

Lie symmetries and exact solutions of a class of thin film equations

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Abstract

A symmetry group classification for fourth-order reaction-diffusion equations, allowing for both second-order and fourth-order diffusion terms, is carried out. The fourth-order equations are treated, firstly, as systems of second-order equations that bear some resemblance to systems of coupled reaction-diffusion equations with cross diffusion, secondly, as systems of a second-order equation and two first-order equations. The paper generalizes the results of Lie symmetry analysis derived earlier for particular cases of these equations. Various exact solutions are constructed using Lie symmetry reductions of the reaction-diffusion systems to ordinary differential equations. The solutions include some unusual structures as well as the familiar types that regularly occur in symmetry reductions, namely, self-similar solutions, decelerating and decaying traveling waves, and steady states.

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1 Introduction

We consider the fourth-order nonlinear partial differential equation (PDE) of the form:

$$u_t = -[K(u)u_{xxx}]_x + [D(u)u_x]_x + F(u), \quad (1.1)$$

where K , D , and F are arbitrary smooth functions (hereafter the subscripts t and x denote differentiation with respect to these variables). Equation (1.1) generalizes a wide range of the known scalar reaction-diffusion equations arising in applications. The case with K identically zero and $D(u) > 0$ for almost all u is the case of second-order reaction-diffusion which has already been widely studied in many practical contexts including combustion, population dynamics, population genetics, biological cellular growth, and adsorptive porous media, for example [9, 11, 15, 38]. Hereinafter, we assume that $K(u)$ is not identically zero, so that the governing equation is of the fourth order, including a fourth-order diffusion term when K is nonnegative. The simplest equation of the form (1.1), with $F = 0$ and $D = 0$, follows from the approximations of lubrication theory to describe thin films of a Newtonian liquid dominated by surface tension effects. The thin film equations are an active area of research (see [6, 7, 26] and papers cited therein). The equation:

$$u_t = -[u^\gamma u_{xxx}]_x \quad (1.2)$$

with the nonnegative parameter γ was introduced in [33]. The case $\gamma = 3$ describes a classical thin film of Newtonian fluid, as reviewed in [41]. $\gamma = 1$ occurs in the dynamics of a Hele-Shaw cell [25], and $\gamma = 2$ arises in a study of wetting films with a free contact line between film and substrate [6].

One important generalization of equation (1.2), which is also a particular case of equation (1.1), can be written as

$$u_t = -[u^\gamma u_{xxx}]_x + [u^\mu u_x]_x, \quad (1.3)$$

where μ is a positive parameter (arbitrary positive coefficients of each term can be set to one by rescaling variables). Equation (1.3) with $\gamma = 0$ can be considered as a semilinear limit of the classical Cahn-Hilliard model of phase separation [13], which is also widely studied (see [34] and the papers cited therein). The linear case $\gamma = \mu = 0$ also follows from a small-slope approximation to metal surface evolution with surface-diffusion and evaporation-condensation represented by fourth-order and second-order diffusion terms [14, 37].

Several papers are devoted to the construction of exact solutions of the thin film equations (1.2) by Lie symmetry reductions [4, 5, 12, 22, 30, 31, 45] or by searching for special invariant finite vector spaces of solutions [29]. The symmetry classification is extended here to include a reaction term. In some circumstances, the reaction term F should naturally arise in fourth-order transport equations with a role similar to that in second-order reaction-diffusion. For example, a particular case of equation (1.1) with $F(u) = u$ occurs as the limiting case of the unstable Cahn-Hilliard equation [27, 39]. In fabricated metal surface evolution, a positive source term may represent ion beam sputtering [46], and a negative source term may represent chemical decay or evaporation [32]. Other examples of equations of the form (1.1) arising in applications and having a reaction term are presented in [27].

The first aim of this paper is to describe all possible Lie symmetries, which equation (1.1) can admit depending on the function triplets $(K, D, \text{ and } F)$, that is, to solve the so-called group classification problem, which was formulated and solved for a class of nonlinear heat equations in the pioneering work in [43]. This problem for the second-order reaction-diffusion equation was solved in [24] (see also [19, 20], where the problem is solved for the general reaction-diffusion-convection equation). Note that the most general results concerning non-classical (Q -conditional) symmetries of reaction-diffusion equations were obtained in [2, 3, 23, 28].

It should be noted that we shall not directly search for Lie symmetries of equation (1.1) but replace one scalar equation by an equivalent cross-diffusion system of equations. Using the symmetries found, we construct exact solutions of equation (1.1) with such triplets $(K, D, \text{ and } F)$, which arise in applications and compare the results obtained with those derived earlier.

Ovsiannikov's method of Lie symmetry classification of differential equations [42] is based on the classical Lie scheme and a set of equivalence transformations of a given equation. The formal application of this method to equations containing several arbitrary functions (equation (1.1) contains three arbitrary functions) usually leads to a large number of equations admitting nontrivial Lie algebras of invariance. Our approach of Lie symmetry classification of differential equations is based on the classical Lie scheme and on finding and then making systematic use of the sets of local transformations that reduce any differential equation with a Lie algebra of invariance, to one given in the relevant list, that is representative of each equivalence class. This approach has earlier been applied also for reaction-diffusion systems [16, 17, 18].

The paper is organized as follows. Section 2 is devoted to a complete description of Lie symmetries of equation (1.1), that is all possible Lie symmetries, which this equation can admit depending on the form the functions K , D , and F , are found. In most applications, K and D are nonnegative, so that the diffusive transport processes are dissipative [10]. However, in solving a group classification problem, it is more common to allow the coefficient functions to be arbitrary smooth functions. In Section 3, the symmetry reductions and some exact solutions are constructed for particular cases of equation (1.1) that are likely to be useful in applications. The main results of the paper are summarized in the last section.

2 Lie symmetry of equation (1.1)

Firstly, we note that equation (1.1) can be reduced to the system:

$$u_t = -[K(u)v_x]_x + [D(u)u_x]_x + F(u), \quad 0 = u_{xx} - v \quad (2.1)$$

by the substitution $v = u_{xx}$, $v = v(t, x)$. Physical motivation of this substitution is quite natural because the thin film equation can be derived in the form of a system containing the equation for pressure $v(t, x)$, which is proportional to u_{xx} (here $u(t, x)$ means the thickness of a film) [29].

From the mathematical point of view, Lie symmetries of this system could include nothing more than extensions (prolongations) of Lie symmetries of the single fourth-order equation for $u(x, t)$ because second-order contact symmetries do not exist [1]. However, it is convenient to analyze system (2.1) as a cross-diffusion system, in which the second equation contains the time variable t as a parameter. The motivation follows from the well-known fact that application of Lie's algorithm in the case of high-order equations leads to very cumbersome formulae. Moreover, each Lie symmetry of equation (1.1) can be easily established from one of system (2.1) and there is one-to-one correspondence between solutions of (2.1) and (1.1). Thus, we shall investigate system (2.1) instead of equation (1.1).

We note that the Lie symmetry classification problem may be solved by iterating a symmetry-finding program, for example, program DESOLV [48]. However, such programs usually produce many special cases of equations with additional symmetries, which are equivalent up to the correctly specified local substitutions. Finding and systematically using the sets of such substitutions often leads to a significant practical reduction in the number of special cases that admit additional invariance. Thus, the problem under investigation here will be tractable without the assistance of a computer, however, the resulting symmetries have been checked by a symmetry-finding program. However, all possible local substitutions will be used to construct the shortest list of "canonical" systems of the form (2.1) with nontrivial Lie algebra of invariance.

Now we formulate the main theorem, which presents the classification of special forms of system (2.1) (with K not identically 0) having additional symmetries.

Theorem 2.1. *All possible maximal algebras of invariance (up to equivalent representations generated by transformations of the form (2.2)) of system (2.1) for any fixed triplet (K , D , and F) are presented in Table 1. Any other system of the form (2.1) with nontrivial Lie symmetry is reduced by a local substitution of the form:*

$$\begin{aligned} t &\longrightarrow C_0t + C_1e^{C_2t}, & x &\longrightarrow C_3x, \\ u &\longrightarrow C_4 + C_5t + C_6e^{C_7t}u, & v &\longrightarrow C_8e^{C_9t}v \end{aligned} \quad (2.2)$$

to one of those given in Table 1 (the constants C with subscripts are determined by the form of the system in question, some of them necessarily being zero in all particular cases).

