Ashdin Publishing
Journal of Physical Mathematics
Vol. 1 (2009), Article ID S090603, 16 pages
doi:10.4303/jpm/S090603

Models of damped oscillators in quantum mechanics

Ricardo CORDERO-SOTO [a], *Erwin SUAZO* [a,b], *and Sergei K. SUSLOV* [a,b]

[a] *Mathematical, Computational and Modeling Sciences Center, Arizona State University, Tempe, AZ 85287-1904, USA*

[b] *School of Mathematical and Statistical Sciences, Arizona State University, Tempe, AZ 85287-1904, USA*

E-mails: ricardojavier81@gmail.com, suazo@mathpost.la.asu.edu, sks@asu.edu

Abstract

We consider several models of the damped oscillators in nonrelativistic quantum mechanics in a framework of a general approach to the dynamics of the time-dependent Schrödinger equation with variable quadratic Hamiltonians. The Green functions are explicitly found in terms of elementary functions and the corresponding gauge transformations are discussed. The factorization technique is applied to the case of a shifted harmonic oscillator. The time evolution of the expectation values of the energy-related operators is determined for two models of the quantum damped oscillators under consideration. The classical equations of motion for the damped oscillations are derived for the corresponding expectation values of the position operator.

2000 MSC: 81Q05, 35C05, 42A38

1 Introduction

We continue an investigation of the one-dimensional Schrödinger equations with variable quadratic Hamiltonians of the form

$$i\frac{\partial\psi}{\partial t} = -a(t)\frac{\partial^2\psi}{\partial x^2} + b(t)x^2\psi - i\left(c(t)x\frac{\partial\psi}{\partial x} + d(t)\psi\right), \tag{1.1}$$

where $a(t)$, $b(t)$, $c(t)$, and $d(t)$ are real-valued functions of time t only; see [8, 9, 22, 23, 25, 34, 35, 36] for a general approach and currently known explicit solutions. Here we discuss elementary cases related to the models of damped oscillators. The corresponding Green functions, or Feynman's propagators, can be found as follows [8, 35]:

$$\psi = G(x,y,t) = \frac{1}{\sqrt{2\pi i\mu(t)}}\, e^{i(\alpha(t)x^2+\beta(t)xy+\gamma(t)y^2)}, \tag{1.2}$$

where

$$\alpha(t) = \frac{1}{4a(t)}\frac{\mu'(t)}{\mu(t)} - \frac{d(t)}{2a(t)}, \tag{1.3}$$

$$\beta(t) = -\frac{h(t)}{\mu(t)}, \quad h(t) = \exp\left(-\int_0^t \big(c(\tau)-2d(\tau)\big)\,d\tau\right), \tag{1.4}$$

$$\gamma(t) = \frac{a(t)h^2(t)}{\mu(t)\mu'(t)} + \frac{d(0)}{2a(0)} - 4\int_0^t \frac{a(\tau)\sigma(\tau)h^2(\tau)}{\big(\mu'(\tau)\big)^2}\,d\tau, \tag{1.5}$$

and the function $\mu(t)$ satisfies the characteristic equation

$$\mu'' - \tau(t)\mu' + 4\sigma(t)\mu = 0 \tag{1.6}$$

with

$$\tau(t) = \frac{a'}{a} - 2c + 4d, \quad \sigma(t) = ab - cd + d^2 + \frac{d}{2}\left(\frac{a'}{a} - \frac{d'}{d}\right) \tag{1.7}$$

subject to the initial data

$$\mu(0) = 0, \quad \mu'(0) = 2a(0) \neq 0. \tag{1.8}$$

More details can be found in [8, 35]. The corresponding Hamiltonian structure is discussed in [9].

The simple harmonic oscillator is of interest in many advanced quantum problems [16, 21, 26, 32]. The forced harmonic oscillator was originally considered by Richard Feynman in his path integrals approach to the nonrelativistic quantum mechanics [12, 13, 14, 15, 16]; see also [23]. Its special and limiting cases were discussed by many authors; see [6, 17, 19, 24, 26, 38] for the simple harmonic oscillator and [1, 7, 18, 27, 31] for the particle in a constant external field and references therein.

The damped oscillations have been analyzed to a great extent in classical mechanics; see, for example, [5, 20]. In the present paper we consider the time-dependent Schrödinger equation

$$i\frac{\partial \psi}{\partial t} = H\psi \tag{1.9}$$

with the following non-self-adjoint Hamiltonians

$$H = H_1 = \frac{\omega_0}{2}\left(p^2 + x^2\right) - \lambda px \tag{1.10}$$

and

$$H = H_2 = \frac{\omega_0}{2}\left(p^2 + x^2\right) - \lambda xp, \tag{1.11}$$

where $p = -i\partial/\partial x$, as quantum analogs of the damped oscillator. A related self-adjoint Hamiltonian

$$H = H_0 = \frac{\omega_0}{2}\left(p^2 + x^2\right) - \frac{\lambda}{2}(px + xp) \tag{1.12}$$

is also analyzed. Although discussion of a quantum damped oscillator is usually missing in the standard classical textbooks [21, 26, 32] among others, we believe that the models presented here have a significant value from the pedagogical and mathematical points of view. For instance, one of these models was crucial for our understanding of a "hidden" symmetry of the quadratic propagators in [9]. Moreover, our models show that fundamentals of quantum mechanics, such as evolution of the expectation values of operators and Ehrenfest's theorem, can be extended to the case of non-self-adjoint Hamiltonians. This provides, in our opinion, a somewhat better understanding of the mathematical foundations of quantum mechanics and can be used in the classroom.

The paper is organized as follows. In Section 2 we derive the propagators for the models of the damped oscillator (1.10) and (1.11) following the method of [8]. The corresponding gauge transformations are discussed in Section 3. The next section is concerned with the separation of the variables for related model of a "shifted" linear harmonic oscillator (1.12). The factorization technique is applied to this oscillator in Section 5. The time evolution of the expectation values of the energy-related operators is determined for these quantum damped oscillators in Section 6. The classical equations for the damped oscillations are derived for the expectation values of the position operator in the next section. One more model of the damped oscillator with a variable quadratic Hamiltonian is introduced in Section 8. The last section contains some remarks on the momentum representation.

