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Hecke pairs of ergodic discrete measured equivalence relations and the Schlichting completion

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Abstract. It is shown that for each Hecke pair of ergodic discrete measured equivalence relations, there exists a Hecke pair of groups determined by an index cocycle associated with the given pair. We clarify that the construction of these groups can be viewed as a generalization of a notion of Schlichting completion for a Hecke pair of groups, and show that the index cocycle cited above arises from "adjusted" choice functions for the equivalence relations. We prove also that there exists a special kind of choice functions, *preferable choice functions*, having the property that the restriction of the corresponding index cocycle to the ergodic subrelation is minimal in the sense of Zimmer. It is then proved that the Hecke von Neumann algebra associated with the Hecke pair of groups obtained above is $*$-isomorphic to the Hecke von Neumann algebra associated with the Hecke pair of equivalence relations with which we start.

1. Introduction.

The present authors initiated in [**3**] an intensive study on a certain type of inclusions of ergodic discrete measured equivalence relations which produce, via the famous Feldman–Moore construction ([**9**]), *discrete* inclusions of factors in the sense of Izumi–Longo–Popa ([**13**]). Later, it was recognized in [**4**] that this special kind of pair of ergodic equivalence relations resembles what is called a *Hecke pair* in group theory. A pair of a group G and a subgroup H of G is said to be a Hecke pair if the subgroup $\{g \in G;\ [H : H \cap g^{-1}Hg] < \infty,\ [H : H \cap gHg^{-1}] < \infty\}$ coincides with the whole G. Once a Hecke pair (G, H) is given, one can construct out of it a von Neumann algebra $W^*(G, H)$, called the Hecke von Neumann algebra of (G, H). One can also construct a C*-algebra, whose theory has attracted many operator-algebraists. To each pair $(\mathcal{R}, \mathcal{S})$ of ergodic equivalence relations of the type mentioned above, we can similarly associate a von Neumann algebra $\mathcal{H}^*(\mathcal{R}, \mathcal{S})$ too (see [**4**, Section 9]), in such a way that if $(\mathcal{R}, \mathcal{S})$ happens to be of the form $(G \ltimes \mathcal{P}, H \ltimes \mathcal{P})$ for some ergodic equivalence relation $\mathcal{P}$ and a Hecke pair (G, H) of groups acting "nicely" on $\mathcal{P}$, then $\mathcal{H}^*(\mathcal{R}, \mathcal{S})$ is precisely the Hecke von Neumann algebra of the Hecke pair (G, H). Because of this, such a pair $(\mathcal{R}, \mathcal{S})$ was also termed a Hecke pair in [**4**]. That a Hecke pair of equivalence relations can be viewed in some sense as a generalization of a Hecke pair of groups was verified also in [**1**], by showing that $(\mathcal{R}, \mathcal{S})$ is a Hecke pair if and only if it admits a distinctive set of choice

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functions. Hence we believe at this point that it is worthwhile to further investigate this analogy between Hecke pairs of groups and Hecke pairs of equivalence relations. This is exactly what we do in this paper. Our first project along this line of investigation is to study the Schlichting completion of a Hecke pair of groups. The Schlichting completion is, as the term suggests, regarded roughly as a completion of a (countable) group. It is known (see [**14**] or [**21**] for example) that for any (reduced) Hecke pair (G, H) of groups, there exists a unique pair $(\widetilde{G}, \widetilde{H})$, up to isomorphism, in which

- $\widetilde{G}$ is a totally disconnected locally compact group;
- $\widetilde{H}$ is a compact open subgroup of $\widetilde{G}$, such that
- $\bigcap_{g\in\widetilde{G}} g^{-1}\widetilde{H}g = \{e\}$ (i.e., $(\widetilde{G}, \widetilde{H})$ is reduced);
- there is an (injective) group homomorphism $\theta\colon G \to \widetilde{G}$ satisfying:
  - $\theta(G)$ is dense in $\widetilde{G}$;
  - $\theta^{-1}(\theta(G) \cap \widetilde{H}) = H$.

The main purpose of this article is to prove that

(1) there does exist a measure-theoretical counterpart of the Schlichting completion $(\mathscr{G}, K)$ in the case of a Hecke pair $(\mathcal{R}, \mathcal{S})$ of ergodic equivalence relations (Theorem 7.1 and Proposition 7.5);

(2) the Hecke von Neumann algebra $W^*(\mathscr{G}, K)$ associated with the Schlichting completion $(\mathscr{G}, K)$ exactly coincides with the von Neumann algebra which appears in the tower of relative commutans of the inclusion of factors $W^*(\mathcal{R}) \supseteq W^*(\mathcal{S})$ (Theorem 10.3).

The organization of this paper is as follows.

In Section 2, we introduce notation and terminology used in this paper.

As already cited above, the first author gave in [**1**] a nice characterization for an inclusion $\mathcal{S} \subseteq \mathcal{R}$ of ergodic equivalence relations to be a Hecke pair from the viewpoint of the choice functions it produces. Although the original definition of a Hecke pair quite involves operator-algebraic arguments, his characterization is purely measure-theoretical. It often provides us with a useful insight into how the subrelation $\mathcal{S}$ sits inside of $\mathcal{R}$. In Section 3, we will give yet another characterization of a Hecke pair in terms of the choice functions. The property we are interested in is the one that Kaiszewski, Landstad and Quigg focused on in [**14**], where they discuss the Schlichting completion of a Hecke pair of (countable) groups from their viewpoint.

In Section 4, as an application of the result obtained in the previous section, we shall prove that the asymptotic range $r^*(\sigma)$ of some index cocycle σ for $\mathcal{S} \subseteq \mathcal{R}$ and the asymptotic range $r^*(\sigma|_{\mathcal{S}})$ of the restriction $\sigma|_{\mathcal{S}}$ together form a Hecke pair of groups.

In Section 5, we will show that if $(\mathcal{R}, \mathcal{S})$ is a Hecke pair of ergodic equivalence relations, then we can always choose a set of choice functions for $\mathcal{S} \subseteq \mathcal{R}$ so that the restriction of the resulting index cocycle to $\mathcal{S}$ is a cocycle into a compact group which is minimal in the sense of Zimmer. We say that a set of choice functions is preferable if it enjoys the property described above. We believe that preferable choice functions are the most *natural* choice functions for Hecke pairs of ergodic measured discrete equivalence relations.

In Section 6, starting with preferable choice functions $\{\psi_i\}_{i\in I}$ for a Hecke pair $(\mathcal{R},\mathcal{S})$, we will construct a pair $(\mathscr{G}(\sigma),K(\sigma))$ of two subsets in the Polish group $\mathrm{Per}(I)$ of all the bijections on the index set I; $\mathscr{G}(\sigma)$ is a locally compact group with the relative topology and $K(\sigma)$ is a compact and open subgroup of $\mathrm{Per}(I)$. We call the pair $(\mathscr{G}(\sigma),K(\sigma))$ a Schlichting completion of the Hecke pair $(\mathcal{R},\mathcal{S})$. This will answer the above-mentioned question as to whether there is a counterpart of the Schlichting completion in the case of a Hecke pair $(\mathcal{R},\mathcal{S})$ of equivalence relations.

In Section 7, it is shown that the pair $(\mathscr{G}(\sigma),K(\sigma))$ constructed in the previous section does not depend on the choice of preferable choice functions, *up to conjugacy in* $\mathrm{Per}(I)$ (Theorem 7.1). We will also demonstrate how any two sets of preferable choice functions are related to each other (Theorem 7.2). Although the construction performed in Section 6 seems different from that considered in Section 4, it will be proved that $(r^*(\sigma),r^*(\sigma|_{\mathcal{S}}))$ coincides with $(\mathscr{G}(\sigma),K(\sigma))$.

In Section 8, we revisit the construction of the Hecke von Neumann algebra $\mathcal{H}^*(\mathcal{R},\mathcal{S})$ associated with a Hecke pair $(\mathcal{R},\mathcal{S})$ of equivalence relations considered in [**4**].

In Section 9, we will briefly review the construction of the Hecke von Neumann algebra $W^*(G,K)$ associated with a Hecke pair (G,K) of groups.

In Section 10, starting with a Hecke pair $(\mathcal{R},\mathcal{S})$ of equivalence relations, we will prove that the von Neumann algebra $\mathcal{H}^*(\mathcal{R},\mathcal{S})$ is $*$-isomorphic to the Hecke von Neumann algebra $W^*(\mathscr{G}(\sigma),K(\sigma))$. Since it was shown in [**4**] that $\mathcal{H}^*(\mathcal{R},\mathcal{S})$ is realized in the tower of relative commutans of the inclusion of factors $W^*(\mathcal{R})\supseteq W^*(\mathcal{S})$, one finds that the Hecke von Neumann algebra $W^*(\mathscr{G}(\sigma),K(\sigma))$ is independent of the choice of a set of choice functions for $\mathcal{S}\subseteq\mathcal{R}$.

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2. Preliminaries.

In this section, we introduce symbols that will be repeatedly used in the whole of this paper. We also collect basic facts about discrete measured equivalence relations and the Jones' basic extension of an inclusion of factors, which are necessary for our later discussion. The readers are referred to [**3**], [**8**], [**9**], [**10**], [**13**] regarding these topics.

We assume that all von Neumann algebras in this paper have separable preduals.

For a faithful normal semifinite weight ϕ on a von Neumann algebra M, we set

$$\mathfrak{n}_\phi:=\{x\in M\colon \phi(x^*x)<\infty\},\qquad \mathfrak{m}_\phi:=\mathfrak{n}_\phi^*\mathfrak{n}_\phi,\qquad \mathfrak{m}_\phi^+:=\mathfrak{m}_\phi\cap M_+.$$

More generally, for an operator valued weight T ([**20**]) from a von Neumann algebra M to a von Neumann subalgebra N, we set

$$\mathfrak{n}_T:=\{x\in M\colon T(x^*x)\in N_+\},\qquad \mathfrak{m}_T:=\mathfrak{n}_T^*\mathfrak{n}_T,\qquad \mathfrak{m}_T^+:=\mathfrak{m}_T\cap M_+.$$

The Hilbert space obtained from ϕ by the GNS-construction will be denoted by H_ϕ, and we let $\Lambda_\phi\colon \mathfrak{n}_\phi \to H_\phi$ stand for the natural injection.

2.1. Discrete measured equivalence relations.

Throughout this paper, we fix a discrete measured equivalence relation $\mathcal{R}$ on a standard probability space $(X, \mathfrak{B}, \mu)$ in which μ is quasi-invariant for $\mathcal{R}$. We denote by ν the (σ-finite) measure on $\mathcal{R}$ given by

$$\nu(E) := \int_X |r^{-1}(\{x\}) \cap E| \, d\mu(x) \qquad (E\text{: Borel subset of } \mathcal{R}),$$

where $r\colon \mathcal{R} \to X$ is the projection onto the first coordinate, and $|S|$ in general stands for the cardinality of a (countable) set S. Replacing r by the map $s\colon \mathcal{R} \to X$ given by $s(x, y) = y$ on the right-hand side of the formula above, we obtain another (σ-finite) measure on $\mathcal{R}$, which we denote by ν^{-1}. The measures ν and ν^{-1} are absolutely continuous with each other, and the Radon–Nikodym derivative $d\nu/d\nu^{-1}$, called the module of $\mathcal{R}$, will be denoted by δ.

We also fix a (normalized) Borel 2-cocycle ω from $\mathcal{R}$ into the one-dimensional torus $\mathbf{T}$ in what follows. We then write $W^*(\mathcal{R}, \omega)$ for the von Neumann algebra on the Hilbert space $L^2(\mathcal{R}, \nu)$ obtained by the Feldman–Moore construction ([**9**]) from $\mathcal{R}$ and ω.

We define $[\mathcal{R}]_*$ to be the set of all bimeasurable nonsingular transformations ρ from a Borel subset $\mathrm{Dom}(\rho)$ of X onto a Borel subset $\mathrm{Im}(\rho)$ of X satisfying $(x, \rho(x)) \in \mathcal{R}$ for μ-a.e. $x \in \mathrm{Dom}(\rho)$.

For a Borel 1-cocycle c from $\mathcal{R}$ into a Polish group K (i.e., a separable, completely metrizable topological group K), the essential range of c is the smallest closed subset $\sigma(c)$ of K such that $c^{-1}(\sigma(c))$ has complement of measure zero. The asymptotic range $r^*(c)$ of c is by definition

$$\bigcap \{\sigma(c_B)\colon B \in \mathfrak{B} \text{ and } \mu(B) > 0\},$$

where c_B stands for the restriction of c to the reduction $\mathcal{R} \cap (B \times B)$ It is known that $r^*(c)$ is a closed subgroup of K.

Assume now that $\mathcal{R}$ is ergodic. Let $\mathcal{S}$ be a Borel subrelation of $\mathcal{R}$. By [**10**], we may choose a countable family $\{\psi_i\}_{i\in I}$ of Borel maps from X into itself such that (i) $(x, \psi_i(x)) \in \mathcal{R}$ for all $i \in I$ and μ-a.e. $x \in X$; (ii) for μ-a.e. $x \in X$, $\{\mathcal{S}(\psi_i(x))\}_{i\in I}$ is a partition of $\mathcal{R}(x)$, where $\mathcal{R}(x) := \{y \in X\colon (x, y) \in \mathcal{R}\}$. The family $\{\psi_i\}_{i\in I}$ is called *choice functions* for $\mathcal{S} \subseteq \mathcal{R}$ ([**10**]). Unless otherwise mentioned, we always agree that I equals $\{0, 1, \ldots, N-1\}$ if I is finite, or equals $\{0, 1, 2, \ldots\}$ when I is infinite, and that $\psi_0 = id_X$. Once choice functions $\{\psi_i\}_{i\in I}$ are fixed, we can define the *index cocycle* $\sigma\colon \mathcal{R} \to \mathrm{Per}(I)$ of the pair $\mathcal{S} \subseteq \mathcal{R}$, where $\mathrm{Per}(I)$ denotes the set of all bijections on I, by the following rule:

$$\sigma(x, y)(i) = j \quad \Longleftrightarrow \quad (\psi_i(y), \psi_j(x)) \in \mathcal{S}.$$

For any $i \in I$, define

$$\mathcal{C}_i := \{(x, y) \in \mathcal{R}\colon \exists z \in X \text{ s.t.} (x, z) \in \mathcal{S} \text{ and } (\psi_i(z), y) \in \mathcal{S}\}.$$

By definition, every $\mathcal{C}_i$ is an $\mathcal{S}$-invariant Borel subset. Clearly, we have $\mathcal{R} = \bigcup_{i\in I} \mathcal{C}_i$.

2.2. Basic extension.

Let $B \subseteq A$ be an inclusion of factors with a faithful normal conditional expectation E_B. (In our situation considered in the following sections, such an expectation always exists uniquely.) Fix a faithful normal state ϕ_0 on B and set $\phi := \phi_0 \circ E_B$. Then the equation $e_B\Lambda_\phi(a) := \Lambda_\phi(E_B(a))$ defines a projection $e_B \in B(H_\phi)$ onto $[\Lambda_\phi(B)]$, where $[S]$ is in general the closed subspace spanned by a set S. We call e_B the Jones projection of the inclusion $B \subseteq A$. *The basic extension* of this inclusion (by E_B) is the factor, denoted by A_1, acting on H_ϕ generated by A and e_B. It is known that $A_1 = J_\phi B' J_\phi$, where J_ϕ is the modular conjugation of ϕ.

According to [**15**] (see also [**13**, Section 2]), there exists a faithful normal semifinite operator valued weight $\widehat{E_B}$, called the operator valued weight dual to E_B, from A_1 to A. It satisfies $\widehat{E_B}(e_B) = 1$ [**15**, Lemma 3.1], so that $Ae_BA \subseteq \mathfrak{m}_{\widehat{E_B}}$. The inclusion $B \subseteq A$ is said to be *discrete* (see [**13**, Definition 3.7]) if $\widehat{E_B}|_{A_1\cap B'}$ is semifinite.

2.3. Hecke pairs of ergodic equivalence relations.

Let us assume that our discrete equivalence relation $\mathcal{R}$ is ergodic, and consider the factor $A := W^*(\mathcal{R}, \omega)$ on the Hilbert space $L^2(\mathcal{R}, \nu)$ for some 2-cocycle ω. We also consider an ergodic Borel subrelation $\mathcal{S}$ of $\mathcal{R}$ and its associated subfactor $B := W^*(\mathcal{S}, \omega)$ of A. There exists a unique faithful normal conditional expectation E_B from A onto B. According to [**3**], the *commensurability groupoid* $\mathcal{CG}(B)$ of B in A is by definition the set of all partial isometries $v \in A$ satisfying the following two conditions:

- Both v^*v and vv^* belong to B.
- The projections z_v and z_{v^*} belong to $\mathfrak{m}^+_{\widehat{E_B}}$, where, for an element $a \in A$, z_a denotes the projection onto $[BaB\xi_0]$ which belongs to $A_1 \cap B'$. Here ξ_0 is the characteristic function of the diagonal set $\{(x, x) \colon x \in X\}$.

It is shown among others in [**3**, Theorem 7.1] that the inclusion $B \subseteq A$ is discrete in the sense explained in Subsection 2.2 if and only if the subfactor generated by $\mathcal{CG}(B)$ coincides with A. We say that $(\mathcal{R}, \mathcal{S})$ is a *Hecke pair* if $\mathcal{CG}(B)'' = A$, that is, $B \subseteq A$ is discrete. We refer the reader to [**1**] as well for a measure-theoretical approach to this notion.

3. Definition of a Hecke pair—revisited.

The purpose of this section is to give a characterization of an inclusion $\mathcal{S} \subseteq \mathcal{R}$ of ergodic equivalence relations being a Hecke pair in terms of the corresponding choice functions. We start with an ergodic $\mathcal{R}$ and an ergodic Borel subrelation $\mathcal{S}$ of $\mathcal{R}$.

3.1. Choice functions for $\mathcal{S} \subseteq \mathcal{R}$ when $(\mathcal{R}, \mathcal{S})$ is a Hecke pair.

We first assume that $(\mathcal{R}, \mathcal{S})$ is a Hecke pair. Thanks to [**1**, Theorem 3.8], we may select choice functions $\{\psi_i\}_{i\in I}$ for $\mathcal{S} \subseteq \mathcal{R}$ which satisfy the following:

(CF1) There exist a countable set Λ and natural numbers $\{n_\lambda\}_{\lambda\in\Lambda}$ such that the index set I is equal to $\{(\lambda, n) \colon \lambda \in \Lambda,\ n = 1, \ldots, n_\lambda\}$.

(CF2) The index $\mathrm{Ind}(\psi_{\lambda,n})$ of $\psi_{\lambda,n}$ ([**1**]) is equal to n_λ for each $(\lambda,n)\in I$.

(CF3) For each $\lambda\in\Lambda$ and $n,m\in\{1,\dots,n_\lambda\}$, $\mathcal{S}(\psi_{\lambda,n}(\mathcal{S}(x)))$ is equal to $\mathcal{S}(\psi_{\lambda,m}(\mathcal{S}(x)))$ for a.e. $x\in X$. Moreover, $\mathcal{S}(\psi_{\lambda,n}(\mathcal{S}(x)))$ is equal to a disjoint union of $\{\mathcal{S}(\psi_{\lambda,k}(x))\}_{k=1}^{n_\lambda}$ for a.e. $x\in X$.

Here refer to [**1**] for the definition and the basic properties of the index of a "nonsingular" map ρ. In what follows, we fix choice functions $\{\psi_{\lambda,n}\}$ as stated above.

