thus

$$\Omega = \omega + d_M F \wedge dy - L dx \wedge dy + a(x,y) dx \wedge dy, \tag{2}$$

where both F and L vanish near $\pi^{-1}(\partial' K)$ and have zero fiberwise means. Since Ω is symplectic it must be positive on the 2-dimensional distribution *Hor*. Hence we must have $-L(x,y,z)+a(x,y)>0$ for all $x,y\in K, z\in M$. Moreover, it is easy to see that area $(P,\Omega)=\int a(x,y)dx\wedge dy$. Hence

$$(3)\qquad \int \max_{z\in M} L(x,y,z)\,dx\wedge dy < \int a(x,y)\,dx\wedge dy = \text{area}\,(P,\Omega).$$

We claim that $-L$ is the curvature of the induced connection Ω_Γ. To see this, consider the vector fields $X=\partial_x, Y=\partial_y - \text{sgrad } F$ on P that are the horizontal lifts of ∂_x,∂_y.[12] It is easy to check that their commutator $[X,Y]=XY-YX$ is vertical and that

$$[X,Y] = -\text{sgrad}\,(\partial_x F) = \text{sgrad } L$$

on each fiber $\pi^{-1}(x,y)$ as claimed. (In fact, the first three terms in (2) make up the coupling form τ_Γ.)

Now let $f_s\in \text{Ham}\,(M)$ be the monodomy of Ω_Γ along the path $t\mapsto (s,t), t\in[0,1]$. (This is well defined because all fibers have a natural identification with M.) The path $s\mapsto f_s$ is a Hamiltonian isotopy from the identity to $\phi=f_1$, and it is easy to see that it is homotopic to the original path $\widetilde{\phi}$ given by parallel transport along $t\mapsto[1,t]$. (As an intermediate path in the homotopy take the monodromies along $t\mapsto(s,t), t\in[0,T]$ for $s\in[0,1]$ followed by the lift of $\widetilde{\phi}_t, t\in[T,1]$, to $\widetilde{\text{Ham}}$.) Therefore $\mathcal{L}^+(f_s)\geq \widetilde{\rho}^+(\widetilde{\phi})$, and we will derive a contradiction by estimating $\mathcal{L}^+(f_s)$.

To this end, let X^s, Y^t be the (partially defined) flows of the vector fields X,Y on P and set $h_{s,t}=Y^tX^s$. Consider the 2-parameter family of (partially defined) vector fields $v_{s,t}$ on P given by

$$v_{s,t} = \partial_s h_{s,t} = Y^t_*(X) \quad \text{on} \quad \text{Im}\, h_{s,t}.$$

In particular $v_{s,1}(x,y)$ is defined when $y=1, s\leq x$. Since $f_s = h_{s,1}$ we are interested in calculating the vertical part of $v_{s,1}(s,1,z)$. Since the points with $y=1$ are in $\text{Im}\, h_{s,t}$ for all (s,t) we may write

$$\begin{aligned} v_{s,1} &= \int_0^1 \partial_t(v_{s,t})\,dt + v_{s,0} \\ &= \int_0^1 Y^t_*([X,Y])\,dt + \partial_x. \end{aligned}$$

[12] Here the symplectic gradient sgrad F is defined by setting $\omega(\text{sgrad } F,\cdot) = -d_M F(\cdot)$.

We saw above that $[X, Y] = \operatorname{sgrad} L$. Hence $Y^t_*([Y, X]) = \operatorname{sgrad}(L \circ (Y^t)^{-1})$ and

$$\begin{aligned} v_{s,1}(s, 1, z) &= \int_0^1 \operatorname{sgrad}(L((Y^t)^{-1}(s, 1, z))\, dt + \partial_x \\ &= \operatorname{sgrad} \int_0^1 L(s, 1-t, (Y^t_v)^{-1}(z))\, dt + \partial_x \end{aligned}$$

where Y^t_v denotes the vertical part of Y^t. Hence the Hamiltonian H_s that generates the path $f_s, s \in [0, 1]$, and has zero mean satisfies the inequality

$$H_s(z) \le \int \left(\max_{z \in M} L(s, t, z) \right) dt < \int a(s, t) dt,$$

since $L(s, t, z) < a(s, t)$ by (3). Thus $\widetilde{\rho}^+(\widetilde{\phi}) \le \operatorname{area} P$, contrary to hypothesis. $\square$

Using this, we can interpret the seminorms $\rho^+ + \rho^-$ and ρ_f in terms of the fibrations $P_{K,H}(\varepsilon) \to S^2$ considered at the end of §2.1.

Corollary 2.3.

(i) *$\rho^+(\phi) + \rho^-(\phi)$ is the minimum of the areas of the fibrations $(P_{K,H}(\varepsilon), \Omega_0) \to S^2$ taken over all $\varepsilon > 0$ and all pairs (H_t, K_t) of Hamiltonians with time 1 map ϕ.*

(ii) *$\rho_f(\phi)$ is the minimum of these areas over the set of pairs that define homotopic flows.*

The proof of Proposition 1.12 is now immediate.

Now let us consider the geometric interpretation of the Hofer norm and compare it with that for ρ_f. The above corollary implies that ρ_f is the minimum of the areas of the symplectically trivial fibrations $(R_{K,H}, \Omega_0)$ that contain a hypersurface $\Gamma_H = \Gamma_K$ with monodromy ϕ. The sets $R_{K,H}$ are constructed as subsets of $(M \times \mathbb{R}^2, \omega + dt \wedge dh)$ that have trivial monodromy round the boundary. This means that the flow lines of the characteristic flow round the boundary are all closed. However, they are not constant in the M direction, i.e. with respect to the given trivialization of the fibers they go round the loops $t \mapsto \phi^K_t \circ (\phi^H_t)^{-1}(x)$. The Hofer norm, on the other hand, is the minimum area of the cylinders $R_H(2\varepsilon)$ which sit inside $(M \times \mathbb{R}^2, \omega + dt \wedge dh)$ as product regions of the form $M \times U_H$. Hence the monodromy of its boundary is constant with respect to the given trivialization. Moreover the hypersurface Γ_H in $R_H(2\varepsilon)$ that has monodromy ϕ is a graph over its front and back faces (defined by $h = \mu_H(t)$, $h = \mu'(t)$). For general M is it not at all clear that these are the same minimizing sets. Even when $M = S^2$ and we know that $(R_{K,H}, \Omega_0)$ is symplectomorphic to a product subset of $(M \times \mathbb{R}^2, \omega + dt \wedge dh)$, it is not obvious that the resulting hypersurfaces in this subset can be assumed to be graphs over the front and back faces.

Proof of Lemma 1.13. By Proposition 1.12 $\widetilde{\rho}^+([\lambda])$ is measured by minimizing the area of fibrations $(P, \Omega) \to D$ with boundary monodromy in the class $[\lambda]$. Therefore, the result will follow if we set up a correspondence between this minimizing set and the set of fibrations $(P_\lambda, \Omega) \to S^2$ with clutching loop $[\lambda]$.

Given $(P, \Omega) \to D$ with monodromy $[\lambda]$ there is an associated fibration $Q \to S^2$ defined by identifying the boundary ∂P to a single fiber via the characteristic flow. Moreover, since the restriction of Ω to ∂P determines Ω near ∂P, Ω descends to Q. We therefore get a fibration $(Q, \Omega) \to S^2 = D/\partial D$, with fiber over the base point $* = \partial D$ identified to M. It remains to observe that the clutching function of this fibration $Q \to S^2$ is $[\lambda] \in \pi_1(\mathrm{Ham})$. Conversely, given such a fibration $(Q, \Omega) \to S^2$ identify $(S^2, *)$ with $(D/\partial D, \partial D)$ and consider the pullback of Q to D. This will be a fibration over D with boundary monodromy in the class $[\lambda]$, as required. □

3. Geodesics.

This section contains proofs of the results in §1.2 assuming the results stated in §1.4 about the nonsqueezing theorem.

Let $\{\phi_t\}_{t \in [0,1]}$ be a path generated by a Hamiltonian H_t that is positively normalized as in §2.1. The following easy result was proved in [7] II Lemma 3.2. Recall that the capacity of a ball of radius r is πr^2.

Lemma 3.1. *If H_t is sufficiently small in the C^2-norm and has a fixed maximum (resp. minimum), then, for all $\varepsilon > 0$ it is possible to embed a ball of capacity $\mathcal{L}(H_t)$ in $R_H^-(\varepsilon)$ (resp. $R_H^+(\varepsilon)$).*

Lemma 3.2. *Suppose that a ball of capacity $\mathcal{L}(H_t)$ can be embedded in $R_H^-(\varepsilon)$ for all ε. Then:*

(i) *the corresponding path $\widetilde{\phi}_t = \phi_t^H$ in $\widetilde{\mathrm{Ham}}$ minimizes $\widetilde{\rho}^+(\widetilde{\phi})$;*

(ii) *the path* ϕ_t^H *also minimizes* $\rho^+(\phi)$ *provided that the nonsqueezing theorem holds for all loops* $\lambda \in \pi_1(\mathrm{Ham}\,)$.

Proof. Suppose that K_t generates a homotopic path ψ_t with $\psi_1 = \phi_1$ and that $\mathcal{L}^+(K_t) < \mathcal{L}^+(H_t)$. Then, the corresponding fibered space $(P_{K,H}(2\varepsilon), \Omega) \to S^2$ has area $< \mathcal{L}(H_t) = \mathcal{L}^-(H_t) + \mathcal{L}^+(H_t)$ provided that ε is sufficiently small. On the other hand it contains a ball of capacity $\mathcal{L}(H_t)$. But this contradicts the nonsqueezing theorem for the symplectically trivial bundle $(P_{K,H}(2\varepsilon), \Omega) \to S^2$. This proves (i). (ii) is also immediate, because the hypothesis means that we can apply the nonsqueezing theorem to any bundle of the form $(P_{K,H}(2\varepsilon), \Omega) \to S^2$. □

We now show that paths that minimize $\widetilde{\rho}^{\,+}$ must have fixed maxima.

Lemma 3.3. *Suppose that* H_t *does not have a fixed maximum. Then the corresponding path* $\phi_t = \phi_t^H$ *in* $\widetilde{\mathrm{Ham}}$ *does not minimize* $\widetilde{\rho}^{\,+}$.

Proof. The analogous result for the two sided norm ρ was proved in [7] I Proposition 2.1 by a simple curve shortening procedure. There we constructed a perturbation Ψ_t^ε such that:

- $\Psi_0^\varepsilon = \Psi_1^\varepsilon = id.$
- if $\phi_t^\varepsilon = \Psi_t^\varepsilon \circ \phi_t$ for all t, then the generating Hamiltonian H_t^ε for ϕ_t^ε satisfies the conditions:

$$\min_{x\in M}(H_t^\varepsilon) = \min_{x\in M}(H_t), \qquad \max_{x\in M}(H_t^\varepsilon) \leq \max_{x\in M}(H_t),$$

  where strict inequality holds for the maximum on some interval $|t - t_0| < \varepsilon$.