2 The first two models

For the time-dependent Schrödinger equation

$$i\frac{\partial \psi}{\partial t} = \frac{\omega_0}{2}\left(-\frac{\partial^2 \psi}{\partial x^2} + x^2\psi \right) + i\lambda \left(x\frac{\partial \psi}{\partial x} + \psi \right) \tag{2.1}$$

with $a = b = \omega_0/2$ and $c = d = -\lambda$, the characteristic equation (1.6) takes the form of the classical equation of motion for the damped oscillator [5, 20]:

$$\mu'' + 2\lambda\mu' + \omega_0^2\mu = 0, \tag{2.2}$$

whose suitable solution is

$$\mu = \frac{\omega_0}{\omega} e^{-\lambda t} \sin \omega t, \quad \omega = \sqrt{\omega_0^2 - \lambda^2} > 0. \tag{2.3}$$

The corresponding propagator is given by

$$\begin{aligned} G(x, y, t) = &\sqrt{\frac{\omega e^{\lambda t}}{2\pi i \omega_0 \sin \omega t}} \exp \left(\frac{i\omega}{2\omega_0 \sin \omega t} \left(\left(x^2 + y^2 \right) \cos \omega t - 2xy \right) \right) \\ &\times \exp \left(\frac{i\lambda}{2\omega_0} \left(x^2 - y^2 \right) \right). \end{aligned} \tag{2.4}$$

Indeed, directly from (1.3)–(1.4),

$$\alpha(t) = \frac{\omega \cos \omega t + \lambda \sin \omega t}{2\omega_0 \sin \omega t}, \quad \beta(t) = -\frac{\omega}{\omega_0 \sin \omega t}. \tag{2.5}$$

The integral in (1.5) can be evaluated with the help of a familiar antiderivative

$$\int \frac{dt}{(A\cos t + B\sin t)^2} = \frac{\sin t}{A(A\cos t + B\sin t)} + C. \tag{2.6}$$

It gives

$$\gamma(t) = \frac{\omega \cos \omega t - \lambda \sin \omega t}{2\omega_0 \sin \omega t} \tag{2.7}$$

with the help of the identity

$$\omega^2 - \omega_0^2 \sin^2 \omega t = \omega^2 \cos^2 \omega t - \lambda^2 \sin^2 \omega t \tag{2.8}$$

and the propagator (2.4) is verified. A “hidden” symmetry of this propagator is discussed in [9].

The time-evolution of the squared norm of the wave function is given by

$$\left\|\psi(x,t)\right\|^2 = \int_{-\infty}^{\infty} \left|\psi(x,t)\right|^2 dx = e^{\lambda t}\left\|\psi(x,0)\right\|^2. \tag{2.9}$$

It is derived in Section 6 among other things. We have discussed here the case $\omega_0^2 > \lambda^2$. Two more cases, when $\omega_0^2 = \lambda^2$ and $\omega_0^2 < \lambda^2$, are similar and the details are left to the reader.

In a similar fashion, the time-dependent Schrödinger equation of the form

$$i\frac{\partial \psi}{\partial t} = \frac{\omega_0}{2}\left(-\frac{\partial^2\psi}{\partial x^2} + x^2\psi\right) + i\lambda x\frac{\partial\psi}{\partial x}, \tag{2.10}$$

with $a = b = \omega_0/2$ and $c = -\lambda$, $d = 0$, has the characteristic equation

$$\mu'' - 2\lambda\mu' + \omega_0^2\mu = 0 \tag{2.11}$$

with the solution

$$\mu = \frac{\omega_0}{\omega}e^{\lambda t}\sin\omega t, \quad \omega = \sqrt{\omega_0^2 - \lambda^2} > 0. \tag{2.12}$$

The corresponding propagator is given by

$$\begin{aligned} G(x,y,t) = {} & \sqrt{\frac{\omega e^{-\lambda t}}{2\pi i\omega_0 \sin\omega t}} \exp\left(\frac{i\omega}{2\omega_0\sin\omega t}\left(\left(x^2+y^2\right)\cos\omega t - 2xy\right)\right) \\ & \times \exp\left(\frac{i\lambda}{2\omega_0}\left(x^2-y^2\right)\right) \end{aligned} \tag{2.13}$$

and the evolution of the squared norm is

$$\left\|\psi(x,t)\right\|^2 = e^{-\lambda t}\left\|\psi(x,0)\right\|^2. \tag{2.14}$$

The solution of the Cauchy initial value problem

$$i\frac{\partial\psi}{\partial t} = H\psi, \quad \psi(x,0) = \chi(x) \tag{2.15}$$

for our models (2.1) and (2.10) is given by the superposition principle in an integral form

$$\psi(x,t) = \int_{-\infty}^{\infty} G(x,y,t)\,\chi(y)\,dy \tag{2.16}$$

for a suitable initial function χ on $\boldsymbol{R}$; a rigorous proof is given in [35].

3 The gauge transformations

The time-dependent Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = \left(\frac{\omega_0}{2}(p-A)^2 + U + (p-A)V + W(p-A)\right)\psi, \tag{3.1}$$

where $p = i^{-1}\partial/\partial x$ is the linear momentum operator and $A = A(x,t)$, $U = U(x,t)$, $V = V(x,t)$, $W = W(x,t)$ are real-valued functions, with the help of the gauge transformation

$$\psi = e^{-if(x,t)}\widetilde{\psi} \tag{3.2}$$

can be transformed into a similar form

$$i\frac{\partial\widetilde{\psi}}{\partial t} = \left(\frac{\omega_0}{2}(p-\widetilde{A})^2 + \widetilde{U} + (p-\widetilde{A})\widetilde{V} + \widetilde{W}(p-\widetilde{A})\right)\widetilde{\psi} \tag{3.3}$$

with the new vector and scalar potentials given by

$$\widetilde{A} = A + \frac{\partial f}{\partial x}, \quad \widetilde{U} = U - \frac{\partial f}{\partial t}, \quad \widetilde{V} = V, \quad \widetilde{W} = W. \tag{3.4}$$

Here we consider the one-dimensional case only and may think of f as being an arbitrary complex-valued differentiable function. Also, the Hamiltonian in the right-hand side of equation (3.1) is not assumed to be self-adjoint; see [21, 26] for discussion of the traditional case, when $V = W \equiv 0$.