Proof. According to the classical Lie scheme [8, 28, 40], we consider system (2.1) as the manifold (S_1, S_2) determined by the restrictions:

$$\begin{aligned} S_1 &\equiv -u_t - [K(u)v_x]_x + [D(u)u_x]_x + F(u) = 0, \\ S_2 &\equiv u_{xx} - v = 0 \end{aligned} \quad (2.3)$$

in the space of the variables $t, x, u, v, u_t, v_t, u_x, v_x, u_{xx}$, and v_{xx} . The maximal algebra of invariance (MAI) of this system is generated by infinitesimal operators of the form:

$$X = \xi^0(t, x, u, v)\partial_t + \xi^1(t, x, u, v)\partial_x + \eta^1(t, x, u, v)\partial_u + \eta^2(t, x, u, v)\partial_v, \quad (2.4)$$

where the functions ξ^0 , ξ^1 , η^1 , and η^2 are to be determined. In order to determine these unknown functions, one needs to use the invariance conditions:

$$\begin{aligned} X^2 S_1 &\equiv X^2(-u_t - [K(u)v_x]_x + [D(u)u_x]_x + F(u))|_{S_1=0, S_2=0} = 0, \\ X^2 S_2 &\equiv X^2(u_{xx} - v)|_{S_1=0, S_2=0} = 0, \end{aligned} \quad (2.5)$$

where X^2 is the second prolongation of the operator X :

$$\begin{aligned} X^2 &= \xi^0(t, x, u, v)\partial_t + \xi^1(t, x, u, v)\partial_x + \eta^1(t, x, u, v)\partial_u + \eta^2(t, x, u, v)\partial_v \\ &\quad + \tau_{10}^1\partial_{u_t} + \tau_{01}^1\partial_{u_x} + \tau_{10}^2\partial_{v_t} + \tau_{01}^2\partial_{v_x} + \tau_{20}^1\partial_{u_{tt}} + \tau_{11}^1\partial_{u_{tx}} \\ &\quad + \tau_{02}^1\partial_{u_{xx}} + \tau_{20}^2\partial_{v_{tt}} + \tau_{11}^2\partial_{v_{xt}} + \tau_{02}^2\partial_{v_{xx}}. \end{aligned} \quad (2.6)$$

The coefficients with relevant subscripts τ_t^1 , τ_x^1 , τ_{10}^2 , τ_{01}^2 , τ_{20}^1 , τ_{11}^1 , τ_{02}^1 , τ_{20}^2 , τ_{11}^2 , and τ_{02}^2 are calculated by the well-known prolongation formulae (see, e.g., [8, 28, 40]).

Substituting operator (2.6) into system (2.5) and carrying out the relevant calculations, we obtain the so-called system of determining equations for finding the functions ξ^0 , ξ^1 , η^1 , and η^2 . This is an overdetermined system of PDEs that can be written in the explicit forms:

$$\xi_x^0 = \xi_u^0 = \xi_v^0 = \xi_t^1 = \xi_u^1 = \xi_v^1 = \xi_{xx}^1 = 0, \quad (2.7)$$

$$\eta_v^1 = \eta_{xu}^1 = \eta_{uu}^1 = 0, \quad (2.8)$$

$$\eta^2 = \eta_{xx}^1 + (\eta_u^1 - 2\xi_x^1)v, \quad (2.9)$$

$$K(u)(\xi_t^0 - 4\xi_x^1) + K'(u)\eta^1 = 0, \quad (2.10)$$

$$D(u)(\xi_t^0 - 2\xi_x^1) + D'(u)\eta^1 = 0, \quad (2.11)$$

$$\eta_t^1 + K(u)\eta_{xxx}^1 + F(u)(\eta_u^1 - \xi_t^0) - D(u)\eta_{xx}^1 - F'(u)\eta^1 = 0. \quad (2.12)$$

Of course, equations (2.7)–(2.9) are linear and do not depend on the functions K , D , and F , hence their general solution can be easily constructed as follows:

$$\begin{aligned} \xi^0 &= \xi^0(t), \quad \xi^1 = \alpha x + x_0, \\ \eta^1 &= R(t)u + P(t, x), \quad \eta^2 = P_{xx}(t, x) + (R(t) - 2\alpha)v, \end{aligned} \quad (2.13)$$

where $P(t, x)$ and $R(t)$ are arbitrary smooth functions, while α and x_0 are arbitrary constants. Equations (2.10)–(2.12) form the system of classification equations. Its general solution under

assumption of arbitrarily given functions K , D , and F generates the invariance algebra that consists only of generators of translations with respect to the x and t , that is, $P_t = \frac{\partial}{\partial t}$, $P_x = \frac{\partial}{\partial x}$. The algebra with this basis is called *the trivial Lie algebra* of the system (2.1) (note that other authors instead use “kernel of the basic Lie groups” [42] or “the principal Lie algebra” [31] in this context). Thus, we aim to find all triplets (K , D , and F) that lead to extensions of the trivial Lie algebra. This means that one needs to solve equations (2.10)–(2.12) with coefficients (2.13). The crucial step in solving this task is to analyze differential consequences of equations (2.10)–(2.11) with respect to the variable x . Since these consequences take the form $K'(u)P_x(t, x) = 0$ and $D'(u)P_x(t, x) = 0$, respectively, one arrives at two basic cases:

- (i) $[K'(u)]^2 + [D'(u)]^2 \neq 0$
- (ii) $K'(u) = D'(u) = 0$.

Consider case (i). In this case $P_x(t, x) = 0$, so that equations (2.10)–(2.12) take the forms:

$$K(u)[\xi_t^0(t) - 4\alpha] + K'(u)[R(t)u + P(t)] = 0, \quad (2.14)$$

$$D(u)[\xi_t^0(t) - 2\alpha] + D'(u)[R(t)u + P(t)] = 0, \quad (2.15)$$

$$R_t(t)u + P_t(t) + F(u)[R(t) - \xi_t^0(t)] - F'(u)[R(t)u + P(t)] = 0. \quad (2.16)$$

Setting $R(t) = P(t) = 0$, we immediately arrive at case 1 of Table 1. In fact, equations (2.14)–(2.16) with $R(t) = P(t) = 0$ and nonzero $K(u)$ are equivalent to

$$\xi_t^0(t) = 4\alpha \neq 0, \quad D(u) = 0, \quad F(u) = 0,$$

and $K(u)$ is an arbitrary smooth function. This means that the triplet $(K(u), 0, 0)$ forms the system from case 1 of Table 1 and the coordinates of the infinitesimal operator (2.4) take the form:

$$\xi^0 = 4\alpha t + t_0, \quad \xi^1 = \alpha x + x_0, \quad \eta^1 = \eta^2 = 0, \quad (2.17)$$

where α , t_0 , and x_0 are arbitrary parameters. Operator (2.4) with coordinates (2.17) generates exactly the operators P_t (for $t_0 = 1, \alpha = x_0 = 0$), P_x (for $t_0 = \alpha = 0, x_0 = 1$), and D_1 (for $\alpha = 1, x_0 = t_0 = 0$) listed in case 1 of Table 1.

If $R^2(t) + P^2(t) \neq 0$ then two possible subcases arise **ia.** $R(t) \neq 0$ and **ib.** $R(t) = 0$, $P(t) \neq 0$.

Consider subcase **ia.** Integrating equation (2.14) as an ODE on the function $K(u)$, one obtains

$$K(u) = k \left[u + \frac{P(t)}{R(t)} \right]^{\frac{4\alpha - \xi_t^0(t)}{R(t)}}, \quad (2.18)$$

where k is a nonzero constant, which can be reduced to $k = 1$ (by scaling time $t \rightarrow kt$) without losing generality. Since the function K must depend only on u , the restrictions

$$\xi_t^0(t) = 4\alpha - \gamma R(t), \quad P(t) = \gamma_0 R(t) \quad (2.19)$$

are obtained. Here, γ and γ_0 are arbitrary constants. Thus, solving equation (2.14), we arrive at the power function:

$$K(u) = (u + \gamma_0)^\gamma, \quad (2.20)$$

and restrictions (2.19).

To solve equations (2.15)-(2.16), we need to analyze two subcases **ia1** $D(u) \neq 0$ and **ia2** $D(u) = 0$.

In subcase **ia1**, equation (2.15) can be solved as an ODE on the function $D(u)$, so that one obtains

$$D(u) = d(u + \gamma_0)^\mu, \quad (2.21)$$

and the condition

$$\xi_t^0(t) = 2\alpha - \mu R(t), \quad (2.22)$$

where $d \neq 0$ and μ are arbitrary constants.