Remark. By the proof of [**1**, Theorem 3.8], the natural numbers $\{n_\lambda\}_{\lambda\in\Lambda}$ stated above are determined as the values of the minimal projections in the relative commutant $A_1\cap B'$ under $\widehat{E_B}|_{A_1\cap B'}$ in the basic extension (see Subsection 2.2). It means that neither the countable set Λ nor the natural numbers $\{n_\lambda\}_{\lambda\in\Lambda}$ depends on the choice of a family of choice functions satisfying (CF1)–(CF3). Namely, if $\{\psi'_{\lambda',n'}\}_{I'}$ is another family of choice functions for $\mathcal{S}\subset\mathcal{R}$ with $I'=\{(\lambda',n')\colon \lambda'\in\Lambda',\ n'=1,\dots,n'_{\lambda'}\}$ satisfying (CF1)–(CF3), then there exists a bijection f from Λ' to Λ which satisfies $n'_{\lambda'}=n_{f(\lambda')}$ for each $\lambda'\in\Lambda'$.

Under this identification, we may and do assume that each family of choice functions in this paper are always indexed by $I=\{(\lambda,n)\colon \lambda\in\Lambda,\ n=1,\dots,n_\lambda\}$.

Let σ be the index cocycle associated with the choice functions $\{\psi_{\lambda,n}\}$. Let $\mathcal{C}_j$ ($j\in I$) be the $\mathcal{S}$-invariant set introduced in Section 2.1. So, under the situation we are now considering, we have, for each $(\lambda,n)\in I$:

$$\mathcal{C}_{\lambda,n}:=\{(x,y)\in\mathcal{R}\colon \exists z\in X \text{ s.t. } (x,z)\in\mathcal{S} \text{ and } (\psi_{\lambda,n}(z),y)\in\mathcal{S}\}.$$

As noted in [**4**, Section 8], we have that, for a.e. $(x,y)\in\mathcal{R}$ and $(\lambda,n)\in I$, (x,y) is in $\mathcal{C}_{\lambda,n}$ if and only if $y\in\bigcup_{m=1}^{n_\lambda}\mathcal{S}(\psi_{\lambda,m}(x))$. It is also true that $\mathcal{C}_{\lambda,n}$ is equal to $\mathcal{C}_{\lambda,m}$ up to a null set. Put $\mathcal{C}_\lambda:=\mathcal{C}_{\lambda,1}$. We note that $\{\mathcal{C}_\lambda\}_{\lambda\in\Lambda}$ is a measurable partition of $\mathcal{R}$.

Let $\lambda\in\Lambda$ and $(\lambda',n')\in I$. For any $x\in X$, define a subset $K^\lambda_{(\lambda',n')}(x)$ of I by

$$K^\lambda_{(\lambda',n')}(x):=\{(\lambda_1,n_1)\in I\colon (\psi_{\lambda',n'}(x),\psi_{\lambda_1,n_1}(x))\in\mathcal{C}_\lambda\}.$$

We regard this assignment $x\in X\mapsto K^\lambda_{(\lambda',n')}(x)$ as a map from X into the family 2^I of all subsets of I, where 2^I is equipped with the Fell topology ([**11**]) by viewing I as a discrete topological space. Note that 2^I is then a Polish space (see [**11**] or [**5**]).

By (the proof of) [**1**, Theorem 3.8], there exists a μ-null subset N_0 of X such that $\mathcal{R}(x)=\bigsqcup_{i\in I}\mathcal{S}(\psi_i(x))$ (disjoint union) and

$$\mathcal{S}(\psi_{\lambda_1,n_1}(\mathcal{S}(x)))=\bigsqcup_{k=1}^{n_{\lambda_1}}\mathcal{S}(\psi_{\lambda_1,k}(x)) \quad \text{for all } x\in N_0^c \text{ and all } (\lambda_1,n_1)\in I. \tag{3.1}$$

Put $N_1:=\bigcup_{i\in I}\psi_i^{-1}(N_0)$. Let $x\in N_1^c$. Then we have

$$\begin{aligned}&(\lambda_1,n_1)\in K^\lambda_{(\lambda',n')}(x)\\ &\qquad\Longleftrightarrow\quad (\psi_{\lambda',n'}(x),\psi_{\lambda_1,n_1}(x))\in\mathcal{C}_\lambda\end{aligned}$$

$$
\begin{aligned}
&\iff \quad \psi_{\lambda_1,n_1}(x) \in \mathcal{S}(\psi_{\lambda,1}(\mathcal{S}(\psi_{\lambda',n'}(x))))\\
&\iff \quad \exists k \in \{1,2,\dots,n_\lambda\} \text{ s.t. } (\psi_{\lambda_1,n_1}(x), \psi_{\lambda,k}(\psi_{\lambda',n'}(x))) \in \mathcal{S} \quad \text{(by Equation (3.1))}\\
&\iff \quad \exists k \in \{1,2,\dots,n_\lambda\} \text{ s.t. } (\lambda_1,n_1) = \sigma(x,\psi_{\lambda',n'}(x))(\lambda,k).
\end{aligned}
$$

Thus we get

$$
K^{\lambda}_{(\lambda',n')}(x) = \{\sigma(x,\psi_{\lambda',n'}(x))(\lambda,k) \colon k \in \{1,2,\dots,n_\lambda\}\}. \tag{3.2}
$$

In particular, we find that

$$
|K^{\lambda}_{(\lambda',n')}(x)| = n_\lambda \qquad \text{for all } x \in N_1^c,\ \lambda \in \Lambda \text{ and all } (\lambda',n') \in I.
$$

By [**1**, Remark], there exists a ν-null subset N_2 of X such that, with $\mathcal{S}_0 := \mathcal{S} \cap (N_2^c \times N_2^c)$, for any $(x,y) \in \mathcal{S}_0$ and for any $(\lambda,n) \in I$, there is a unique $m \in \{1,2,\dots,n_\lambda\}$ satisfying

$$
(\psi_{\lambda,n}(x), \psi_{\lambda,m}(y)) \in \mathcal{S}.
$$

Set $N_3 := \bigcup_{i\in I} \psi_i^{-1}(N_2)$ and $\underline{\mathrm{N}} := N_1 \cup N_3$. To sum up, we have the following.

Lemma 3.1. *There exists a μ-null subset $\underline{\mathrm{N}}$ of X such that*

(1) $\mathcal{R}(x) = \bigsqcup_{i\in I} \mathcal{S}(\psi_i(x))$ *and* $\mathcal{S}(\psi_{\lambda_1,n_1}(\mathcal{S}(x))) = \bigsqcup_{k=1}^{n_{\lambda_1}} \mathcal{S}(\psi_{\lambda_1,k}(x))$ *for all* $x \in \underline{\mathrm{N}}^c$ *and all* $(\lambda_1,n_1) \in I$;

(2) *We have*

$$
\begin{aligned}
K^{\lambda}_{(\lambda',n')}(x) &= \{(\lambda_1,n_1) \in I \colon \exists k \in \{1,2,\dots,n_\lambda\} \ s.t.\ (\psi_{\lambda_1,n_1}(x), \psi_{\lambda,k}(\psi_{\lambda',n'}(x))) \in \mathcal{S}\}\\
&= \{\sigma(x,\psi_{\lambda',n'}(x))(\lambda,k) \colon k \in \{1,2,\dots,n_\lambda\}\},
\end{aligned}
$$

and $|K^{\lambda}_{(\lambda',n')}(x)| = n_\lambda$ *for all* $x \in \underline{\mathrm{N}}^c$, $\lambda \in \Lambda$ *and all* $(\lambda',n') \in I$;

(3) *With* $\mathcal{S}_0 := \mathcal{S} \cap (\underline{\mathrm{N}}^c \times \underline{\mathrm{N}}^c)$, *for any* $(x,y) \in \mathcal{S}_0$ *and for any* $(\lambda,n) \in I$, *there is a unique* $m \in \{1,2,\dots,n_\lambda\}$ *satisfying* $(\psi_{\lambda,n}(x), \psi_{\lambda,m}(y)) \in \mathcal{S}$;

(4) *The properties listed in* (1)–(3) *are enjoyed if x is replaced by $\psi_i(x)$ for any $i \in I$.*

For any $x \in X$, let

$$
L^{(\lambda',n')}_{(\lambda,n)}(x) := \Big\{(\lambda_1,n_1) \in I \colon\ (\lambda',n') \in K^{\lambda_1}_{(\lambda,n)}(x),\ n_1 \in \{1,2,\dots,n_{\lambda_1}\}\Big\}.
$$

Lemma 3.2. *The function* $x \in X \mapsto L^{(\lambda',n')}_{(\lambda,n)}(x) \in 2^I$ *defined above is Borel.*

Proof. First, we briefly review the definition of the Fell topology. For a subset E of I, one defines the following subsets of 2^I:

$$
E^- := \{A \in 2^I \colon A \cap E \neq \emptyset\}, \qquad E^+ := \{A \in 2^I \colon A \subseteq E\}.
$$

Keeping in mind that I is equipped with the discrete topology, the Fell topology on 2^I has, by definition, as a subbase all sets of the form V^-, where V is a subset of I, plus all sets of the form $(K^c)^+$, where K is a finite subset of I. Hence, in order to prove our claim, it suffices to show that both $\left(L_{(\lambda,n)}^{(\lambda',n')}\right)^{-1}(V^-)$ and $\left(L_{(\lambda,n)}^{(\lambda',n')}\right)^{-1}((K^c)^+)$ are Borel in X for any subset V and any finite subset K. We simply write L for $L_{(\lambda,n)}^{(\lambda',n')}$. Let $g_0\colon X \to X \times X$ be $g_0(x) := (x, x)$. Then we have

$$\begin{aligned} x \in L^{-1}(V^-) &\iff \exists(\lambda_1, n_1) \in V \text{ s.t. } (\lambda', n') \in K_{(\lambda,n)}^{\lambda_1}(x) \\ &\iff \exists(\lambda_1, n_1) \in V \text{ s.t. } (\psi_{\lambda,n}(x), \psi_{\lambda',n'}(x)) \in \mathcal{C}_{\lambda_1} \\ &\iff \exists(\lambda_1, n_1) \in V \text{ s.t. } x \in ((\psi_{\lambda,n} \times \psi_{\lambda',n'}) \circ g_0)^{-1}(\mathcal{C}_{\lambda_1}). \end{aligned}$$

This shows that

$$L^{-1}(V^-) = \bigcup_{(\lambda_1,n_1)\in V} ((\psi_{\lambda,n} \times \psi_{\lambda',n'}) \circ g_0)^{-1}(\mathcal{C}_{\lambda_1}),$$

which in turn implies, since V is countable, that $L^{-1}(V^-)$ is a Borel subset. Similarly, we easily obtain

$$L^{-1}((K^c)^+) = \left(\bigcup_{(\lambda_1,n_1)\in K} ((\psi_{\lambda,n} \times \psi_{\lambda',n'}) \circ g_0)^{-1}(\mathcal{C}_{\lambda_1}) \right)^c.$$

Therefore, $L^{-1}((K^c)^+)$ is Borel. □

Lemma 3.3. *Let* $\underline{\mathrm{N}}$ *be the* μ*-null set in Lemma* 3.1. *Then*
(i) $L_{(\lambda,n)}^{(\lambda',n')}(x)$ *is not an empty set for all* $x \in \underline{\mathrm{N}}^c$;
(ii) *For each* $x \in \underline{\mathrm{N}}^c$, $\left|L_{(\lambda,n)}^{(\lambda',n')}(x)\right| = n_{\lambda_1}$ *if* $(\lambda_1, n_1) \in L_{(\lambda,n)}^{(\lambda',n')}(x)$.

Proof. Fix any $x \in \underline{\mathrm{N}}^c$.

Since $\psi_{\lambda',n'}(x) \in \mathcal{R}(\psi_{\lambda,n}(x)) = \bigsqcup_{i\in I} \mathcal{S}(\psi_i(\psi_{\lambda,n}(x)))$, there is a unique $(\lambda_1, n_1) \in I$ such that

$$(\psi_{\lambda',n'}(x), \psi_{\lambda_1,n_1}(\psi_{\lambda,n}(x))) \in \mathcal{S}.$$

Thus $(\lambda', n') \in K_{(\lambda,n)}^{\lambda_1}(x)$, i.e., $(\lambda_1, n_1) \in L_{(\lambda,n)}^{(\lambda',n')}(x)$. In particular, $L_{(\lambda,n)}^{(\lambda',n')}(x) \neq \emptyset$.

Let $(\lambda_1,\ n_1)$, $(\lambda_2, n_2) \in L_{(\lambda,n)}^{(\lambda',n')}(x)$. Then $(\psi_{\lambda,n}(x), \psi_{\lambda',n'}(x)) \in \mathcal{C}_{\lambda_1}$ and $(\psi_{\lambda,n}(x), \psi_{\lambda',n'}(x)) \in \mathcal{C}_{\lambda_2}$. So there are $u, v \in X$ such that $(x, u) \in \mathcal{S}$, $(x, v) \in \mathcal{S}$ and $(\psi_{\lambda_1,1}(u), \psi_{\lambda',n'}(x)) \in \mathcal{S}$, $(\psi_{\lambda_2,1}(v), \psi_{\lambda',n'}(x)) \in \mathcal{S}$. Hence

$$\psi_{\lambda',n'}(x) \in \mathcal{S}(\psi_{\lambda_1,1}(\mathcal{S}(x))) = \bigsqcup_{k=1}^{n_{\lambda_1}} \mathcal{S}(\psi_{\lambda_1,k}(x)) \quad \text{and}$$

$$\psi_{\lambda',n'}(x) \in \mathcal{S}(\psi_{\lambda_2,1}(\mathcal{S}(x))) = \bigsqcup_{k=1}^{n_{\lambda_2}} \mathcal{S}(\psi_{\lambda_2,k}(x)).$$

From this, we find that $\lambda_1 = \lambda_2$. By the definition of $L^{(\lambda' n')}_{(\lambda,n)}(x)$, $(\lambda_1, n_1) \in L^{(\lambda',n')}_{(\lambda,n)}(x)$ if and only if $(\lambda_1, n_2) \in L^{(\lambda',n')}_{(\lambda,n)}(x)$ for $n_1, n_2 \in \{1, 2, \ldots, n_{\lambda_1}\}$. Therefore, we conclude that $\left|L^{(\lambda',n')}_{(\lambda,n)}(x)\right| = n_{\lambda_1}$ once we know that one (λ_1, n_1) belongs to $L^{(\lambda',n')}_{(\lambda,n)}(x)$. $\square$

Lemma 3.4. *Let $\lambda, \lambda' \in \Lambda$. There exists a finite subset $L_{\lambda,\lambda'}$ of I such that $\bigcup_{n=1}^{n_\lambda} \bigcup_{n'=1}^{n_{\lambda'}} L^{(\lambda',n')}_{(\lambda,n)}(x) = L_{\lambda,\lambda'}$ for a.e. $x \in X$.*

Proof. For any $x \in X$, set $L(x) := \bigcup_{n=1}^{n_\lambda} \bigcup_{n'=1}^{n_{\lambda'}} L^{(\lambda',n')}_{(\lambda,n)}(x)$. Take the null set $\underline{\mathrm{N}}$ in Lemma 3.1. From Lemmas 3.1 and 3.2, we find that the map $x \in \underline{\mathrm{N}}^c \mapsto L(x) \in 2^I$ is also Borel (cf. [**19**, Section 3.3]). Note also that $L(x)$ is a finite set for all $x \in \underline{\mathrm{N}}^c$, due to Lemma 3.3.

With $\mathcal{S}_0$ defined in Lemma 3.1, let $(x, y) \in \mathcal{S}_0$. Take any $(\lambda_1, n_1) \in L^{(\lambda',n')}_{(\lambda,n)}(x)$. So $(\psi_{\lambda,n}(x), \psi_{\lambda',n'}(x)) \in \mathcal{C}_{\lambda_1}$. Since $(x, y) \in \mathcal{S}_0$, there are $m \in \{1, 2, \ldots, n_\lambda\}$ and $m' \in \{1, 2, \ldots, n_{\lambda'}\}$ such that $(\psi_{\lambda,n}(x), \psi_{\lambda,m}(y)) \in \mathcal{S}$ and $(\psi_{\lambda',n'}(x), \psi_{\lambda',m'}(y)) \in \mathcal{S}$. From this, we obtain $(\psi_{\lambda,m}(y), \psi_{\lambda',m'}(y)) \in \mathcal{C}_{\lambda_1}$. This yields $(\lambda_1, n_1) \in L^{(\lambda',m')}_{(\lambda,m)}(x) \subseteq L(y)$. Hence $L(x)$ is contained in $L(y)$. By reversing the roles of x and y in the above arguments, we obtain the reverse inclusion. Consequently, we have $L(x) = L(y)$. Since $x \in X \mapsto L(x)$ is a Borel map from X into the Polish space 2^I, it follows form the ergodicity of $\mathcal{S}$ that there exists a finite subset $L_{\lambda,\lambda'}$ of I such that $L(x) = L_{\lambda,\lambda'}$ for a.e. $x \in X$. $\square$

For each $(\lambda, n) \in I$, set

$$\mathcal{R}_{\lambda,n} := \{(x, y) \in \mathcal{R} \colon \sigma(x, y)(\lambda, n) = (\lambda, n)\}.$$

It is easy to see that $(\psi_{\lambda,n} \times \psi_{\lambda,n})^{-1}(\mathcal{S}) \cap \mathcal{R} = \mathcal{R}_{\lambda,n}$.

Proposition 3.5. *Let $(\lambda,n), (\lambda',n') \in I$. There exists a ν-conull subset $\mathcal{R}_{\lambda,n}(\lambda',n')$ of $\mathcal{R}_{\lambda,n}$ such that the set*

$$\{\sigma(x,y)(\lambda',n') \colon (x,y) \in \mathcal{R}_{\lambda,n}(\lambda',n')\}$$

is finite.