An interesting special case of the gauge transformation related to this paper is given by

$$A = 0, \quad U = \frac{\omega_0}{2}x^2, \quad V = -\lambda x, \quad W = 0, \quad f = \frac{i\lambda t}{2}, \tag{3.5}$$

$$\widetilde{A} = 0, \quad \widetilde{U} = \frac{\omega_0}{2}x^2 - \frac{i\lambda}{2}, \quad \widetilde{V} = -\lambda x, \quad \widetilde{W} = 0, \tag{3.6}$$

when the new Hamiltonian is

$$\widetilde{H} = \frac{\omega_0}{2}(p-\widetilde{A})^2 + \widetilde{U} + p\widetilde{V} = \frac{\omega_0}{2}\left(-\frac{\partial^2}{\partial x^2} + x^2\right) + i\frac{\lambda}{2}\left(2x\frac{\partial}{\partial x} + 1\right), \tag{3.7}$$

and equation (2.1) takes the form

$$i\frac{\partial\psi}{\partial t} = \frac{\omega_0}{2}\left(-\frac{\partial^2\psi}{\partial x^2} + x^2\psi\right) + i\frac{\lambda}{2}\left(2x\frac{\partial\psi}{\partial x} + \psi\right). \tag{3.8}$$

The corresponding Green function is given by

$$\begin{aligned} G(x,y,t) = {} & \sqrt{\frac{\omega}{2\pi i\omega_0 \sin\omega t}} \exp\left(\frac{i\omega}{2\omega_0 \sin\omega t}\left(\left(x^2+y^2\right)\cos\omega t - 2xy\right)\right) \\ & \times \exp\left(\frac{i\lambda}{2\omega_0}\left(x^2-y^2\right)\right), \quad \omega = \sqrt{\omega_0^2 - \lambda^2} > 0, \end{aligned} \tag{3.9}$$

and the norm of the wave function is conserved with time. This can be established once again directly from our equations (1.2)–(1.8). We leave the details to the reader. A traditional method of separation of the variables and using the Mehler formula for Hermite polynomials is discussed in the next section. The factorization technique is applied to this Hamiltonian in Section 5.

Equation (3.8), in turn, admits another local gauge transformation:

$$A = 0, \quad U = \frac{\omega_0}{2}x^2, \quad V = W = -\frac{\lambda x}{2}, \quad f = -\frac{\lambda x^2}{2\omega_0}, \tag{3.10}$$

$$\widetilde{A} = -\frac{\lambda x}{\omega_0}, \quad \widetilde{U} = \frac{\omega_0}{2}x^2, \quad \widetilde{V} = \widetilde{W} = -\frac{\lambda x}{2} \tag{3.11}$$

and the Hamiltonian becomes

$$\begin{aligned}\widetilde{H} &= \frac{\omega_0}{2}(p-\widetilde{A})^2 + \widetilde{U} + (p-\widetilde{A})\widetilde{V} + \widetilde{W}(p-\widetilde{A}) \\ &= \frac{\omega_0}{2}\left(p + \frac{\lambda x}{\omega_0}\right)^2 + \frac{\omega_0}{2}x^2 \\ &\quad + \left(p + \frac{\lambda x}{\omega_0}\right)\left(-\frac{\lambda x}{\omega_0}\right) + \left(-\frac{\lambda x}{\omega_0}\right)\left(p + \frac{\lambda x}{\omega_0}\right) \\ &= \frac{\omega_0}{2}p^2 + \frac{\omega_0^2 - \lambda^2}{2\omega_0}x^2.\end{aligned} \tag{3.12}$$

As a result, equation (3.8) takes the form of equation for the harmonic oscillator:

$$i\frac{\partial \psi}{\partial t} = \frac{\omega_0}{2}\left(-\frac{\partial^2 \psi}{\partial x^2} + \frac{\omega^2}{\omega_0^2}x^2\psi\right), \quad \omega^2 = \omega_0^2 - \lambda^2 > 0 \tag{3.13}$$

and can be solved, once again, by the traditional method of separation of the variables or by the factorization technique.

4 Separation of variables for a shifted harmonic oscillator

We will refer to the case (3.8) as one of a shifted linear harmonic oscillator. The Ansatz

$$\psi(x,t) = e^{-iEt}\varphi(x) \tag{4.1}$$

in the time-dependent Schrödinger equation results in the stationary Schrödinger equation

$$H\varphi = E\varphi \tag{4.2}$$

with the Hamiltonian (3.7). The last equation, namely,

$$-\varphi'' + x^2\varphi + \frac{i\lambda}{\omega_0}(2x\varphi' + \varphi) = \frac{2E}{\omega_0}\varphi, \tag{4.3}$$

with the help of the substitution

$$\varphi = \exp\left(\frac{i\lambda x^2}{2\omega_0}\right)u(x) \tag{4.4}$$

is reduced to the equation

$$-u'' + \frac{\omega^2}{\omega_0^2}x^2 u = \frac{2E}{\omega_0}u. \tag{4.5}$$

The change of the variable

$$u(x) = v(\xi), \quad x = \xi\sqrt{\frac{\omega_0}{\omega}} \tag{4.6}$$

gives us the stationary Schrödinger equation for the simple harmonic oscillator [21, 26, 29, 32]:

$$v'' + \left(2\varepsilon - \xi^2\right)v = 0 \tag{4.7}$$

with $\varepsilon = E/\omega$, whose eigenfunctions are given in terms of the Hermite polynomials

$$v_n = C_n e^{-\xi^2/2} H_n(\xi), \tag{4.8}$$

and the corresponding eigenvalues are

$$\varepsilon_n = n + \frac{1}{2}, \quad E_n = \omega\left(n + \frac{1}{2}\right) \quad (n = 0, 1, 2, \ldots). \tag{4.9}$$

Thus the normalized wave functions of our shifted oscillator (3.8) are given by

$$\psi_n(x,t) = e^{-i\omega(n+1/2)t} \varphi_n(x), \tag{4.10}$$

where

$$\varphi_n(x) = C_n \exp\left(\frac{i\lambda x^2}{2\omega_0}\right) e^{-\xi^2/2} H_n(\xi), \quad \xi = x\sqrt{\frac{\omega}{\omega_0}} \tag{4.11}$$

and

$$\left|C_n\right|^2 = \sqrt{\frac{\omega}{\omega_0}} \frac{1}{\sqrt{\pi} 2^n n!} \tag{4.12}$$

in view of the orthogonality relation

$$\int_{-\infty}^{\infty} \varphi_n^*(x) \varphi_m(x)\, dx = \delta_{nm}. \tag{4.13}$$

We use the star for complex conjugate.