If $\gamma \neq \mu$, then solving equations (2.19) and (2.22) and substituting the found functions $R(t)$, $P(t)$, and ξ^0 into equation (2.16), we arrive at the ODE:

$$(u + \gamma_0)F'(u) + (\gamma - 2\mu - 1)F(u) = 0. \quad (2.23)$$

The general solution of equation (2.23) and the functions $K(u)$ and $D(u)$ found above form the system:

$$u_t = -[(u + \gamma_0)^\gamma v_x]_x + d[(u + \gamma_0)^\mu u_x]_x + \lambda(u + \gamma_0)^{2\mu - \gamma + 1}, \quad u_{xx} - v = 0, \quad (2.24)$$

which is reduced to the system listed in case 3 of Table 1 by renaming $u + \gamma_0 \rightarrow u$. Substituting the functions $R(t)$, $P(t)$, and ξ^0 into (2.13), we obtain the infinitesimal operator (2.4), which generates three basic operators listed in case 3 of Table 1.

If $\gamma = \mu$, then equations (2.19) and (2.22) are compatible only under restriction $\alpha = 0$. It turns out that this restriction does not lead to any new cases but to case 3 of Table 1 with $\gamma = \mu$. In fact, equations (2.14) and (2.15) have identical structure, hence $D(u) = d(u + \gamma_0)^\gamma$, $d = \text{const}$. The general solution of (2.16) has the form $F(u) = \lambda(u + \gamma_0)^{1+\gamma} + \lambda_1(u + \gamma_0)$, where λ and $\lambda_1 \neq 0$ are arbitrary constants. The relevant coordinates of infinitesimal operator (2.4) take the forms:

$$\begin{aligned} \xi^0 &= \frac{r_1}{\lambda_1} e^{-\lambda_1 \gamma t} + t_0, \quad \xi^1 = x_0, \\ \eta^1 &= r_1 e^{-\lambda_1 \gamma t} (u + \gamma_0), \quad \eta^2 = r_1 e^{-\lambda_1 \gamma t} v, \end{aligned} \quad (2.25)$$

where t_0 , r_1 , and x_0 are arbitrary parameters. Thus, the system:

$$\begin{aligned} u_t &= -[(u + \gamma_0)^\gamma v_x]_x + d[(u + \gamma_0)^\gamma u_x]_x + \lambda(u + \gamma_0)^{1+\gamma} + \lambda_1(u + \gamma_0), \\ u_{xx} - v &= 0 \end{aligned} \quad (2.26)$$

is invariant under three-dimensional MAI with the basic operators:

$$P_t, P_x, \quad Q^* = e^{-\gamma \lambda_1 t} (\partial_t + \lambda_1 ((u + \gamma_0) \partial_u + v \partial_v)). \quad (2.27)$$

However, we found the local substitution:

$$t^* = \frac{1}{\lambda_1 \gamma} e^{\lambda_1 \gamma t}, \quad x^* = x, \quad u^* = (u + \gamma_0) e^{-\lambda_1 t}, \quad v^* = v e^{-\lambda_1 t}, \quad (2.28)$$

which reduces system (2.26) and Lie algebra (2.27) to the system and Lie algebra listed in case 3 of Table 1 with $\gamma = \mu$.

The analysis of subcase **ia2** is straightforward because equation (2.15) vanishes for $D(u) = 0$, while equation (2.16) can be treated in a similar way. In conclusion, we found that the system:

$$u_t = -[(u + \gamma_0)^\gamma v_x]_x + \lambda_1(u + \gamma_0), \quad u_{xx} - v = 0, \quad (2.29)$$

is invariant under four-dimensional MAI with the basic operators:

$$P_t, P_x, Q^*, \quad D^* = x\partial_x + \frac{4}{\gamma}(u + \gamma_0)\partial_u + \left(\frac{4}{\gamma} - 2\right)v\partial_v. \quad (2.30)$$

Direct checking shows that system (2.29) and Lie algebra (2.30) are reduced to the system and Lie algebra listed in case 6 of Table 1 if one applies substitution (2.28).

Examination of subcase **ib** is much simpler because equations (2.14)–(2.16) with $R(t) = 0$ can be easily integrated. Finally, one obtains cases 2 and 5 of Table 1.

To complete the proof, we need to examine case (ii) $K'(u) = D'(u) = 0$, that is $K(u) = k = \text{const}$, $D(u) = d = \text{const}$. Since $K(u) \neq 0$, we can again set $k = 1$ without losing generality.

Thus, the classification equations (2.10)–(2.11) with coefficients (2.13) can be essentially simplified and one obtains

$$\xi^0(t) = 4\alpha t + t_0, \quad \alpha d = 0. \quad (2.31)$$

The third classification equation (2.12) takes the form:

$$\begin{aligned} R_t(t)u + P_t(t, x) + P_{xxxx}(t, x) - dP_{xx}(t, x) + F(u)[R(t) - 4\alpha] \\ - F'(u)[R(t)u + P(t, x)] = 0. \end{aligned} \quad (2.32)$$

Differentiating equation (2.32) with respect to x and u , we find the condition $F''(u)P_x(t, x) = 0$. Hence, two different subcases should be examined:

ia $F'' = 0$

ib $F'' \neq 0, P_x(t, x) = 0$.

Consider subcase **ia**. Since $F''(u) = 0$, we immediately obtain $F(u) = \lambda_1 u + \lambda_0$, so that the triplet of functions (K , D , and F) is known.

Substituting the function F into equation (2.32) and splitting the obtained expression into two equations (with the variable u and without it), we arrive at $R(t) = 4\alpha\lambda_1 t + r_1$ and the linear PDE:

$$P_t(t, x) + P_{xxxx}(t, x) - dP_{xx}(t, x) - \lambda_1 P(t, x) + \lambda_0(4\alpha\lambda_1 t + r_1 - 4\alpha) = 0 \quad (2.33)$$

to find the function $P(t, x)$. Thus, the coordinates of infinitesimal operator (2.4) take the form:

$$\begin{aligned} \xi^0 = 4\alpha t + t_0, \quad \xi^1 = \alpha x + x_0, \\ \eta^1 = (4\alpha\lambda_1 t + r_1)u + P(t, x), \quad \eta^2 = (4\alpha\lambda_1 t + r_1 - 2\alpha)v + P_{xx}(t, x), \end{aligned} \quad (2.34)$$

where $P(t, x)$ is the general solution of equation (2.33).

The last step is to take into account the second condition from (2.31). If $d \neq 0$, then $\alpha = 0$ and, applying the relevant simplifications, we arrive at the case 7 of Table 1.

Case	Systems of the form (2.1)	Restrictions	Basic operators of MAI
1.	$u_t = -[K(u)v_x]_x$ $u_{xx} - v = 0$		$P_t, P_x,$ $D_1 = 4t\partial_t + x\partial_x - 2v\partial_v$
2.	$u_t = -[e^{\gamma u}v_x]_x + d[e^{\mu u}u_x]_x$ $+ \lambda e^{(2\mu-\gamma)u}$ $u_{xx} - v = 0$	$\gamma^2 + \mu^2 \neq 0$ $d^2 + \lambda^2 \neq 0$	$P_t, P_x,$ $D_2 = 2(\gamma-2\mu)t\partial_t + (\gamma-\mu)x\partial_x$ $+ 2(\partial_u - (\gamma-\mu)v\partial_v)$
3.	$u_t = -[u^\gamma v_x]_x + d[u^\mu u_x]_x$ $+ \lambda u^{2\mu-\gamma+1}$ $u_{xx} - v = 0$	$\gamma^2 + \mu^2 \neq 0$ $d^2 + \lambda^2 \neq 0$	$P_t, P_x,$ $D_3 = 2(\gamma-2\mu)t\partial_t + (\gamma-\mu)x\partial_x$ $+ 2(u\partial_u + (\mu-\gamma+1)v\partial_v)$
4.	$u_t = -v_{xx} + du_{xx} + \lambda u \ln u$ $u_{xx} - v = 0$	$\lambda \neq 0$	$P_t, P_x,$ $Q_1 = e^{\lambda t}(u\partial_u + v\partial_v)$
5.	$u_t = -[e^{\gamma u}v_x]_x$ $u_{xx} - v = 0$	$\gamma \neq 0$	$P_t, P_x,$ D_1, D_2 with $\mu = \gamma$
6.	$u_t = -[u^\gamma v_x]_x$ $u_{xx} - v = 0$	$\gamma \neq 0$	$P_t, P_x,$ D_1, D_3 with $\mu = \gamma$
7.	$u_t = -v_{xx} + du_{xx}$ $u_{xx} - v = 0$	$d \neq 0$	$P_t, P_x, I = u\partial_u + v\partial_v,$ $X^\infty = P(t, x)\partial_u + P_{xx}(t, x)\partial_v,$ where $P_t + P_{xxxx} - dP_{xx} = 0$
8.	$u_t = -v_{xx}$ $u_{xx} - v = 0$		$P_t, P_x, I, D_1,$ $X^\infty = P(t, x)\partial_u + P_{xx}(t, x)\partial_v,$ where $P_t + P_{xxxx} = 0$

Table 1. Lie symmetries of (2.1).