Proof. Consider $\underline{\mathrm{N}}$ and $\mathcal{S}_0$ defined in Lemma 3.1. Then $(\psi_{\lambda,n} \times \psi_{\lambda,n})^{-1}(\mathcal{S}_0)$ is a ν-conull subset of $\mathcal{R}_{\lambda,n}$. By Lemma 3.4, there exist a μ-null subset N' of X and a finite set $L_{\lambda,\lambda'}$ of I such that $\bigcup_{n=1}^{n_\lambda} \bigcup_{n'=1}^{n_{\lambda'}} L^{(\lambda',n')}_{(\lambda,n)}(x) = L_{\lambda,\lambda'}$ for all $x \in (N')^c$. Let $L_{\lambda,\lambda'} = \{(\bar{\lambda}_k, \bar{n}_k) \colon 1 \le k \le t\}$ be an enumeration of $L_{\lambda,\lambda'}$. We also know that there exist a μ-null subset N'' of X and finite subsets K^k_λ $(1 \le k \le t)$ of I such that $\bigcup_{m=1}^{n_\lambda} K^{\bar{\lambda}_k}_{(\lambda,m)}(x) \subseteq K^k_\lambda$ for all $x \in (N'')^c$. Put $\tilde{N} := \bigcup_{i \in I} \psi_i^{-1}(\underline{\mathrm{N}} \cup N' \cup N'')$. Set

$$\mathcal{R}_{\lambda,n}(\lambda', n') := (\psi_{\lambda,n} \times \psi_{\lambda,n})^{-1}(\mathcal{S}_0) \cap (\tilde{N}^c \times \tilde{N}^c).$$

Clearly, $\mathcal{R}_{\lambda,n}(\lambda', n')$ is ν-conull in $\mathcal{R}_{\lambda,n}$. Let $(x, y) \in \mathcal{R}_{\lambda,n}(\lambda', n')$. Then

$$\sigma(x, y)(\lambda', n') = \sigma(x, \psi_{\lambda,n}(x))\sigma(\psi_{\lambda,n}(x), \psi_{\lambda,n}(y))\sigma(\psi_{\lambda,n}(y), y)(\lambda', n'). \tag{3.3}$$

With $(\lambda_y, n_y) := \sigma(\psi_{\lambda,n}(y), y)(\lambda', n')$. We have

$$\begin{aligned}\sigma(\psi_{\lambda,n}(y), y)(\lambda', n') = (\lambda_y, n_y) \quad &\Longleftrightarrow \quad (\lambda', n') = \sigma(y, \psi_{\lambda,n}(y))(\lambda_y, n_y)\\ &\Longrightarrow \quad (\lambda', n') \in K^{\lambda_y}_{(\lambda,n)}(y)\\ &\Longrightarrow \quad (\lambda_y, n_y) \in L^{(\lambda',n')}_{(\lambda,n)}(y).\end{aligned}$$

In particular, $(\lambda_y, n_y) = \sigma(\psi_{\lambda,n}(y), y)(\lambda', n')$ belongs to $L_{\lambda,\lambda'}$. Since $(\psi_{\lambda,n}(x), \psi_{\lambda,n}(y)) \in \mathcal{S}_0$, it follows that there exists a unique $m_y \in \{1, 2, \ldots, n_{\lambda_y}\}$ such that $\sigma(\psi_{\lambda,n}(x), \psi_{\lambda,n}(y))$ $(\lambda_y, n_y) = (\lambda_y, m_y)$. Finally, thanks to Lemma 3.1 (2), one has $\sigma(x, \psi_{\lambda,n}(x))(\lambda_y, m_y) \in K^{\lambda_y}_{(\lambda,n)}(x)$. With the results obtained above, we get

$$\begin{aligned}\{\sigma(x,y)(\lambda', n') \colon (x,y) \in \mathcal{R}_{\lambda,n}(\lambda', n')\} &\subseteq \bigcup_{k=1}^{t} \left\{K^{\bar{\lambda}_k}_{(\lambda,n)}(x) \colon x \in \tilde{N}^c\right\}\\ &\subseteq \bigcup_{k=1}^{t} \bigcup_{m=1}^{n_\lambda} \left\{K^{\bar{\lambda}_k}_{(\lambda,m)}(x) \colon x \in \tilde{N}^c\right\}\\ &\subseteq \bigcup_{k=1}^{t} K^k_\lambda.\end{aligned}$$

This proves our proposition. $\square$

3.2. Characterization of a Hecke pair in terms of choice functions.

As in the preceding section, let $\mathcal{S}$ be an ergodic subrelation of $\mathcal{R}$. As usual, put $A := W^*(\mathcal{R}, \omega)$ and $B = W^*(\mathcal{S}, \omega)$. Motivated by Proposition 3.5, we consider the following condition for a set of choice functions $\{\psi_i\}_{i\in I}$ for $\mathcal{S} \subseteq \mathcal{R}$:

with $\mathcal{R}_i := \{(x,y) \in \mathcal{R} \colon \sigma(x,y)(i) = i\}$ $(\forall i \in I)$, there exists, for each $j \in I$, a ν-conull subset $\mathcal{R}_i(j)$ of $\mathcal{R}_i$ such that $\{\sigma(x,y)(j) \colon (x,y) \in \mathcal{R}_i(j)\}$ is finite. $(\spadesuit)$

Lemma 3.6. *Suppose that there exist a set of choice functions $\{\psi_i\}_{i\in I}$ for $\mathcal{S} \subseteq \mathcal{R}$ satisfying $(\spadesuit)$. Then $(\mathcal{R}, \mathcal{S})$ is a Hecke pair.*

Proof. Let $\mathcal{C}_i$ $(i \in I)$ be the $\mathcal{S}$-invariant set introduced in Section 2.1. Since every $\mathcal{C}_i$ is $\mathcal{S}$-invariant, $\chi_{\mathcal{C}_i}$ belongs to $A_1 \cap B'$, where A_1 is the basic extension of $B \subseteq A$. Here χ_E in general stands for the characteristic function of a set E.

Let N_0 be a μ-null subset of X such that $\mathcal{R}(x) = \bigsqcup_{i\in I} \mathcal{S}(\psi_i(x))$ for all $x \in N_0^c$. We may and do assume that N_0 is $\mathcal{S}$-invariant.

Fix any $k \in I$. We know that

$$\sum_{i\in I} \chi_{\mathcal{C}_k}(x, \psi_i(x))$$

is constant for a.e. $x \in X$. So there exist a μ-null subset N_1 of X and a $C \in \mathbb{N} \cup \{\infty\}$ such that

$$\sum_{i\in I}\chi_{\mathcal{C}_k}(x,\psi_i(x))=C \qquad \text{for all } x\in N_1^c.$$

In the meantime, by assumption, there exists a μ-null subset N_2 of X such that, if $\mathcal{E}:=\{(x,y)\in\mathcal{R}_0\colon x\in N_2^c\}$, then $\mathcal{E}\subseteq\mathcal{R}_0(k)$ and $\nu(\mathcal{R}_0\setminus(\mathcal{R}_0\cap(N_2^c\times N_2^c)))=0$.

Let $I_k:=\{i\in I\colon \nu(\mathcal{C}_k\cap\Gamma(\psi_i))>0\}$. For any $i\in(I_k)^c$, there is a μ-null set $N(i)$ of X such that $\chi_{\mathcal{C}_k}(x,\psi_i(x))=0$ for all $x\in N(i)^c$. Put $N_i':=\bigcup_{j\in I}\psi_j^{-1}\Big(\bigcup_{i\in(I_k)^c}N(i)\cup N_0\cup N_1\cup N_2\Big)$. We clearly have

$$\sum_{i\in I_k}\chi_{\mathcal{C}_k}(x,\psi_i(x))=C \qquad \text{for all } x\in(N')^c.$$

Thus, if we put $I_k(x):=\{i\in I_k\colon (x,\psi_i(x))\in\mathcal{C}_k\}$ for each $x\in(N')^c$, then we have $C=|I_k(x)|$ for any $x\in(N')^c$. Fix any $a\in(N')^c$. For each $i\in I_k(a)$, there is a $z_i\in X$ such that $(a,z_i)\in\mathcal{S}$ and $(\psi_k(z_i),\psi_i(a))\in\mathcal{S}$. Since a belongs to $(N_0\cup N_2)^c$ and $(a,z_i)\in\mathcal{S}$, it follows that (a,z_i) is in $\mathcal{E}\subseteq\mathcal{R}_0(k)$. Similarly, we find that $(\psi_k(z_i),\psi_i(a))$ belongs to $\mathcal{E}\subseteq\mathcal{R}_0(k)$. Hence we have

$$\begin{aligned}
i&=\sigma(a,\psi_i(a))(0)\\
&=\sigma(a,z_i)\sigma(z_i,\psi_k(z_i))\sigma(\psi_k(z_i),\psi_i(a))(0)\\
&=\sigma(a,z_i)\sigma(z_i,\psi_k(z_i))(0) \qquad (\because\ (\psi_k(z_i),\psi_i(a))\in\mathcal{E}\subseteq\mathcal{R}_0)\\
&=\sigma(a,z_i)(k)\in\{\sigma(x,y)(k)\colon (x,y)\in\mathcal{R}_0(k)\} \qquad (\because\ (a,z_i)\in\mathcal{E}\subseteq\mathcal{R}_0(k)).
\end{aligned}$$

Hence $I_k(a)$ is contained in the finite set $\{\sigma(x,y)(k)\colon (x,y)\in\mathcal{R}_0(k)\}$. In particular, C is a finite number.

By the result of the previous paragraph, we find that $\widehat{E_B}(\chi_{\mathcal{C}_i})$ is finite for all $i\in I$. Since $\mathcal{R}=\bigcup_{i\in I}\mathcal{C}_i$, it follows that $\widehat{E_B}|_{A_1\cap B'}$ is semifinite. Therefore, the inclusion $B\subseteq A$ is discrete. Namely, $(\mathcal{R},\mathcal{S})$ is a Hecke pair. □

Theorem 3.7. *Let $\mathcal{R}$ be an ergodic discrete measured equivalence relation on a standard Borel probability space $(X,\mathfrak{B},\mu)$ and $\mathcal{S}$ be an ergodic Borel subrelation of $\mathcal{R}$. Then the following are equivalent*:

(1) *$(\mathcal{R},\mathcal{S})$ is a Hecke pair.*

(2) *There exists a set of choice functions $\{\psi_i\}_{i\in I}$ of $\mathcal{S}\subseteq\mathcal{R}$ satisfying (♠).*

Proof. Theorem follows from Proposition 3.5 and Lemma 3.6. □

4. Index cocycles associated with Hecke pairs.

4.1. The range of an index cocycle.

The arguments given in this subsection overlap the exposition given in [**14**], but we have decided to present them here for the reader's convenience.

Let X be a set equipped with the discrete topology, and let $\mathrm{Map}(X)$ denote the set of maps from X into itself, equipped with the product topology. Namely, we identify $\mathrm{Map}(X)$ with the product space $\prod_{x\in X}X_x$, where $X_x:=X$ for all $x\in X$. For each $a\in X$, let p_a denote the projection from $\mathrm{Map}(X)$ onto X: $p_a(f) := f(a)(\forall f = \{f(x)\}_{x\in X} \in \mathrm{Map}(X))$. By definition, a fundamental system of (open) neighborhoods of $f \in \mathrm{Map}(X)$ consists of subsets of X of the form

$$p_{x_1}^{-1}(U_1) \cap p_{x_2}^{-1}(U_2) \cap \cdots \cap p_{x_n}^{-1}(U_n),$$

where $x_1, x_2, \ldots, x_n \in X$, $n \in \mathbb{N}$, and $U_1, U_2, \ldots, U_n$ are subsets of X with $f(x_k) \in U_k$ for any $k = 1, 2, \ldots, n$. (Note that $p_a^{-1}(U)$ is also closed, since $p_a^{-1}(U) = (p_a^{-1}(U^c))^c$.) Hence each point in $\mathrm{Map}(X)$ admits a fundamental system of open and closed neighborhoods. In this topology, a net $\{f_i\}_{i\in I}$ in $\mathrm{Map}(X)$ converges to $f \in \mathrm{Map}(X)$ if and only if, for any $a \in X$, there exists an $i_0 \in I$ such that $f_i(a) = f(a)$ for all $i \geq i_0$. In other words, the topology on $\mathrm{Map}(X)$ we are considering is just the topology of pointwise convergence arising from the discrete topology on X. It then easily follows that the topological space $\mathrm{Map}(X)$ is Hausdorff and totally disconnected.

Let $\mathrm{Per}(X)$ be the set of bijections of X onto itself, with the relative topology from $\mathrm{Map}(X)$. One can readily check that $\mathrm{Map}(X)$ is a topological semigroup, and that $\mathrm{Per}(X)$ is a topological group.

Note that the discrete topology on X is given by the metric d_0 on X defined by

$$d_0(a, b) := \begin{cases} 0 & (a = b), \\ 1 & (a \neq b). \end{cases}$$

Let us assume for the moment that X is a countable set. So X is of second countable. Let $X = \{x_n\}$ be an enumeration of the elements of X. Then $\mathrm{Map}(X)$ can be equipped with a metric d, called the product metric [**19**, Section 2.1], given by

$$d(\phi, \psi) := \sum_n \frac{1}{2^{n+1}} d_0(\phi(x_n), \psi(x_n)) = \sum_{n\colon \phi(x_n)\neq\psi(x_n)} \frac{1}{2^{n+1}} \qquad (\phi, \psi \in \mathrm{Map}(X)).$$

Lemma 4.1. *The metric topology on* $\mathrm{Map}(X)$ *introduced above coincides with the product topology.*

Proof. It suffices to prove the case where X is an infinite countable set. Thus $X = \{x_n\}_{n=1}^{\infty}$. Suppose that $\phi_k \to \phi$ in the product topology of $\mathrm{Map}(X)$ (note that the product topology is of second countable). By assumption, there exists a $k_1 \in \mathbb{N}$ such that $\phi_k(x_1) = \phi(x_1)$ for all $k \geq k_1$. Suppose that we have constructed natural numbers $k_1 < k_2 < \cdots < k_l$ satisfying

$$\phi_k = \phi \text{ on the set } \{x_1, x_2, \ldots, x_j\} \quad (\forall k \geq k_j),$$

for any $j = 1, 2, \ldots, l$. Since $\phi_k \to \phi$ pointwise, we may choose k_{l+1} with $k_{l+1} > k_l$ so that $\phi_k = \phi$ on $\{x_1, \ldots, x_l, x_{l+1}\}$ for all $k \geq k_{l+1}$. Thus we obtain a strictly increasing sequence $\{k_l\}_{l=1}^{\infty} \subseteq \mathbb{N}$ satisfying

$$\phi_k = \phi \text{ on the set } \{x_1, x_2, \ldots, x_l\} \quad (\forall k \geq k_l),$$

for any $l \in \mathbb{N}$. Fix any $\varepsilon > 0$. Choose an $m \in \mathbb{N}$ such that $2^{-m-1} < \varepsilon$. If $k \geq k_m$, then

$\phi_k = \phi$ on $\{x_1, x_2, \ldots, x_m\}$, so that

$$d(\phi_k, \phi) = \sum_{n\colon \phi_k(x_n) \neq \phi(x_n)} \frac{1}{2^{n+1}} \leq \sum_m \frac{1}{2^{m+2}} < \varepsilon.$$

Hence $\phi_k \to \phi$ in the metric topology.

Suppose conversely that $\phi_k \to \phi$ in the metric topology. Set $J := \{k \in \mathbb{N} \colon \phi_k = \phi\}$. It suffices to assume that $\mathbb{N} \setminus J$ contains an infinite number of elements. Otherwise, there is a $k_0 \in \mathbb{N}$ such that $\phi_k = \phi$ for all $k \geq k_0$. In particular, $\phi_k \to \phi$ in the product topology. So let us suppose that $\mathbb{N} \setminus J = \{k_1, k_2, \ldots, k_l, \ldots\}$. Fix any $x = x_m \in X$. For each $l \in \mathbb{N}$, define

$$I_l := \{n \in \mathbb{N} \colon \phi_{k_l}(x_n) \neq \phi(x_n)\}, \qquad n_l := \min I_l.$$

Then we have

$$0 < \frac{1}{2^{n_l+1}} < \sum_{n \in I_l} \frac{1}{2^{n+1}} = d(\phi_{k_l}, \phi) \longrightarrow 0 \qquad (l \to \infty).$$

This implies that $\lim_{l\to\infty} n_l = \infty$. Hence there is an $l_0 \in \mathbb{N}$ such that $n_l > m$ for all $l \geq l_0$. This means that $m \notin I_k$ for all $k \geq k_{l_0}$. Particularly, $\phi_k(x) = \phi_k(x_m) = \phi(x_m) = \phi(x)$ for any $k \geq k_{l_0}$. Therefore, ϕ_k converges to ϕ in the product topology. $\square$

The next result can be found in [**19**, Sections 2.2 and 2.4].

Corollary 4.2. *If X is countable, then* $\mathrm{Map}(X)$ *is a Polish space*; $\mathrm{Per}(X)$ *is a Polish group.*

Let us return to an ergodic discrete measured equivalence relation $\mathcal{R}$ and an ergodic Borel subrelation $\mathcal{S}$ of $\mathcal{R}$. We choose choice functions for $\mathcal{S} \subseteq \mathcal{R}$. Throughout the rest of this note, we always assume that the index set I is an infinite (countable) set. Thus $\mathrm{Per}(I)$ is a Polish group thanks to Corollary 4.2. We assert that the index cocycle $\sigma\colon \mathcal{R} \to \mathrm{Per}(I)$ associated with $\{\psi_i\}$ is Borel. Indeed, for each $i \in I$, then $f_i := p_i \circ \sigma$ is given by $f_i(x,y) = \sigma(x,y)(i)$. So $f_i^{-1}(J) = \bigcup_{j \in J} (\psi_i \times \psi_j)^{-1}(\mathcal{S}) \cap \mathcal{R}$ for any subset J of I. It follows that f_i is Borel. Hence we find that σ is Borel, as asserted. It thus makes sense to consider the asymptotic range $r^*(\sigma)$ of the index cocycle σ.

4.2. Asymptotic range of an index cocycle associated with a Hecke pair.

In this subsection, we assume that $(\mathcal{R}, \mathcal{S})$ is a Hecke pair. From Theorem 3.7, we may choose a set of choice functions $\{\psi_i\}_{i \in I}$ for $\mathcal{S} \subseteq \mathcal{R}$ whose associated index cocycle σ satisfies the condition (♠) defined in Section 3.2.

For each $i \in I$, set $\mathcal{N}(i) := \bigcup_{j \in I} \mathcal{R}_i \setminus \mathcal{R}_i(j)$, which is a ν-null set. Put $\widetilde{\mathcal{R}}_i := \mathcal{R}_i \setminus \mathcal{N}(i)$. By definition, $\widetilde{\mathcal{R}}_i$ is a ν-conull subset of $\mathcal{R}_i$ with the property that $\{\sigma(x,y)(j) \colon (x,y) \in \widetilde{\mathcal{R}}_i\}$ is a finite set for all $j \in J$.

Theorem 4.3. *The asymptotic range $r^*(\sigma)$ of the index cocycle σ obtained from the special choice functions $\{\psi_i\}_{i \in I}$ as above is a locally compact, totally disconnected,* *closed subgroup of the Polish group* $\mathrm{Per}(I)$*. The stabilizer subgroup* $r^*(\sigma)_0$ *at* 0 *is a compact open subgroup of* $r^*(\sigma)$*. Therefore,* $(r^*(\sigma), r^*(\sigma)_0)$ *is a Hecke pair of groups.*

Proof. Fix an arbitrary $i \in I$ and consider the stabilizer subgroup $r^*(\sigma)_i$ at i. Let $\phi \in r^*(\sigma)_i$. Take any $j \in I$. Since $p_i^{-1}(\{i\}) \cap p_j^{-1}(\{\phi(j)\})$ is an open neighborhood of ϕ, it follows from the definition of $r^*(\sigma)$ that $\sigma^{-1}(p_i^{-1}(\{i\}) \cap p_j^{-1}(\{\phi(j)\}))$ has positive measure. Clearly, $\sigma^{-1}(p_i^{-1}(\{i\}) \cap p_j^{-1}(\{\phi(j)\}))$ is contained in $\mathcal{R}_i$. So $\sigma^{-1}(p_i^{-1}(\{i\}) \cap p_j^{-1}(\{\phi(j)\})) \cap \widetilde{\mathcal{R}}_i$ still has positive measure. We pick any $(x,y) \in \sigma^{-1}(p_i^{-1}(\{i\}) \cap p_j^{-1}(\{\phi(j)\})) \cap \widetilde{\mathcal{R}}_i$. Then $\sigma(x,y)(i) = i = \phi(i)$ and $\sigma(x,y)(j) = \phi(j)$. This means that $\phi = (\phi(j))_{j \in I}$ belongs to $\prod_{j \in I} \{\sigma(x,y)(j) \colon (x,y) \in \widetilde{\mathcal{R}}_i\}$, which is compact by the Tychonoff theorem. Thus $r^*(\sigma)_i$ is contained in the compact set $\prod_{j \in I} \{\sigma(x,y)(j) \colon (x,y) \in \widetilde{\mathcal{R}}_i\}$. Since $r^*(\sigma)_i$ is closed, we find that $r^*(\sigma)_i$ is compact. Because $r^*(\sigma)$ has a compact neighborhood of the identity id_I, namely any $r^*(\sigma)_i$, it is locally compact. Total connectedness of $r^*(\sigma)$ follows from that of $\mathrm{Map}(I)$.

As we saw in the preceding paragraph, the stabilizer subgroup $r^*(\sigma)_0$ at the point $0 \in I$ is compact and open. Hence $(r^*(\sigma), r^*(\sigma)_0)$ forms a Hecke pair. □

Proposition 4.4. *The asymptotic range of the cocycle* $\sigma|_{\mathcal{S}}$ *obtained by restricting* σ *to the ergodic subrelation* $\mathcal{S}$ *coincides with the stabilizer subgroup* $r^*(\sigma)_0$ *at* 0*.*

Proof. Let us denote $\sigma|_{\mathcal{S}}$ by c.