Solution of the initial value problem (2.15) can be found by the superposition principle in the form

$$\psi(x,t) = \sum_{n=0}^{\infty} c_n\, \psi_n(x,t), \tag{4.14}$$

where

$$\psi(x,0) = \chi(x) = \sum_{n=0}^{\infty} c_n\, \varphi_n(x) \tag{4.15}$$

and

$$c_n = \int_{-\infty}^{\infty} \varphi_n^*(y) \chi(y)\, dy \tag{4.16}$$

in view of the orthogonality property (4.13). Substituting (4.16) into (4.14) and changing the order of the summation and integration, one gets

$$\psi(x,t) = \int_{-\infty}^{\infty} G(x,y,t) \chi(y)\, dy, \tag{4.17}$$

where the Green function is given as the eigenfunction expansion:

$$G(x,y,t) = \sum_{n=0}^{\infty} e^{-i\omega(n+1/2)t} \varphi_n(x) \varphi_n^*(y). \tag{4.18}$$

This infinite series is summable with the help of the Poisson kernel for the Hermite polynomials (Mehler's formula) [30]:

$$\sum_{n=0}^{\infty} \frac{H_n(x) H_n(y)}{2^n n!} r^n = \frac{1}{\sqrt{1-r^2}} \exp\left(\frac{2xyr - \left(x^2+y^2\right) r^2}{1-r^2}\right), \quad |r| < 1. \tag{4.19}$$

The result is given, of course, by equation (3.9).

5 The factorization method for shifted harmonic oscillator

It is worth applying the well-known factorization technique (see, e.g., [2, 3, 4, 10, 26]) to the Hamiltonian (3.7). The corresponding ladder operators can be found in the forms

$$a = (\alpha + i\beta)x + \gamma\frac{\partial}{\partial x}, \tag{5.1}$$

$$a^\dagger = (\alpha - i\beta)x - \gamma\frac{\partial}{\partial x}, \tag{5.2}$$

where α, β, and γ are real numbers to be determined as follows. One gets

$$aa^\dagger\psi = (\alpha^2+\beta^2)x^2\psi + (\alpha - i\beta)\gamma\psi - 2i\beta\gamma x\frac{\partial\psi}{\partial x} - \gamma^2\frac{\partial^2\psi}{\partial x^2}, \tag{5.3}$$

$$a^\dagger a\psi = (\alpha^2+\beta^2)x^2\psi - (\alpha + i\beta)\gamma\psi - 2i\beta\gamma x\frac{\partial\psi}{\partial x} - \gamma^2\frac{\partial^2\psi}{\partial x^2}, \tag{5.4}$$

whence

$$\left(aa^\dagger - a^\dagger a\right)\psi = 2\alpha\gamma\psi \tag{5.5}$$

and

$$\frac{1}{2}\left(aa^\dagger + a^\dagger a\right)\psi = -\gamma^2\frac{\partial^2\psi}{\partial x^2} + \left(\alpha^2+\beta^2\right)x^2\psi - i\beta\gamma\left(2x\frac{\partial\psi}{\partial x} + \psi\right). \tag{5.6}$$

The canonical commutation relation occurs and the Hamiltonian (3.7) takes the standard form

$$H = \frac{\omega}{2}\left(aa^\dagger + a^\dagger a\right), \tag{5.7}$$

if

$$2\alpha\gamma = 1, \quad \omega\left(\alpha^2+\beta^2\right) = \omega\gamma^2 = \frac{1}{2}\omega_0, \quad \omega\beta\gamma = -\frac{1}{2}\lambda. \tag{5.8}$$

The relation $\omega_0^2 = \omega^2 + \lambda^2$, which defines the new oscillator frequency, holds. As a result, the explicit form of the annihilation and creation operators is given by

$$\sqrt{2}a = \left(\sqrt{\frac{\omega}{\omega_0}} - \frac{i\lambda}{\sqrt{\omega_0\omega}}\right)x + \sqrt{\frac{\omega_0}{\omega}}\frac{\partial}{\partial x}, \tag{5.9}$$

$$\sqrt{2}a^\dagger = \left(\sqrt{\frac{\omega}{\omega_0}} + \frac{i\lambda}{\sqrt{\omega_0\omega}}\right)x - \sqrt{\frac{\omega_0}{\omega}}\frac{\partial}{\partial x}. \tag{5.10}$$

The special case $\lambda = 0$ and $\omega = \omega_0$ gives a traditional form of these operators.

The oscillator spectrum (4.9) and the corresponding stationary wave functions (4.11) can be obtained now in a standard way by using the Heisenberg-Weyl algebra of the rasing and lowering operators. In addition, the n-dimensional oscillator wave functions form a basis of the irreducible unitary representation of the Lie algebra of the noncompact group SU$(1,1)$ corresponding to the discrete positive series $\mathcal{D}_+^j$; see [25, 28, 33]. Our operators (5.9)–(5.10) allow us to extend these group-theoretical properties for the case of the shifted oscillators. We leave the details to the reader.

6 Dynamics of energy-related expectation values

The expectation value of an operator A in quantum mechanics is given by the formula

$$\langle A\rangle = \int_{-\infty}^{\infty} \psi^*(x,t)\, A(t)\psi(x,t)\, dx, \tag{6.1}$$

where the wave function satisfies the time-dependent Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = H\psi. \tag{6.2}$$

The time derivative of this expectation value can be written as

$$i\frac{d}{dt}\langle A\rangle = i\left\langle \frac{\partial A}{\partial t}\right\rangle + \left\langle AH - H^\dagger A\right\rangle, \tag{6.3}$$

where $H^\dagger$ is the Hermitian adjoint of the Hamiltonian operator H. Our formula is a simple extension of the well-known expression [21, 26, 32] to the case of a non-self-adjoint Hamiltonian.