If $d = 0$, then equation (2.33) takes the form:

$$P_t(t, x) + P_{xxxx}(t, x) - \lambda_1 P(t, x) + \lambda_0(4\alpha\lambda_1 t + r_1 - 4\alpha). \quad (2.35)$$

The corresponding system is

$$u_t = -v_{xx} + \lambda_1 u + \lambda_0, \quad u_{xx} - v = 0. \quad (2.36)$$

It turns out that system (2.35) is reduced to the form listed in case 8 of Table 1 if one applies the local substitutions:

$$u^* = u - \lambda_0 t, \quad \lambda_1 = 0, \quad v^* = v, \quad (2.37)$$

and

$$u^* = e^{-\lambda_1 t} \left(u + \frac{\lambda_0}{\lambda_1} \right), \quad \lambda_1 \neq 0, \quad v^* = e^{-\lambda_1 t} v. \quad (2.38)$$

Simultaneously, operator (2.4) with coordinates (2.34) is transformed in such a way that the basic operators P_t , P_x , I , D_1 , and $X^\infty = P(t, x)\partial_u + P_{xx}(t, x)\partial_v$, listed in case 8 of Table 1, can be easily derived.

Consider subcase **iib**. Since $P_x(t, x) = 0$, equation (2.32) takes the form:

$$R_t(t)u + P_t(t) + F(u)[R(t) - 4\alpha] - F'(u)[R(t)u + P(t)] = 0. \quad (2.39)$$

Differentiating equation (2.39) with respect to the variables t and u , we find

$$F''(u)[R_t(t)u + P_t(t)] = R_{tt}(t). \quad (2.40)$$

Because equation (2.40) has a simple structure, we prefer to solve this equation and check when the solution obtained will satisfy equation (2.39). We note that the special case $R_t(t) = P_t(t) = 0$ does not lead to new results, so that $R_t^2(t) + P_t^2(t) \neq 0$. Moreover, since the function F depends only on u , the relations:

$$P_t(t) = \gamma R_t(t), \quad R_{tt}(t) = \lambda R_t(t), \quad (2.41)$$

where γ and $\lambda \neq 0$ are arbitrary constants, should take place. The general solution of equation (2.40) with the coefficients (2.41) is

$$F(u) = \lambda(u + \gamma) \ln(u + \gamma) + \lambda_1 u + \lambda_0, \quad (2.42)$$

where λ_0 and λ_1 are arbitrary constants. Now, we substitute (2.42) and the general solution of the linear ODEs system (2.41) into equation (2.39) and find conditions when the obtained expression can be fulfilled. The simple calculations give

$$R(t) = r_1 e^{\lambda t}, \quad P(t) = \gamma r_1 e^{\lambda t}, \quad \alpha = 0, \quad \lambda_0 = \lambda_1 \gamma. \quad (2.43)$$

So the system:

$$u_t = -v_{xx} + du_{xx} + \lambda(u + \gamma) \ln(u + \gamma) + \lambda_1(u + \gamma), \quad u_{xx} - v = 0 \quad (2.44)$$

admits MAI generated by operator (2.4) with coordinates:

$$\xi^0 = t_0, \quad \xi^1 = x_0, \quad \eta^1 = r_1 e^{\lambda t}(u + \gamma), \quad \eta^2 = r_1 e^{\lambda t} v. \quad (2.45)$$

Finally, the system (2.44) and operator (2.4) with (2.45) are simplified by the substitution:

$$u^* = e^{\frac{\lambda_1}{\lambda}}(u + \gamma), \quad v^* = e^{\frac{\lambda_1}{\lambda}} v, \quad (2.46)$$

so that the system and MAI listed in case 4 of Table 1 are obtained.

Thus, the system of determining equations (2.7)–(2.12) is completely solved, and eight different systems of the form (2.1) have been found, which admit three- and higher-dimensional Lie algebras. Simultaneously, we have shown that all other systems admitting nontrivial Lie algebra are reduced to those listed in Table 1 by the substitutions of the form (2.28), (2.37), (2.38), and (2.46). One notes that all of these substitutions can be united to the form (2.2).

The proof is now completed. \square

One easily notes that cases 2 and 3 of Table 1 generalize the results of Lie symmetry analysis for the Cahn-Hilliard equation derived in [22, 30]. For example, the systems:

$$u_t = -v_{xx} + d[e^{\mu u} u_x]_x, \quad u_{xx} - v = 0, \quad (2.47)$$

and

$$u_t = -v_{xx} + d[u^\mu u_x]_x, \quad u_{xx} - v = 0, \quad (2.48)$$

which are the particular cases of the corresponding systems from Table 1, admit the Lie symmetry operators:

$$4\mu t\partial_t + \mu x\partial_x - 2(\partial_u + \mu v\partial_v), \quad (2.49)$$

and

$$4\mu t\partial_t + \mu x\partial_x - 2(u\partial_u + (\mu + 1)v\partial_v), \quad (2.50)$$

respectively. The table also includes the results obtained in [30] for equation (1.1) with $K(u) = \text{const}$ and $F(u) = 0$.

A natural question is can we claim that eight systems listed in Table 1 are inequivalent up to any local substitutions (not only of the form (2.2))? It turns out that, using the theorem presented below, one easily checks that answer is positive. Thus, Table 1 contains a complete list of canonical systems of the form (2.1) with nontrivial MAI.

Consider the most general form of local substitutions:

$$\tau = a(t, x, u, v), \quad y = b(t, x, u, v), \quad w = f(t, x, u, v), \quad z = g(t, x, u, v) \quad (2.51)$$

with the restrictions:

$$\Delta_1 = \begin{vmatrix} a_x & a_t \\ b_x & b_t \end{vmatrix} \neq 0, \quad \Delta_2 = \begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix} \neq 0 \quad (2.52)$$

on the smooth functions a , b , f , and g , which guarantee that (2.51) is nondegenerate.

Theorem 2.2. *Any nonlinear system of the form (2.1) can be reduced to another system of the same form:*

$$w_\tau = -(\tilde{K}(w)z_y)_y + (\tilde{D}(w)w_y)_y + \tilde{F}(w), \quad w_{yy} - z = 0 \quad (2.53)$$

by the substitution (2.51) if and only if

$$\tau = a = a(t), \quad y = b = \beta_1 x + \beta_0, \quad w = f = \xi(t)u + \zeta(t), \quad z = g = \frac{\xi(t)}{\beta_1^2} v, \quad (2.54)$$

where $a(t)$, $\xi(t)$, and $\zeta(t)$ are the correctly-specified functions satisfying the following relations:

$$a_t(t)\tilde{K}(w) = \beta_1^4 \cdot K\left(\frac{1}{\xi(t)}w - \frac{\zeta(t)}{\xi(t)}\right), \quad (2.55)$$

$$a_t(t)\tilde{D}(w) = \beta_1^2 \cdot D\left(\frac{1}{\xi(t)}w - \frac{\zeta(t)}{\xi(t)}\right), \quad (2.56)$$

$$a_t(t)\tilde{F}(w) = \xi(t) \cdot F\left(\frac{1}{\xi(t)}w - \frac{\zeta(t)}{\xi(t)}\right) + \frac{\xi_t(t)}{\xi(t)}[w - \zeta(t)] + \zeta_t(t) \quad (2.57)$$

provided $a_t\beta_1\xi \neq 0$.

Proof. It is quite similar to the proof presented in [21] (see Theorem 4 therein) for the general reaction-diffusion-convection equation and omitted here because its bulk. \square

Remark 2.3. The transformations presented in Theorem 2.2 are nothing else but the set of the so-called form-preserving point transformations [36] for the class of systems (2.1). These transformations present the most general form of local substitutions, which can map an equation (system) from a given class to another one belonging to the same class. In the particular case, transformation (2.51) with (2.54) contain those arising in Theorem 2.1. Form-preserving transformations also contain as particular cases the well-known equivalence transformations and discrete transformations used in Ovsiannikov's method of group (Lie symmetry) classification.