Take any $\phi \in r^*(\sigma)_0$. Let B be a Borel subset of X of positive measure. We also let $i_1, i_2, \ldots, i_n$ be any points in I and $U_1, U_2, \ldots, U_n$ be any subsets of I satisfying $\phi(i_k) \in U_k$ for $1 \le k \le n$. Then, since $p_0^{-1}(\{0\}) \cap \bigcap_{k=1}^n p_{i_k}^{-1}(U_k)$ is an open neighborhood of ϕ, it follows that $(\sigma_B)^{-1}(p_0^{-1}(\{0\}) \cap \bigcap_{k=1}^n p_{i_k}^{-1}(U_k))$ has positive measure (with respect to ν). If $(x,y) \in \mathcal{R}$ belongs to $(\sigma_B)^{-1}(p_0^{-1}(\{0\}) \cap \bigcap_{k=1}^n p_{i_k}^{-1}(U_k))$, then we have $\sigma(x,y)(0) = 0$ and $\sigma(x,y)(i_k) \in U_k$ for $1 \le k \le n$. So (x,y) particularly lies in $\mathcal{S} \cap (B \times B)$. Hence we obtain

$$(\sigma_B)^{-1}\left(p_0^{-1}(\{0\}) \cap \bigcap_{k=1}^n p_{i_k}^{-1}(U_k)\right) \subseteq (c_B)^{-1}\left(\bigcap_{k=1}^n p_{i_k}^{-1}(U_k)\right).$$

Hence $\phi \in r^*(c)$. Therefore, we have $r^*(\sigma)_0 \subseteq r^*(c)$.

Suppose conversely that $\phi \in r^*(c)$. Because $p_0^{-1}(\{\phi(0)\})$ is an open neighborhood of ϕ, it follows that $\nu(c^{-1}(p_0^{-1}(\{\phi(0)\}))) > 0$. Let us take any $(x,y) \in c^{-1}(p_0^{-1}(\{\phi(0)\}))$. Then we have $\phi(0) = c(x,y)(0) = \sigma(x,y)(0) = 0$, since (x,y) is in $\mathcal{S}$. Thus we find that ϕ belongs to the stabilizer subgroup $\mathrm{Per}(I)_0$. Let B, $\{i_k\}$ and $\{U_k\}$ be as in the previous paragraph. By definition, we have

$$\nu\left((c_B)^{-1}\left(\bigcap_{k=1}^n p_{i_k}^{-1}(U_k)\right)\right) > 0.$$

If (x,y) belongs to $(c_B)^{-1}(\bigcap_{k=1}^n p_{i_k}^{-1}(U_k))$, then $(x,y) \in \mathcal{S} \cap (B \times B)$ and $\sigma(x,y)(i_k) \in U_k$ for all k. In particular, we have $\sigma(x,y)(0) = 0$. Hence (x,y) is in $(\sigma_B)^{-1}(p_0^{-1}(\{0\}) \cap \bigcap_{k=1}^n p_{i_k}^{-1}(U_k))$. So we get

$$(c_B)^{-1}\left(\bigcap_{k=1}^{n} p_{i_k}^{-1}(U_k)\right) \subseteq (\sigma_B)^{-1}\left(p_0^{-1}(\{0\}) \cap \bigcap_{k=1}^{n} p_{i_k}^{-1}(U_k)\right).$$

This in turn implies that $(\sigma_B)^{-1}(\bigcap_{k=1}^{n} p_{i_k}^{-1}(U_k))$ is of positive measure. Thus $\phi \in r^*(\sigma)$. Therefore, we have $\phi \in r^*(\sigma)_0$. We conclude that $r^*(c) = r^*(\sigma)_0$. □

Let us compute the Hecke pair $(r^*(\sigma), r^*(\sigma)_0)$ obtained above in the case where $(\mathcal{R}, \mathcal{S})$ is of a special form.

For a discrete measured equivalence relation $\mathcal{P}$ on a standard Borel probability space X, we define $[\mathcal{P}] := \{\phi \in [\mathcal{P}]_*\colon \mathrm{Dom}(\phi) = \mathrm{Im}(\phi) = X\}$ and call it the *full group* of $\mathcal{P}$. The *normalizer* $N[\mathcal{P}]$ of the full group $[\mathcal{P}]$ is by definition the set of all Borel automorphisms ϕ of X satisfying $\phi[\mathcal{P}]\phi^{-1} = [\mathcal{P}]$.

Suppose now that there exist an ergodic Borel subrelation $\mathcal{P}$ contained in $\mathcal{S}$, a countable discrete group G in $N[\mathcal{P}]$ and a subgroup H of G such that

(1) $G \cap [\mathcal{P}] = \{e\}$, i.e., the action of G on $\mathcal{P}$ is outer;

(2) $(\mathcal{S} \subseteq \mathcal{R}) = (H \ltimes \mathcal{P} \subseteq G \ltimes \mathcal{P})$;

(3) (G, H) is a Hecke pair of groups, i.e., $G = \{g \in G\colon [H : H \cap g^{-1}Hg] < \infty\}$.

Thus we have

$$\mathcal{S} = \{(x, y) \in X \times X\colon \exists h \in H \text{ s.t.}\, (x, h(y)) \in \mathcal{P}\},$$
$$\mathcal{R} = \{(x, y) \in X \times X\colon \exists g \in G \text{ s.t.}\, (x, g(y)) \in \mathcal{P}\}.$$

In this setting, it is known (see [**4**, Section 11]) that $(\mathcal{R}, \mathcal{S})$ is a Hecke pair.

Let $\{t_q\}_{q \in H\backslash G} \subseteq G$ be a set of representatives of the right coset space $H\backslash G$ with $t_H = e$. For each $q \in H\backslash G$, set $\psi_q := t_q$. We see that $\{\psi_q\}_{q \in H\backslash G}$ is a set of choice functions for $\mathcal{S} \subseteq \mathcal{R}$. Hence the index set I of the choice functions for $\mathcal{S} \subseteq \mathcal{R}$ is in this special case the quotient space $I = H\backslash G$, and the distinguished point $0 \in I$ is $H \in H\backslash G$. Thus the index cocycle σ is a Borel 1-cocycle from $\mathcal{R}$ into the Polish space $\mathrm{Per}(H\backslash G)$. By outerness of the action of G, we may and do assume that, for each $(x, y) \in \mathcal{R}$, the mapping $\sigma(x, y)\colon H\backslash G \to H\backslash G$ is the right translation $\theta(g)$: $\theta(g)q := qg^{-1} (q \in H\backslash G)$, where g is determined by the condition $(g(x), y) \in \mathcal{P}$. In particular, $\mathcal{P}$ is included in $\mathrm{Ker}(\sigma)$.

Proposition 4.5. *In the situation considered above, we have* $(r^*(\sigma), r^*(\sigma)_H) = (\overline{\theta(G)}, \overline{\theta(H)})$. *Therefore,* $(r^*(\sigma), r^*(\sigma)_H)$ *is the Schlichting completion of the Hecke pair* (G, H) *in the sense of* [**14**].

Proof. Let $g \in G$ and $B \in \mathfrak{B}$ be such that $\mu(B) > 0$. Since $\mathcal{P}$ is ergodic, there exists a $\gamma \in [\mathcal{P}]_*$ such that $\mathrm{Dom}(\gamma) \subseteq g(B)$ and $\mathrm{Im}(\gamma) \subseteq B$. So $g^{-1}(\mathrm{Dom}(\gamma)) \subseteq B$. Put

$$\mathcal{E} := \{(x, \gamma(g(x)))\colon x \in g^{-1}(\mathrm{Dom}(\gamma))\},$$

which is included in $B \times B$. If $x \in g^{-1}(\mathrm{Dom}(\gamma))$, then $(g(x), \gamma(g(x))) \in \mathcal{P}$. This implies that $(x, \gamma(g(x)))$ belongs to $\mathcal{R}$. Hence $\mathcal{E} \subseteq \mathcal{R} \cap (B \times B)$. Note that $\nu(\mathcal{E})$ has positive measure, because $\nu(\mathcal{E})$ equals $\mu(g^{-1}(\mathrm{Dom}(\gamma)))$. For any $(x, \gamma(g(x))) \in \mathcal{E}$ with $x \in g^{-1}(\mathrm{Dom}(\gamma))$, we have $\sigma(x, \gamma(g(x))) = \theta(g)$, since $(g(x), \gamma(g(x))) \in \mathcal{P}$. This means that $\theta(g)$ belongs to $r^*(\sigma)$. It follows that $\theta(G) \subseteq r^*(\sigma)$. In particular, $\overline{\theta(G)} \subseteq r^*(\sigma)$.

Let $\phi \in r^*(\sigma)$. Also let $F := \{q_1, \ldots, q_n\}$ be any finite subset of $H \backslash G$. Set

$$V_F := \bigcap_{k=1}^{n} p_{q_k}^{-1}(\{\phi(q_k)\}).$$

We know by definition that $\{V_F \colon F \text{ is a finite subset of } H\backslash G\}$ forms a fundamental system of neighborhoods of ϕ. Since $\phi \in r^*(\sigma)$, we have $\nu(\mathcal{E}_F) > 0$, where

$$\mathcal{E}_F := \{(x, y) \in \mathcal{R} \colon \sigma(x, y)(q_k) = \phi(q_k) \ \ (1 \leq k \leq n)\}.$$

For each $g \in G$, set $\mathcal{E}_g := \{(x, y) \in \mathcal{E} \colon (g(x), y) \in \mathcal{P}\}$. Since $\mathcal{E} = \bigcup_{g \in G} \mathcal{E}_g$, there is a $g_F \in G$ such that $\nu(\mathcal{E}_{g_F}) > 0$. By definition, if $(x, y) \in \mathcal{E}_{g_F}$, then $\sigma(x, y) = \theta(g_F)$. Hence we have $\theta(g_F)|_F = \phi|_F$. This implies that $\theta(g_F)$ belongs to V_F. So $V_F \cap \theta(G) \neq \emptyset$, which shows that ϕ is in $\overline{\theta(G)}$. Consequently, $r^*(\sigma)$ is contained in $\overline{\theta(G)}$. Therefore, $r^*(\sigma) = \overline{\theta(G)}$.

The identity $r^*(\sigma)_H = \overline{\theta(H)}$ can be proven similarly. So we leave the verification to the reader. $\square$

5. Preferable choice functions for Hecke pairs.

We shall show in this section that any index cocycle which arises from a Hecke pair can be changed, within its cohomology class, into a new one which behaves nicely in our context.

Let us begin with an ergodic discrete measured equivalence relation $\mathcal{R}$ on a standard Borel probability space $(X, \mathfrak{B}, \mu)$ and an ergodic Borel subrelation $\mathcal{S}$ of $\mathcal{R}$. As in Section 3, let us fix a set of choice functions $\{\psi_{\lambda,n}\}_{(\lambda,n) \in I}$ for $\mathcal{S} \subseteq \mathcal{R}$ satisfying (CF1)–(CF3), and let σ be the associated index cocycle. For each $\lambda \in \Lambda$, denote by $P(\lambda)$ the permutation group $\mathrm{Per}(\{1, 2, \ldots, n_\lambda\})$. Consider the direct product compact group $K := \prod_{\lambda \in \Lambda} P(\lambda)$. Take any $\boldsymbol{f} = (f_\lambda)_{\lambda \in \Lambda} \in K$. It induces a map $\tilde{\boldsymbol{f}} \colon I \to I$ given by

$$\tilde{\boldsymbol{f}}(\lambda, n) = (\lambda, f_\lambda(n)) \qquad (\forall (\lambda, n) \in I).$$

It is easy to see that $\tilde{\boldsymbol{f}}$ belongs to $\mathrm{Per}(I)$.

Lemma 5.1. *The map $\boldsymbol{f} \in K \longmapsto \tilde{\boldsymbol{f}} \in \mathrm{Per}(I)$ is a topological isomorphism onto its image.*

Proof. The map defined above is clearly a homomorphism.

Suppose that a net $\{\boldsymbol{f}_n = (f_\lambda^{(n)})_{\lambda \in \Lambda}\}_n \in K$ converges to $\boldsymbol{f} = (f_\lambda)_{\lambda \in \Lambda} \in K$. Let $(\lambda, k) \in I$. By assumption, we have $\lim_n f_\lambda^{(n)} = f_\lambda$ in $P(\lambda)$. So there is an n_0 such that for any $n \geq n_0$, one has $f_\lambda^{(n)}(k) = f_\lambda(k)$. This means that $\widetilde{\boldsymbol{f}_n}(\lambda, k) = \tilde{\boldsymbol{f}}(\lambda, k)$. Hence $\lim_n \widetilde{\boldsymbol{f}_n} = \tilde{\boldsymbol{f}}$. Namely, the map is continuous.

Conversely, suppose that $\lim_n \widetilde{\boldsymbol{f}_n} = \tilde{\boldsymbol{f}}$ for a net $\{\boldsymbol{f}_n = (f_\lambda^{(n)})_{\lambda \in \Lambda}\}_n$ and an element

$\boldsymbol{f} = (f_\lambda)_{\lambda\in\Lambda}$. Take any $\lambda \in \Lambda$. By assumption, there exists an n_1 such that for any $n \geq n_1$, we have $\widetilde{\boldsymbol{f}_n}(\lambda, k) = \tilde{\boldsymbol{f}}(\lambda, k)$ for all $k \in \{1, 2, \ldots, n_\lambda\}$. This means that $(\lambda, f_\lambda^{(n)}(k)) = (\lambda, f_\lambda(k))$ for all k, which yields $f_\lambda^{(n)} = f_\lambda$ for any $n \geq n_1$. Since $\lambda \in \Lambda$ is arbitrary, we obtain $\lim_n \boldsymbol{f}_n = \boldsymbol{f}$. $\square$

By this lemma, we may and do identify K with its image in $\mathrm{Per}(I)$, so that K is regarded as a totally disconnected compact subgroup of $\mathrm{Per}(I)$. Since $\tilde{\boldsymbol{f}}(0) = 0$ for every $\boldsymbol{f} = (f_\lambda)_{\lambda\in\Lambda} \in K$, K is contained in the stabilizer $\mathrm{Per}(I)_0$ at $0 \in I$.

Let $\mathcal{S}_0$ be the Borel subset of $\mathcal{S}$ defined in Lemma 3.1. Take any $(x, y) \in \mathcal{S}_0$. By Lemma 3.1 (3), for a fixed $\lambda \in \Lambda$, there exists a unique $f \in P(\lambda)$ such that $\sigma(y, x)(\lambda, n) = (\lambda, f(n))$ for any $n \in \{1, 2, \ldots, n_\lambda\}$. Motivated by this, we define, for any $\lambda \in \Lambda$ and any $f \in P(\lambda)$, a subset $\mathcal{S}_0(\lambda, f)$ of $\mathcal{S}_0$ by

$$\mathcal{S}_0(\lambda, f) := \{(x, y) \in \mathcal{S}_0 \colon \sigma(y, x)(\lambda, n) = (\lambda, f^{-1}(n)) \ (1 \leq \forall n \leq n_\lambda)\}.$$

Since

$$\mathcal{S}_0(\lambda, f) = \bigcap_{n=1}^{n_\lambda} \left(\psi_{\lambda,n} \times \psi_{\lambda,f^{-1}(n)}\right)^{-1} (\mathcal{S}) \cap \mathcal{S}_0,$$

$\mathcal{S}_0(\lambda, f)$ is Borel. Clearly, we have

$$\mathcal{S}_0 = \bigsqcup_{f\in P(\lambda)} \mathcal{S}_0(\lambda, f).$$

From the argument given above, we find that for each $(x, y) \in \mathcal{S}_0$, there exists an element $(f_\lambda(x, y))_{\lambda\in\Lambda}$ of K such that

$$\sigma(y, x)(\lambda, n) = (\lambda, f_\lambda(x, y)^{-1}(n)) \qquad (\forall(\lambda, n) \in I).$$

For $(x, y), (y, z) \in \mathcal{S}_0$ and $(\lambda, n) \in I$, we have

$$\begin{aligned}(\lambda, f_\lambda(x, z)^{-1}(n)) &= \sigma(z, x)(\lambda, n) = \sigma(z, y)\sigma(y, x)(\lambda, n)\\ &= \sigma(z, y)(\lambda, f_\lambda(x, y)^{-1}(n))\\ &= (\lambda, f_\lambda(y, z)^{-1}(f_\lambda(x, y)^{-1}(n))).\end{aligned}$$

This shows that

$$f_\lambda(x, z) = f_\lambda(x, y) f_\lambda(y, z) \tag{5.1}$$

for all $\lambda \in \Lambda$. In particular, $f_\lambda(y, x) = f_\lambda(x, y)^{-1}$ and $f_\lambda(x, x) = id$. Hence, under the identification of K with its image in $\mathrm{Per}(I)$, the restriction of σ to $\mathcal{S}$ coincides almost everywhere with the map $c \colon \mathcal{S} \to K$ given by

$$c(x, y) = \begin{cases} (f_\lambda(x, y))_{\lambda\in\Lambda} \in K & \text{if } (x, y) \in \mathcal{S}_0,\\ e_K & \text{otherwise,}\end{cases}$$

where e_K stands for the identity of K. For any $\lambda \in \Lambda$, let π_λ denote the projection from K onto $P(\lambda)$. If F is a subset of $P(\lambda)$, then we have

$$(\pi_\lambda \circ c)^{-1}(F) \cap \mathcal{S}_0 = \bigcup_{f \in F} \mathcal{S}_0(\lambda, f).$$

This implies that c is a Borel map. Moreover, by (5.1), we have

$$c(x, y)c(y, z) = c(x, z)$$

for all $(x, y), (y, z) \in \mathcal{S}_0$. From [**18**, Theorem 3.2], it follows that there exists a Borel 1-cocycle c' from $\mathcal{S}$ into K such that $c = c'$ a.e. Let us denote c' by c again.

By [**23**, Corollary 3.8 (i)], c is equivalent (cohomologous) to a *minimal* cocycle c'. Recall (see [**23**, Definition 3.7]) that a Borel cocycle α from a Borel equivalence relation $\mathcal{T}$ into a compact group H is said to be *minimal* if there is no Borel cocycle $\beta\colon \mathcal{T} \to H$ cohomologous to α such that $H_\beta \subsetneq H_\alpha$, where for a cocycle $\gamma\colon \mathcal{T} \to H$, H_γ stands for the closed subgroup generated by $\gamma(\mathcal{T})$. Hence there exists a Borel map $\phi\colon X \to K$ such that the cocycle $c'(x, y) := \phi(x)c(x, y)\phi^{-1}(y)$ is minimal in the sense stated above. Set $\phi_\lambda := \pi_\lambda \circ \phi$ for each $\lambda \in \Lambda$, which is clearly Borel. It is easy to check that $\pi_\lambda \circ c$ is a Borel 1-cocycle on $\mathcal{S}$ into $P(\lambda)$ for any $\lambda \in \Lambda$.