We apply formula (6.3) to the Hamiltonian

$$H = \frac{\omega_0}{2}\left(p^2 + x^2\right) - \lambda px, \quad p = -i\frac{\partial}{\partial x} \tag{6.4}$$

in equation (2.1). A few examples will follow. In the case of the identity operator $A = 1$, one gets

$$AH - H^\dagger A = \lambda(xp - px) = i\lambda \tag{6.5}$$

by the Heisenberg commutation relation

$$[x,p] = xp - px = i. \tag{6.6}$$

As a result,

$$\frac{d}{dt}\|\psi\|^2 = \lambda\|\psi\|^2, \tag{6.7}$$

and time evolution of the squared norm of the wave function for our model of the damped quantum oscillator is given by equation (2.9).

In a similar fashion, if $A = H$, then

$$H^2 - H^\dagger H = \left(H - H^\dagger\right)H = i\lambda H, \tag{6.8}$$

and

$$\frac{d}{dt}\langle H\rangle = \lambda\langle H\rangle, \quad \langle H\rangle = \langle H\rangle_0 e^{\lambda t}. \tag{6.9}$$

Moreover,

$$\frac{d}{dt}\left\langle H^n\right\rangle = \lambda\left\langle H^n\right\rangle, \quad \left\langle H^n\right\rangle = \left\langle H^n\right\rangle_0 e^{\lambda t} \quad \left(n = 0, 1, 2, \dots\right), \tag{6.10}$$

which unifies both of the previous cases.

Now we choose $A = p^2$, $A = x^2$ and $A = px + xp$, respectively, in order to obtain the following system:

$$\begin{aligned} &\frac{d}{dt}\langle p^2\rangle = 3\lambda\langle p^2\rangle - \omega_0\langle px + xp\rangle, \\ &\frac{d}{dt}\langle x^2\rangle = -\lambda\langle x^2\rangle + \omega_0\langle px + xp\rangle, \\ &\frac{d}{dt}\langle px + xp\rangle = 2\omega_0\big(\langle p^2\rangle - \langle x^2\rangle\big) + \lambda\langle px + xp\rangle. \end{aligned} \tag{6.11}$$

Indeed,

$$p^2 H - H^\dagger p^2 = \frac{\omega_0}{2}\left[p^2, x^2\right] + \lambda\left[x, p^3\right] = 3i\lambda p^2 - i\omega_0(px + xp), \tag{6.12}$$

$$x^2 H - H^\dagger x^2 = \frac{\omega_0}{2}\left[x^2, p^2\right] - \lambda x[x,p]x = i\omega_0(px + xp) - i\lambda x^2, \tag{6.13}$$

and

$$\begin{aligned} &(px + xp)H - H^\dagger(px + xp) \\ &\qquad = \frac{\omega_0}{2}\left(\left[p, x^3\right] + \left[x, p^3\right]\right) + \frac{\omega_0}{2}\big(p[x,p]p - x[x,p]x\big) + \lambda\big((xp)^2 - (px)^2\big) \\ &\qquad = 2i\omega_0\big(p^2 - x^2\big) + i\lambda(px + xp), \end{aligned} \tag{6.14}$$

which results in (6.11).

The system can be solved explicitly, thus providing the complete dynamics of these expectation values. The eigenvalues are given by $r_0 = \lambda$, $r_\pm = \lambda \pm 2i\omega$ and the corresponding linearly independent eigenvectors are

$$\boldsymbol{x}_0 = \begin{pmatrix} \omega_0 \\ \omega_0 \\ 2\lambda \end{pmatrix}, \quad \boldsymbol{x}_\pm = \begin{pmatrix} (\lambda \pm i\omega)^2 \\ \omega_0^2 \\ 2\omega_0(\lambda \pm i\omega) \end{pmatrix} \tag{6.15}$$

with the determinant

$$\begin{vmatrix} \omega_0 & (\lambda + i\omega)^2 & (\lambda - i\omega)^2 \\ \omega_0 & \omega_0^2 & \omega_0^2 \\ 2\lambda & 2\omega_0(\lambda + i\omega) & 2\omega_0(\lambda - i\omega) \end{vmatrix} = -8i\omega_0^2\omega^3 \neq 0. \tag{6.16}$$

The general solution of the system (6.11) can be obtained in a complex form

$$\begin{aligned} \begin{pmatrix} \langle p^2\rangle \\ \langle x^2\rangle \\ \langle px + xp\rangle \end{pmatrix} &= C_0 e^{\lambda t}\begin{pmatrix} \omega_0 \\ \omega_0 \\ 2\lambda \end{pmatrix} + C_+ e^{(\lambda + 2i\omega)t}\begin{pmatrix} (\lambda + i\omega)^2 \\ \omega_0^2 \\ 2\omega_0(\lambda + i\omega) \end{pmatrix} \\ &\quad + C_- e^{(\lambda - 2i\omega)t}\begin{pmatrix} (\lambda - i\omega)^2 \\ \omega_0^2 \\ 2\omega_0(\lambda - i\omega) \end{pmatrix}, \end{aligned} \tag{6.17}$$

where C_0 and $C_\pm$ are constants. The corresponding solution of the initial value problem is given by

$$
\begin{aligned}
\begin{pmatrix} \langle p^2\rangle \\ \langle x^2\rangle \\ \langle px+xp\rangle \end{pmatrix} &= \frac{1}{2\omega^2}\Big(\omega_0\big(\langle p^2\rangle_0+\langle x^2\rangle_0\big)-\lambda\langle px+xp\rangle_0\Big)e^{\lambda t}\begin{pmatrix}\omega_0\\ \omega_0\\ 2\lambda\end{pmatrix} \\
&+\frac{1}{2\omega^2}\left(\frac{\lambda}{\omega_0}\langle px+xp\rangle_0+\frac{\omega^2-\lambda^2}{\omega_0^2}\langle x^2\rangle_0-\langle p^2\rangle_0\right) \\
&\times e^{\lambda t}\begin{pmatrix} \big(\lambda^2-\omega^2\big)\cos 2\omega t-2\lambda\omega\sin 2\omega t \\ \omega_0^2\cos 2\omega t \\ 2\lambda\omega_0\cos 2\omega t-2\omega_0\omega\sin 2\omega t\end{pmatrix} \\
&+\frac{1}{2\omega_0\omega}\left(\langle px+xp\rangle_0-\frac{2\lambda}{\omega_0}\langle x^2\rangle_0\right) \\
&\times e^{\lambda t}\begin{pmatrix} 2\lambda\omega\cos 2\omega t+\big(\lambda^2-\omega^2\big)\sin 2\omega t \\ \omega_0^2\sin 2\omega t \\ 2\omega_0\omega\cos 2\omega t+2\lambda\omega_0\sin 2\omega t\end{pmatrix}.
\end{aligned}
\tag{6.18}
$$