To finish the Lie symmetry description, we note that equation (1.1) can be reduced to an equivalent system of PDEs in different ways. One sees that the system:

$$u_t = -[K(u)v_x]_x + [D(u)u_x]_x + F(u), \quad 0 = u_x - w, \quad 0 = w_x - v \quad (2.58)$$

by introducing new unknown functions $v = v(t, x)$, $w = w(t, x)$ can be obtained from equation (1.1). Since the system includes the first-order variable $w = u_x$, point symmetries of this system include contact symmetries of the original single fourth-order equation. System (2.58) is nothing else but a cross-diffusion system, in which the second and third equations contain the time variable t as a parameter. Thus, we may investigate also system (2.58) instead of equation (1.1). There is an essential difference between systems (2.1) and (2.58) because the second system contains the equations of different orders. In fact, according to the classical Lie scheme, MAI of this system is generated by the infinitesimal operator:

$$X = \xi^0(t, x, u, v, w)\partial_t + \xi^1(t, x, u, v, w)\partial_x + \eta^1(t, x, u, v, w)\partial_u \\ + \eta^2(t, x, u, v, w)\partial_v + \eta^3(t, x, u, v, w)\partial_w, \quad (2.59)$$

where the functions ξ^0 , ξ^1 , η^1 , η^2 , and η^3 are to be determined. Applying the second prolongation of the operator (2.59) to system (2.58) and using the invariance conditions, one can derive the determining equations for finding the functions ξ^0 , ξ^1 , η^1 , η^2 , and η^3 . It should be stressed that the relevant invariance conditions must take into account the differential consequences (with respect to the variable x) of the second and third equations of system (2.58). We omit cumbersome calculations and present the final result in the explicit forms:

$$\xi_x^0 = \xi_u^0 = \xi_v^0 = \xi_w^0 = \xi_t^1 = \xi_u^1 = \xi_v^1 = \xi_w^1 = \xi_{xx}^1 = 0, \quad (2.60)$$

$$\eta_v^1 = \eta_w^1 = \eta_{xu}^1 = \eta_{uu}^1 = 0, \quad (2.61)$$

$$\eta^2 = \eta_{xx}^1 + (\eta_u^1 - 2\xi_x^1)v, \quad (2.62)$$

$$\eta^3 = \eta_x^1 + (\eta_u^1 - \xi_x^1)w, \quad (2.63)$$

$$K(u)(\xi_t^0 - 4\xi_x^1) + K'(u)\eta^1 = 0, \quad (2.64)$$

$$D(u)(\xi_t^0 - 2\xi_x^1) + D'(u)\eta^1 = 0, \quad (2.65)$$

$$\eta_t^1 + K(u)\eta_{xxxx}^1 + F(u)(\eta_u^1 - \xi_t^0) - D(u)\eta_{xx}^1 - F'(u)\eta^1 = 0. \quad (2.66)$$

Now we note that the determining equations obtained (of course, without equation (2.63)) are equivalent to those (2.7)–(2.12) for the system (2.1), so that no new Lie point symmetries or contact symmetries can be found.

3 Symmetry reduction and exact solutions

Some of the systems presented in Table 1 are equivalent to known fourth-order PDEs arising in applications. For example, the system listed in case 6 is nothing else but the thin film equation (1.2). Since the motivation to this study is to consider this equations with nonzero reaction terms, which naturally arise in some processes [27, 32, 39, 46]; henceforth we restrict our attention mainly to case 3 of Table 1, involving the most common nonlinearities.

It is well known that a Lie symmetry allows one to reduce the given PDE (system of PDEs) to an equation (system of equations) of lower dimensionality. Here, we reduce systems arising in case 3 of Table 1 to systems of ordinary differential equations (ODEs); furthermore, these ODE systems are solved in particular cases, and exact solutions of the initial PDE systems are constructed. Finally, these solutions are compared with those obtained by other authors.

Consider the system arising in the case 3 of Table 1:

$$u_t = -[u^\gamma v_x]_x + d[u^\mu u_x]_x + \lambda u^{2\mu-\gamma+1}, \quad u_{xx} - v = 0, \quad (3.1)$$

where $\gamma^2 + \mu^2 \neq 0$. The most general Lie symmetry operator of system (3.1) has the form:

$$X = \alpha_1 P_t + \alpha_2 P_x + \alpha_3 D_3 = [\alpha_1 + 2\alpha_3(\gamma - 2\mu)t] \partial_t + [\alpha_2 + \alpha_3(\gamma - \mu)x] \partial_x \\ + [2\alpha_3 u] \partial_u + [2\alpha_3(1 - \gamma + \mu)v] \partial_v, \quad (3.2)$$

where $\alpha_i, i = 1, 2, 3$ are arbitrary constants. To construct the relevant ansatz, one needs to solve the Pfaffian system of characteristic equations:

$$\frac{dt}{\alpha_1 + 2\alpha_3(\gamma - 2\mu)t} = \frac{dx}{\alpha_2 + \alpha_3(\gamma - \mu)x} = \frac{du}{2\alpha_3 u} = \frac{dv}{2\alpha_3(1 - \gamma + \mu)v}. \quad (3.3)$$

The general solution of (3.3) essentially depends on the parameters $\alpha_1, \alpha_2, \alpha_3, \gamma$, and μ , and five different cases occur.

Case 1. $\alpha_3 = 0$ leads to the plane wave solutions of the form:

$$\omega = \alpha_1 x - \alpha_2 t, \quad u = \phi(\omega), \quad v = \psi(\omega), \quad (3.4)$$

where ϕ and ψ are new unknown functions. These functions should satisfy the ODE system:

$$-\alpha_2 \phi' = -\alpha_1^2 \phi^{\gamma-1} [\gamma \phi' \psi' + \phi \psi''] + d \alpha_1^2 \phi^{\mu-1} [\mu (\phi')^2 + \phi \phi''] + \lambda \phi^{1-\gamma+2\mu}, \\ \psi = \alpha_1^2 \phi''. \quad (3.5)$$

It should be noted that this system is equivalent to the fourth-order ODE:

$$\alpha_1^4 \phi^\gamma \phi^{iv} + \gamma \alpha_1^4 \phi^{\gamma-1} \phi' \phi''' - d \alpha_1^2 \phi^\mu \phi'' - d \alpha_1^2 \mu \phi^{\mu-1} (\omega) (\phi')^2 - \alpha_2 \phi' - \lambda \phi^{1-\gamma+2\mu} = 0. \quad (3.6)$$

This equation is not integrable because there are no general solutions for this equation in terms of elementary functions and known special functions [44]. However, it can be noted that the special case with $\gamma = 3\mu$ possesses the particular solution:

$$\phi(\omega) = \alpha \omega^{\frac{1}{\mu}}, \quad (3.7)$$

where α is a solution of algebraic equation:

$$\alpha_1^4 (1 - \mu) (1 - 2\mu) \alpha^{4\mu} - d \alpha_1^2 \mu^2 \alpha^{2\mu} - \alpha_2 \mu^3 \alpha^\mu - \lambda \mu^4 = 0. \quad (3.8)$$

Hence the system:

$$u_t = -[u^{3\mu}v_x]_x + d[u^\mu u_x]_x + \lambda u^{1-\mu}, \quad u_{xx} - v = 0 \quad (3.9)$$

possesses the exact solution:

$$u = \alpha(\alpha_1 x - \alpha_2 t)^{\frac{1}{\mu}}, \quad v = \alpha_1^2 \alpha \frac{1}{\mu} \left(\frac{1}{\mu} - 1 \right) (\alpha_1 x - \alpha_2 t)^{\frac{1}{\mu} - 2}, \quad (3.10)$$

where α satisfies (3.8).

Case 2. $\alpha_3 \neq 0$, $\gamma = 2\mu \neq 0$, $\alpha_1 = 0$ lead to the ansatz:

$$\omega = t, \quad u = \phi(t)x^{\frac{2}{\mu}}, \quad v = \psi(t)x^{\frac{2}{\mu} - 2}. \quad (3.11)$$

The corresponding ODE system takes the form:

$$\begin{aligned} \phi' &= -2 \left(\frac{1}{\mu} - 1 \right) \left(1 + \frac{2}{\mu} \right) \phi^{2\mu} \psi + d \frac{2}{\mu} \left(\frac{2}{\mu} + 1 \right) \phi^{\mu+1} + \lambda \phi, \\ \psi &= \frac{2}{\mu} \left(\frac{2}{\mu} - 1 \right) \phi, \end{aligned} \quad (3.12)$$

and can be rewritten as the single ODE:

$$\phi' = -\frac{4}{\mu} \left(\frac{1}{\mu} - 1 \right) \left(\frac{4}{\mu^2} - 1 \right) \phi^{2\mu+1} + d \frac{2}{\mu} \left(\frac{2}{\mu} + 1 \right) \phi^{\mu+1} + \lambda \phi. \quad (3.13)$$

It should be noted that this ODE with $\lambda = 0$ coincides with the one derived in the recently published paper [31] for the fourth-order PDE, which is equivalent to (3.1) with $\lambda = 0$. However, system (3.1) with $\lambda \neq 0$ cannot be reduced to the one with $\lambda = 0$, so that the solutions presented below cannot be obtained from [31].