Now we introduce a family of maps $\{\psi'_{\lambda,n}\}_{(\lambda,n)\in I}$ from X into itself by

$$\psi'_{\lambda,n}(x) := \psi_{\lambda,\phi_\lambda(x)^{-1}(n)}(x) \qquad (x \in X,\ (\lambda, n) \in I).$$

Lemma 5.2. *The maps $\psi'_{\lambda,n}$ defined above are Borel for all $(\lambda, n) \in I$.*

Proof. Take an arbitrary $(\lambda, n) \in I$. Since ϕ_λ is Borel, $\{\phi_\lambda^{-1}(\{\tau\})\colon \tau \in P(\lambda)\}$ is a Borel partition of X. Take any $E \in \mathfrak{B}$ and $\tau \in P(\lambda)$. For $x \in \phi_\lambda^{-1}(\{\tau\})$, we have

$$\begin{aligned} x \in (\psi'_{\lambda,n})^{-1}(E) \quad &\Longleftrightarrow \quad \psi_{\lambda,\phi_\lambda(x)^{-1}(n)}(x) \in E \\ &\Longleftrightarrow \quad \psi_{\lambda,\tau^{-1}(n)}(x) \in E \\ &\Longleftrightarrow \quad x \in (\psi_{\lambda,\tau^{-1}(n)})^{-1}(E). \end{aligned}$$

From this, it follows that one has

$$(\psi'_{\lambda,n})^{-1}(E) = \bigsqcup_{\tau \in P(\lambda)} (\psi'_{\lambda,n})^{-1}(E) \cap \phi_\lambda^{-1}(\{\tau\}) = \bigsqcup_{\tau \in P(\lambda)} (\psi_{\lambda,\tau^{-1}(n)})^{-1}(E) \cap \phi_\lambda^{-1}(\{\tau\}).$$

This shows that $(\psi'_{\lambda,n})^{-1}(E)$ is a Borel subset. Therefore, $\psi'_{\lambda,n}$ is Borel. $\square$

Lemma 5.3. *The functions $\{\psi'_{\lambda,n}\}_{(\lambda,n)\in I}$ defined above are choice functions for $\mathcal{S} \subseteq \mathcal{R}$ satisfying* (CF1)–(CF3). *Moreover, the index cocycle σ' which comes from these functions satisfies the following*:

$$\sigma'(x, y)(\lambda, n) = (\lambda, \pi_\lambda \circ c'(x, y)(n)) \qquad ((x, y) \in \mathcal{S}),$$

where c' is the minimal cocycle from $\mathcal{S}$ into K that appeared above. In particular, the *restriction of σ' to $\mathcal{S}$ is a minimal cocycle into the compact group K.*

PROOF. By definition, $\phi_\lambda(x) \in P(\lambda)$ is bijective for each $x \in X$ and $\lambda \in \Lambda$. So we have that the set $\{\psi'_{\lambda,n}(x)\}_{n=1}^{n_\lambda}$ coincides with $\{\psi_{\lambda,n}(x)\}_{n=1}^{n_\lambda}$. Hence $\{\psi'_{\lambda,n}\}_{(\lambda,n)\in I}$ are also choice functions for $\mathcal{S} \subseteq \mathcal{R}$.

The second half assertion follows from a direct computation, using $c'(x,y) = \phi(x)c(x,y)\phi(y)^{-1}$. □

We regard the cocycle c' as a (minimal) cocycle from $\mathcal{S}$ into the compact group $K_{c'}$. Then the skew-product relation $K_{c'} \ltimes X$ is ergodic by [**23**, Corollary 3.8 (ii)]. This is equivalent to saying that the crossed product $\widehat{K_{c'}} \ltimes W^*(\mathcal{S})$ by $\alpha_{c'}$ is a factor, where $\alpha_{c'}$ is a coaction of $K_{c'}$ on $W^*(\mathcal{S})$ induced by c' (see [**2**] for coactions induced by 1-cocycles on measured equivalence relations).

It follows from [**17**, Chapter IV, Corollary 1.6] and [**17**, Chapter V, Corollary 2.7] (see [**2**] also) that the Connes spectrum $\Gamma(\alpha_{c'})$, or equivalently the asymptotic range $r^*(c')$, equals $K_{c'}$.

LEMMA 5.4. *The* 1*-cocycle $c'_\lambda := \pi_\lambda \circ c'$ is a minimal* 1*-cocycle on $\mathcal{S}$ onto a* (*finite*) *subgroup* (*we denote it by L_λ*) *of $P(\lambda)$ for each $\lambda \in \Lambda$. In particular, each* $\mathrm{Ker}(c'_\lambda)$ *is an ergodic subrelation of $\mathcal{S}$ with the index* $[\mathcal{S} : \mathrm{Ker}(c_\lambda)]$ *equals* $|L_\lambda|$.

PROOF. This follows from [**23**, Proposition 3.10] and the fact that c'_λ is a 1-cocycle onto a finite group. □

From the results we have established so far, we get the following:

PROPOSITION 5.5. *For each Hecke pair $(\mathcal{R}, \mathcal{S})$ of ergodic equivalence relations, there exist choice functions for this pair satisfying* (CF1)–(CF3) *such that the restriction of the associated index cocycle to $\mathcal{S}$, which we denote by c, is a minimal cocycle into a compact group K. The asymptotic range $r^*(c)$ equals K_c. Moreover, $\pi_\lambda \circ c$ is a minimal cocycle whose kernel is ergodic for each $\lambda \in \Lambda$.*

DEFINITION 5.6. Let $(\mathcal{R}, \mathcal{S})$ be a Hecke pair of ergodic discrete measured equivalence relations on a standard Borel probability space $(X, \mathfrak{B}, \mu)$. We say that a set of choice functions for $\mathcal{S} \subseteq \mathcal{R}$ is *preferable* if they enjoy the property mentioned in Proposition 5.5.

We note that if the Hecke pair $(\mathcal{R}, \mathcal{S})$ has the form $(\mathcal{R} = G \ltimes \mathcal{P}, \mathcal{S} = H \ltimes \mathcal{P})$ as discussed just before Proposition 4.6, then the set of choice functions $\{\psi_q\}_{q \in H\backslash G}$ defined there is preferable with $\Lambda = H\backslash G/H$.

6. Construction of the sets $\mathscr{G}(\sigma)$ and $K(\sigma)$.

Throughout this section, unless stated otherwise, we fix a set of preferable choice functions $\{\psi_{\lambda,n}\}_{(\lambda,n)\in I}$ of a Hecke pair $(\mathcal{R}, \mathcal{S})$. As usual, we denote by $\sigma\colon \mathcal{R} \to \mathrm{Per}(I)$ the index cocycle derived from $\{\psi_{\lambda,n}\}_{(\lambda,n)\in I}$. From this cocycle, we will construct two subsets $\mathscr{G}(\sigma)$ and $K(\sigma)$ of $\mathrm{Per}(I)$. By preferability, the restriction $c := \sigma|_{\mathcal{S}}$ of σ to $\mathcal{S}$ is a minimal cocycle into the compact group $\prod_{\lambda\in\Lambda} P(\lambda)$. Write $K(\sigma)$ or simply K for the closed (hence compact) subgroup of $\prod_{\lambda\in\Lambda} P(\lambda)$ generated by the image $c(\mathcal{S})$.

Lemma 6.1. *Let $\{\psi_i\}_{i\in I}$ be general choice functions for $\mathcal{S}\subseteq\mathcal{R}$. Let $\{\mathcal{C}_i\}_{i\in I}$ be the subset defined in Subsection 2.1. Suppose that N is a null subset of X. For each $i\in I$, set*

$$\mathcal{C}_i' := \{(x,y)\in\mathcal{R}\colon \exists z\in N^c \textit{ s.t. } (x,z)\in\mathcal{S} \textit{ and } (\psi_i(z),y)\in\mathcal{S}\}.$$

Then $\nu(\mathcal{C}_i\setminus\mathcal{C}_i')=0$ for any $i\in I$.

Proof. Fix an arbitrary $i\in I$. By the definition of the measure ν, we have

$$\nu(\mathcal{C}_i\setminus\mathcal{C}_i') = \int_X \left|r^{-1}(x)\cap(\mathcal{C}_i\setminus\mathcal{C}_i')\right| d\mu(x).$$

Put

$$E = \left\{x\in X\colon \left|r^{-1}(x)\cap(\mathcal{C}_i\setminus\mathcal{C}_i')\right| > 0\right\}.$$

Take any $a\in E$. Then there is a $b\in X$ such that $(a,b)\in\mathcal{C}_i\setminus\mathcal{C}_i'$. This means that there exists a $z\in N$ such that $(a,z)\in\mathcal{S}$ and $(\psi_i(z),b)\in\mathcal{S}$. Choose a countable subgroup H of the full group $[\mathcal{R}]$ such that $\mathcal{S}=\{(x,hx)\colon x\in X,\ h\in H\}$. Since $(a,z)\in\mathcal{S}$, it follows that $a\in\bigcup_{h\in H} hN$. Hence we obtain $E\subseteq\bigcup_{h\in H} hN$. Because N is a null set, so is $\bigcup_{h\in H} hN$. This implies that E is also a null set. Therefore, we conclude that $\nu(\mathcal{C}_i\setminus\mathcal{C}_i')=0$. □

For any $x\in X$ and $\lambda\in\Lambda$, define a subset $\mathscr{A}(x,\sigma,\lambda)$ of $\mathrm{Per}(I)$ by

$$\mathscr{A}(x,\sigma,\lambda) = \bigcup_{n=1}^{n_\lambda} K\sigma(x,\psi_{\lambda,n}(x))K.$$

Since K is compact, so is $\mathscr{A}(x,\sigma,\lambda)$. As before, we regard this assignment $x\in X\mapsto\mathscr{A}(x,\sigma,\lambda)$ as a map from X into the family $\mathscr{F}(\mathrm{Per}(I))$ of all the closed subsets of $\mathrm{Per}(I)$ equipped with the Fell topology.

Lemma 6.2. *For each $\lambda\in\Lambda$, the function $x\in X\mapsto\mathscr{A}(x,\sigma,\lambda)\in\mathscr{F}(\mathrm{Per}(I))$ is Borel.*

Proof. Recall that the Fell topology on $\mathscr{F}(\mathrm{Per}(I))$ has as a subbase all sets of the form V^-, where V is an open subset of $\mathrm{Per}(I)$, plus all sets of the form $(C^c)^+$, where C is a compact subset of $\mathrm{Per}(I)$. Hence it suffices to shows that the inverse images of V^- and $(C^c)^+$ under the map under consideration are Borel subsets in X for any open set V and any compact set C of $\mathrm{Per}(I)$. As before, let $g_0\colon X\to X\times X$ be $g_0(x)=(x,x)$. Then we have

$$\begin{aligned}\mathscr{A}(x,\sigma,\lambda)\in V^- &\iff \exists\theta\in\mathscr{A}(x,\sigma,\lambda)\cap V\\ &\iff \exists k_1,k_2\in K \text{ and } \exists n\in\{1,\dots,n_\lambda\} \text{ s.t. } k_1\sigma(x,\psi_{\lambda,n}(x))k_2\in V\end{aligned}$$

$$\iff \exists k_1, k_2 \in K \text{ and } \exists n \in \{1, \dots, n_\lambda\} \text{ s.t. } \sigma(x, \psi_{\lambda,n}(x)) \in k_1^{-1} V k_2^{-1}$$
$$\iff x \in \bigcup_{n=1}^{n_\lambda} (\sigma \circ (id_X \times \psi_{\lambda,n}) \circ g_0)^{-1}(KVK).$$

This shows that

$$\{x \in X \colon \mathscr{A}(x, \sigma, \lambda) \in V^-\} = \bigcup_{n=1}^{n_\lambda} (\sigma \circ (id_X \times \psi_{\lambda,n}) \circ g_0)^{-1}(KVK).$$

Clearly, the set on the right-hand side is a Borel subset of X. Similarly, one can show that

$$\{x \in X \colon \mathscr{A}(x, \sigma, \lambda) \in (C^c)^+\} = \bigcup_{n=1}^{n_\lambda} (\sigma \circ (id_X \times \psi_{\lambda,n}) \circ g_0)^{-1}(KC^cK).$$

This completes the proof. □

Lemma 6.3. *Let $\lambda \in \Lambda$. There exists a compact subset $\mathscr{A}(\sigma, \lambda)$ of* $\mathrm{Per}(I)$ *such that $\mathscr{A}(x, \sigma, \lambda) = \mathscr{A}(\sigma, \lambda)$ for a.e. $x \in X$. We have $\mathscr{A}(\sigma, 0) = K$.*

Proof. Assume that $(x, y) \in \mathcal{S}_0$, where $\mathcal{S}_0$ is the set defined in Lemma 3.1. Take any $\theta \in \mathscr{A}(x, \sigma, \lambda)$. By definition, there exist $k_1, k_2 \in K$ and an $n \in \{1, \dots, n_\lambda\}$ such that $\theta = k_1 \sigma(x, \psi_{\lambda,n}(x)) k_2$. By Lemma 3.1 (3), there is a unique $m \in \{1, \dots, n_\lambda\}$ such that $(\psi_{\lambda,n}(x), \psi_{\lambda,m}(y)) \in \mathcal{S}$. Then

$$\begin{aligned}\theta &= k_1 \sigma(x, \psi_{\lambda,n}(x)) k_2 \\ &= k_1 \underbrace{\sigma(x, y)}_{\text{in } K} \sigma(y, \psi_{\lambda,m}(y)) \underbrace{\sigma(\psi_{\lambda,m}(y), \sigma_{\lambda,n}(x))}_{\text{in } K} k_2 \in K \sigma(y, \psi_{\lambda,m}(y)) K \subseteq \mathscr{A}(y, \sigma, \lambda).\end{aligned}$$

Thus $\mathscr{A}(x, \sigma, \lambda) \subseteq \mathscr{A}(y, \sigma, \lambda)$. By changing the roles of x and y in the argument above, we obtain the reverse inclusion. Hence we have $\mathscr{A}(x, \sigma, \lambda) = \mathscr{A}(y, \sigma, \lambda)$. From this and Lemma 6.2, it follows that the Borel map $x \in X \mapsto \mathscr{A}(x, \sigma, \lambda) \in \mathscr{F}(\mathrm{Per}(I))$ is $\mathcal{S}$-invariant. From the ergodicity of $\mathcal{S}$, we find that this Borel map is constant up to a null set.

By definition, we have $\mathscr{A}(x, \sigma, 0) = K\sigma(x, \psi_0(x))K = KK = K$ for all $x \in X$, which implies that $\mathscr{A}(\sigma, 0) = K$. □

By Lemma 6.3, there exists a null subset N_σ of X such that $\mathscr{A}(\sigma, \lambda) = \mathscr{A}(x, \sigma, \lambda)$ for all $x \in (N_\sigma)^c$ and all $\lambda \in \Lambda$. We choose an $x_0 \in (N_\sigma)^c$ such that $\mathscr{A}(\sigma, \lambda) = \mathscr{A}(x_0, \sigma, \lambda)$ for all $\lambda \in \Lambda$. For each (λ, n), define $\theta_{\lambda,n} = \sigma(x_0, \psi_{\lambda,n}(x_0))$. So we have $\mathscr{A}(\sigma, \lambda) = \bigcup_{n=1}^{n_\lambda} K\theta_{\lambda,n}K$.

Lemma 6.4. *We have $\mathscr{A}(\sigma, \lambda) \cap \mathscr{A}(\sigma, \lambda') = \emptyset$ whenever $\lambda \neq \lambda'$.*

Proof. Assume that $\lambda \neq \lambda'$. Suppose that $\mathscr{A}(\sigma, \lambda) \cap \mathscr{A}(\sigma, \lambda') \neq \emptyset$. This means that there are elements $k_1, k_2, k_1', k_2' \in K$, $n \in \{1, \dots, n_\lambda\}$ and $n' \in \{1, \dots, n_{\lambda'}\}$ such that $k_1 \theta_{\lambda,n} k_2 = k_1' \theta_{\lambda',n'} k_2'$. Then the map on the left transforms $0 \in I$ to (λ, m) for some $m \in \{1, \ldots, n_\lambda\}$, while the map on the right transforms 0 to (λ', m') for some $m \in \{1, \ldots, n_{\lambda'}\}$. This is a contradiction. $\square$

Define $\mathscr{G}(\sigma) = \bigcup_{\lambda \in \Lambda} \mathscr{A}(\sigma, \lambda) \subseteq \mathrm{Per}(I)$.

Lemma 6.5. *For each $\lambda \in \Lambda$, the subset $\mathscr{A}(\sigma, \lambda)$ is compact and open in $\mathscr{G}(\sigma)$ with respect to the relative topology.*

Proof. We already know that every $\mathscr{A}(\sigma, \lambda)$ is compact. It suffices to prove that it is open in $\mathscr{G}(\sigma)$.

Fix an arbitrary $\lambda \in \Lambda$. Define $U_\lambda = \{(\lambda, 1), \ldots, (\lambda, n_\lambda)\}$. We agree that $U_0 = \{0\}$ if $\lambda = 0$. By the definition of the topology of $\mathrm{Map}(I)$, $p_0^{-1}(U_\lambda)$ is open (and closed) in $\mathrm{Map}(I)$. As we saw in the proof of Lemma 6.4, every element in $\mathscr{A}(\sigma, \lambda)$ sends $0 \in I$ to (λ, n) for some $n \in \{1, \ldots, n_\lambda\}$. This shows that (i) $\mathscr{A}(\sigma, \lambda) \subseteq p_0^{-1}(U_\lambda)$; (ii) $\mathscr{A}(\sigma, \lambda') \cap p_0^{-1}(U_\lambda) = \emptyset$ whenever $\lambda' \neq \lambda$. From this and Lemma 6.4, we find that

$$\mathscr{G}(\sigma) \cap p_0^{-1}(U_\lambda) = \bigcup_{\lambda' \in \Lambda} \mathscr{A}(\sigma, \lambda') \cap p_0^{-1}(U_\lambda) = \mathscr{A}(\sigma, \lambda) \cap p_0^{-1}(U_\lambda) = \mathscr{A}(\sigma, \lambda). \tag{6.1}$$

This proves that $\mathscr{A}(\sigma, \lambda)$ is open in $\mathscr{G}(\sigma)$ with respect to the relative topology. $\square$

Lemma 6.6. *The set $\mathscr{G}(\sigma)$ is closed in* $\mathrm{Per}(I)$.

Proof. Take any γ in the closure $\overline{\mathscr{G}(\sigma)}$ of $\mathscr{G}(\sigma)$. Thus there is a sequence $\{\gamma_j\}_{j=1}^\infty$ in $\mathscr{G}(\sigma)$ such that $\lim_{j\to\infty} \gamma_j = \gamma$. Suppose that $\gamma(0) = (\lambda, n) \in I$. Then $p_0^{-1}(\{(\lambda, n)\})$ is an open subset of $\mathrm{Per}(I)$ that contains γ. Hence there exists $j_0 \in \mathbb{N}$ such that $\gamma_j \in p_0^{-1}(\{(\lambda, n)\})$ for all $j \geq j_0$. From the proof of Lemma 6.5, we know that $\mathscr{G}(\sigma) \cap p_0^{-1}(\{(\lambda, n)\})$ is included in the compact set $\mathscr{A}(\sigma, \lambda)$. So $\gamma_j \in \mathscr{A}(\sigma, \lambda)$ for all $j \geq j_0$. Compactness of $\mathscr{A}(\sigma, \lambda)$ now implies that $\gamma \in \mathscr{A}(\sigma, \lambda) \subseteq \mathscr{G}(\sigma)$. Therefore, $\mathscr{G}(\sigma)$ is closed. $\square$

Proposition 6.7. *The set $\mathscr{G}(\sigma)$ is a locally compact (Hausdorff) space with respect to the relative topology from* $\mathrm{Map}(I)$.

Proof. By Lemma 6.5, every element $\theta \in \mathscr{G}(\sigma)$ has a compact and open neighborhood $\mathscr{A}(\sigma, \lambda)$ when $\theta \in \mathscr{A}(\sigma, \lambda)$. $\square$

Lemma 6.8. *We have $\sigma(x, y) \in \mathscr{G}(\sigma)$ for a.e. $(x, y) \in \mathcal{R}$.*

Proof. For each $\lambda \in \Lambda$, consider the subset $\mathcal{C}_\lambda$. By definition, we have

$$\mathcal{C}_\lambda = \{(x, y) \in \mathcal{R} \colon \exists z \in X \text{ s.t.} (x, z) \in \mathcal{S},\ (\psi_{\lambda,1}(z), y) \in \mathcal{S}\}.$$

Set

$$\mathcal{C}_\lambda(\sigma) = \{(x, y) \in \mathcal{R} \colon \exists z \in (N_\sigma)^c \text{ s.t.} (x, z) \in \mathcal{S},\ (\psi_{\lambda,1}(z), y) \in \mathcal{S}\}.$$

By Lemma 6.1, we have $\mathcal{C}_\lambda(\sigma)$ is conull in $\mathcal{C}_\lambda$. This then implies that $\bigcup_{\lambda \in \Lambda} \mathcal{C}_\lambda(\sigma)$ is conull in $\mathcal{R}$.