This implies that the two sided length $\mathcal{L}$ is smaller on ϕ_t^ε. To ensure that $\mathcal{L}^+(\phi_t^\varepsilon) < \mathcal{L}^+(\phi_t)$ it suffices to arrange in addition that

$$\int H_t^\varepsilon\,\omega^n = \int H_t\,\omega^n = 0, \qquad t \in [0,1].$$

Clearly we can arrange that H_t has zero mean, and H_t^ε will have too as long as we choose the functions K_j that generate the perturbation Ψ_t^ε to have zero mean.

The K_j were constructed as follows. We chose $t_0 < t_1 < \cdots < t_k$ so that $\cap_j X_j = \emptyset$, where $X_j = \operatorname{maxset} H_{t_j} \neq M$, and then chose a partition of unity $\{\beta_j\}$ subordinate to the cover $M - \mathcal{N}_\kappa(X_0), M - X_1, \ldots, M - X_k$, where $\mathcal{N}_\kappa(X_0)$ is the κ-neighborhood of X_0. Fix $\delta > 0$ so that

$$\mathcal{N}_{\kappa/2}(X_0) \subset \cup_{j\geq 1}\beta_j^{-1}([\delta, 1]),$$

and choose functions $K_j \leq 0$ with support in $\beta_j^{-1}([\delta/2, 1])$ that are constant and < 0 on $\beta_j^{-1}([\delta, 1])$. Then define Ψ_t^ε as a smoothing of the following path, where the flow of K_j is denoted ψ_t^j:

- $\Psi_t^\varepsilon = id$ for $t < t_0 - \varepsilon$ and then equals $\psi_s^1 \circ \cdots \circ \psi_s^k$ for $0 \leq s = t - t_0 + \varepsilon \leq 2\varepsilon$;
- Ψ_t^ε is constant when $|t - t_j| > \varepsilon$;
- when $|t - t_j| \leq \varepsilon$ for some $j > 0$, Ψ_t^ε has the form

$$(\psi_s^j)^{-1}\Psi_{t_j-\varepsilon}, \quad \text{where } s = t - t_j + \varepsilon.$$

It is not hard to see that this satisfies all the requirements for small enough ε. In particular, because $\sum_{j\geq 1} K_j(x) < 0$ for $x \in \mathcal{N}_\kappa(X_0)$, $\max H_t^\varepsilon < \max H_t$ for $|t - t_0| < \varepsilon$. (Details are in [7] I.)

In the present situation, we need to allow each K_j to be positive somewhere so that it can have zero mean. If $M - \mathcal{N}_{\kappa/2}(X_j \cup X_0)$ is nonempty for small κ, then there is no problem; the argument goes through as before if K_j is small and positive on such a set. Conceivably, we have to choose some of the X_j so that $M = X_0 \cup X_j$ in order to achieve that $\cap_{j=0}^k X_j = \emptyset$. In this case the frontier of X_0 lies entirely in X_j. So there must be an open set $U_j \subset X_0 \cap X_j$ that lies outside some other X_k. So for each such j we let K_j be positive in this open set U_j and make K_k sufficiently negative in U_j to compensate. □

Corollary 3.4. *Proposition* 1.5 *holds.*

Proof. It follows from Lemmas 3.1, 3.2 above that a sufficiently short piece γ of any path with a fixed maximum (resp. minimum) minimizes $\widetilde{\rho}^+$ (resp. $\widetilde{\rho}^-$). Conversely, if a path does not have a fixed maximum it cannot minimize $\widetilde{\rho}^+$. □

Proof of Lemma 1.16. To prove (i) we have to show that there is some C^2-neighborhood of id in Ham such that every $\widetilde{\nu}$-minimizing path γ in $\mathcal{N}$ actually minimizes ν, where $(\widetilde{\nu}, \nu)$ is $(\widetilde{\rho}, \rho)$ or $(\widetilde{\rho}_f, \rho_f)$. Clearly, it suffices to prove this for paths that start at id.

In both cases, we know from Lemma 3.3 above that $\gamma = \{\phi_t^H\}$ has fixed extrema. Therefore, by Lemma 3.1 we may choose $\mathcal{N}$ so that a ball of capacity $\mathcal{L}(\gamma)$ embeds in $R_H^{\pm}(\delta)$ for all $\delta > 0$. We may also choose $\mathcal{N}$ so that $\mathcal{L}(\gamma) = \widetilde{\rho}(\gamma) < \varepsilon/2$. We claim that such γ must minimize both ρ and ρ_f. We will carry out the argument for ρ_f since it is slightly more complicated.

Suppose that γ does not minimize ρ_f. Then there are paths $\psi_t^{\pm}$ from id to $\phi = \phi_1^H$ generated by $K_t^{\pm}$ such that

$$\mathcal{L}^+(K_t^+) + \mathcal{L}^-(K_t^-) \ < \ \mathcal{L}(\gamma) \ = \ \mathcal{L}^+(H_t) + \mathcal{L}^-(H_t).$$

Therefore at least one of the following inequalities must hold:

$$\mathcal{L}^+(K_t^+) < \mathcal{L}^+(H_t), \qquad \mathcal{L}^-(K_t^-) < \mathcal{L}^-(H_t),$$

say the former. Note that the two functions K_t^+ and K_t^- may be different. (In the case of ρ they would be the same.) However, because we are dealing with the norm ρ_f rather than $\rho^+ + \rho^-$ the paths $\beta = \beta^{\pm}$ that they generate are homotopic. Hence the fibrations $P_{K^+,H}, P_{H,K^-}$ correspond to loops $\lambda, -\lambda$ that are mutual inverses, where $\lambda = \beta * (-\gamma)$. Note also that for suitably small δ

$$\widetilde{\rho}^+([\lambda]) \leq \text{area}\,(P_{K^+,H}(\delta), \Omega_0) < \mathcal{L}(\gamma) < \varepsilon/2,$$

while

$$\begin{aligned}
\widetilde{\rho}_f([\lambda]) &= \widetilde{\rho}^+([\lambda]) + \widetilde{\rho}^-([\lambda]) \\
&\leq \text{area}\,(P_{K^+,H}(\delta), \Omega_0) + \text{area}\,(P_{H,K^-}(\delta), \Omega_0) \\
&= \mathcal{L}(H) + \mathcal{L}^+(K_t^+) + \mathcal{L}^-(K_t^-) + 2\delta \\
&\leq \varepsilon.
\end{aligned}$$

Therefore by hypothesis the nonsqueezing theorem holds for $\pm\lambda$. Hence $(P_{K^+,H}(\delta), \Omega_0)$ cannot contain a ball of capacity $\mathcal{L}(\gamma)$. This contradiction shows that γ must minimize ρ_f. Hence (i) holds. The proof of (ii) is similar: compare Lemma 1.15. □

Proof of Theorem 1.6. When (M, ω) is spherically rational (i) follows from Proposition 1.5 by Lemma 1.16 and Proposition 1.17. For the general case see §4.4. The statement in (ii) about weakly exact M also follows by a similar argument since, by Proposition 1.20, the nonsqueezing theorem now holds for all loops. Therefore, it remains to consider the case $M = \mathbb{C}P^n$.

Let $\{\phi_t\}_{t\in[0,1]}$ be a path in $\mathrm{Ham}\,(\mathbb{C}P^n)$ with a fixed maximum and minimum. We must show that sufficiently short pieces of it minimize $\rho^+ + \rho^-$. In the following we denote by $H_t * K_t$ the Hamiltonian $H_t + K_t \circ (\phi_t^H)^{-1}$ that generates the composite $\phi_t^H \circ \phi_t^K$. Further we write $m = n+1$.

First choose ε so that each piece

$$\gamma_a = \{\phi_{a+t}\phi_a^{-1}\}_{t\in[a,a+\varepsilon]}$$

is generated by a Hamiltonian H_t^a for which the m-fold composite

$$(H_t^a)^{*m} = H_t^a * \cdots * H_t^a$$

satisfies the conditions of Lemma 3.1. Since H_t^a has fixed extrema,

$$\mathcal{L}((H_t^a)^{*m}) = m\mathcal{L}(\gamma_a).$$

Therefore for each γ_a a ball of capacity $m\mathcal{L}(\gamma_a)$ can be embedded in $R^{\pm}_{(H^a)^{*m}}$. If some such piece γ_a does not minimize $\mathcal{L}^+(\gamma_a)$, then there is a shorter path γ' with the same endpoints generated by some K_t. Let λ be the loop $\gamma' * (-\gamma_a)$ and consider the fibration

$$(P_{m\lambda}, \Omega) = (P_{(K^{*m},(H^a)^{*m})}, \Omega_0) \ \to \ S^2.$$

The nonsqueezing theorem holds for this fibration by Proposition 1.20. On the other hand it has area strictly less than $m\mathcal{L}(\gamma_a)$ while containing embedded balls of this capacity, a contradiction. A similar argument applies if $\mathcal{L}^-$ is not minimized. Hence result. □

Proof of Proposition 1.8. This states that there is a C^2-neighborhood $\mathcal{N}$ of id on which the two norms ρ_f and ρ agree. It is shown in [7] II Proposition 5.11 that we may suppose that every $\phi \in \mathcal{N}$ is generated by some Hamiltonian H_t with a fixed maximum and minimum. Moreover, the length of this path is given by a Banach space norm. If in addition H_t is sufficiently C^2-small we may apply the arguments above to conclude that the path minimizes both ρ and ρ_f. □

Remark 3.5. Here is an explicit description for the size of this neighborhood $\mathcal{N}$. First, to get an explicit choice of generating Hamiltonian H_t for each $\phi \in \mathcal{N}$, choose an identification of a neighborhood of the diagonal in $M \times M$ with the zero section in T^*M. Secondly, to get uniform bounds for the embedded balls of Lemma 3.1, we can argue as follows. Choose for some $\varepsilon > 0$ a family of symplectic embeddings $\iota_x : B^{2n}(\varepsilon) \to M$ such that $\iota_x(0) = x$ for all $x \in M$. This family should have uniformly bounded second derivatives, though it need not depend smoothly (or even continuously) on x. It can be constructed from a finite covering of M by Darboux charts $B^{2n}(r_i) \to M$ chosen so that the images of the subballs $B^{2n}(r_i - \varepsilon)$ also cover M. Then, we require $\mathcal{L}(H_t) \le \varepsilon$. Further, for each H_t there must be at least one fixed maximum x^+ and one fixed minimum x^- such that the functions $H_t \circ \iota_{x^\pm}$ have sufficiently small C^2-norm for the embedding techniques of [7] II to work. Finally, we need $\mathcal{L}(H_t) < \hbar/4$ in order for Corollary 1.19 to hold.

Proof of Proposition 1.9. The first statement is proved in [13]. Although that paper only mentions the usual Hofer norm, the capacity–area inequality proved there shows that $\{\phi_t^H\}_{0\le t\le 1}$ minimizes both $\widetilde{\rho}^+$ and $\widetilde{\rho}^-$. (Argue as in Lemma 4.13 above using [13] Proposition 2.4.)