The mechanical energy operator E can be conveniently introduced as the Hamiltonian of our shifted linear harmonic oscillator (3.7):

$$E=H_0=\frac{\omega_0}{2}\big(p^2+x^2\big)-\frac{\lambda}{2}(px+xp), \tag{6.19}$$

so that

$$H=H_0+i\frac{\lambda}{2}. \tag{6.20}$$

Then

$$
\begin{aligned}
\frac{d}{dt}\langle E\rangle &= \frac{\omega_0}{2}\left(\frac{d}{dt}\langle p^2\rangle+\frac{d}{dt}\langle x^2\rangle\right)-\frac{\lambda}{2}\frac{d}{dt}\langle px+xp\rangle \\
&=\lambda\left\langle\frac{\omega_0}{2}\big(p^2+x^2\big)-\frac{\lambda}{2}(px+xp)\right\rangle
\end{aligned}
\tag{6.21}
$$

with the help of our system (6.11). Therefore,

$$\frac{d}{dt}\langle E\rangle=\lambda\langle E\rangle,\quad \langle E\rangle=\langle E\rangle_0e^{\lambda t} \tag{6.22}$$

for the expectation value of the mechanical energy of the damped oscillator under consideration.

The case of the second Hamiltonian,

$$H=\frac{\omega_0}{2}\big(p^2+x^2\big)-\lambda xp=H_0-i\frac{\lambda}{2}, \tag{6.23}$$

which is the Hermitian adjoint of the Hamiltonian (6.4), is similar. Here

$$H^{n+1}-H^\dagger H^n=\big(H-H^\dagger\big)H^n=\lambda[p,x]H^n=-i\lambda H^n$$

and

$$\frac{d}{dt}\langle H^n\rangle = -\lambda\langle H^n\rangle, \quad \langle H^n\rangle = \langle H^n\rangle_0 e^{-\lambda t} \quad (n=0,1,2,\ldots). \tag{6.24}$$

Moreover,

$$p^2H - H^\dagger p^2 = \frac{\omega_0}{2}\left[p^2,x^2\right] + \lambda p[x,p]p = i\lambda p^2 - i\omega_0(px+xp), \tag{6.25}$$

$$x^2H - H^\dagger x^2 = \frac{\omega_0}{2}\left[x^2,p^2\right] + \lambda\left[p,x^3\right] = -3i\lambda x^2 + i\omega_0(px+xp), \tag{6.26}$$

$$\begin{aligned}(px+xp)H - H^\dagger(px+xp) &= \frac{\omega_0}{2}\left(\left[p,x^3\right]+\left[x,p^3\right]\right)\\ &\quad+\frac{\omega_0}{2}\left(p[x,p]p - x[x,p]x\right) - \lambda\left((xp)^2-(px)^2\right)\\ &= 2i\omega_0\left(p^2-x^2\right) - i\lambda(px+xp),\end{aligned} \tag{6.27}$$

and the corresponding system has the form

$$\begin{aligned}&\frac{d}{dt}\langle p^2\rangle = \lambda\langle p^2\rangle - \omega_0\langle px+xp\rangle,\\ &\frac{d}{dt}\langle x^2\rangle = -3\lambda\langle x^2\rangle + \omega_0\langle px+xp\rangle,\\ &\frac{d}{dt}\langle px+xp\rangle = 2\omega_0\left(\langle p^2\rangle - \langle x^2\rangle\right) - \lambda\langle px+xp\rangle.\end{aligned} \tag{6.28}$$

The change $p\leftrightarrow x$, $\lambda\to-\lambda$, $\omega_0\to-\omega_0$ transforms formally this system back into (6.11). This observation allows us to obtain solution of the initial value problem from the previous solution given by (6.18). For the mechanical energy operator E introduced by equation (6.19), one gets

$$\frac{d}{dt}\langle E\rangle = -\lambda\langle E\rangle, \quad \langle E\rangle = \langle E\rangle_0 e^{-\lambda t} \tag{6.29}$$

with the help of (6.28).

The case of a general variable quadratic Hamiltonian of the form

$$H = a(t)p^2 + b(t)x^2 + c(t)px + d(t)xp, \tag{6.30}$$

where $a(t)$, $b(t)$, $c(t)$, $d(t)$ are real-valued functions of time only, is considered in a similar fashion. One gets

$$H^{n+1} - H^\dagger H^n = \left(H-H^\dagger\right)H^n = (c-d)[p,x]H^n = i(d-c)H^n \tag{6.31}$$

and

$$\frac{d}{dt}\langle H^n\rangle = \left\langle\frac{\partial H^n}{\partial t}\right\rangle + \left(d(t)-c(t)\right)\langle H^n\rangle. \tag{6.32}$$

The cases $n=0$ and $n=1$ result in

$$\langle 1\rangle = \langle 1\rangle_0 \exp\left(\int_0^t \left(d(\tau)-c(\tau)\right)d\tau\right) \tag{6.33}$$

and

$$\frac{d}{dt}\langle H\rangle = \left\langle \frac{\partial H}{\partial t} \right\rangle + \big(d(t) - c(t)\big)\langle H\rangle, \tag{6.34}$$

respectively.