Let $(x, y) \in \bigcup_{\lambda \in \Lambda} \mathcal{C}_\lambda(\sigma)$. Then there is a unique $\lambda \in \Lambda$ such that $(x, y) \in \mathcal{C}_\lambda(\sigma)$. Hence there exists a $z \in (N_\sigma)^c$ such that $(x, z) \in \mathcal{S}$ and $(\psi_{\lambda,1}(z), y) \in \mathcal{S}$. Since $z \in (N_\sigma)^c$, we find that $\mathscr{A}(z, \sigma, \lambda) = \mathscr{A}(\sigma, \lambda)$. This implies that $\sigma(z, \psi_{\lambda,1}(z))$ belongs to $\mathscr{A}(\sigma, \lambda)$. Since $\mathscr{A}(\sigma, \lambda)$ is a two-sided K-invariant set, we have

$$\sigma(x, y) = \underbrace{\sigma(x, z)}_{\text{in } K} \cdot \underbrace{\sigma(z, \psi_{\lambda,1}(z))}_{\text{in } \mathscr{A}(\sigma,\lambda)} \cdot \underbrace{\sigma(\sigma_{\lambda,1}(z), y)}_{\text{in } K} \in \mathscr{A}(\sigma, \lambda).$$

Thus we are done. $\square$

As before, let H be a countable subgroup of the full group $[\mathcal{R}]$ such that $\mathcal{S} = \{(x, hx) \colon x \in X,\ h \in H\}$. Then define $X_\sigma := \left(\bigcup_{i \in I} \psi_i^{-1}(HN_\sigma)\right)^c$. Since N_σ is null and ψ_is are non-singular in the sense that $\psi_i^{-1}(N)$ is null whenever N is null, it follows that X_σ is a conull subset of X. Because $X_\sigma \subseteq (N_\sigma)^c$, we see that $\mathscr{A}(x, \sigma, \lambda) = \mathscr{A}(\sigma, \lambda)$ for all $x \in X_\sigma$ and all $\lambda \in \Lambda$.

In the next lemma, recall that for each $\lambda \in \Lambda$, there is a unique $\lambda^{-1} \in \Lambda$ such that $(\mathcal{C}_\lambda)^{-1} = \mathcal{C}_{\lambda^{-1}}$.

Lemma 6.9. *If $\theta \in \mathscr{A}(\sigma, \lambda)$ for some $\lambda \in \Lambda$, then $\theta^{-1} \in \mathscr{A}(\sigma, \lambda^{-1})$. In particular, if $\theta \in \mathscr{G}(\sigma)$, then $\theta^{-1} \in \mathscr{G}(\sigma)$.*

Proof. Let $a \in X_\sigma$ and $\theta \in \mathscr{A}(a, \sigma, \lambda)$. Thus there exists $k_1, k_2 \in K$ and an $n \in \{1, \dots, n_\lambda\}$ such that $\theta = k_1 \sigma(a, \psi_{\lambda,n}(a)) k_2$. So $\theta^{-1} = k_2^{-1} \sigma(\psi_{\lambda,n}(a), a) k_1^{-1}$. Since $(a, \psi_{\lambda,n}(a))$ belongs to $\mathcal{C}_{\lambda,n} = \mathcal{C}_\lambda$, we find that $(\psi_{\lambda,n}(a), a) \in (\mathcal{C}_\lambda)^{-1} = \mathcal{C}_{\lambda^{-1}}$. So there is a $z \in X$ such that $(\psi_{\lambda,n}(a), z) \in \mathcal{S}$ and $(\psi_{\lambda^{-1},1}(z), a) \in \mathcal{S}$. If $z \in N_\sigma$, then the fact that $(\psi_{\lambda,n}(a), z) \in \mathcal{S}$ would imply that $a \in \psi_{\lambda,n}^{-1}(HN_\sigma)$, which leads to a contradiction that a belongs to X_σ. Hence we have $z \in (N_\sigma)^c$. So $\mathscr{A}(z, \sigma, \lambda^{-1}) = \mathscr{A}(\sigma, \lambda^{-1})$. From this, it follows that

$$\begin{aligned}
\theta^{-1} &= k_2^{-1} \sigma(\psi_{\lambda,n}(a), a) k_1^{-1} \\
&= k_2^{-1} \sigma(\psi_{\lambda,n}(a), z) \sigma(z, \psi_{\lambda^{-1},1}(z)) \sigma(\psi_{\lambda^{-1},1}(z), a) k_1^{-1} \\
&= \underbrace{k_2^{-1} \sigma(\psi_{\lambda,n}(a), z)}_{\text{in } K} \cdot \sigma(z, \psi_{\lambda^{-1},1}(z)) \cdot \underbrace{\sigma(\psi_{\lambda^{-1},1}(z), a) k_1^{-1}}_{\text{in } K} \in \mathscr{A}(z, \sigma, \lambda^{-1}) = \mathscr{A}(\sigma, \lambda^{-1}).
\end{aligned}$$

This completes the proof. $\square$

For each $(\lambda, n) \in I$ and each $x \in X$, define a subset $F_{(\lambda,n)}(x)$ of $\{1, \dots, n_\lambda\}$ by

$$F_{(\lambda,n)}(x) := \{m \in \{1, \dots, n_\lambda\} \colon \sigma(x, \psi_{\lambda,m}(x)) \in K\theta_{\lambda,n}K\}.$$

By the definition of the set N_σ, we have for any $x \in (N_\sigma)^c$:

$$K\theta_{\lambda,n}K \subseteq \bigcup_{m=1}^{n_\lambda} K\theta_{\lambda,m}K = \mathscr{A}(x_0, \sigma, \lambda) = \mathscr{A}(x, \sigma, \lambda) = \bigcup_{m=1}^{n_\lambda} K\sigma(x, \psi_{\lambda,m}(x))K.$$

Thus there is at least one $m_x \in \{1, \ldots, n_\lambda\}$ such that

$$K\theta_{\lambda,n}K \cap K\sigma(x, \psi_{\lambda,m_x}(x))K \neq \emptyset.$$

From this, we immediately see that $\sigma(x, \psi_{\lambda,m_x}(x)) \in K\theta_{\lambda,n}K$. In particular, we obtain $m_x \in F_{(\lambda,n)}(x)$. It follows that $F_{(\lambda,n)}(x)$ is non-empty for all $x \in (N_\sigma)^c$.

We now set

$$f_{\lambda,n}(x) := |F_{(\lambda,n)}(x)| \qquad (x \in X).$$

By the result of the previous paragraph, we see that $f_{\lambda,n}(x) \geq 1$ for a.e. $x \in X$. We claim that $f_{\lambda,n}$ is constant almost everywhere. In fact, let $(x, y) \in \mathcal{S}_0$, where $\mathcal{S}_0$ is the subset of $\mathcal{S}$ defined in Lemma 3.1. Take any $l \in F_{(\lambda,n)}(x)$. Thus $\sigma(x, \psi_{\lambda,l}(x)) \in K\theta_{\lambda,n}K$. Meanwhile, there exists a unique $q(l) \in \{1, \ldots, n_\lambda\}$ such that $(\psi_{\lambda,l}(x), \psi_{\lambda,q(l)}(y)) \in \mathcal{S}$. Then we have

$$\sigma(y, \psi_{\lambda,q(l)}(y)) = \underbrace{\sigma(y, x)}_{\text{in } K} \cdot \sigma(x, \psi_{\lambda,l}(x)) \cdot \underbrace{\sigma(\psi_{\lambda,l}(x), \psi_{\lambda.q(l)}(y))}_{\text{in } K} \in K \cdot K\theta_{\lambda,n}K \cdot K = K\theta_{\lambda,n}K.$$

This shows that $q(l) \in F_{(\lambda,n)}(y)$. Thus we obtain a map $q\colon F_{(\lambda,n)}(x) \to F_{(\lambda,n)}(y)$. Suppose that $q(l_1) = q(l_2)$ for some $l_1, l_2 \in F_{(\lambda,n)}(x)$. Since

$$(\psi_{\lambda,l_1}(x), \psi_{\lambda,q(l_1)}(y)) \in \mathcal{S} \quad \text{and} \quad (\psi_{\lambda,l_2}(x), \psi_{\lambda,q(l_2)}(y)) \in \mathcal{S},$$

it follows that $(\psi_{\lambda,l_1}(x), \psi_{\lambda,l_2}(x)) \in \mathcal{S}$. Because $\{\psi_i\}$ are choice functions, we have to have $l_1 = l_2$, which shows that q is injective. Let $m \in F_{(\lambda,n)}(y)$. By changing the roles of x and y in the argument above, we get an element $l \in F_{(\lambda,n)}(x)$ such that $m = q(l)$. So q is surjective as well. Hence $q\colon F_{(\lambda,n)}(x) \to F_{(\lambda,n)}(y)$ is bijective. In particular, we have $f_{\lambda,n}(x) = |F_{(\lambda,n)}(x)| = |F_{(\lambda,n)}(y)| = f_{\lambda,n}(y)$. We have shown that $f_{\lambda,n}$ is $\mathcal{S}_0$-invariant. By the ergodicity of $\mathcal{S}$, we find that $f_{\lambda,n}$ is constant almost everywhere, as claimed.

By the result of the preceding paragraph, there exists a conull subset $Y(\sigma)$ of X contained in $(N_\sigma)^c$ such that $f_{\lambda,n}$ is constant on $Y(\sigma)$ for each $(\lambda, n) \in I$. We assert that we may assume from the outset that x_0 belongs to $Y(\sigma)$. Indeed, choose one element $x_1 \in Y(\sigma)$ and set

$$\begin{aligned} \theta'_{\lambda,n} &:= \sigma(x_1, \psi_{\lambda,n}(x_1)), \\ F'_{(\lambda,n)}(x) &:= \{m \in \{1, \ldots, n_\lambda\} : \sigma(x, \psi_{\lambda,m}(x)) \in K\theta'_{\lambda,n}K\}, \\ f'_{\lambda,n}(x) &:= |F'_{(\lambda,n)}(x)|. \end{aligned}$$

Since $x_1 \in Y(\sigma) \subseteq (N_\sigma)^c$, we have that $\mathscr{A}(x_1, \sigma, \lambda) = \mathscr{A}(x_0, \sigma, \lambda)$. Thus, for each $n \in \{1, \ldots, n_\lambda\}$, there exists an $n' \in \{1, \ldots, n_\lambda\}$ such that $K\theta'_{\lambda,n'}K = K\theta_{\lambda,n}K$. It follows that for each $n \in \{1, \ldots, n_\lambda\}$, there exists $n' \in \{1, \ldots, n_\lambda\}$ such that $f_{\lambda,n}$ is equal to $f'_{\lambda,n'}$. Then we obtain $f'_{\lambda,n'}(x) = f_{\lambda,n}(x) = f_{\lambda,n}(x_1) = f'_{\lambda,n'}(x_1)$ for any $x \in Y(\sigma)$. Hence what we have established concerning $\theta_{\lambda,n}$, $F_{(\lambda,n)}$ and $f_{\lambda,n}$ is also true for $\theta'_{\lambda,n}$, $F'_{(\lambda,n)}$ and $f'_{\lambda,n}$. From this, we see that we may assume that x_0 belongs to $Y(\sigma)$ by replacing x_0 by x_1.

We shall show that for each $\lambda \in \Lambda$, there exists a Borel map $\phi_\lambda \colon X \to P(\lambda)$ which satisfies the following:

$$\sigma(x, \psi_{\lambda,\phi_\lambda(x)(n)}(x)) \in K\theta_{\lambda,n}K \qquad (n \in \{1, \dots, n_\lambda\}).$$

In what follows, we fix an arbitrary $\lambda \in \Lambda$, unless otherwise stated. We then choose natural numbers $1 = m_1 < \cdots < m_{n'_\lambda} \le n_\lambda$ so that $K\theta_{\lambda,m_1}K, \dots, K\theta_{\lambda,m_{n'_\lambda}}K$ are all distinct and satisfy $\mathscr{A}(\sigma, \lambda) = \bigcup_{j=1}^{n'_\lambda} K\theta_{\lambda,m_j}K$. Then we have for any $x \in Y(\sigma)$:

$$\{1, \dots, n_\lambda\} = \bigcup_{j=1}^{n'_\lambda} F_{(\lambda,m_j)}(x). \tag{6.2}$$

Note that the union is disjoint, because $K\theta_{\lambda,m_1}K, \dots, K\theta_{\lambda,m_{n'_\lambda}}K$ are distinct. We set $F_j = F_{(\lambda,m_j)}(x_0)$ for each $j = 1, \dots, n'_\lambda$. So $\{F_j\}_{j=1}^{n'_\lambda}$ is a partition of $\{1, \dots, n_\lambda\}$.

Lemma 6.10. *Let $n \in \{1, \dots, n_\lambda\}$ and $k \in \{1, \dots, n'_\lambda\}$. Then n is in F_k if and only if $K\theta_{\lambda,n}K = K\theta_{\lambda,m_k}K$.*

Proof. Suppose that n is in F_k. So we have that $n \in F_{(\lambda,m_k)}(x_0)$. This means that $\theta_{\lambda,n} = \sigma(x_0, \psi_{\lambda,n}(x_0))$ belongs to $K\theta_{\lambda,m_k}K$. It follows that the two-sided cosets $K\theta_{\lambda,n}K$ and $K\theta_{\lambda,m_k}K$ with respect to the subgroup K are equal to each other.

Conversely, suppose that $K\theta_{\lambda,n}K$ is equal to $K\theta_{\lambda,m_k}K$. This implies that $\theta_{\lambda,n} = \sigma(x_0, \psi_{\lambda,n}(x_0))$ is in $K\theta_{\lambda,m_k}K$. It follows that $n \in F_{(\lambda,m_k)}(x_0) = F_k$. □

For each $k \in \{1, \dots, n'_\lambda\}$ and each subset F of $\{1, \dots, n_\lambda\}$, define $X_{k,F} = \{x \in X\colon F_{(\lambda,m_k)}(x) = F\}$.

Lemma 6.11. *$X_{k,F}$ is Borel.*

Proof. Note that $K\theta_{\lambda,n}K$ is a compact subset of $\mathrm{Per}(I)$ for each $n \in \{1, \dots, n_\lambda\}$, because K is. Since both σ and $\psi_{\lambda,n}$ are Borel for any n, so is the map $\alpha_n \colon X \to \mathrm{Per}(I)$ given by $\alpha_n(x) := \sigma(x, \psi_{\lambda,n}(x))$. It follows that $X_{k,n} := \alpha_n^{-1}(K\theta_{\lambda,m_k}K)$ is a Borel subset of X. We claim that $X_{k,F}$ is equal to $\bigcap_{n\in F} X_{k,n} \cap \bigcap_{n\in F^c}(X_{k,n})^c$. For this, suppose first that $x \in X_{k,F}$. So we have $F_{(\lambda,m_k)}(x) = F$. It follows that for each $n \in \{1, \dots, n_\lambda\}$, $\sigma(x, \psi_{\lambda,n}(x))$ is in $K\theta_{\lambda,m_k}K$ if and only if n is in F. This means that $x \in X_{k,n}$ for each $n \in F$ and $x \notin X_{k,n'}$ for each $n' \notin F$.

Conversely, suppose that $x \in \bigcap_{n\in F} X_{k,n} \cap \bigcap_{n\in F^c}(X_{k,n})^c$. Then $\sigma(x, \psi_{\lambda,n}(x)) \in K\theta_{\lambda,m_k}K$ for any $n \in F$, and $\sigma(x, \psi_{\lambda,n'}(x)) \notin K\theta_{\lambda,m_k}K$ for each $n' \in F$. It means that $F_{(\lambda,m_k)}(x)$ is equal to F, so that $x \in X_{k,F}$. Thus our claim has been proven.

Since each $X_{k,n}$ is Borel, we conclude that $X_{k,F}$ is also Borel. □

Lemma 6.12. *For any fixed $k \in \{1, \dots, n'_\lambda\}$, the family*

$$\{X_{k,F}\colon F \subseteq \{1, \dots, n_\lambda\},\ |F| = |F_k|\}$$

is a Borel partition of X.

Proof. Fix $k \in \{1, \ldots, n'_\lambda\}$. By Lemma 6.11, we know that each $X_{k,F}$ is Borel. It is obvious that $x \in X_{k,F_{(\lambda,m_k)}(x)}$ for all $x \in Y(\sigma)$. Moreover, we have $|F_{(\lambda,m_k)}(x)| = f_{\lambda,m_k}(x) = |F_k|$.

We next show that $X_{k,F} \cap X_{k,F'} = \emptyset$ whenever $F \neq F'$ with $|F| = |F'| = |F_k|$. Indeed, if x belongs to $X_{k,F} \cap X_{k,F'}$, then $F = F_{(\lambda,m_k)}(x) = F'$. $\square$

For each subset F of $\{1, \ldots, n_\lambda\}$ with $|F_k|$ elements, there exists $|F_k|!$ bijections from F_k to F. Let us choose and fix one such map from them and denote it by $\kappa_{k,F}$.

Lemma 6.13. *Fix any $k \in \{1, \ldots, n'_\lambda\}$ and any $F \subseteq \{1, \ldots, n'_\lambda\}$ with $|F| = |F_k|$. Then $\sigma(x, \psi_{\lambda,\kappa_{k,F}(n)}(x))$ is in $K\theta_{\lambda,m_k}K = K\theta_{\lambda,n}K$ for any $n \in F_k$ and any $x \in X_{k,F}$.*

Proof. Fix $k \in \{1, \ldots, n'_\lambda\}$ and $F \subseteq \{1, \ldots, n'_\lambda\}$ with $|F| = |F_k|$. Let $n \in F_k$ and $x \in X_{k,F}$. Since $\kappa_{k,F}(n)$ is in $F = F_{(\lambda,m_k)}(x)$, it follows from Lemma 6.10 that $\sigma(x, \psi_{\lambda,\kappa_{k,F}(n)}(x))$ belongs to $K\theta_{\lambda,m_k}K = K\theta_{\lambda,n}K$. $\square$

For each $x \in X$, we define a map $\phi_\lambda(x)$ from $\{1, \ldots, n_\lambda\}$ into itself by the following rule:

- If $x \in Y(\sigma)^c$, then define $\phi_\lambda(x) = id$. Otherwise, proceed to the next step.
- Take any $n \in \{1, \ldots, n_\lambda\}$. Then there is a unique $k \in \{1, \ldots, n'_\lambda\}$ such that $n \in F_k$. By Lemma 6.12, there exists a unique $F \subseteq \{1, \ldots, n_\lambda\}$ with $|F| = |F_k|$ such that $x \in X_{k,F}$. Remark that F is realized as $F_{(\lambda,m_k)}(x)$.
- Define $\phi_\lambda(x)(n) = \kappa_{k,F}(n)$. Equivalently, $\phi_\lambda(x)(n) = \kappa_{k,F_{(\lambda,m_k)}(x)}(n)$ by the remark above.

We will show below that $\phi_\lambda(x)$ thus defined is a bijection.