To prove the next statement it suffices to show that if M is weakly exact then all the norms $\rho^+ + \rho^-$, ρ_f and ρ are minimized as well. To do this we simply have to show that the arguments in [13] apply when the fibration $(P_{K,H}, \Omega_0) \to S^2$ is not symplectically trivial. However all we used about this fibration is that there is a section σ_A with $u_\lambda(\sigma_A) \le 0$ such that the Gromov–Witten invariant $n_P([M],[M],pt;\sigma_A)$ is nonzero: see [13] §3.1. Since this holds for weakly exact M by Lemma 4.13, the result follows. $\square$

Remark 3.6.

(i) The above argument shows that Proposition 1.9 holds for every M for which all loops λ have good sections of positive weight in the sense of Definition 4.6 below. It was based on [13] but one could equally use Entov [4].

(ii) The proof of Theorem 1.6(ii) for $\mathbb{C}P^n$ also applies to any manifold M that is not spherically monotone[13] and such that $\pi_1(\mathrm{Ham}\,(M))$ is finite, or more generally has finite image under the representation Ψ defined below. See Remark 4.14(i).

[13] (M, ω) is said to be spherically monotone if there is κ such that $c_1 - \kappa[\omega]$ vanishes on all classes in $\pi_2(M)$.

4. Quantum homology and the nonsqueezing theorem.

In [11] Proposition 1.1 we established the nonsqueezing theorem for some fibrations $P_\gamma \to S^2$ by using the modified Seidel representation

$$\Psi : \pi_1(\mathrm{Ham}\,(M)) \to (QH_{\mathrm{ev}}\,(M,\Lambda))^\times.$$

Here $(QH_{\mathrm{ev}}\,(M,\Lambda))^\times$ denotes the group of units in the even part of the small quantum homology of M with coefficients in a real Novikov ring Λ. Since we use the same approach in this paper, we will begin by recalling some definitions from [10, 11].

4.1. Preliminaries.

Set $c_1 = c_1(TM) \in H^2(M,\mathbb{Z})$. Let Λ be the Novikov ring of the group $\mathcal{H} = H_2^S(M,\mathbb{R})/\!\sim$ with valuation I_ω where $B \sim B'$ if $\omega(B-B') = c_1(B-B') = 0$. Thus Λ is the completion of the rational group ring[14] of $\mathcal{H}$ with elements of the form

$$\sum_{B\in\mathcal{H}} q_B\, e^B$$

where for each κ there are only finitely many nonzero $q_B \in \mathbb{Q}$ with $\omega(B) > -\kappa$. Set

$$QH_*(M) = QH_*(M,\Lambda) = H_*(M) \otimes \Lambda.$$

We may define an $\mathbb{R}$ grading on $QH_*(M,\Lambda)$ by setting

$$\deg(a \otimes e^B) = \deg(a) + 2c_1(B),$$

and can also think of $QH_*(M,\Lambda)$ as $\mathbb{Z}/2\mathbb{Z}$-graded with

$$QH_{\mathrm{ev}} = H_{\mathrm{ev}}\,(M) \otimes \Lambda, \quad QH_{odd} = H_{odd}(M) \otimes \Lambda.$$

Recall that the quantum intersection product

$$a *_M b \in QH_{i+j-2n}(M), \qquad \text{for } \ a \in H_i(M), b \in H_j(M)$$

is defined as follows:

$$a *_M b = \sum_{B\in\mathcal{H}} (a *_M b)_B \otimes e^{-B}, \tag{4}$$

[14] In [11, 10] we distinguished between the integral version of Λ which is generated by the integral elements of $\mathcal{H}$ and the real Novikov ring that we are now calling Λ. It is not necessary to do that here.

where $(a *_M b)_B \in H_{i+j-2n+2c_1(B)}(M)$ is defined by the requirement that

$$(a *_M b)_B \cdot_M c = n_M(a,b,c;B) \quad \text{for all } c \in H_*(M). \tag{5}$$

Here $n_M(a,b,c;B)$ denotes the Gromov–Witten invariant that counts the number of B-spheres in M meeting the cycles $a,b,c \in H_*(M)$, and we have written $\cdot_M$ for the usual intersection pairing on $H_*(M) = H_*(M,\mathbb{Q})$. Thus $a \cdot_M b = 0$ unless $\dim(a) + \dim(b) = 2n$ in which case it is the algebraic number of intersection points of the cycles. The product $*_M$ is extended to $QH_*(M)$ by linearity over Λ, and is associative. Moreover, it preserves the $\mathbb{R}$-grading.

This product $*_M$ gives $QH_*(M)$ the structure of a graded commutative ring with unit $\mathbb{1} = [M]$. Further, the invertible elements in $QH_{\rm ev}(M)$ form a commutative group $QH_{\rm ev}(M,\Lambda)^\times$ that acts on $QH_*(M)$ by quantum multiplication.

Now consider the fibration $P_\lambda \to S^2$ constructed from a loop λ as in §1.3. As noted in [10], the manifold P_λ carries two canonical cohomology classes, the first Chern class of the vertical tangent bundle

$$c_{vert} = c_1(TP_\lambda^{vert}) \in H^2(P_\lambda, \mathbb{Z}),$$

and the coupling class u_λ, i.e. the unique class in $H^2(P_\lambda,\mathbb{R})$ such that

$$i^*(u_\lambda) = [\omega], \qquad u_\lambda^{n+1} = 0,$$

where $i : M \to P_\lambda$ is the obvious inclusion.

The next step is to choose a canonical (generalized) section class[15] $\sigma_\lambda \in H_2(P_\lambda,\mathbb{R})/\sim$. In the general case, when c_1 and $[\omega]$ induce linearly independent homomorphisms $H_2^S(M) \to \mathbb{R}$, σ_λ is defined by the requirement that

$$c_{vert}(\sigma_\lambda) = u_\lambda(\sigma_\lambda) = 0, \tag{6}$$

which has a unique solution modulo the given equivalence. We show in [11] that when M is weakly exact such a class σ_λ still exists and moreover is integral. (The proof is included in Lemma 4.9 below.) In the remaining spherically monotone case, we choose σ_λ so that $c_{vert}(\sigma_\lambda) = 0$.

We then set

$$\Psi(\lambda) = \sum_{B \in \mathcal{H}} a_B \otimes e^B \tag{7}$$

[15] By section class, we mean one that projects onto the positive generator of $H_2(S^2,\mathbb{Z})$.

where, for all $c \in H_*(M)$,

$$a_B \cdot_M c = n_{P_\lambda}([M],[M],c\,;\sigma_\lambda - B). \tag{8}$$

Note that $\Psi(\lambda)$ belongs to the strictly commutative part QH_{ev} of $QH_*(M)$. Moreover $\deg(\Psi(\lambda)) = 2n$ because $c_{vert}(\sigma_\lambda) = 0$. It is shown in [11] (using ideas from [22, 10]) that for all $[\lambda_1],[\lambda_2] \in \pi_1(\mathrm{Ham}\,(M))$

$$\Psi(\lambda_1+\lambda_2) = \Psi(\lambda_1) * \Psi(\lambda_2), \qquad \Psi(0) = \mathbb{1},$$

where 0 denotes the constant loop. Therefore $\Psi(\lambda)$ is invertible for all λ and we get a representation[16]

$$\Psi : \pi_1(\mathrm{Ham}\,(M,\omega)) \;\to\; (QH_{\mathrm{ev}}\,(M,\Lambda))^\times.$$

Since all ω-compatible forms are deformation equivalent, Ψ is independent of the choice of Ω.

This is a mild reformulation of Seidel's representation from [22, 23]. Our choice of canonical section class allows us to define Ψ on the group $\pi_1(\mathrm{Ham}\,(M))$ itself rather than on the extension considered by Seidel. The cost is that we have to allow B to range in the real group $\mathcal{H}$ rather than in its integral lattice. For σ_λ might have real coefficients, and if it does, since we need to sum over classes B such that $\sigma_\lambda - B$ is integral, we must allow B to have real coefficients.

Note also that the fact that $\Psi(\lambda) \neq 0$ implies that the fibration $P_\lambda \to S^2$ has a section (which, moreover, can be taken to be symplectic); equivalently, the loop $t \mapsto \lambda_t(x)$ contracts in M for all $x \in M$. This is an easy consequence of the proof of the Arnold conjecture: see for example [9]§1.3. However, it can be proved more simply, using only the compactness theorem and not gluing, by considering the limit of J-holomorphic sections in class $[pt \times S^2]$ of the trivial bundle $M \times S^2 = P_\lambda \# P_{-\lambda}$ as this space degenerates into the singular union $P_\lambda \cup_M P_{-\lambda}$ as described in [11] §2.3.2.

4.2. Using Ψ to estimate area.

Consider the fibration $\pi : (P_\lambda, \Omega) \to S^2$ corresponding to the class $[\lambda] \in \pi_1(\mathrm{Ham}\,(M))$. Here Ω is any ω-compatible symplectic form on $P = P_\lambda$. Hence, as remarked in §1.3, its cohomology class has the form

$$[\Omega] = u_\lambda + \pi^*([\alpha])$$

[16] Ψ is called ρ in [11], but we have changed the notation here, for obvious reasons. Note also that the formula for Ψ has been written using sign conventions for B that are different from those in [10, 11], to clarify the inequalities considered later.

where area $(P_\lambda, \Omega) = \int_{S^2} \alpha$.

Following Seidel, consider the valuation $v : QH_*(M) \to \mathbb{R}$ defined by

$$v\left(\sum_{B\in\mathcal{H}} a_B \otimes e^B\right) = \sup\{\omega(B) : a_B \neq 0\}). \tag{9}$$

It follows from the definition of the quantum intersection product in (4), (5) that $v(a * b) \le v(a) + v(b)$. As Seidel points out, the following stronger statement is true. Here we set

$$\hbar = \hbar(M) = \min(\{\omega(B) > 0 : \text{ some } n_M(a,b,c;B) \neq 0\},$$

and $\cap$ denotes the usual intersection product, so that $a * b - a \cap b$ is the quantum correction to the usual product. Note that if all the invariants $n_M(a,b,c;B)$ with $a,b,c \in H_*(M)$ and $B \neq 0$ vanish, then $\hbar = \infty$.

Lemma 4.1. *For all $a, b \in QH_*(M)$, $v(a * b - a \cap b) \le v(a) + v(b) - \hbar(M)$.*

Proof. This follows immediately from the definitions. □

Seidel's results in [23] are based on the following observation.