If $\mu = 1$ or $\mu = 2$, then the known solutions of the reaction-diffusion equations:

$$u_t = d[uu_x]_x + \lambda u, \quad \mu = 1,$$

and

$$u_t = d[u^2u_x]_x + \lambda u, \quad \mu = 2$$

are obtained because $u_{xxx} = 0$ (see (3.11)). If $\mu = -2$, then

$$\phi(t) = Ce^{\lambda t}, \quad (3.14)$$

and there follows an exact solution:

$$u = \frac{Ce^{\lambda t}}{x}, \quad v = \frac{2Ce^{\lambda t}}{x^3} \quad (3.15)$$

of the system:

$$u_t = -\left[\frac{1}{u^4} v_x \right]_x + d \left[\frac{1}{u^2} u_x \right]_x + \lambda u, \quad u_{xx} - v = 0. \quad (3.16)$$

If $\mu \neq 1, \pm 2$, then two subcases, $\lambda = 0$ and $\lambda \neq 0$ should be separately considered. Both of them lead to the function $\phi(t)$ in the implicit form, and one can be found from the transcendental equation:

$$\ln \left| 1 + \frac{d}{2(1 - \frac{1}{\mu})(\frac{2}{\mu} - 1)} \frac{1}{\phi^\mu} \right| - \frac{d}{2(1 - \frac{1}{\mu})(\frac{2}{\mu} - 1)} \frac{1}{\phi^\mu} = \frac{d^2(\frac{2}{\mu} + 1)}{(1 - \frac{1}{\mu})(\frac{2}{\mu} - 1)} (t - t_0), \quad (3.17)$$

if $\lambda = 0$ and from the equation:

$$\ln \frac{\phi^{2\mu}(t)}{|\phi^{2\mu}(t) + \alpha\phi^\mu(t) + \beta|} - \alpha \int_0^{\phi^\mu(t)} \frac{dz}{z^2 + \alpha z + \beta} = 2\lambda\mu(t - t_0), \quad (3.18)$$

$$\alpha = \frac{d\mu^2}{2(1-\mu)(\mu-2)}, \quad \beta = \frac{\lambda\mu^4}{4(1-\mu)(\mu^2-4)} \text{ if } \lambda \neq 0.$$

Thus, the system:

$$u_t = -[u^{2\mu}v_x]_x + d[u^\mu u_x]_x, \quad u_{xx} - v = 0 \quad (3.19)$$

possesses the exact solution:

$$u = \phi(t)x^{\frac{2}{\mu}}, \quad v = \frac{2}{\mu} \left(\frac{2}{\mu} - 1 \right) \phi(t)x^{\frac{2}{\mu}-2}, \quad (3.20)$$

where $\phi(t)$ satisfies the equation (3.17). Note that the function $\phi(t)$ tends to 0 if $t \rightarrow \infty$ and this function blows up if $t \rightarrow t_0$. These properties follow from the simple analysis of (3.17). Analogously, the system:

$$u_t = -[u^{2\mu}v_x]_x + d[u^\mu u_x]_x + \lambda u, \quad u_{xx} - v = 0 \quad (3.21)$$

possesses the exact solution (3.20) with $\phi(t)$ satisfying equation (3.18).

Case 3. $\alpha_3 \neq 0$, $\gamma = 2\mu \neq 0$, $\alpha_1 \neq 0$ lead to the ansatz:

$$\omega = x e^{-\frac{\alpha_3 \mu}{\alpha_1} t}, \quad u = \phi(\omega) e^{\frac{2\alpha_3}{\alpha_1} t}, \quad v = \psi(\omega) e^{\frac{2\alpha_3}{\alpha_1} (1-\mu)t}, \quad (3.22)$$

which reduces the initial system to the ODE system:

$$\begin{aligned} -\alpha_3 \mu \omega \phi' + 2\alpha_3 \phi &= -\alpha_1 \phi^{2\mu-1} [2\mu \phi' \psi' + \phi \psi''] \\ &+ d\alpha_1 \phi^{\mu-1} [\mu(\phi')^2 + \phi \phi''] + \alpha_1 \lambda \phi, \quad \psi = \phi''. \end{aligned} \quad (3.23)$$

System (3.23) is equivalent to the 4th-order equation:

$$\begin{aligned} \alpha_1 \phi^{2\mu} \phi^{iv} + 2\mu \alpha_1 \phi^{2\mu-1} \phi' \phi''' - d\alpha_1 \phi^\mu \phi'' - d\alpha_1 \mu \phi^{\mu-1}(\omega)(\phi')^2 \\ - \alpha_3 \mu \omega \phi' - (\alpha_1 \lambda - 2\alpha_3) \phi = 0, \end{aligned} \quad (3.24)$$

which possesses the particular solution:

$$\phi(\omega) = \alpha \omega^{\frac{2}{\mu}}. \quad (3.25)$$

Here, α must be a solution of the algebraic equation:

$$4(4 - \mu^2)(1 - \mu)\alpha^{2\mu} - 2d\mu^2(\mu + 2)\alpha^\mu - \lambda\mu^4 = 0, \quad (3.26)$$

which is simply a quadratic equation for α^μ .

Thus, the cross-diffusion system:

$$u_t = -[u^{2\mu}v_x]_x + d[u^\mu u_x]_x + \lambda u, \quad u_{xx} - v = 0 \quad (3.27)$$

has the stationary solution:

$$u = \alpha x^{\frac{2}{\mu}}, \quad v = 2\alpha \frac{2-\mu}{\mu^2} x^{\frac{2}{\mu}-2}, \quad (3.28)$$

where α satisfies (3.26).

Case 4. $\alpha_3 \neq 0$, $\gamma = \mu \neq 0$ lead to the ansatz:

$$\omega = x + \frac{\alpha_2}{2\mu\alpha_3} \ln t, \quad u = \phi(\omega)t^{-\frac{1}{\mu}}, \quad v = \psi(\omega)t^{-\frac{1}{\mu}}. \quad (3.29)$$

The corresponding ODE system takes the form:

$$\begin{aligned} \alpha_2\phi' - 2\alpha_3\phi &= -2\alpha_3\mu\phi^{\mu-1}[\mu\phi'\psi' + \phi\psi''] + 2\alpha_3\mu d\phi^{\mu-1}[\mu(\phi')^2 + \phi\phi''] \\ &+ 2\alpha_3\mu\lambda\phi^{\mu+1}, \quad \psi = \phi'' \end{aligned} \quad (3.30)$$

and is equivalent to the 4th-order equation:

$$\begin{aligned} 2\alpha_3\mu\phi^\mu\phi^{iv} + 2\alpha_3\mu^2\phi^{\mu-1}\phi'\phi''' - 2\alpha_3\mu d\phi^\mu\phi'' - 2d\alpha_3\mu^2\phi^{\mu-1}(\phi')^2 \\ + \alpha_2\phi' - 2\alpha_3\mu\lambda\phi^{\mu+1} - 2\alpha_3\phi = 0. \end{aligned} \quad (3.31)$$

For $\alpha_2 \neq 0$, the solutions $u(x, t)$ are traveling waves whose speed decreases in proportion to t^{-1} and whose amplitude decreases in proportion to $t^{-1/\mu}$.