Lemma 6.14. *Let $n \in \{1, \ldots, n_\lambda\}$ and $x \in Y(\sigma)$. Then $\sigma(x, \psi_{\lambda,\phi_\lambda(x)(n)}(x)) \in K\theta_{\lambda,n}K$.*

Proof. Let n and x be as above. If $n \in F_k$, we have that $\phi_\lambda(x)(n) = \kappa_{k,F_{(\lambda,m_k)}(x)(n)}(n) \in F_{(\lambda,m_k)}(x)$. It follows from the definition of $F_{(\lambda,m_k)}(x)$ and Lemma 6.10 that $\sigma(x, \psi_{\lambda,\phi_\lambda(x)(n)}(x)) \in K\theta_{\lambda,m_k}K = K\theta_{\lambda,n}K$. $\square$

Lemma 6.15. *For any $x \in X$, the map $\phi_\lambda(x)$ is in $P(\lambda)$.*

Proof. By definition, $\phi_\lambda(x)$ is bijective if $x \in Y(\sigma)^c$. So let $x \in Y(\sigma)$. It suffices to show that $\phi_\lambda(x)$ is injective. Suppose that $\phi_\lambda(x)(n) = \phi_\lambda(x)(n')$ with $n \in F_k$ and $n' \in F_{k'}$. This implies that

$$\kappa_{k,F_{(\lambda,m_k)(x)(n)}}(n) = \kappa_{k',F_{(\lambda,m_{k'})(x)(n)}}(n') \text{ and } \sigma(x, \psi_{\phi_\lambda(x)(n)}(x)) \in K\theta_{\lambda,m_k}K \cap K\theta_{\lambda,m_{k'}}K.$$

Thus $k = k'$. It follows that $\kappa_{k,F_{(\lambda,m_k)}(x)}(n) = \kappa_{k,F_{(\lambda,m_k)}(x)}(n')$. Since $\kappa_{k,F_{(\lambda,m_k)}(x)}$ is a bijection, we conclude that $n = n'$. $\square$

Lemma 6.16. *The map $\phi_\lambda \colon X \to P(\lambda)$ is Borel.*

Proof. It suffices to show that $\{x \in X : \phi_\lambda(x)(n) = m\}$ is Borel for any $n, m \in \{1, \ldots, n_\lambda\}$. Note that

$$\{x \in X : \phi_\lambda(x)(n) = m\} = \{x \in Y(\sigma)^c : \phi_\lambda(x)(n) = m\} \cup \{x \in Y(\sigma) : \phi_\lambda(x)(n) = m\}.$$

The first subset on the right-hand side equals either $Y(\sigma)^c$ or $\emptyset$, depending upon $n = m$ or not. In any case, it is Borel. So we examine the second subset. Suppose that $n \in F_k$. We then claim the following identity:

$$\{x \in Y(\sigma) : \phi_\lambda(x)(n) = m\} = \bigcup_{F \subseteq \{1,\ldots,n_\lambda\},\, \kappa_{k,F}(n)=m} X_{k,F} \cap Y(\sigma). \tag{6.3}$$

Indeed, if $x \in Y(\sigma)$ satisfies $\phi(x)(n) = m$, then we have that $\kappa_{k,F_{(\lambda,m_k)}(x)}(n) = m$. Moreover, we clearly have $x \in X_{k,F_{(\lambda,m_k)}(x)}$. Conversely, if $x \in X_{k,F} \cap Y(\sigma)$ for some F which satisfies $\kappa_{k,F}(n) = m$, then, by Lemma 6.12, we have that $F = F_{(\lambda,m_k)}(x)$ and $\phi_\lambda(x)(n) = \kappa_{k,F}(n) = m$. So we our claim has been proven.

Since each $X_{k,F}$ is a Borel subset of X by Lemma 6.11, it follows from the claim stated above that $\{x \in Y(\sigma) : \phi_\lambda(x)(n) = m\}$ is also Borel. □

For each $(\lambda, n) \in I$, we define a map $\psi'_{\lambda,n} : X \to X$ by $\psi'_{\lambda,n}(x) := \psi_{\lambda,\phi_\lambda(x)(n)}(x)$.

Lemma 6.17. *For any $(\lambda, n) \in I$, the map $\psi'_{\lambda,n}$ defined above is Borel and nonsingular, and satisfies $\Gamma(\psi'_{\lambda,n}) \subseteq \mathcal{R}$.*

Proof. Let $(\lambda, n) \in I$. We immediately see that $\Gamma(\psi'_{\lambda,n}) \subseteq \mathcal{R}$, due to the fact that $\Gamma(\psi_{\lambda,n}) \subseteq \mathcal{R}$.

Take any Borel subset E of X. Clearly, we have

$$(\psi'_{\lambda,n})^{-1}(E) = \big((\psi'_{\lambda,n})^{-1}(E) \cap Y(\sigma)^c\big) \cup \big((\psi'_{\lambda,n})^{-1}(E) \cap Y(\sigma)\big).$$

From the definition of $\psi'_{\lambda,n}$, we see that $(\psi'_{\lambda,n})^{-1}(E) \cap Y(\sigma)^c = \psi^{-1}_{\lambda,n}(E) \cap Y(\sigma)^c$, which is Borel. In the meantime, it is easy to check the following identity:

$$(\psi'_{\lambda,n})^{-1}(E) \cap Y(\sigma) = \bigcup_{m=1}^{n_\lambda} \{x \in Y(\sigma) : \phi_\lambda(x)(n) = m\} \cap \psi^{-1}_{\lambda,m}(E).$$

The subset on the right-hand side is Borel, as we saw in the proof of Lemma 6.16 (see Equation (6.3)). Therefore, $\psi'_{\lambda,n}$ is Borel, as desired. From the two identities displayed above (and the nonsingularity of $\psi_{\lambda,n}$), we easily find that $(\psi'_{\lambda,n})^{-1}(E)$ is null if E is. □

Lemma 6.18. *If $n_1, n_2 \in \{1, \ldots, n_\lambda\}$ satisfy $n_1 \neq n_2$, then $\mathcal{S}(\psi'_{\lambda,n_1}(x)) \cap \mathcal{S}(\psi'_{\lambda,n_2}(x)) = \emptyset$ for a.e. $x \in X$.*

Proof. Since $\{\psi_{\lambda,n}\}_{(\lambda,n)\in I}$ are choice functions for $\mathcal{S} \subseteq \mathcal{R}$, there exists a conull set X_0 of X such that for each $x \in X_0$, one has $\mathcal{S}(\psi_{\lambda,l}(x)) \cap \mathcal{S}(\psi_{\lambda,m}(x)) = \emptyset$ if $l, m \in \{1, \ldots, n_\lambda\}$ with $l \neq m$. Now let $x \in X_0$. By Lemma 6.15, we know that the map $\phi_\lambda(x)$ is injective. So $\phi_\lambda(x)(n_1) \neq \phi_\lambda(x)(n_2)$. Hence we conclude that

$$\mathcal{S}(\psi'_{\lambda,n_1}(x)) \cap \mathcal{S}(\psi'_{\lambda,n_2}(x)) = \mathcal{S}(\psi_{\lambda,\phi_\lambda(x)(n_1)}(x)) \cap \mathcal{S}(\psi_{\lambda,\phi_\lambda(x)(n_2)}(x)) = \emptyset.$$

Thus we are done. □

Lemma 6.19. *We have*

$$\{\psi'_{\lambda,n}(x) \colon 1 \le n \le n_\lambda\} = \{\psi_{\lambda,n}(x) \colon 1 \le n \le n_\lambda\}$$

for any $x \in X$.

Proof. Fix any $x \in X$. Since $\phi_\lambda(x)$ is bijective by Lemma 6.15, it follows that

$$\{\psi'_{\lambda,n}(x) \colon 1 \le n \le n_\lambda\} = \{\psi_{\lambda,\phi_\lambda(x)(n)}(x) \colon 1 \le n \le n_\lambda\} = \{\psi_{\lambda,n}(x) \colon 1 \le n \le n_\lambda\}.$$

So we complete the proof. □

Lemma 6.20. *The maps $\{\psi'_{\lambda,n}\}_{(\lambda,n)\in I}$ are choice functions for $\mathcal{S} \subseteq \mathcal{R}$.*

Proof. This follows from Lemma 6.17, Lemma 6.18 and Lemma 6.19. □

Lemma 6.21. *For each $n \in \{1,\dots,n_\lambda\}$, there exist an $m \in \{1,\dots,n_\lambda\}$ and a nonnull Borel subset F of X such that $\phi_\lambda(x)(n) = m$ for all $x \in F$.*

Proof. For any $n, m \in \{1,\dots,n_\lambda\}$, put $X_{n,m} := \{x \in X \colon \phi_\lambda(x)(n) = m\}$. By the proof of Lemma 6.16, $\{X_{n,m}\}_{m=1}^{n_\lambda}$ is a Borel finite partition of X for each $n \in \{1,\dots,n_\lambda\}$. So there exists an $m \in \{1,\dots,n_\lambda\}$ such that $X_{n,m}$ is nonnull. Put $F := X_{n,m}$ and we get the conclusion. □

Remark. Lemma 6.21 can be strengthened as follows. Let E be a nonnull Borel subset of X. Then, for each $n \in \{1,\dots,n_\lambda\}$, there exist an $m \in \{1,\dots,n_\lambda\}$ and a nonnull Borel subset F of E such that $\phi_\lambda(x)(n) = m$ for all $x \in F$. Indeed, if we consider $X'_{n,m} = \{x \in E \colon \phi_\lambda(x)(n) = m\}$ instead of $X_{n,m}$ in the proof of Lemma 6.21, then $\{X'_{n,m}\}_{m=1}^{n_\lambda}$ is a Borel finite partition of E for each $n \in \{1,\dots,n_\lambda\}$.

Lemma 6.22. *For any $n_1, n_2 \in \{1,\dots,n_\lambda\}$, we have $\mathcal{S}(\psi'_{\lambda,n_1}(\mathcal{S}(x))) = \mathcal{S}(\psi'_{\lambda,n_2}(\mathcal{S}(x)))$ for a.e. $x \in X$.*

Proof. By [**1**, Theorem 3.8 (4)], there exists a null subset N_0 of X such that

$$\mathcal{S}(\psi_{\lambda,l}(\mathcal{S}(x))) = \mathcal{S}(\psi_{\lambda,m}(\mathcal{S}(x))) \text{ for all } l, m \in \{1,\dots,n_\lambda\} \text{ and all } x \in (N_0)^c.$$

Meanwhile, it follows from Lemma 6.21 that for each $l \in \{1,\dots,n_\lambda\}$, there exist an $n_l \in \{1,\dots,n_\lambda\}$ and a nonnull subset F_l of X such that $\phi_\lambda(z)(l) = n_l$ for all $x \in F_l$. By [**1**, Theorem 3.8 (3)], for each $l \in \{1,\dots,n_\lambda\}$, there is a null subset N_l of X such that $\mathcal{S}(\psi_{\lambda,n_l}(\mathcal{S}(x))) = \mathcal{S}(\psi_{\lambda,n_l}|_{F_l}(\mathcal{S}(x)))$ for all $x \in (N_l)^c$. We then set $X_\lambda = \left(N_0 \cup \bigcup_{l=1}^{n_\lambda} N_l\right)^c$, which is a conull subset of X.

Let $x \in X_\lambda$ be an arbitrary element, and let $l \in \{1,\dots,n_\lambda\}$.

Take any $y \in \mathcal{S}(\psi'_{\lambda,l}(\mathcal{S}(x)))$. So there is a $z \in X$ such that $(x,z) \in \mathcal{S}$ and $(\psi'_{\lambda,l}(z), y) \in \mathcal{S}$. Thus $(\psi_{\lambda,\phi_\lambda(z)(l)}(z), y) \in \mathcal{S}$, which implies that $y \in \mathcal{S}(\psi_{\lambda,\phi_\lambda(z)(l)}(\mathcal{S}(x)))$. Hence

y is in $\bigcup_{m=1}^{n_\lambda}\mathcal{S}(\psi_{\lambda,m}(\mathcal{S}(x)))=\mathcal{S}(\psi_{\lambda,1}(\mathcal{S}(x)))$. Therefore, we obtain $\mathcal{S}(\psi'_{\lambda,l}(\mathcal{S}(x)))\subseteq\mathcal{S}(\psi_{\lambda,1}(\mathcal{S}(x)))$.

Conversely, we have

$$\begin{aligned}\mathcal{S}(\psi'_{\lambda,l}(\mathcal{S}(x))) &= \{y\in X\colon \exists z\in X \text{ s.t.} (x,z)\in\mathcal{S} \text{ and } (\psi'_{\lambda,l}(z),y)\in\mathcal{S}\}\\ &\supseteq \{y\in X\colon \exists z\in F_l \text{ s.t.} (x,z)\in\mathcal{S} \text{ and } (\psi'_{\lambda,l}(z),y)\in\mathcal{S}\}\\ &= \{y\in X\colon \exists z\in F_l \text{ s.t.} (x,z)\in\mathcal{S} \text{ and } (\psi_{\lambda,\phi_\lambda(z)(l)}(z),y)\in\mathcal{S}\}\\ &= \{y\in X\colon \exists z\in F_l \text{ s.t.} (x,z)\in\mathcal{S} \text{ and } (\psi_{\lambda,n_l}(z),y)\in\mathcal{S}\}\\ &= \mathcal{S}(\psi_{\lambda,n_l}|_{F_l}(\mathcal{S}(x)))\\ &= \mathcal{S}(\psi_{\lambda,n_l}(\mathcal{S}(x)))\\ &= \mathcal{S}(\psi_{\lambda,1}(\mathcal{S}(x))).\end{aligned}$$

Therefore, we obtain $\mathcal{S}(\psi'_{\lambda,l}(\mathcal{S}(x)))=\mathcal{S}(\psi_{\lambda,1}(\mathcal{S}(x)))$ for all $l\in\{1,\dots,n_\lambda\}$ and all $x\in X_\lambda$. □

Corollary 6.23. *Let $(\lambda,l)\in I$ and E be a nonnull Borel subset of X. Then we have $\mathcal{S}(\psi'_{\lambda,l}(\mathcal{S}(x)))=\mathcal{S}(\psi'_{\lambda,l}|_E(\mathcal{S}(x)))$ for a.e. $x\in X$.*

Proof. Fix an arbitrary $(\lambda,l)\in I$. Let N_0 be the null subset in the proof of Lemma 6.22. By the remark just before Lemma 6.22, we see that for each $n\in\{1,\dots,n_\lambda\}$, there exist an $m_n\in\{1,\dots,n_\lambda\}$ and a nonnull subset E_n of E such that $\phi_\lambda(z)(n)=m_n$ for all $x\in E_n$. As in the proof of Lemma 6.22, we may choose a null subset N'_n of X such that $\mathcal{S}(\psi_{\lambda,m_n}(\mathcal{S}(x)))=\mathcal{S}(\psi_{\lambda,m_n}|_{E_n}(\mathcal{S}(x)))$ for all $x\in(N'_n)^c$. We then set $X'_\lambda=\left(N_0\cup\bigcup_{n=1}^{n_\lambda}N'_n\right)^c$, which is a conull subset of X. Let $x\in X'_\lambda$. From the proof of Lemma 6.22 again, we find that

$$\begin{aligned}\mathcal{S}(\psi_{\lambda,1}(\mathcal{S}(x))) &= \mathcal{S}(\psi'_{\lambda,l}(\mathcal{S}(x)))\\ &\supseteq \mathcal{S}(\psi'_{\lambda,l}|_E(\mathcal{S}(x)))\\ &\supseteq \mathcal{S}(\psi'_{\lambda,l}|_{E_l}(\mathcal{S}(x)))\\ &= \mathcal{S}(\psi_{\lambda,m_l}|_{E_l}(\mathcal{S}(x)))\\ &= \mathcal{S}(\psi_{\lambda,m_l}(\mathcal{S}(x)))\\ &= \mathcal{S}(\psi_{\lambda,1}(\mathcal{S}(x))).\end{aligned}$$

Therefore, we obtain $\mathcal{S}(\psi'_{\lambda,l}(\mathcal{S}(x)))=\mathcal{S}(\psi'_{\lambda,l}|_E(\mathcal{S}(x)))$. □

Proposition 6.24. *Let $\lambda\in\Lambda$. For any $n_1,n_2\in\{1,\dots,n_\lambda\}$, we have $K\theta_{\lambda,n_1}K=K\theta_{\lambda,n_2}K$*

Proof. It suffices to show that $K\theta_{\lambda,n_1}K\cap K\theta_{\lambda,n_2}K$ is not empty. By Corollary 6.23, there is a conull subset X_0 of X such that $\mathcal{S}(\psi'_{\lambda,n_1}(\mathcal{S}(x)))=\mathcal{S}(\psi'_{\lambda,n_2}|_{Y(\sigma)}(\mathcal{S}(x)))$ for any $x\in X_0$. Set $Z_0:=Y(\sigma)\cap X_0$, which is again conull. Take any $x\in Z_0$. So $\mathcal{S}(\psi'_{\lambda,n_1}(\mathcal{S}(x)))=\mathcal{S}(\psi'_{\lambda,n_2}|_{Y(\sigma)}(\mathcal{S}(x)))$. Since $\psi'_{\lambda,n_1}(x)\in\mathcal{S}(\psi'_{\lambda,n_1}(\mathcal{S}(x)))=\mathcal{S}(\psi'_{\lambda,n_2}|_{Y(\sigma)}(\mathcal{S}(x)))$, there is some $z\in Y(\sigma)$ such that

$$(x,z)\in\mathcal{S} \text{ and } (\psi'_{\lambda,n_2}(z),\psi'_{\lambda,n_2}(x))\in\mathcal{S}.$$

Remark that by Lemma 6.14, $\sigma(x,\psi'_{\lambda,n_1}(x))$ belongs to $K\theta_{\lambda,n_1}K$, while $\sigma(z,\psi'_{\lambda,n_2}(z))$ belongs to $K\theta_{\lambda,n_2}K$. It follows that

$$K\theta_{\lambda,n_1}K\ni\sigma(x,\psi'_{\lambda,n_1}(x))=\underbrace{\sigma(x,z)}_{\in K}\cdot\underbrace{\sigma(z,\psi'_{\lambda,n_2}(z))}_{\in K\theta_{\lambda,n_2}K}\cdot\underbrace{\sigma(\psi'_{\lambda,n_2}(z),\psi'_{\lambda,n_2}(x))}_{\in K}\in K\theta_{\lambda,n_2}K.$$

Thus we complete the proof. □

Thanks to Proposition 6.24, we immediately obtain the following

Corollary 6.25. *For any $\lambda\in\Lambda$, we have $\mathscr{A}(\sigma,\lambda)=K\theta_{\lambda,1}K=\cdots=K\theta_{\lambda,n_\lambda}K$.*

Lemma 6.26. *Let $\lambda\in\Lambda$. Then the following are equivalent:*

(1) *For any $k\in K$ and any $n\in\{1,\ldots,n_\lambda\}$, $k\theta_{\lambda,n}$ is contained in $\theta_{k(\lambda,n)}K$.*

(2) *For any $n\in\{1,\ldots,n_\lambda\}$, $K\theta_{\lambda,n}K$ is equal to $\bigcup_{m=1}^{n_\lambda}\theta_{\lambda,m}K$.*

(3) *For any $n\in\{1,\ldots,n_\lambda\}$, $\sigma(x,\psi_{\lambda,n}(x))$ is contained in $\theta_{\lambda,n}K$ for a.e. $x\in X$.*

Proof. ((1) ⇒ (2)): Let $n\in\{1,\ldots,n_\lambda\}$. By assumption, we see that $K\theta_{\lambda,n}$ is included in $\bigcup_{m=1}^{n_\lambda}\theta_{\lambda,m}K$ (recall that every element of K moves only n in (λ,n)). Thus we obtain $K\theta_{\lambda,n}K\subseteq\bigcup_{m=1}^{n_\lambda}\theta_{\lambda,m}K$. For the reverse inclusion, just note that we have $\theta_{\lambda,m}K\subseteq K\theta_{\lambda,m}K=K\theta_{\lambda,n}K$, due to Proposition 6.24.

((2) ⇒ (3)): Let $n\in\{1,\ldots,n_\lambda\}$. Take any $x\in(N_\sigma)^c$. By Corollary 6.25, we find that

$$\bigcup_{m=1}^{n_\lambda}\theta_{\lambda,m}K=K\theta_{\lambda,n}K=\mathscr{A}(\sigma,\lambda)=\mathscr{A}(x,\sigma,\lambda)=\bigcup_{m=1}^{n_\lambda}K\sigma(x,\psi_{\lambda,m}(x))K.$$

It follows that $\sigma(x,\psi_{\lambda,n}(x))$ belongs to $\theta_{\lambda,m}K$ for some $m\in\{1,\ldots,n_\lambda\}$. Note that $\sigma(x,\psi_{\lambda,n}(x))(0)=(\lambda,n)$, while every element in $\theta_{\lambda,m}K$ sends $0\in I$ to (λ,m). Thus we must have $m=n$.