Proposition 4.2 (Seidel). *Suppose that (M, ω) is not spherically monotone. Then, for each loop λ in* Ham (M)

$$\text{area}\,(P_\lambda, \Omega) \quad \ge \quad v(\Psi(\lambda)).$$

Proof. Since $[\omega]$ and c_1 are linearly independent on $H_2^S(M)$ we may define Ψ using a section σ_λ that satisfies (6). Let $\Psi(\lambda) = \sum_B a_B \otimes e^B$. Then by definition a_B is derived from a count of J-holomorphic curves in (P_λ, Ω) in the class $\sigma_\lambda - B$. If $a_B \neq 0$ then this moduli space cannot be empty. Hence

$$\begin{aligned} 0 < [\Omega](\sigma_\lambda - B) \quad &= \quad \pi^*([\alpha])(\sigma_\lambda) - \omega(B) \\ &= \quad \textstyle\int_{S^2} \alpha - \omega(B) \\ &= \quad \text{area}\,(P_\lambda, \Omega) - \omega(B), \end{aligned}$$

as required. □

Corollary 4.3. *In these circumstances $\ell^+(\lambda) \ge v(\Psi(\lambda))$.*

Proof. Since $\ell^+(\lambda) = \widetilde{\rho}^+([\lambda])$, this follows by combining Lemma 1.13 with Proposition 4.2. □

We will apply this in §5 to calculate the lengths of loops. Note that the above argument applies when Ψ is normalized using σ for which $u_\lambda(\sigma) = 0$: the value of $c_{vert}(\sigma)$ is irrelevant here. Hence it can apply in the spherically monotone case when $[\omega] = \kappa c_1$ on $\pi_2(M)$ for $\kappa \neq 0$.

Remark 4.4. Another closely related way of getting an estimate in the monotone case when $[\omega] = \kappa c_1$ for $\kappa > 0$ was observed by Polterovich in [17] Thm 2.A. He defined a homomorphism $I : \pi_1(\mathrm{Ham}\,(M,\omega)) \to \mathbb{R}$ as follows: for each λ choose a section class σ'_λ in P_λ such that $c_{vert}(\sigma'_\lambda) = 0$ and then set

$$I(\lambda) = u_\lambda(\sigma'_\lambda).$$

A dimension count shows that the only section classes σ that contribute to $\Psi(\lambda)$ have $c_{vert}(\sigma) \leq 0$. Hence, given any such σ and any symplectic form Ω on P_λ,

$$\begin{aligned} 0 < \Omega(\sigma) &= \mathrm{area}\,(P_\lambda, \Omega) + u_\lambda(\sigma) \\ &= \mathrm{area}\,(P_\lambda, \Omega) + u_\lambda(\sigma'_\lambda + B) \leq \mathrm{area}\,(P_\lambda, \Omega) + I(\lambda), \end{aligned}$$

since $u_\lambda(B) = \kappa c_1(B)$ for $\kappa > 0$ is nonpositive. Therefore, since $I(-\lambda) = -I(\lambda)$,

$$\widetilde{\rho}_f(\lambda) = \inf(\mathrm{area}\,(P_\lambda, \Omega)) + \inf(\mathrm{area}\,(P_{-\lambda}, \Omega)) \geq |I(\lambda)|.$$

This estimate is weaker than the previous one if $c_1(B) < 0$. However, because I is a homomorphism, one immediately finds $\widetilde{\rho}_f(k\lambda) \geq k|I(\lambda)|$. See also [23].

4.3. The nonsqueezing theorem.

Let $QH_+(M)$ denote the set

$$QH_+(M) = \left\{ x \in \sum_{i \leq 2n-2} H_i(M) \otimes \Lambda \subset QH_*(M) \right\}.$$

We first give a simple criterion for the nonsqueezing theorem to hold. Recall from above that every fibration $P_\lambda \to S^2$ admits a generalized section class σ_λ on which the coupling class u_λ vanishes, except possibly when (M,ω) is weakly exact and c_1 does not vanish on $H_2^S(M)$.

Lemma 4.5. *Suppose that* $(P_\lambda, \Omega) \to S^2$ *admits a generalized section class* σ_λ *on which* u_λ *vanishes, and that the corresponding element* $\Psi(\lambda)$ *has the form*

$$\Psi(\lambda) = \mathbb{1} \otimes \mu + x,$$

where $x \in QH_+(M)$ *and* $\mu = \sum q_B\, e^B$ *has some nonzero coefficient* q_B *with* $\omega(B) \geq 0$. *Then the nonsqueezing theorem holds for* (P_λ, Ω).

Proof. The hypotheses imply that $n_P([M],[M],pt;\sigma_\lambda - B) = q_B \neq 0$. Since this invariant counts perturbed J-holomorphic stable maps in class $\sigma_\lambda - B$ through an arbitrary point, it follows that there is such a curve through every point in P. Since the perturbation can be taken arbitrarily small, it follows from Gromov's compactness theorem that there has to be some J-holomorphic stable map in this class through every point in P. Hence the usual arguments (cf [6], for example) imply that the radius r of any embedded ball satisfies the inequality:

$$\pi r^2 \leq [\Omega](\sigma_\lambda - B) \leq [\Omega](\sigma_\lambda) = \text{area}\,(P_\lambda, \Omega).$$

The result follows. □

It will be convenient to make the following definition.

Definition 4.6. We say that the fibration $(P, \Omega) \to S^2$ with fiber M has a **good section of weight** κ if there is a class $\sigma_A \in H_2(P)$ such that

(i) $n_P([M],[M],pt;\sigma_A) \neq 0$;

(ii) $u(\sigma_A) = -\kappa$ where u is the coupling class.

Note that κ could be positive or negative.

The previous lemma shows that any fibration with a good section of weight 0 has the nonsqueezing property. More generally, the same argument proves the following weighted nonsqueezing property.

Lemma 4.7. *Suppose that* (P_λ, Ω) *has a good section of weight* κ. *Then the radius* r *of an embedded ball in* (P, Ω) *is constrained by the inequality:*

$$\pi r^2 \leq \text{area}\ (P, \Omega) - \kappa.$$

In particular, if Ψ *is defined relative to a section class* σ_λ *on which* u_λ *vanishes, we may take* κ *to be the maximum of* $\omega(B)$ *where* $q_B \neq 0$ *in the expression for* $\Psi(\lambda)$.

It is not hard to find conditions under which $\Psi(\lambda) = \mathbb{1} \otimes \mu + x$, where $\mu \neq 0$. The tricky point is to find ways of estimating the maximal weight of a good section. In this subsection we will describe situations in which there is a good section of weight 0 so that the nonsqueezing theorem holds, leaving the discussion of the more general case to §4.4.

To find good sections of weight 0, we can use arguments in [11] that give conditions under which the (usual) cohomology ring $H^*(P)$ splits. The idea is the following.

Suppose that for some $A \in H_2(M)$ we can define a map $s_A : H_*(M) \to H_{*+2}(P)$ such that

$$s_A(pt) = \sigma_\lambda - A, \quad s_A(a) \cap [M] = a, \quad s_A(a \cap b) = s_A(a) \cap s_A(b), \tag{10}$$

for all $a, b \in H_*(M)$. Then the Poincaré dual map

$$r : H^*(M) \to H^*(P), \quad \alpha \mapsto \mathrm{PD}_P(s_A(\mathrm{PD}_M(\alpha)))$$

is a ring homomorphism such that $\iota^* \circ r = \mathrm{id}$, where $\iota : M \to P$ is the inclusion. In particular, $r([\omega]) = u_\lambda$, since $r([\omega])^{n+1} = 0$ and u_λ is the unique extension of ω such that $u_\lambda^{n+1} = 0$. Further,

$$\mathrm{PD}_P(u_\lambda^n) = s_A(\mathrm{PD}_M[\omega]^n) = s_A(pt)/n! = (\sigma_\lambda - A)/n!.$$

Hence $u_\lambda(\sigma_\lambda - A) = 0$. Since $u_\lambda(\sigma_\lambda) = 0$ by construction, we have $\omega(A) = 0$ as required. Thus

Lemma 4.8. *Suppose that there is a splitting $s_A : H_*(M) \to H_{*+2}(P)$ satisfying* (10)*. Then (P, Ω) has a good section $\sigma_A := \sigma_\lambda - A$ of weight* 0*.*

There are two ways to construct such a splitting s_A. First suppose that there is an Ω-tame almost complex structure J on (P, Ω) such that the moduli space $\mathcal{M}_J$ of J-holomorphic curves of class $\sigma_A = \sigma_\lambda - A$ is compact (where we assume them to be parametrized as sections) and of dimension $2n$. Then, there are evaluation maps

$$e : \mathcal{M}_J \times S^2 \to P, \qquad e_0 : \mathcal{M}_J \to M$$

of equal degree q. If $q \neq 0$ we define

$$s_A : H_*(M) \to H_{*+2}(P) : \quad a \;\mapsto\; \frac{1}{q}\, e_*(e_0^!(a) \times [S^2]), \tag{11}$$

where $(e_0)^! : H_*(M) \to H_*(\mathcal{M})$ is the homology transfer (defined as the Poincaré dual $\mathrm{PD}_{\mathcal{M}} \circ e_0^* \circ \mathrm{PD}_M$ of the pullback in cohomology.) More generally, if all we know is that the invariant

$$n_P([M],[M],pt;\sigma_A) = q_A \neq 0,$$

we define $s_A(a)$ to be the unique class in $H_*(P)$ such that

$$s_A(a) \cdot_P v = \frac{1}{q_A} n_P(a,[M],v;\sigma_A), \qquad v \in H_*(P).$$

(Here, as elsewhere, H_* denotes rational homology.)

The next lemma describes situations in which s_A satisfies the conditions in (10). Statement (i) is due to Seidel[17] and suffices to prove all our main results, including Proposition 1.17 and Theorem 1.6. We include the proof because it is completely elementary, even though one could equally well argue using the other parts of the lemma below. Note that (ii) is a corrected version of [11] Proposition 3.4(i).[18] As in [11], the letters a, b, c denote either elements of $H_*(M)$ or their images in $H_*(P)$, and u, v, w denote general elements in $H_*(P)$. Also $B \in H_2(M,\mathbb{Z})/\sim = \mathcal{H}$.

Lemma 4.9. *With notation as above,*

$$(12) \quad s_A(a) \cap [M] = a, \quad s_A(a \cap b) = s_A(a) \cap s_A(b), \ \textit{for all} \ \ a,b \in H_*(M)$$

under each of the following conditions:

(i) *s_A is defined as above from a compact moduli space $\mathcal{M} := \mathcal{M}_J$.*

(ii) *The only nonzero Gromov–Witten invariants of the form $n_P([M],a,v;\sigma_A - B)$, with $\omega(B) \geq 0$ have $B = 0$.*

(iii) *The 3-point invariants $n_M(a,b,c;B), B \neq 0$, vanish and $n_P([M],[M],pt;\sigma_A - B) = 0$ when $\omega(B) > 0$.*

(iv) *All 3-point vertical invariants $n_P(u,v,w;B)$, $B \in H_2(M) - \{0\}$, in P vanish, as do all 4-point invariants $n_M(a,b,c,d;B)$, $B \neq 0$, in M.*

Proof. (i) It is slightly easier to prove the cohomological version of (i). However, we phrase the argument in homology to make this argument closer to the proof of (ii)–(iv).

[17] Private communication.

[18] In the preprint version of [11], Proposition 3.4 is numbered as 3.21.