Equation (3.31) is not integrable but we were able to find the particular solutions if $\mu = 1$, $\alpha_2 = 0$, and $\lambda \neq 0$:

$$\phi(\omega) = -\frac{2}{3\lambda} + \frac{2}{3|\lambda|} \sin(\theta\omega + \theta_0), \quad \theta^4 + d\theta^2 - \frac{\lambda}{2} = 0, \quad (3.32)$$

and

$$\phi(\omega) = -\frac{2}{3\lambda} + C_1 e^{\theta\omega} + \frac{1}{9C_1\lambda^2} e^{-\theta\omega}, \quad \theta^4 - d\theta^2 - \frac{\lambda}{2} = 0. \quad (3.33)$$

Thus, using ansatz (3.29), we arrive at the solutions:

$$u = \frac{-\frac{2}{3\lambda} + \frac{2}{3|\lambda|} \sin(\theta x + \theta_0)}{t}, \quad v = \frac{-\frac{2}{3|\lambda|} \theta^2 \sin(\theta x + \theta_0)}{t}, \quad (3.34)$$

which is a spatial sinusoid for which amplitude varies in proportion to $1/t$ and

$$u = \frac{-\frac{2}{3\lambda} + C_1 e^{\theta x} + \frac{1}{9C_1\lambda^2} e^{-\theta x}}{t}, \quad v = \frac{C_1 \theta^2 e^{\theta x} + \frac{\theta^2}{9C_1\lambda^2} e^{-\theta x}}{t} \quad (3.35)$$

of the system:

$$u_t = -[uv_x]_x + d[uv_x]_x + \lambda u^2, \quad u_{xx} - v = 0. \quad (3.36)$$

In formulae (3.34) and (3.35), we can also shift the time t to $t - t_0$ or $t + t_0$ with the positive parameter t_0 . The first shift leads to a solution having blow up at $t = t_0$, the second leads to a solution that avoids singularity at $t = 0$ and tends to 0 as t approaches ∞ . Blow up and extinction are interesting phenomena in some applications.

Case 5. $\alpha_3 \neq 0$, $\gamma \neq 2\mu$, $\gamma \neq \mu$ lead to the similarity reduction:

$$\omega = xt^{\frac{\mu-\gamma}{2(\gamma-2\mu)}}, \quad u = \phi(\omega)t^{\frac{1}{\gamma-2\mu}}, \quad v = \psi(\omega)t^{\frac{1-\gamma+\mu}{\gamma-2\mu}}, \quad (3.37)$$

and to the ODE system:

$$\begin{aligned} (\mu - \gamma)\omega\phi' + 2\phi &= -2(\gamma - 2\mu)\phi^{\gamma-1}[\gamma\phi'\psi' + \phi\psi''] \\ &+ 2(\gamma - 2\mu)d\phi^{\mu-1}[\mu(\phi')^2 + \phi\phi''] \\ &+ 2(\gamma - 2\mu)\lambda\phi^{1-\gamma+2\mu}, \quad \psi = \phi''. \end{aligned} \quad (3.38)$$

The equivalent 4th-order equation has the form:

$$\begin{aligned} 2(\gamma - 2\mu)\phi^\gamma\phi^{iv} + 2(\gamma - 2\mu)\gamma\phi^{\gamma-1}\phi'\phi''' - 2(\gamma - 2\mu)d\phi^\mu\phi'' \\ - 2d(\gamma - 2\mu)\mu\phi^{\mu-1}(\phi')^2 + (\mu - \gamma)\omega\phi' - 2(\gamma - 2\mu)\lambda\phi^{1-\gamma+2\mu} + 2\phi = 0, \end{aligned} \quad (3.39)$$

whose solutions are self-similar by a scaling invariance. Although this equation is again not integrable, one may try to construct particular solutions in the form of a high-order polynomial. For example, setting $d = 0$, $\gamma = 1$, $\mu = 0$, the exact solution:

$$\phi(\omega) = \frac{1}{120}\omega^4 + \frac{5}{6}\lambda \quad (3.40)$$

is obtained. Thus, we arrive at the solution:

$$u = \frac{1}{120} \frac{x^4}{t} + \frac{5}{6}\lambda t, \quad v = \frac{1}{10} \frac{x^2}{t} \quad (3.41)$$

of the system:

$$u_t = -[uv_x]_x + \lambda, \quad u_{xx} - v = 0. \quad (3.42)$$

Remark 3.1. Following [35], let us consider the ad hoc ansatz:

$$\begin{aligned} u &= \phi_0(t) + \phi_1(t)x + \phi_2(t)x^2 + \phi_3(t)x^3 + \phi_4(t)x^4, \\ v &= 2\phi_2(t) + 6\phi_3(t)x + 12\phi_4(t)x^2. \end{aligned} \quad (3.43)$$

Using this ansatz, solution (3.41) can be generalized to the form:

$$\begin{aligned} u &= \frac{C_0}{t^{\frac{1}{5}}} + 30\frac{C_1C_3}{t^{\frac{2}{5}}} + 900\frac{C_2C_3^2}{t^{\frac{3}{5}}} + 6750\frac{C_3^4}{t} + \frac{5}{6}\lambda t \\ &+ \left(\frac{C_1}{t^{\frac{2}{5}}} + 60\frac{C_2C_3}{t^{\frac{3}{5}}} + 900\frac{C_3^3}{t}\right)x + \left(\frac{C_2}{t^{\frac{3}{5}}} + 45\frac{C_3^2}{t}\right)x^2 + \frac{C_3}{t}x^3 + \frac{1}{120}\frac{1}{t}x^4, \\ v &= 2\left(\frac{C_2}{t^{\frac{3}{5}}} + 45\frac{C_3^2}{t}\right) + \frac{6C_3}{t}x + \frac{1}{10}\frac{1}{t}x^2, \end{aligned} \quad (3.44)$$

where C_i ($i = 0, \dots, 3$) are arbitrary constants.

Similarly, setting $d = 0$, $\gamma = 1$, $\mu = \frac{1}{4}$ in (3.39), the solution:

$$\phi(\omega) = \left(\frac{1}{\sqrt{120}}\omega^2 + \frac{5}{11}\lambda\right)^2 \quad (3.45)$$

can be derived. Thus, the system:

$$u_t = -[uv_x]_x + \lambda\sqrt{u}, \quad u_{xx} - v = 0 \quad (3.46)$$

possesses the exact solution:

$$u = \left(\frac{1}{\sqrt{120}} \frac{x^2}{\sqrt{t}} + \frac{5}{11} \lambda t \right)^2, \quad v = \frac{1}{10} \frac{x^2}{t} + \frac{10}{11\sqrt{30}} \lambda \sqrt{t}. \quad (3.47)$$

It should be noted that solutions (3.41) and (3.47) have been earlier obtained in [29] via the method of invariant subspaces. Indeed, the formulas (3.29) and (3.76) [29] contain (3.41) and (3.47), respectively.

4 Conclusions

In this paper, the Lie symmetry classification of a class of fourth-order reaction-diffusion equations was carried out. Equation (1.1) has been treated, firstly, as a system of second-order equations that bears some resemblance to a system of coupled reaction-diffusion equations with cross diffusion, secondly, as a system of a second-order equation and two first-order equations. It turns out that both systems lead to the same result of symmetry group classification. Our paper generalizes the results of Lie symmetry analysis derived earlier for particular cases of equation (1.1). Moreover, we were able to construct all possible Lie symmetries, in which equation (1.1) can admit depending on the function triplets $(K, D, \text{ and } F)$. This distinguishes our investigation from those that have focussed on Lie symmetry of particular cases of the given fourth-order evolution equation. However, our result is analogous to those derived in recent papers [20, 21, 47] for different classes of second-order evolution equations, where form-preserving point transformations [36] were used to construct shortest lists of the relevant equations with nontrivial MAI.

To the best of our knowledge, there is only the recently published book [29], where exact solutions have been found for some equations of the form (1.1) with $F \neq 0$. Thus, a fourth-order nonlinear equation with the nonzero reaction term in the form of system (3.1) was examined by applying the Lie symmetry reduction, where possible, we have constructed exact solutions to the ordinary differential equations that were obtained from this reaction-diffusion system. The solutions include some unusual structures as well as the familiar types that regularly occur in symmetry reductions, namely, self-similar solutions, decelerating and decaying traveling waves, and steady states. Many of the functional relationships between the two symmetry invariants are quite simple, involving polynomials, algebraic functions, logarithms, exponentials, and sinusoids. However, there are some that have been reduced only to the solutions of transcendental equations (see formulas (3.17) and (3.18)).

Finally, it should be also noted that the nonlinear fourth-order ODEs obtained in Section 3 can be solved by numerical methods, and therefore solutions of the relevant generalized thin film equations will be constructed.