Moreover,

$$p^2 H - H^\dagger p^2 = b\big[p^2, x^2\big] + c\big[p^3, x\big] + dp[p,x]p = -i(3c+d)p^2 - 2ib(px+xp), \tag{6.35}$$

$$x^2 H - H^\dagger x^2 = a\big[x^2, p^2\big] + cx[x,p]x + d\big[x^3, p\big] = i(3d+c)x^2 + 2ia(px+xp), \tag{6.36}$$

$$\begin{aligned}(px+xp)H - H^\dagger(px+xp) &= a\big(\big[x,p^3\big] + p[x,p]p\big) + b\big(\big[p,x^3\big] + x[p,x]x\big)\\ &\quad + (c-d)\big((px)^2 - (xp)^2\big)\\ &= 4iap^2 - 4ibx^2 - i(c-d)(px+xp),\end{aligned} \tag{6.37}$$

and the corresponding system has the form

$$\begin{aligned}\frac{d}{dt}\big\langle p^2\big\rangle &= -(3c+d)\big\langle p^2\big\rangle - 2b\langle px+xp\rangle,\\ \frac{d}{dt}\big\langle x^2\big\rangle &= (c+3d)\big\langle x^2\big\rangle + 2a\langle px+xp\rangle,\\ \frac{d}{dt}\langle px+xp\rangle &= 4a\big\langle p^2\big\rangle - 4b\big\langle x^2\big\rangle + (d-c)\langle px+xp\rangle.\end{aligned} \tag{6.38}$$

We have used the familiar identities

$$[x,p] = i, \quad (xp)^2 - (px)^2 = i(px+xp), \tag{6.39}$$

$$\big[x^2, p^2\big] = 2i(px+xp), \quad \big[x,p^3\big] = 3ip^2, \quad \big[x^3,p\big] = 3ix^2 \tag{6.40}$$

once again.

7 A relation with the classical damped oscillations

Application of formula (6.3) to the position x and momentum p operators allows to modify the Ehrenfest theorem [11, 26, 32] for the models of damped oscillators under consideration. For the Hamiltonian (6.4), one gets

$$xH - H^\dagger x = \frac{\omega_0}{2}\big[x, p^2\big] = i\omega_0 p, \tag{7.1}$$

$$pH - H^\dagger p = \frac{\omega_0}{2}\big[p, x^2\big] + \lambda\big[x, p^2\big] = -i\omega_0 x + 2i\lambda p \tag{7.2}$$

and

$$\frac{d}{dt}\langle x\rangle = \omega_0\langle p\rangle, \quad \frac{d}{dt}\langle p\rangle = -\omega_0\langle x\rangle + 2\lambda\langle p\rangle. \tag{7.3}$$

Elimination of the expectation value $\langle p\rangle$ from this system results in

$$\frac{d^2}{dt^2}\langle x\rangle - 2\lambda\frac{d}{dt}\langle x\rangle + \omega_0^2\langle x\rangle = 0, \tag{7.4}$$

which is a classical equation of motion for a damped oscillator [5, 20].

For the second Hamiltonian (6.23), we obtain

$$\frac{d}{dt}\langle x\rangle = \omega_0\langle p\rangle - 2\lambda\langle x\rangle, \qquad \frac{d}{dt}\langle p\rangle = -\omega_0\langle x\rangle, \tag{7.5}$$

which gives

$$\frac{d^2}{dt^2}\langle x\rangle + 2\lambda\frac{d}{dt}\langle x\rangle + \omega_0^2\langle x\rangle = 0 \tag{7.6}$$

in a similar fashion.

Finally, our model of the shifted harmonic oscillator (3.8), when the Hamiltonian is given by (6.19), results in

$$\frac{d^2}{dt^2}\langle x\rangle + \left(\omega_0^2 - \lambda^2\right)\langle x\rangle = 0. \tag{7.7}$$

We leave the details to the reader.

8 The third model

For the time-dependent Schrödinger equation with variable quadratic Hamiltonian:

$$i\frac{\partial\psi}{\partial t} = \frac{\omega_0}{2}\left(-e^{-2\lambda t}\frac{\partial^2\psi}{\partial x^2} + e^{2\lambda t}x^2\psi\right), \tag{8.1}$$

where $a = (\omega_0/2)e^{-2\lambda t}$, $b = (\omega_0/2)e^{2\lambda t}$ and $c = d = 0$, the characteristic equation takes the form (2.2) with the same solution (2.3). The corresponding propagator has the form (1.2) with

$$\alpha(t) = \frac{\omega\cos\omega t - \lambda\sin\omega t}{2\omega_0\sin\omega t}e^{2\lambda t}, \tag{8.2}$$

$$\beta(t) = -\frac{\omega}{\omega_0\sin\omega t}e^{\lambda t}, \tag{8.3}$$

$$\gamma(t) = \frac{\omega\cos\omega t + \lambda\sin\omega t}{2\omega_0\sin\omega t}. \tag{8.4}$$

This can be derived directly from equations (1.2)–(1.8) with the help of identity (2.8). We leave the details to the reader. It is worth noting that equation (8.1) can be obtained by introducing a variable unit of length $x \to xe^{\lambda t}$ in the Hamiltonian of the linear oscillator.

9 Momentum representation

The time-dependent Schrödinger equations for the damped oscillators are also solved in the momentum representation. One can easily verify that under the Fourier transform our first Hamiltonian (6.4) takes the form of the second Hamiltonian (6.23) with $\lambda \to -\lambda$ and vice versa (see, e.g., [9] for more details). Moreover, the inverses of the corresponding time evolution operators are obtained by the time reversal. Therefore, all identities of the commutative evolution diagram introduced in [9] for the modified oscillators are also valid for the quantum damped oscillators under consideration. We leave further details to the reader.

Acknowledgments

We thank professor Carlos Castillo-Chávez for support, valuable discussions, and encouragement. We are grateful to professor Valeriy N. Tolstoy for valuable comments and suggestions to improve the presentation, which were taken into account. One of the authors (R. Cordero-Soto) is supported by the following National Science Foundation programs: Louis Stokes Alliances for Minority Participation (LSAMP): NSF Cooperative Agreement No. HRD-0602425 (WAESO LSAMP Phase IV); Alliances for Graduate Education and the Professoriate (AGEP): NSF Cooperative Agreement No. HRD-0450137 (MGE@MSA AGEP Phase II).

References

[1] G. P. Arrighini and N. L. Durante. More on the quantum propagator of a particle in a linear potential. *Amer. J. Phys.*, **64** (1996), 1036–1041.