If $e : X \to Y$ is any map between two closed manifolds of the same dimension, then the homology transfer

$$e^! : H_*(Y) \to H_*(X) : \quad a \mapsto \mathrm{PD}_X(e^*(\mathrm{PD}_Y a))$$

has the following properties:

$$\begin{array}{lrcl} \text{(a)} & e^!(v \cap w) & = & e^!(v) \cap e^!(w), \\ \text{(b)} & e_*(v) \cap w & = & e_*(v \cap e^!(w)), \\ \text{(c)} & e_*(v) \cdot w & = & v \cdot e^!(w), \\ \text{(d)} & e_* e^!(v) & = & (deg\, e)\, v, \end{array}$$

Further, if $e_0 : \mathcal{M} \to M, e : \mathcal{M} \times S^2 \to P$ are as above, it is not hard to check that

$$\text{(e)} \qquad e^!(v) \cap [\mathcal{M}] = e_0^!(v \cap [M]).$$

Thus, by (11)

$$\begin{aligned} s_A(a) \cap [M] &= \frac{1}{q}\, e_*\left(e_0^!(a) \times [S^2]\right) \cap [M] \\ &= \frac{1}{q}\, e_*\left(e_0^!(a) \times [pt]\right) \\ &= \frac{1}{q}\, (e_0)_* e_0^!(a) = a. \end{aligned}$$

It remains to prove the second half of condition (12). In view of (a) above, this would be obvious if e_* respected the cap product. Since this is not the case, we must use a different approach.

Let us write $e^!(s_A(a)) = [a_0 \times S^2] + \iota(a_1)$ where $a_1 \in H_*(\mathcal{M})$ and $\iota : H_*(\mathcal{M}) \to H_*(\mathcal{M} \times S^2)$ is induced by the inclusion. It follows easily from (e) above that $a_0 = e_0^!(a)$. Hence, by (d), $(e_0)_*(a_1) = 0$. By (c), this means that $a_1 \cdot e_0^!(b) = 0$ for all b. Hence

$$\begin{aligned} q\, s_A(b) \cdot s_A(a) &= e_*([e_0^!(b) \times S^2]) \cdot s_A(a) \\ &= [e_0^!(b) \times S^2] \cdot (a_0 \times [S^2] + \iota(a_1)) \\ &= a_1 \cdot e_0^!(b) \;=\; 0 \end{aligned}$$

for all a, b. Again using $s_A(a) \cap [M] = a$, we have that

$$s_A(a) \cap s_A(b) = s_A(a \cap b) + \iota(x)$$

for some $x \in H_*(M)$. Further $x = 0$ if and only if

$$(s_A(a) \cap s_A(b)) \cdot_P s_A(c) = \iota(x) \cdot_P s_A(c) = x \cdot_M c = 0$$

for all c. But, as above,

$$\begin{aligned} q\, s_A(c) \cdot_P (s_A(a) \cap s_A(b)) \\ &= [e_0^!(c) \times S^2] \cdot \Big(e^!(s_A(a)) \cap e^!(s_A(b)) \Big) \\ &= [e_0^!(c) \times S^2] \cdot \Big([e_0^!(a) \times S^2] \cap \iota(b_1) + \iota(a_1) \cap [e_0^!(b) \times S^2] \Big) \\ &= e_0^!(c \cap a) \cdot b_1 \;\; \pm \;\; e_0^!(c \cap b) \cdot a_1 \\ &= 0. \end{aligned}$$

This completes the proof of (i). □

The proofs of (ii), (iii) and (iv) can be found in [11]; (iv) is Proposition 3.4, while (ii) is a corrected form of Proposition 3.5 (i). We have added the extra assumption that there are no nonzero invariants of the form $n_P(v, [M], a; \sigma_A - B)$ with $B \neq 0$ in order for the proofs of [11] Lemmas 3.10 and 3.11 to hold. (iii) is a slightly more general version of [11] Proposition 3.5 (ii). We now allow invariants of the form $n_P([M], [M], pt; \sigma_A - B)$ with $\omega(B) < 0$ to be nonzero. But clearly these do not contribute to the sums considered in [11] Lemmas 3.9, 3.10, 3.11. □

Corollary 4.10. *Consider a fibration $(P_\lambda, \Omega) \to S^2$ such that $n_P([M], [M], pt; \sigma_A) \neq 0$ for some class $\sigma_A \in H_2(P)$. If one of the conditions in Lemma 4.9 also holds then σ_A is a good section of weight 0 in (P_λ, Ω).*

Proof. This holds by Lemma 4.8. Observe that the conclusion holds even when we cannot assume that $u_\lambda(\sigma_\lambda) = 0$ since we prove in all cases that $u_\lambda(\sigma_A) = 0$. □

With these preliminaries, we are now ready to prove Proposition 1.17. In view of Lemma 4.5 it is an immediate consequence of the next result. Here

$$\Lambda^- = \Big\{ \mu \in \Lambda : \mu = q_0 e^0 + \sum q_C e^C, \;\; \omega(C) < 0, \; q_C \in \mathbb{Q}, \; q_0 \neq 0. \Big\}$$

Proposition 4.11. *Suppose that (M, ω) is spherically rational with index of rationality $q(M)$ and let λ be a loop in $\mathrm{Ham}\,(M, \omega)$. If $\ell^+(\lambda) + \ell^-(\lambda) < q(M)$ then $\Psi(\lambda)$ and $\Psi(-\lambda)$ both have the form $\mathbb{1} \otimes \mu + x$ with $\mu \in \Lambda^-$, $x \in QH_+(M)$. In particular, they both have a good section of weight 0.*

Proof. We will assume that M is not weakly exact since this case is dealt with in Lemma 4.13. Hence we can choose σ_λ so that $u_\lambda(\sigma_\lambda) = 0$.

Choose $\varepsilon > 0$ so that $\ell^+(\lambda) + \ell^-(\lambda) < q(M) - 2\varepsilon$. By Proposition 1.12, there is a ω-compatible symplectic form Ω_λ on P_λ with area $< \ell^+(\lambda) + \varepsilon$, and a similar form $\Omega_{-\lambda}$ on $P_{-\lambda}$ with area $< \ell^-(\lambda) + \varepsilon$.

Write

$$\Psi(\lambda) = \sum_{B\in\mathcal{H}} \mathbb{1} \otimes q_B e^B + x, \quad \Psi(-\lambda) = \sum_{B'\in\mathcal{H}} \mathbb{1} \otimes q'_{B'} e^{B'} + x'$$

where $q_B, q'_{B'} \in \mathbb{Q}$ and $x, x' \in QH^+$. Note first that for all e^B (resp. $e^{B'}$) that occur in $\Psi(\lambda)$ (resp. $\Psi(-\lambda)$) with nonzero coefficient,

$$\omega(B) < \ell^+(\lambda) + \varepsilon, \qquad \omega(B') < \ell^-(\lambda) + \varepsilon.$$

This holds from the definition of the coefficients via Gromov–Witten invariants $n_P([M], [M], c; A)$ where $A = \sigma_\lambda - B$ (resp. $\sigma_{-\lambda} - B$), and the fact that Ω_λ (resp. $\Omega_{-\lambda}$) must have positive integral on A: see equations (7), (8).

Next apply the valuation v in (9) to the identity

$$\Psi(\lambda) * \Psi(-\lambda) = \Psi(0) = \mathbb{1}.$$

We claim that at least one of $\Psi(\lambda), \Psi(-\lambda)$ has a term $\mathbb{1} \otimes q_B e^B$ with $q_B \neq 0, \omega(B) \geq 0$. For otherwise the product $x * x'$ must contain the term $\mathbb{1} \otimes e^0$ with a nonzero coefficient. Because this term appears in $x * x' - x \cap x'$ we find from Lemma 4.1 that

$$\begin{aligned} 0 = v(\mathbb{1} \otimes e^0) &\leq v(\Psi(\lambda)) + v(\Psi(-\lambda)) - q(M) \\ &\leq \ell^+(\lambda) + \ell^-(\lambda) + 2\varepsilon - q(M) < 0, \end{aligned}$$

a contradiction.

Therefore, replacing λ by $-\lambda$ if necessary we may suppose that

$$\Psi(\lambda) = \mathbb{1} \otimes \mu e^A + x$$

where $x \in QH_+(M), 0 \leq \omega(A) < q(M) - \varepsilon$, and $\mu \in \Lambda^-$. The lemma will follow if we show that $\omega(A) = 0$.

To do this, consider the class $\sigma_A = \sigma_\lambda - A$ as above. In view of Corollary 4.10 it suffices to prove the following claim.

Claim. There is an Ω-tame J on P such that the moduli space $\mathcal{M}_J$ of unparametrized J-holomorphic curves in (P, Ω) of class σ_A is a compact manifold of dimension $2n$.

Proof of Claim. We first show that $\mathcal{M}_J$ is compact for all fibered J. (Recall that an almost complex structure J on a symplectically fibered space $P \to S^2$ is called fibered if each fiber is J-holomorphic.) If not, there is a sequence of σ_A-curves that converges to a stable map. One component of this stable map will be a section and the others will each lie entirely in a fiber. There must be at least one bubble in a fiber, which will use up a minimum of $q(M)$ in energy. Since

$$[\Omega](\sigma_A) = \varepsilon + \omega(A) < q(M)$$

this is impossible. Next observe that the curves in $\mathcal{M}_J$ are all embedded so that they can be regularized by choosing a generic J. It is not hard to see that this can be chosen to be fibered: compare Lemma 4.3 of [11].[19] Alternatively, use the compactness theorem again to conclude that $\mathcal{M}_J$ is compact for every J that is sufficiently close to a fibered J, and then choose a regular one from among these. This proves the claim and hence the Proposition. □

Remark 4.12. Using [11] Proposition 3.4 one can conclude from the above argument that if (M, ω) is spherically rational then whenever $[\lambda] \in \pi_1(\mathrm{Ham}\,(M))$ and its inverse $[-\lambda]$ have sufficiently $\widetilde{\rho}$-short representatives they are in the kernel of the homomorphisms $\overline{I}_c$ and $\overline{I}_u$ in [10, 11]. Moreover the rational cohomology rings $H^*(P_\lambda)$ and $H^*(P_{-\lambda})$ split as products.

Further extensions of the proof of Proposition 4.11 are discussed in §4.4. We end this section by proving Proposition 1.20.

Lemma 4.13.