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References

- [1] R. Anderson and N. Ibragimov. *Lie-Bäcklund Transformations in Applications*. SIAM Studies in Applied Mathematics **1**, SIAM, Philadelphia, 1979.
- [2] D. Arrigo and J. Hill. Nonclassical symmetries for nonlinear diffusion and absorption. *Stud. Appl. Math.*, **94** (1995), 21–39.
- [3] D. Arrigo, J. Hill, and P. Broadbridge. Nonclassical symmetry reductions of the linear diffusion equation with a nonlinear source. *IMA J. Appl. Math.*, **52** (1994), 1–24.
- [4] F. Bernis, J. Hulshof, and J. King. Dipoles and similarity solutions of the thin film equation in the half-line. *Nonlinearity*, **13** (2000), 413–439.
- [5] F. Bernis and J. McLeod. Similarity solutions of a higher order nonlinear diffusion equation. *Nonlinear Anal.*, **17** (1991), 1039–1068.
- [6] A. Bertozzi. The mathematics of moving contact lines in thin liquid films. *Notices Amer. Math. Soc.*, **45** (1998), 689–697.
- [7] A. Bertozzi and M. Pugh. Long-wave instabilities and saturation in thin film equations. *Comm. Pure Appl. Math.*, **51** (1998), 625–661.
- [8] G. Bluman and J. Cole. *Similarity Methods for Differential Equations*. Springer, Berlin, 1974.
- [9] B. Bradshaw-Hajek and P. Broadbridge. A robust cubic reaction-diffusion system for gene propagation. *Math. Comput. Modelling*, **39** (2004), 1151–1163.
- [10] P. Broadbridge. Entropy diagnostics for fourth order partial differential equations in conservation form. *Entropy*, **10** (2008), 365–379.
- [11] K. Brown and A. Lacey. *Reaction-Diffusion Equations*. Oxford University Press, New York, 1990.
- [12] M. Bruzón, M. Gandarias, E. Medina, and E. Muriel. New symmetry reductions for a lubrication model. In “*Nonlinear Physics: Theory and Experiment, II*”. M. J. Ablowitz, M. Boiti, F. Pempinelli, and B. Prinari (Eds.), World Scientific, Singapore, (2002), pp. 143–148.
- [13] J. Cahn and J. Hilliard. Free energy of a nonuniform system. I. Interfacial energy. *J. Chem. Phys.*, **28** (1958), 258–266.
- [14] J. Cahn and J. Taylor. Surface motion by surface diffusion. *Acta Metall. Mater.*, **42** (1994), 1045–1063.
- [15] R. Cantrell and C. Cosner. *Spatial Ecology via Reaction-Diffusion Equations*. John Wiley & Sons, Chichester, 2003.
- [16] R. Cherniha and J. King. Lie symmetries and conservation laws of non-linear multidimensional reaction-diffusion systems with variable diffusivities. *IMA J. Appl. Math.*, **71** (2006), 391–408.
- [17] R. Cherniha and J. King. Lie symmetries of nonlinear multidimensional reaction-diffusion systems. I. *J. Phys. A*, **33** (2000), 267–282, 7839–7841.
- [18] R. Cherniha and J. King. Lie symmetries of nonlinear multidimensional reaction-diffusion systems. II. *J. Phys. A*, **36** (2003), 405–425.
- [19] R. Cherniha and M. Serov. Symmetries, ansätze and exact solutions of nonlinear second-order evolution equations with convection terms. *European J. Appl. Math.*, **9** (1998), 527–542.
- [20] R. Cherniha and M. Serov. Symmetries, ansätze and exact solutions of nonlinear second-order evolution equations with convection terms. II. *European J. Appl. Math.*, **17** (2006), 597–605.
- [21] R. Cherniha, M. Serov, and I. Rassokha. Lie symmetries and form-preserving transformations of reaction-diffusion-convection equations. *J. Math. Anal. Appl.*, **342** (2008), 1363–1379.
- [22] S. Choudhury. General similarity reductions of a family of Cahn-Hilliard equations. *Nonlinear Anal.*, **24** (1995), 131–146.
- [23] P. Clarkson and E. Mansfield. Symmetry reductions and exact solutions of a class of nonlinear heat equations. *Phys. D*, **70** (1994), 250–288.
- [24] V. Dorodnitsyn. On invariant solutions of non-linear heat conduction with a source. *USSR Comput. Math. and Math. Phys.*, **22** (1982), 115–122.
- [25] T. Dupont, R. Goldstein, L. Kadanoff, and S.-M. Zhou. Finite-time singularity formation in Hele-Shaw systems. *Phys. Rev. E (3)*, **47** (1993), 4182–4196.

- [26] J. Evans, V. Galaktionov, and J. King. Blow-up similarity solutions of the fourth-order unstable thin film equation. *European J. Appl. Math.*, **18** (2007), 195–231.
- [27] J. Evans, V. Galaktionov, and J. Williams. Blow-up and global asymptotics of the limit unstable Cahn-Hilliard equation. *SIAM J. Math. Anal.*, **38** (2006), 64–102.
- [28] W. Fushchich, W. Shtelen, and M. Serov. *Symmetry Analysis and Exact Solutions of Equations of Nonlinear Mathematical Physics*. Kluwer, Dordrecht, 1993.
- [29] V. Galaktionov and S. Svirshchevskii. *Exact Solutions and Invariant Subspaces of Nonlinear Partial Differential Equations in Mechanics and Physics*. Chapman & Hall/CRC Applied Mathematics and Nonlinear Science Series, Chapman & Hall/CRC, Boca Raton, FL, 2007.
- [30] M. Gandarias and M. Bruzón. Symmetry analysis and solutions for a family of Cahn-Hilliard equations. *Rep. Math. Phys.*, **46** (2000), 89–97.
- [31] M. Gandarias and N. Ibragimov. Equivalence group of a fourth-order evolution equation unifying various non-linear models. *Comm. in Nonlin. Sci. and Num. Simulation*, **13** (2008), 259–268.
- [32] P. Gaskell, P. Jimack, M. Sellier, and H. Thompson. Flow of evaporating, gravity-driven thin liquid films over topography. *Phys. Fluids*, **18** (2006), 013601.1–013601.14.
- [33] H. P. Greenspan. On the motion of a small viscous droplet that wets a surface. *J. Fluid Mech.*, **84** (1978), 125–143.
- [34] M. Grinfeld and A. Novick-Cohen. The viscous Cahn-Hilliard equation: morse decomposition and structure of the global attractor. *Trans. Amer. Math. Soc.*, **351** (1999), 2375–2406.
- [35] J. R. King. Mathematical analysis of a model for substitutional diffusion. *Proc. Roy. Soc. London Ser. A*, **430** (1990), 377–404.
- [36] J. G. Kingston and C. Sophocleous. On form-preserving point transformations of partial differential equations, *J. Phys. A*, **31** (1998), 1597–1619.
- [37] W. Mullins. Theory of thermal grooving. *J. Appl. Phys.*, **28** (1957), 333–339.
- [38] J. D. Murray. *Mathematical Biology*. Springer, Berlin, 1989.
- [39] A. Novick-Cohen and L. Segel. Nonlinear aspects of the Cahn-Hilliard equation. *Phys. D*, **10** (1984), 277–298.
- [40] P. Olver. *Applications of Lie Groups to Differential Equations*. Springer, Berlin, 1986.
- [41] A. Oron, S. Davis, and S. Bankoff. Long-scale evolution of thin liquid films. *Rev. Mod. Phys.*, **69** (1997), 931–980.
- [42] L. Ovsiannikov. *Group Analysis of Differential Equations*. Academic Press, New York, 1982
- [43] L. Ovsyannikov. Group relations of the equation of non-linear heat conductivity, *Dokl. Akad. Nauk SSSR*, **125** (1959), 492–495.
- [44] A. Polyanin and V. Zaitsev. *Handbook of Exact Solutions for Ordinary Differential Equations*. Chapman & Hall/CRC, Boca Raton, FL, 2003.
- [45] N. Smyth and J. Hill. High-order nonlinear diffusion. *IMA J. Appl. Math.*, **40** (1988), 73–86.
- [46] U. Valbusa, C. Boragno, and F. Buatier de Mongeot. Nanostructuring surfaces by ion sputtering, *J. Phys.: Condens. Matter*, **14** (2002), 8153–8175.
- [47] O. O. Vaneeva, A. G. Johnpillai, R. O. Popovych, and C. Sophocleous, Enhanced group analysis and conservation laws of variable coefficient reaction-diffusion equations with power nonlinearities. *J. Math. Anal. Appl.*, **330** (2007), 1363–1386.
- [48] K. Vu, J. Butcher, and J. Carminati. Similarity solutions of partial differential equations using DESOLV. *Comput. Phys. Comm.*, **176** (2007), 682–693.

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