[2] R. Askey and S. K. Suslov. The q-harmonic oscillator and an analogue of the Charlier polynomials. *J. Phys. A*, **26** (1993), L693–L698.

[3] R. Askey and S. K. Suslov. The q-harmonic oscillator and the Al-Salam and Carlitz polynomials. *Lett. Math. Phys.*, **29** (1993), 123–132.

[4] N. M. Atakishiev and S. K. Suslov. Difference analogues of the harmonic oscillator. *Theoret. and Math. Phys.*, **85** (1990), 1055–1062.

[5] H. Bateman. *Partial Differential Equations of Mathematical Physics*. Dover Publications, New York, 1944.

[6] L. A. Beauregard. Propagators in nonrelativistic quantum mechanics. *Amer. J. Phys.*, **34** (1966), 324–332.

[7] L. S. Brown and Y. Zhang. Path integral for the motion of a particle in a linear potential. *Amer. J. Phys.*, **62** (1994), 806–808.

[8] R. Cordero-Soto, R. M. Lopez, E. Suazo, and S. K. Suslov. Propagator of a charged particle with a spin in uniform magnetic and perpendicular electric fields. *Lett. Math. Phys.*, **84** (2008), 159–178.

[9] R. Cordero-Soto and S. K. Suslov. The time inversion for modified oscillators. To appear in Theoret. and Math. Phys.

[10] Shi-H. Dong. *Factorization Method in Quantum Mechanics*. Springer-Verlag, Dordrecht, 2007.

[11] P. Ehrenfest. Bemerkung über die angenäherte Gültigkeit der klassischen Mechanik innerhalb der Quantenmechanik. *Z. Phys. A*, **45** (1927), 455–457.

[12] R. P. Feynman. The principle of least action in quantum mechanics. Ph.D. thesis, Princeton University, 1942. Reprinted in *"Feynman's Thesis – A New Approach to Quantum Theory"*. L. M. Brown (Ed.). World Scientific Publishers, Singapore, 2005, pp. 1–69.

[13] R. P. Feynman. Space-time approach to non-relativistic quantum mechanics. *Rev. Modern Physics*, **20** (1948), 367–387. Reprinted in *"Feynman's Thesis – A New Approach to Quantum Theory"*. L. M. Brown (Ed.). World Scientific Publishers, Singapore, 2005, pp. 71–112.

[14] R. P. Feynman. The theory of positrons. *Phys. Rev.*, **76** (1949), 749–759.

[15] R. P. Feynman. Space-time approach to quantum electrodynamics. *Phys. Rev.*, **76** (1949), 769–789.

[16] R. P. Feynman and A. R. Hibbs. *Quantum Mechanics and Path Integrals*. McGraw-Hill, New York, 1965.

[17] K. Gottfried and T.-M. Yan. *Quantum Mechanics: Fundamentals*. 2nd ed. Springer-Verlag, New York, 2004.

[18] B. R. Holstein. The linear potential propagator. *Amer. J. Phys.*, **65** (1997), 414–418.

[19] B. R. Holstein. The harmonic oscillator propagator. *Amer. J. Phys.*, **67** (1998), 583–589.

[20] L. D. Landau and E. M. Lifshitz. *Mechanics.* Pergamon Press, Oxford, 1976.

[21] L. D. Landau and E. M. Lifshitz. *Quantum Mechanics: Non-relativistic Theory.* Pergamon Press, Oxford, 1977.

[22] N. Lanfear and S. K. Suslov. The time-dependent Schrödinger equation, Riccati equation and Airy functions. Preprint arXiv:0903.3608v5 [math-ph], 2009.

[23] R. M. Lopez and S. K. Suslov. The Cauchy problem for a forced harmonic oscillator. Preprint arXiv:0707.1902v8 [math-ph], 2007.

[24] V. P. Maslov and M. V. Fedoriuk. *Semiclassical Approximation in Quantum Mechanics.* Reidel Publishing, Dordrecht, 1981.

[25] M. Meiler, R. Cordero-Soto, and S. K. Suslov. Solution of the Cauchy problem for a time-dependent Schrödinger equation. *J. Math. Phys.*, **49** (2008), 072102 (27 pages).

[26] E. Merzbacher. *Quantum Mechanics.* 3rd ed. John Wiley & Sons, New York, 1998.

[27] P. Nardone. Heisenberg picture in quantum mechanics and linear evolutionary systems. *Amer. J. Phys.*, **61** (1993), 232–237.

[28] A. F. Nikiforov, S. K. Suslov, and V. B. Uvarov. *Classical Orthogonal Polynomials of a Discrete Variable.* Springer-Verlag, Berlin, 1991.

[29] A. F. Nikiforov and V. B. Uvarov. *Special Functions of Mathematical Physics.* Birkhäuser Verlag, Basel, 1988.

[30] E. D. Rainville. *Special Functions.* The Macmillan Co., New York, 1960.

[31] R. W. Robinett. Quantum mechanical time-development operator for the uniformly accelerated particle. *Amer. J. Phys.*, **64** (1996), 803–808.

[32] L. I. Schiff. *Quantum Mechanics.* 3rd ed. McGraw-Hill, New York, 1968.

[33] Yu. F. Smirnov and K. V. Shitikova. The method of K harmonics and the shell model. *Sov. J. Part. Nucl.*, **8** (1977), 344–370.

[34] E. Suazo and S. K. Suslov. An integral form of the nonlinear Schrödinger equation with wariable coefficients. Preprint arXiv:0805.0633v2 [math-ph], 2008.

[35] E. Suazo and S. K. Suslov. Cauchy problem for Schrödinger equation with variable quadratic Hamiltonians. In preparation.

[36] E. Suazo, S. K. Suslov, and J. M. Vega. The Riccati differential equation and a diffusion-type equation. Preprint arXiv:0807.4349v4 [math-ph], 2008.

[37] S. K. Suslov and B. Trey. The Hahn polynomials in the nonrelativistic and relativistic Coulomb problems. *J. Math. Phys.*, **49** (2008), 012104 (51 pages).

[38] N. S. Thomber and E. F. Taylor. Propagator for the simple harmonic oscillator. *Amer. J. Phys.*, **66** (1998), 1022–1024.

Received June 22, 2009
Revised September 12, 2009