(i) *When $c_1 = 0$ on $\pi_2(M)$ and all 3-point Gromov–Witten invariants on M vanish, every fibration $(P, \Omega) \to S^2$ has a good section of weight 0.*

(ii) *The same statement holds if (M, ω) is weakly exact.*

(iii) *If $M = \mathbb{C}P^n$ the $(n+1)$st multiple $(n+1)[\lambda]$ of each loop λ has a good section of weight 0.*

Proof. (i) Assume first that c_1 vanishes on $\pi_2(M)$ but ω does not, and choose σ'_λ so that $u_\lambda(\sigma'_\lambda) = 0$. We claim that $c_{vert}(\sigma'_\lambda)$ must also be zero. For c_{vert} takes the same value on all section classes $\sigma'_\lambda - B$, and since at least one

[19] Lemma 4.9 in the preprint.

invariant $n_P([M],[M],a;\sigma'_\lambda - B)\neq 0$ this value must be ≤ 0. On the other hand

$$c_{vert}(\sigma'_\lambda) + c_{vert}(\sigma'_{-\lambda}) = c_{vert}(\sigma'_\lambda\#\sigma'_{-\lambda}) = 0,$$

since the concatenation $\sigma'_\lambda\#\sigma'_{-\lambda}$ is the canonical section in the product $M\times S^2$: see [10]. Hence $c_{vert}(\sigma'_\lambda)=0$ so that $\sigma'_\lambda=\sigma_\lambda$, and $\Psi(\lambda)=\mathbb{1}\otimes\mu$ for some $\mu\in\Lambda$. Therefore, there are classes B so that $n_P([M],[M],pt;\sigma_\lambda - B)\neq 0$ and we choose A from among them so that $\omega(A)$ is a maximum. This means that condition (iii) in Lemma 4.9 holds. Therefore the result follows from Corollary 4.10. □

To prove (ii), suppose that (M,ω) is weakly exact. We said before that by the results of [11] we may choose σ_λ so that both classes c_{vert} and u_λ vanish on it. For the sake of completeness, we will give this argument now. Suppose first that c_1 vanishes on $H_2^S(M)$ as well. Then there is only one section class (up to equivalence) and we call it σ_λ. The argument in (i) above shows that c_{vert} must vanish on this class. Further, the moduli space $\mathcal{M}_J$ of curves in this class must be compact (since there are no J-holomorphic curves in M). Hence $u_\lambda(\sigma_\lambda)=0$ as well. Since $\Psi(\lambda)=q\mathbb{1}\neq 0$ the result follows.

When $c_1\neq 0$ on $H_2^S(M)$, we choose σ_λ so that $c_{vert}(\sigma_\lambda)=0$. As in (i), we choose A so that $\omega(A)$ is maximal among the classes B for which $n_P([M],[M],pt;\sigma_\lambda - B)\neq 0$. The argument then continues either as in (i) or as in the proof of Proposition 4.11. In particular, it shows that $u_\lambda(\sigma_A)=0$. □

When $M=\mathbb{C}P^n$, we again choose σ_λ so that $c_{vert}(\sigma_\lambda)=0$, so there is no control over $u_\lambda(\sigma_\lambda)$. Because $c_1(L)=n+1$, where $L=[\mathbb{C}P^1]$, and $-n\le c_{vert}(\sigma_\lambda - B)\le 0$ whenever

$$n_P([M],[M],c;\sigma_\lambda - B)\neq 0, \qquad c\in H_*(M), \tag{13}$$

there can be at most one class $B=rL$ with nonzero invariant. Hence $\Psi(\lambda)$ has the form $a\otimes e^{rL}$, where $a\in H_{2k}(\mathbb{C}P^n)$ is homogeneous. Since $\deg\Psi(\lambda)=2n$ we must have $r(n+1)=n-k$. Therefore

$$\Psi((n+1)\lambda) = a^{(n+1)}\otimes e^{(n-k)L} = q\mathbb{1}, \quad q\in\mathbb{Q}-\{0\},$$

since the hyperplane class $b\in QH_{2n-2}(\mathbb{C}P^n)$ satisfies the relation $b^{(n+1)} = \mathbb{1}\otimes e^{-L}$ and $a=q'b^{n-k}$. Therefore, if $Q\to S^2$ denotes the fibration corresponding to the loop $(n+1)\lambda$,

$$n_Q([M],[M],pt;\sigma_{(n+1)\lambda})\neq 0.$$

Again using the fact that $c_1(L) = n+1$ and taking $\sigma_A = \sigma_{(n+1)\lambda}$, we find that condition (ii) in Lemma 4.9 holds. Hence $u_{(n+1)\lambda}(\sigma_{(n+1)\lambda}) = 0$ and the result follows. Observe that in this case condition (i) in Lemma 4.9 also holds for generic J on P since for each $k \geq 1$ the moduli space of curves in P of class $\sigma_A - kL$ must vanish for reasons of dimension. □

Remark 4.14.

(i) The above argument about $\mathbb{C}P^n$ applies to any manifold M for which $\Psi(\pi_1(\mathrm{Ham}))$ is a finite subgroup of $QH_{ev}(M)^\times$. Here we should assume that M is not spherically monotone so that we always have $u_\lambda(\sigma_\lambda) = 0$. This was unnecessary for $M = \mathbb{C}P^n$ since $c_1(L)$ is so large.

(ii) Lemma 4.7 shows that there is some nonsqueezing inequality for any fibration $(P_\lambda, \Omega) \to S^2$ that has good sections, i.e., has $\Psi(\lambda) = 1\!\!1 \otimes \mu + x$ for some $\mu \neq 0$. One cannot say anything for general fibrations $P_\lambda \to S^2$ unless the embedded ball is disjoint from one fiber. In the latter case, one can make P_λ symplectically trivial by taking the fiber sum with $P_{-\lambda}$, and deduce that the radius r of any symplectic ball in (P_λ, Ω) that misses a fiber is constrained by the inequality

$$\pi r^2 \leq \mathrm{area}\,(P_\lambda, \Omega) + \ell^-(\lambda).$$

More generally, as suggested by Polterovich,[20] one can consider nonsqueezing for fibrations

$$(P_{\widetilde{\phi}}, \Omega) \to D$$

with nontrivial boundary monodromy $\widetilde{\phi}$. Completing these to symplectically trivial fibrations over S^2 by adding a fibration with boundary monodromy $\widetilde{\phi}^{-1}$, one finds that the radius r of any symplectic ball in $(P_{\widetilde{\phi}}, \Omega)$ satisfies

$$\pi r^2 \leq \mathrm{area}\,(P_{\widetilde{\phi}}, \Omega) + \widetilde{\rho}^-(\widetilde{\phi}).$$

Further, if the nonsqueezing theorem holds for all loops in $\mathrm{Ham}\,(M)$, one has the inequality

$$\pi r^2 \leq \mathrm{area}\,(P_\phi, \Omega) + \rho^-(\phi).$$

[20] Private communication.

4.4. Weighted nonsqueezing and geodesics.

In this section we prove Proposition 1.18 and Corollary 1.19, and hence complete the proof of Theorem 1.6.

Proof of Proposition 1.18. By hypothesis there are fibrations $(P_{\pm\lambda}, \Omega)$ with area $< \hbar/2$. Apply the valuation v in (9) to the identity

$$\Psi(\lambda) * \Psi(-\lambda) = \Psi(0) = \mathbb{1},$$

as in the proof of Proposition 4.11. As there, because area $(P_{\pm\lambda}, \Omega) < \hbar/2$, the product $x * x'$ cannot contain the term $\mathbb{1} \otimes e^0$ with a nonzero coefficient. Therefore for λ' equal to at least one of λ or $-\lambda$, $\Psi(\lambda')$ has a term $q_B \mathbb{1} \otimes e^B$ with $q_B \neq 0$ and $0 \leq \omega(B) <$ area $(P_{\lambda'}, \Omega)$.

Set

$$\varepsilon(\lambda') = \max\{\omega(B) : q_B \neq 0 \ \text{ in } \ \Psi(\lambda')\}, \quad \lambda' = \pm\lambda.$$

The equation $\Psi(\lambda) * \Psi(-\lambda) = \mathbb{1}$ implies that $\varepsilon(\lambda) = -\varepsilon(-\lambda)$. Moreover, by Lemma 4.7, the radius r of any embedded ball in $(P_{\lambda'}, \Omega)$ satisfies

$$\pi r^2 \leq \text{area}\,(P_{\lambda'}, \Omega) - \varepsilon(\lambda').$$

Hence we may take $\varepsilon = -\varepsilon(\lambda)$. □

Remark 4.15. The above argument uses only the first half of the proof of Proposition 4.11 since it is not clear what hypothesis would guarantee that the Claim holds, i.e. that an appropriate moduli space of sections is compact. Although for each individual ω-tame J the minimum energy $\hbar(M, J)$ of a nontrivial J-holomorphic bubble is positive, the minimum of $\hbar(M, J)$ over all such J will not in general be strictly positive since any symplectically embedded 2-sphere is J-holomorphic for some tame J. We cannot restrict ourselves to a compact set of J since we must consider all loops λ in $\pi_1(\text{Ham})$, each of which gives rise to some 2-parameter family of J on the fibers of P_λ. One might also try to establish nonsqueezing by using Lemma 4.9 (iv). For this one would need a constant $\hbar > 0$ such that all vertical Gromov–Witten invariants $n_P(u, v, w; B)$ vanish, where u, v, w are arbitrary elements in $H_*(P_\lambda)$ and $0 < \omega(B) < \hbar$. Again, because we are considering the Gromov–Witten invariants of an arbitrary fibration P_λ rather than those of a compact manifold, it is not clear that such $\hbar$ exists. Therefore, our present methods do not suffice to show that the nonsqueezing theorem holds for all $\widetilde{\rho}$-short loops.

Proof of Corollary 1.19. If $(\widetilde{\nu}, \nu)$ is one of the pairs $(\widetilde{\rho}, \rho), (\widetilde{\rho}_f, \rho_f)$, we must show that if a path γ has $\widetilde{\nu}$-length $< \hbar(M)/4$ and minimizes $\widetilde{\nu}$ in $\widetilde{\mathrm{Ham}}(M)$, it is ν-minimizing in $\mathrm{Ham}(M)$.

As in the proof of Lemma 1.16 we will carry out the argument for the pair $(\widetilde{\rho}_f, \rho_f)$. Suppose that γ does not minimize ρ_f. Then there are paths $\psi_t^{\pm}$ from id to $\phi = \phi_1^H$ generated by $K_t^{\pm}$ and $\delta > 0$ such that

$$\mathcal{L}^+(K_t^+) + \mathcal{L}^-(K_t^-) = \mathcal{L}(\gamma) - \delta < \mathcal{L}(\gamma) = \mathcal{L}^+(H_t) + \mathcal{L}^-(H_t).$$

As before, we may assume that:

$$\mathcal{L}^+(K_t^+) = \mathcal{L}^+(H_t) - \delta' < \mathcal{L}^+(H_t), \qquad \mathcal{L}^-(K_t^-) = \mathcal{L}^-(H_t) - \delta + \delta',$$

for some $\delta' > 0$. Let $\lambda = \beta * (-\gamma)$ as before so that $P_{K^+,H} = P_\lambda$, $P_{H,K^-} = P_{-\lambda}$. Then for small ε

$$\begin{array}{rcl} \mathrm{area}\,(P_\lambda(\varepsilon), \Omega_0) & = & \mathcal{L}(\gamma) - \delta' + \varepsilon \;<\; \mathcal{L}(\gamma) < \hbar/4, \\ \mathrm{area}\,(P_{-\lambda}(\varepsilon), \Omega_0) & = & \mathcal{L}(\gamma) - \delta + \delta' + \varepsilon \le 2\mathcal{L}(\gamma) < \hbar/2. \end{array}$$

By Proposition 1.18 there is κ with $|\kappa| \le \hbar/2$ such that embedded balls satisfy

$$\pi r^2 \le \mathrm{area}\,(P_\lambda(\varepsilon), \Omega) + \kappa, \quad \pi r^2 \le \mathrm{area}\,(P_{-\lambda}(\varepsilon), \Omega) - \kappa.$$

But, by construction, both $(P_\lambda(\varepsilon), \Omega)$ and $(P_{-\lambda}(\varepsilon), \Omega)$ contain embedded balls of capacity $\pi r^2 = \mathcal{L}(\gamma) > \mathrm{area}\,(P_\lambda(\varepsilon), \Omega)$. Hence $\kappa > 0$. Further,

$$\begin{array}{rcccl} \mathcal{L}(\gamma) & \le & \mathrm{area}\,(P_\lambda(\varepsilon), \Omega) + \kappa & = & \mathcal{L}(\gamma) - \delta' + \varepsilon + \kappa \\ \mathcal{L}(\gamma) & \le & \mathrm{area}\,(P_{-\lambda}(\varepsilon), \Omega) - \kappa & = & \mathcal{L}(\gamma) - \delta + \delta' + \varepsilon - \kappa. \end{array}$$

Adding, we find $0 \le -\delta + 2\varepsilon$. Since δ is positive and ε can be arbitrarily small, this is impossible. Hence result. □

5. Estimating lengths of loops.

Let M_* be the one point blow-up of $\mathbb{C}P^2$. We may think of this as the region

$$\{(z_1, z_2) \in \mathbb{C}^2 : a^2 \le |z_1|^2 + |z_2|^2 \le 1\}$$

with boundaries collapsed along the Hopf flow, and give it the corresponding symplectic form ω_a. We are considering the action

$$(z_1, z_2) \mapsto (e^{-2\pi i t} z_1, e^{-2\pi i t} z_2), \quad 0 \le t \le 1.$$

It is not hard to check that its normalized Hamiltonian is

$$H = \pi(c - |z_1|^2 - |z_2|^2), \qquad c = \frac{2(1-a^6)}{3(1-a^4)}.$$

Since $\max H = \pi(c - a^2)$ and $-\min H = \pi(1-c)$, we find $\max H > -\min H$ whenever $a^2 < 1$. (A similar example was given in Example 1.C in [17].)

The next task is to calculate $QH_*(M_*)$. This ring is generated over Λ by elements $p = pt$, the exceptional divisor E, the fiber class $F = L - E$ and the fundamental class $[M_*]$. (Here $L = [\mathbb{C}P^1]$.) The quantum multiplication has $1\!\!1 = [M_*]$ as a unit, and is derived from the following nontrivial Gromov–Witten invariants:

$$\begin{array}{ll} n(p,p,F;E+F) = 1; & n(p,E,E;F) = 1; \\ n(A_1,A_2,A_3;E) = \pm 1 & \text{where} \quad A_i = E \text{ or } F. \end{array}$$

One finds

$$\begin{array}{rclrcl} p*p &=& (E+F)\otimes e^{-E-F} & E*p &=& F\otimes e^{-F} \\ p*F &=& 1\!\!1\otimes e^{-E-F}, & E*E &=& -p + E\otimes e^{-E} + 1\!\!1\otimes e^{-F} \\ E*F &=& p - E\otimes e^{-E} & F*F &=& E\otimes e^{-E}. \end{array}$$

We will be particularly interested in the element $Q = F\otimes e^{E/2+F/4}$, since, as we shall see, this is the part of $\Psi(\alpha)$ that is independent of the choice of symplectic form ω_a. Note that

$$Q*Q = Q^2 = E\otimes e^{F/2}, \qquad Q^{-1} = p\otimes e^{3F/4+E/2},$$

where the multiplication is $*$. Recall that v is defined by:

$$v\left(\sum_{B\in\mathcal{H}} a_B\otimes e^B\right) = \sup\{\omega(B) : a_B \neq 0\}.$$

Lemma 5.1. *$v(Q^k) + v(Q^{-k}) \geq \omega(F)$ for all $k > 1$.*

Proof. First consider $v(Q^{-k}), k \geq 1$. The first few terms are

$$\begin{array}{rcl} Q^{-1} &=& p\otimes e^{3F/4+E/2}, \\ Q^{-2} &=& (E+F)\otimes e^{F/2}, \\ Q^{-3} &=& F\otimes e^{F/4+E/2} + 1\!\!1\otimes e^{F/4-E/2}, \\ Q^{-4} &=& p\otimes e^F + 1\!\!1. \end{array}$$

We claim that for all $m \geq 1$ and $1 \leq i \leq 4$,

$$v(Q^{-i-4m}) \geq v(Q^{-i}). \tag{14}$$

To see this note that multiplication by p has the effect:

$$\begin{array}{cccc} p & E & F & \mathbb{1} \\ \downarrow & \downarrow & \downarrow & \downarrow \\ E+F & F & \mathbb{1} & p \end{array}$$

Since all coefficients are positive, no terms can cancel. The p-term in Q^{-k} contributes to the p-term in $Q^{-(k+4)}$ with unchanged valuation via the cycle $p \mapsto E \mapsto F \mapsto \mathbb{1} \mapsto p$. Similarly, the F-term in Q^{-k} contributes to the F-term in $Q^{-(k+4)}$ with unchanged valuation. Hence result.

Next consider $Q^k, k > 0$. The first few terms are

$$\begin{array}{rcl} Q & = & F \otimes e^{E/2+F/4}, \\ Q^2 & = & E \otimes e^{F/2}, \\ Q^3 & = & p \otimes e^{3F/4+E/2} - E \otimes e^{3F/4-E/2}, \\ Q^4 & = & -p \otimes e^{F} + E \otimes e^{F-E} + \mathbb{1}, \\ Q^5 & = & p \otimes e^{5F/4-E/2} - E \otimes e^{5F/4-3E/2} + F \otimes e^{E/2+F/4} - \mathbb{1} \otimes e^{F/4-E/2}. \end{array}$$

Thus the lemma holds for $1 \leq i \leq 4$ by inspection. We will write

$$Q^k = p \otimes \lambda_{k,p} + E \otimes \lambda_{k,E} + F \otimes \lambda_{k,F} + \mathbb{1} \otimes \lambda_{k,\mathbb{1}},$$

where $\lambda_{k,\cdot} \in \Lambda$, and $\lambda_{k,\cdot} = 0$ for $k \leq 0$, and will prove that for all $m \geq 1$

$$\begin{array}{rcl} v(\lambda_{i+4m,E}) & \geq & v(\lambda_{i,E}), \quad i = 2,3 \\ v(\lambda_{1+4m,F}) & \geq & v(\lambda_{1,F}), \\ v(\lambda_{4+4m,\mathbb{1}}) & \geq & v(\lambda_{4,\mathbb{1}}). \end{array}$$

In view of (14), this will prove the lemma.

Multiplying by F has the following effect:

$$\begin{array}{cccc} p & E & F & \mathbb{1} \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \mathbb{1} & p, -E & E & F \end{array}$$

The term in E in Q^k comes either from the term in E in Q^{k-1} or from the term in F in Q^{k-1}. The latter contribution traces directly back to the term in E in Q^{k-4} via multiplication by the term $\mathbb{1}$ in Q^4. Hence its valuation is $v(\lambda_{k-4,E})$, while the valuation of the former term is $v(\lambda_{k-1,E}) + \omega(F/4 - E/2)$. Hence

$$v(\lambda_{k,E}) = \max\{v(\lambda_{k-4,E}), v(\lambda_{k-1,E}) + \omega(F/4 - E/2)\}.$$

Therefore $v(\lambda_{k,E}) \geq v(\lambda_{k-4,E})$ provided that no terms cancel. Cancellations could theoretically occur since the transition $E \mapsto E$ occurs with a $-$ sign; however we claim that they do not. To see this, it suffices to look at the terms that contribute to $\lambda_{2+4m,E}$. Since Q^2 is a multiple of E each term that involves E in Q^{2+4m} must involve some number of the period 4 cycle $E \mapsto p \mapsto \mathbb{1} \mapsto F \mapsto E$ interspersed with some $E \to -E$ transitions. But there must always be an even number of these transitions if we are to arrive back at E after $4m$ multiplications. Therefore all the contributions to the E term in Q^{2+4m} occur with positive sign. This shows that the coefficients $\lambda_{k,E}$ have the claimed behavior. Similar arguments prove the statements about the other coefficients. □

Our next aim is to calculate $\Psi(\alpha)$. This depends on ω_a, since $\omega_a(E) = a^2, \omega_a(F) = 1 - a^2$. The case when $\omega_a(F - 2E) = 1 - 3a^2 = 0$ is special, since in this case ω_a is a multiple of c_1, i.e. (M_*, ω_a) is monotone. We will see that when $3a^2 \neq 1$ then $\Psi_\alpha = Qe^{\delta(F-2E)}$ for an appropriate constant δ. Since $c_1(F - 2E) = 0$ the constant $\delta = \delta(a)$ is determined by the requirement that $u_\alpha(\sigma_\alpha) = 0$. In fact, the exact value of δ is irrelevant for our main argument since we just need to estimate $\ell^+(k\alpha) + \ell^+(-k\alpha)$: see Corollary 5.4 below.

It turns out to be easier first to calculate $\Psi(\lambda)$ where $\lambda = 2\alpha$.[21]

Lemma 5.2. *If* $1 - 3a^2 \neq 0$ *then*

$$\Psi(\lambda) = E \otimes e^{2\delta(F-2E)+F/2}, \quad \textit{where } \delta = \frac{(1-a^2)^2}{12(1+a^2)(1-3a^2)}.$$

Proof. Since the circle action $(z_1, z_2) \mapsto (e^{-2\pi it}z_1, e^{-2\pi it}z_2)$ preserves the fibers of the projection $M_* \to S^2$, the space P_λ fibers over $S^2 \times S^2$ with fiber S^2. Let us denote the generators of $H_2(S^2 \times S^2)$ by $A = [S^2 \times pt], \sigma = [pt \times S^2]$. Here we are thinking that the original fibration $\pi : P_\lambda \to S^2$ is the composite with projection pr_2 to the second factor, so that lifts of σ to P_λ correspond to sections of the original fibration π. It is then not hard to see that the fibration over $S^2 \times S^2$ has the form

$$\pi_1 : P(L \oplus \mathbb{C}) \to S^2 \times S^2$$

[21] A simpler way of doing these calculations is developed in [14].

where $c_1(L) = A + \sigma$. Further, this fibration has an obvious complex structure J as well as two natural J-holomorphic sections, Z_- with normal bundle L^{-1} and Z_+ with normal bundle L. Note that $M_* = \pi_1^{-1}(S^2 \times pt)$. Moreover the class of the exceptional divisor E in M_* is represented by the intersection $Z_- \cap M_*$. To check this, recall that P_λ is made from the region R_H^+ above the graph of H by collapsing the boundary Γ_H to a single fiber. Moreover, the set of points $F_{\max}$ where the Hamiltonian H takes its maximum is precisely the exceptional divisor, while the corresponding set $F_{\min}$ where H is a minimum is a complex line. It is not hard to see that in this realization the sections Z_+ and Z_- corrrespond to the intersection of R_H^+ with the slices $F_{\min} \times [0,1] \times \mathbb{R}$ and $F_{\max} \times [0,1] \times \mathbb{R}$, respectively, where Z_+ corresponds to $F_{\min}$ because this slice has larger volume (with respect to Ω_0^2.)

A dimension count shows that the only section classes $\sigma_\lambda + B$ that contribute to $\Psi(\lambda)$ are those with vertical Chern class $c_{vert} \le 0$. Let σ_- denote a lift to Z_- of the sphere $[pt \times S^2]$, Then $c_{vert}(\sigma_-) = -1$, and so the only classes that could contribute $\Psi(\lambda)$ are σ_- and $\sigma_- + E$. Since $c_{vert}(\sigma_- + E) = 0$, this class would contribute with coefficient

$$n_{P_\lambda}([M_*], [M_*], pt\,; \sigma_- + E).$$

But this invariant is zero since all the J-holomorphic sections in class $\sigma_- + E$ lie in $Z_- = S^2 \times S^2$ and so do not meet an arbitrary point. (This follows by positivity of intersections since Z_- is holomorphic and has "negative" normal bundle.) For similar reasons, the J-holomorphic sections in class σ_- also lie in Z_- and it is not hard to see that

$$n_{P_\lambda}([M_*], [M_*], c\,; \sigma_-) = E \cdot c.$$

It remains to calculate the canonical section class σ_λ. We claim that

$$\sigma_\lambda = \sigma_- + F/2 + 2\delta(F - 2E)$$

for the δ given above. First observe that, as in [11] Example 3.1, the cohomology ring $H^*(P_\lambda)$ is generated by the vertical Chern class ν of $\pi_1 : P \to S^2 \times S^2$ together with the pullbacks μ_1, μ_2 via π_1 of the two obvious generators of $H^2(S^2 \times S^2)$. Thus

$$\mu_1(E) = 1, \;\; \nu(E) = -1, \;\; \mu_1(F) = \mu_2(F) = 0, \;\; \nu(F) = 2.$$

Clearly $\mu_i^2 = 0$. It is also not hard to see that $\nu^2 = 2\mu_1\mu_2$. (As in [11] Example 3.1, look at the Poincaré duals.) The vertical Chern class c_{vert} for $P_\lambda \to S^2$ takes the value 1 on E and so is $\nu + 2\mu_1$. But

$\mu_1(\sigma_-) = 0$. Therefore $c_{vert}(\sigma_- + F/2) = 0$. Since $c_{vert}(F - 2E) = 0$ and $u_\lambda(F - 2E) = \omega_a(F - 2E) \neq 0$ when $3a^2 \neq 1$, σ_λ must have the stated form for some δ. Now observe that

$$u_\lambda = \frac{1-a^2}{2}\nu + \frac{1+a^2}{2}\mu_1 + \varepsilon\mu_2,$$

and use the fact that $u_\lambda^3 = 0$ to conclude $\varepsilon = -(1-a^2)^2/6(1+a^2)$. Further, because $u_\lambda(\sigma_\lambda) = 0$ we must have $\varepsilon = 2\delta(3a^2 - 1)$, as required.

Finally, the fact that $\sigma_- = \sigma_\lambda - F/2 - 2\delta(F - 2E)$ implies

$$\Psi(\lambda) = E \otimes e^{F/2+2\delta(F-2E)}$$

as claimed. (Compare Seidel [22], where a similar calculation is made for $M_* = S^2 \times S^2$.) □

Proposition 5.3. *If* $3a^2 \neq 1$, *then* $\Psi(\alpha) = \pm F \otimes e^{E/2+F/4+\delta(F-2E)}$ *where* δ *is as before.*

Proof. We show that $\Psi(\alpha)$ has the form $E \otimes \lambda_E + F \otimes \lambda_F$, i.e., that the coefficients of p and $\mathbb{1}$ vanish. The desired result then follows by an easy calculation, using our knowledge of $(\Psi(\alpha))^2$.

The action

$$(z_1, z_2) \mapsto (e^{-2\pi i\theta} z_1, z_2)$$

permutes the lines through $(0,0) \in \mathbb{C}^2$. Hence its action on M_* is the lift of an action α^* on the base S^2 of the obvious fibration $M_* \to S^2$. It is easy to see that α^* is a rotation with two fixed points, one corresponding to the line $z_1 = 0$ (whose points are all fixed) and the other corresponding to $z_2 = 0$. Hence the fibration $\pi : P_\alpha \to S^2$ factors as

$$P_\alpha \xrightarrow{\pi_1} P_{\alpha*} \xrightarrow{\pi_2} S^2,$$

where P_{α^*} is a one point blow up of S^2 and $\pi_1 : P_\alpha \to P_{\alpha^*}$ is a fibration with fiber S^2. All these spaces have natural complex structures that are preserved by these maps. Further there are two natural sections $Z_\pm$ of π_1 given by the restricting the action α to the two canonical sections of $M_* \to S^2$ (the images of the two boundary components of the region $\{a^2 \leq |z_1|^2 + |z_2|^2 \leq 1\}$.)

The fibration $\pi_2 : P_{\alpha^*} \to S^2$ has two natural sections given by the two fixed lines in M_*. The exceptional divisor E^* in $P_{\alpha*}$ corresponds to the line

$z_1 = 0$ which is pointwise fixed by α, while the other section L^* has self intersection $+1$. It follows that the fibration

$$\pi : \pi_1^{-1}(E_*) \to S^2$$

is holomorphically trivial: it has the form $S^2 \to P_\gamma \to S^2$ where γ is the induced action on the line $z_1 = 0$. On the other hand the fibration $\pi : \pi_1^{-1}(L_*) \to S^2$ is nontrivial, and its total space is the one point blow up of $\mathbb{C}P^2$. Indeed given any other line $L \in P_{\alpha^*}$, the total space $\pi_1^{-1}(L)$ is also the one point blow up of $\mathbb{C}P^2$ with exceptional divisor $Z_- \cap (\pi_1)^{-1}(L)$.

Now observe that the coefficient of p in $\Psi(\alpha)$ is nonzero only if there is a nonzero invariant of the form $n_{P_\alpha}([M_*],[M_*],[M_*];\sigma)$. Hence $c_{vert}(\sigma) = -2$ and σ is represented (for generic J on P_α) by isolated curves. The natural complex structure on J_α may not be regular for σ. Nevertheless, since it is a limit of generic structures, Gromov compactness implies that it would have to contain some holomorphic representative of σ which would be a section S together perhaps with some bubbles in the fibers. Since each fiberwise bubble has positive Chern class, this means that $c_{vert}(S) \leq -2$. But such a section S does not exist. It would have to project to a section S_2 of $\pi_2 : P_{\alpha^*} \to S^2$. We cannot have $S_2 = E_*$ since the only lifts of E_* to P_α have $c_{vert} = -1$. Thus S_2 would have to be a line L in P_{α^*}, and, because $c_{vert}(L) = 1$, S would have to have self intersection -3 in $\pi_1^{-1}(S_2)$. This is impossible since, as we saw above, $\pi_1^{-1}(S_2)$ is the blow up of $\mathbb{C}P^2$.

A similar argument shows that the coefficient of $\mathbb{1}$ in $\Psi(\alpha)$ must vanish. Otherwise there would be a section class σ in P_α with $c_{vert} = 0$ with holomorphic representatives through every point $x \in P_\alpha$. Again these representatives would have to be the union of a section S' together with some fiberwise bubbles. Hence if $\pi_1(x) \notin E_*$, S' would have to project to a line L in P_{α^*}. Such a line has one lift to a curve with $c_{vert} = 0$, namely $\pi_1^{-1}(L) \cap Z_-$, but such a lift does not exist through an arbitrary point. $\square$

Corollary 5.4. $r_{\widetilde{\rho}}(M_*,\omega_a) = r_{\widetilde{\rho}_f}(M_*,\omega_a) = \omega(F) = \pi(1-a^2)$.

Proof. $\pi_1(\mathrm{Ham}\,(M_*,\omega))$ is isomorphic to $\mathbb{Z}$ with generator α. It follows from Corollary 4.3 and Lemma 5.1 that when $3a^2 \neq 1$

$$\begin{aligned} \widetilde{\rho}(k\alpha) \geq \widetilde{\rho}_f(k\alpha) &= \widetilde{\rho}^+(k\alpha) + \widetilde{\rho}^+(-k\alpha) \\ &\geq v(Q^k e^{k\delta(F-2E)}) + v(Q^{-k} e^{-k\delta(F-2E)}) \\ &= v(Q^k) + v(Q^{-k}) \geq \omega(F), \end{aligned}$$

where the second equality holds because the terms involving δ cancel. But the loop 2α is generated by the Hamiltonian

$$H = \pi(c - |z_1^2| - |z_2^2|)$$

which has length $\pi(1-a^2) = \omega(F)$. Hence result.

This conclusion is still valid when $3a^2 = 1$ since the inequality $\widetilde{\rho}(k\alpha) \geq \omega(F)$ still holds. Because it is uniform one can prove this by continuity in a: the set of a for which this inequality does not hold must be open. Alternatively, one can argue using the remarks after Corollary 4.3. □

Proof of Proposition 1.23. The above corollary shows that the loop 2α on (M_*, ω_a) satisfies all the required conditions. □

Remark 5.5.

(i) It follows from [13, 4] that the circle action generated by $K = -\pi|z_1^2|$ is also a $\widetilde{\rho}$-minimizing representative of its homotopy class α. Thus

$$\ell_{\widetilde{\rho}}(\alpha) = \pi \quad > \quad \ell_{\widetilde{\rho}}(2\alpha) = \pi(1-a^2).$$

(ii) It is also interesting to try to understand the asymptotic growth of the loops $\pm\alpha$ by calculating the limits of the one sided measures $\ell^+(k\alpha)/k$ as $k \to \pm\infty$ for all values of $a < 1$. Polterovich showed in [17] that in the monotone case $\ell^+(k\alpha)/k \geq |I(\widetilde{\rho})| = 1/18$ when $k \geq 1$ and I is as in Remark 4.4, but his method gave nothing for $k \to -\infty$. Our methods extend his result for the case $k \to \infty$ to all values of a, but also have no information about the limit as $k \to -\infty$. To see this, note first that $\omega(\delta(F - 2E)) > 0$ for all a. The arguments in Lemma 5.1 show that $v(Q^{-k})$ is bounded as $k \to \infty$ when $3a^2 \geq 1$ and grows as $k\omega(F/4 - E/2)/3$ for $3a^2 < 1$. (This growth comes from the cycle $p \mapsto F \mapsto 1\!\!1 \mapsto p$ that increases valuation by $\omega(F/4 - E/2)$.) Since $\omega(F/4 - E/2)/3 < \omega(\delta(F - 2E))$ for all a, $v(\Psi(-k\alpha)) \to -\infty$ as $k \to \infty$. Thus we get no information on $\ell^+(-k\alpha)$. On the other hand, $v(Q^k)$ is either bounded, or, if $3a^2 < 1$, grows as a multiple of $\omega(F/4 - E/2)$. Hence

$$\ell^+(k\alpha)/k \;\geq\; v(\Psi(k\alpha))/k \;\to\; \infty \quad \text{as} \quad k \to \infty$$

for all a. See Polterovich [20] for a further discussion of this question.

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