((3) ⇒ (1)): Define

$$K_0:=\{\tau\in K\colon\tau\theta_{\lambda,n}\in\theta_{\tau(\lambda,n)}K\,(\forall n\in\{1,\ldots,n_\lambda\})\}.$$

Let $\tau_1,\tau_2\in K_0$ and $n\in\{1,\ldots,n_\lambda\}$. By definition, there is some $k_2\in K$ such that $\tau_2\theta_{\lambda,n}=\theta_{\tau_2(\lambda,n)}k_2$. Note that $\tau_2(\lambda,n)$ is of the form $\tau_2(\lambda,n)=(\lambda,m)$ for some $m\in\{1,\ldots,n_\lambda\}$. Because τ_1 is in K_0, it follows that there exists a $k_1\in K$ such that $\tau_1\theta_{\lambda,m}=\theta_{\tau_1(\lambda,m)}k_1$. Thus

$$\tau_1\tau_2\theta_{\lambda,n}=\tau_1\theta_{\tau_2(\lambda,n)}k_2=\tau_1\theta_{\lambda,m}k_2=\theta_{\tau_1(\lambda,m)}k_1k_2=\theta_{\tau_1\tau_2(\lambda,n)}(k_1k_2).$$

This proves that $\tau_1\tau_2$ belongs to K_0.

Fix any $n\in\{1,\ldots,n_\lambda\}$. Since τ_1^{-1} is in K, $\tau_1^{-1}(\lambda,n)$ has the form $\tau_1^{-1}(\lambda,n)=(\lambda,l)$ for some $l\in\{1,\ldots,n_\lambda\}$. For this l, k_1 being in K_0, there is some $k\in K$ such that

$\tau_1\theta_{\lambda,l} = \theta_{\tau_1(\lambda,l)}k$. From this, we find that $\tau_1^{-1}\theta_{\lambda,n} = \theta_{\tau_1^{-1}(\lambda,n)}k^{-1}$. This proves that τ_1^{-1} is in K_0. From the current and the previous paragraphs, we see that K_0 is a subgroup of K. One can show without difficulty that K_0 is closed.

Now we proceed to the proof of the implication (3) $\Rightarrow$ (1). Since K is by definition generated by $\sigma(\mathcal{S})$, it suffices by the result of the preceding paragraph to show that K_0 contains $\sigma(\mathcal{S})$. By assumption, there is a null subset N of X such that $\sigma(x, \psi_{\lambda,n}(x)) \in \theta_{\lambda,n}K$ for all $x \in N^c$ and all $n \in \{1, \ldots, n_\lambda\}$. Choose a countable subgroup H of the full group $[\mathcal{R}]$ such that $\mathcal{S} = \{(x, hx) \colon x \in X,\ h \in H\}$. With $\underline{\mathrm{N}}$ defined just before Lemma 3.1, set $X_0 = \left(\bigcup_{h\in H} h(N \cup \underline{\mathrm{N}})\right)^c$, which is a conull $\mathcal{S}$-invariant subset of X. Then we consider the essential reduction $\mathcal{S}_1 := \mathcal{S} \cap (X_0 \times X_0)$ of $\mathcal{S}$ to X_0. From the minimality of the 1-cocycle $\sigma|_{\mathcal{S}} \colon \mathcal{S} \to K$, we find that the closed subgroup generated by $\sigma(\mathcal{S}_1)$ coincides with K itself. Hence it is enough to show that K_0 contains $\sigma(\mathcal{S}_1)$. Take any $(x, y) \in \mathcal{S}_1$ and any $n \in \{1, \ldots, n_\lambda\}$. Since $y \in X_0 \subseteq N^c$, we have $\sigma(y, \psi_{\lambda,n}(y)) \in \theta_{\lambda,n}K$, so that there exists a $k \in K$ such that $\theta_{\lambda,n} = \sigma(y, \psi_{\lambda,n}(y))k$. Hence we obtain

$$\sigma(x, y)\theta_{\lambda,n} = \sigma(x, y)\sigma(y, \psi_{\lambda,n}(y))k = \sigma(x, \psi_{\lambda,n}(y)) \cdot k.$$

Because $(x, y) \in \mathcal{S}_1 \subseteq \mathcal{S}_0$, where $\mathcal{S}_0$ is defined in Lemma 3.1, it follows that there is an $l \in \{1, \ldots, n_\lambda\}$ such that $(\psi_{\lambda,l}(x), \psi_{\lambda,n}(y)) \in \mathcal{S}$. This means that $\sigma(x, y)(\lambda, n) = (\lambda, l)$. Note that since $x \in X_0 \subseteq N^c$, we have $\sigma(x, \psi_{\lambda,l}(x)) = \theta_{\lambda,l}k'$ for some $k' \in K$. Thus

$$\begin{aligned}\sigma(x, y)\theta_{\lambda,n} &= \sigma(x, \psi_{\lambda,l}(x))\sigma(\psi_{\lambda,l}(x), \psi_{\lambda,n}(y))k \\ &= \theta_{\lambda,l}k'\sigma(\psi_{\lambda,l}(x), \psi_{\lambda,n}(y))k \\ &= \theta_{\sigma(x,y)(\lambda,n)} \cdot \underbrace{k'\sigma(\psi_{\lambda,l}(x), \psi_{\lambda,n}(y))k}_{\text{in } K} \in \theta_{\sigma(x,y)(\lambda,n)}K.\end{aligned}$$

Therefore, $\sigma(x, y)$ belongs to K_0. $\square$

Lemma 6.27. *If the equivalent conditions in Lemma* 6.26 *hold for each $\lambda \in \Lambda$, then the set $\mathscr{G}(\sigma)$ is a subgroup of* $\mathrm{Per}(I)$.

Proof. Thanks to Lemma 6.9, we already know that $\mathscr{G}(\sigma)$ is closed under taking the inverse operation. Hence we show below that $\mathscr{G}(\sigma)$ is closed under the group multiplication. For this, it suffices by (2) to show that $\theta_{i_1}k\theta_{i_2} \in \mathscr{G}(\sigma)$ for any $k \in K$ and any $i_1, i_2 \in I$. So let us fix arbitrary $k \in K$ and $i_1, i_2 \in I$. By (3), there exists a null subset N of X such that $\sigma(x, \psi_i(x)) \in \theta_i K$ for all $x \in N^c$ and all $i \in I$. Set $X_0 := \left(\bigcup_{i\in I} \psi_i^{-1}(N)\right)^c$, which is conull in X. Take any $z \in X_0$. Then $\sigma(z, \psi_{i_1}(z)) \in \theta_{i_1}K$. So θ_{i_1} has the form

$$\theta_{i_1} = \sigma(z, \psi_{i_1}(z))k_1$$

for some k_1. Then, by (1), $k_1k\theta_{i_2}$ is in $\theta_{k_1k(i_2)}K$, so that there is a $k_2 \in K$ such that $k_1k\theta_{i_2} = \theta_{k_1k(i_2)}k_2$. Since $\sigma(\psi_{i_1}(z), \psi_{k_1k(i_2)}(\psi_{i_1}(z)))$ belongs to $\theta_{k_1k(i_2)}K$ thanks to (3), we find that

$$\theta_{k_1k(i_2)} = \sigma(\psi_{i_1}(z), \psi_{k_1k(i_2)}(\psi_{i_1}(z)))k_3$$

for some $k_3 \in K$. Define $i_3 := \sigma(z, \psi_{i_1}(z))(k_1 k(i_2)) \in I$. By definition, we have $(\psi_{i_3}(z), \psi_{k_1 k(i_2)}(\psi_{i_1}(z))) \in \mathcal{S}$. Hence it follows that

$$\begin{aligned}
\theta_{i_1} k \theta_{i_2} &= \sigma(z, \psi_{i_1}(z)) k_1 k \theta_{i_2} \\
&= \sigma(z, \psi_{i_1}(z)) \cdot \theta_{k_1 k(i_2)} k_2 \\
&= \sigma(z, \psi_{i_1}(z)) \cdot \sigma(\psi_{i_1}(z), \psi_{k_1 k(i_2)}(\psi_{i_1}(z))) k_3 k_2 \\
&= \sigma(z, \psi_{k_1 k(i_2)}(\psi_{i_1}(z))) k_3 k_2 \\
&= \underbrace{\sigma(z, \psi_{i_3}(z))}_{\text{in } \theta_{i_3} K \text{ by (3)}} \cdot \underbrace{\sigma(\psi_{i_3}(z), \psi_{k_1 k(i_2)}(\psi_{i_1}(z))) k_3 k_2}_{\text{in } K} \in \theta_{i_3} K \subseteq \mathscr{G}(\sigma).
\end{aligned}$$

So $\mathscr{G}(\sigma)$ is a subgroup of $\mathrm{Per}(I)$. □

Let Λ_0 be a finite subset of Λ and denote by π_{Λ_0} the projection $\prod_{\lambda \in \Lambda} P(\lambda)$ onto $\prod_{\lambda \in \Lambda_0} P(\lambda)$. Define a 1-cocycle $c_{\Lambda_0} \colon \mathcal{S} \to \prod_{\lambda \in \Lambda_0} P(\lambda)$ by

$$c_{\Lambda_0} = \pi_{\Lambda_0} \circ c.$$

Thanks to [**23**, Proposition 3.10], c_{Λ_0} is a minimal cocycle. Write L_{Λ_0} for the subgroup of $\prod_{\lambda \in \Lambda_0} P(\lambda)$ generated by $c_{\Lambda_0}(\mathcal{S})$. As explained just before Lemma 5.4, we have $r^*(c_{\Lambda_0}) = L_{\Lambda_0}$. It follows that for each $g \in L_{\Lambda_0}$, there exists $\rho \in [\mathcal{S}]_*$ such that $c_{\Lambda_0}(x, \rho(x))$ is equal to g for all $x \in \mathrm{Dom}(\rho)$.

Put $\mathcal{P}_{\Lambda_0} := \mathrm{Ker}(c_{\Lambda_0})$. Since L_{Λ_0} is a finite group, we have that $\mathcal{P}_{\Lambda_0}$ is an ergodic subrelation of $\mathcal{S}$.

Lemma 6.28. *For a.e. $x \in X$, we have*

$$\{c_{\Lambda_0}(x, z) \colon z \in \mathcal{S}(x)\} = \{c_{\Lambda_0}(y, x) \colon y \in \mathcal{S}(x)\} = L_{\Lambda_0}.$$

Proof. For each $x \in X$, set $L_{\Lambda_0}(x) := \{c_{\Lambda_0}(x, z) \colon z \in \mathcal{S}(x)\}$. A direct computation shows that $L_{\Lambda_0}(x)$ is equal to $L_{\Lambda_0}(y)$ for a.e. $(x, y) \in \mathcal{P}_{\Lambda_0}$. By the ergodicity of $\mathcal{P}_{\Lambda_0}$, there exist a subset S of L_{Λ_0} and a conull Borel subset X_0 of X such that $L_{\Lambda_0}(x) = S$ for all $x \in X_0$. Take any $g \in L_{\Lambda_0}$. As remarked just before this lemma, there is $\rho \in [\mathcal{S}]_*$ such that $c_{\Lambda_0}(x, \rho(x)) = g$ for all $x \in \mathrm{Dom}(\rho)$. Choose one $x_0 \in X_0 \cap \mathrm{Dom}(\rho)$. Then

$$S = L_{\Lambda_0}(x_0) \ni c_{\Lambda_0}(x_0, \rho(x_0)) = g.$$

Therefore, $S = L_{\Lambda_0}$. □

Proposition 6.29. *The set $\mathscr{G}(\sigma)$ is a subgroup of* $\mathrm{Per}(I)$.

Proof. It suffices to show that Lemma 6.26 (3) always holds.

Fix any $(\lambda, n) \in I$ and set $E := \{x \in X \colon \sigma(x, \psi_{\lambda,n}(x)) \notin \theta_{\lambda,n} K\}$. We will prove below that E is a null set in X.

Suppose that E is not null. Since $\theta_{\lambda,n} K$ is closed, the condition $\sigma(x, \psi_{\lambda,n}(x)) \notin \theta_{\lambda,n} K$ is equivalent to the one that some (open) neighborhood of $\sigma(x, \psi_{\lambda,n}(x))$ does not intersect with $\theta_{\lambda,n} K$, which is in turn equivalent to saying that there is a finite subset F of I such that, with the notation $p_a\colon \mathrm{Map}(I) \to I$ in Section 4, we have $\bigcap_{i\in F} p_i^{-1}(\{\sigma(x,\psi_{\lambda,n}(x))(i)\}) \cap \theta_{\lambda,n}K = \emptyset$. Hence, if we define $\mathcal{F}(I)$ to be the family of finite subsets of I, and if we define, for any $F \in \mathcal{F}(I)$, a Borel set X_F by

$$X_F := \{x \in X\colon \forall k \in K,\, \exists i_k \in F \text{ such that } \sigma(x,\psi_{\lambda,n}(x))(i_k) \neq \theta_{\lambda,n}k(i_k)\},$$

then we obtain

$$E = \bigcup_{F\in\mathcal{F}(I)} X_F.$$

Because E is non-null, there exists $F_0 \in \mathcal{F}(I)$ such that X_{F_0} is non-null. Note that if F_0 is of the form $F_0 = \{(\lambda_1, l_1), \ldots, (\lambda_m, l_m)\}$, then, for every $k \in K$, kF_0 has the form $kF_0 = \{(\lambda_1, l_1'), \ldots, (\lambda_m, l_m')\}$. This observation ensures that we may assume if necessary that $kF_0 = F_0$ for any $k \in K$. With the map $q_\Lambda\colon I \to \Lambda$ given by $q_\Lambda((\lambda', n')) = \lambda'$, define a finite subset Λ_0 of Λ by the following:

$$\Lambda_0 := q_\Lambda(\{\theta_{\lambda,n}(i)\colon i \in F_0\} \cup \{(\lambda, n)\}).$$

We assert that X_{F_0} is $\mathcal{P}_{\Lambda_0}$-invariant. To verify this, assume that $x \in X_{F_0}$ and $(x,y) \in \mathcal{P}_{\Lambda_0}$. Since $(x,y) \in \mathcal{P}_{\Lambda_0}$, we have $\sigma(x,y)|_{\{(\lambda,l)\colon l\in\{1,\ldots,n_\lambda\}\}} = id$, which implies that $\sigma(x,y)(\lambda,n) = (\lambda,n)$, i.e., $(\psi_{\lambda,n}(y), \psi_{\lambda,n}(x)) \in \mathcal{S}$. Thus $\sigma(\psi_{\lambda,n}(y), \psi_{\lambda,n}(x)) \in K$. Take any $k \in K$. Because $x \in X_{F_0}$, there exists $i_0 \in F_0$ such that $\sigma(x,\psi_{\lambda,n}(x))(i_0) \neq \theta_{\lambda,n}k\sigma(\psi_{\lambda,n}(y), \psi_{\lambda,n}(x))(i_0)$. Set $j_0 := \sigma(\psi_{\lambda,n}(y), \psi_{\lambda,n}(x))(i_0)$, which also belongs to F_0 since F_0 is K-invariant. Then we have that

$$\begin{aligned}
\sigma(y,\psi_{\lambda,n}(y))(j_0) &= \sigma(y,x)\sigma(x,\psi_{\lambda,n}(x))\sigma(\psi_{\lambda,n}(x),\psi_{\lambda,n}(y))(j_0)\\
&= \sigma(y,x)\sigma(x,\psi_{\lambda,n}(x))(i_0)\\
&\neq \sigma(y,x)\theta_{\lambda,n}k\sigma(\psi_{\lambda,n}(y),\psi_{\lambda,n}(x))(i_0)\\
&= \sigma(y,x)\theta_{\lambda,n}k(j_0)\\
&= \theta_{\lambda,n}k(j_0) \qquad (\because q_\Lambda(\theta_{\lambda,n}(j_0)) \in \Lambda_0).
\end{aligned}$$

So we conclude that y is also in X_{F_0}, which proves the $\mathcal{P}_{\Lambda_0}$-invariance of X_{F_0}, as asserted. From the ergodicity of $\mathcal{P}_{\Lambda_0}$, it follows that X_{F_0} is conull.

On the other hand, we already have shown in the present section that $\sigma(x,\psi_{\lambda,n}(x)) \in K\theta_{\lambda,n}K$ for a.e. $x \in X$. Thus there is a conull Borel subset X_0 such that $\sigma(x,\psi_{\lambda,n}(x)) \in K\theta_{\lambda,n}K$ for all $x \in X_0$. Let $z = \theta_{\lambda,n}K$ be the point in the quotient Polish space $\mathrm{Per}(I)/K$. We think of $\mathrm{Per}(I)/K$ as a Polish K-space. Then the K-orbit S of the point z is Borel (see [**22**, Corollary 5.8] for example). Let K_z be the stabilizer subgroup of z. Since K is compact, the canonical map $kK_z \in K/K_z \longmapsto kz \in S$ is a homeomorphism (refer to [**24**, Chapter 2]). We denote by f the inverse of this homeomorphism. Next we choose a Borel cross section $s\colon K/K_z \to K$ with $s(K_z) = e$. Then define a Borel map $\xi\colon X_0 \to K$ by

$$\xi(x) := s(f(\sigma(x,\psi_{\lambda,n}(x))K)) \qquad (x \in X_0).$$

Because $\sigma(x, \psi_{\lambda,n}(x)) \in K\theta_{\lambda,n}K$ for all $x \in X_0$, the equation

$$\eta(x) := \theta_{\lambda,n}^{-1}\xi(x)^{-1}\sigma(x, \psi_{\lambda,n}(x)) \qquad (x \in X_0)$$

defines a Borel map η from X_0 into K. Thus we have

$$\sigma(x, \psi_{\lambda,n}(x)) = \xi(x)\theta_{\lambda,n}\eta(x) \qquad (x \in X_0)$$

Set $\pi_{\Lambda_0}(\xi(x)) := \prod_{\lambda\in\Lambda_0} \pi_\lambda(\xi(x))$ $(x \in X_0)$. Note that $\pi_{\Lambda_0}(K)$ is a finite group, and let $\{g_0 = id, \ldots, g_M\}$ be its enumeration. If $X_0(j) := \{x \in X_0 \colon \pi_{\Lambda_0}(\xi(x)) = g_j\}$ $(j = 0, \ldots, M)$, then at least one of $X_0(j)$ is a non-null Borel set, because $X_0 = \bigcup_{j=0}^{M} X_0(j)$. Let $X_0(j_1)$ be such a set. By Lemma 6.28, there exists $\rho \in [\mathcal{S}]_*$ such that $\mathrm{Dom}(\rho) \subseteq X_0(j_1)$ and the equation

$$c_{\Lambda_0}(x, \rho(x)) = \pi_{\Lambda_0}(\xi(x)) = g_{j_1}$$

holds for each $x \in \mathrm{Dom}(\rho)$. Hence if $i \in F_0$ and $x \in \mathrm{Dom}(\rho)$, then we have

$$\begin{aligned}\sigma(\rho(x), \psi_{\lambda,n}(x))(i) &= \sigma(\rho(x), x)\xi(x)\theta_{\lambda,n}\eta(x)(i)\\ &= g_{j_1}^{-1}g_{j_1}(\theta_{\lambda,n}\eta(x)(i)) \qquad (\because\ q_\Lambda(\theta_{\lambda,n}\eta(x)(i)) \in \Lambda_0)\\ &= \theta_{\lambda,n}\eta(x)(i).\end{aligned}$$

Thus we have shown

$$\sigma(\rho(x), \psi_{\lambda,n}(x))(i) = \theta_{\lambda,n}\eta(x)(i) \qquad \text{for all } i \in F_0. \tag{6.4}$$

This holds true even if $i=0$, since $q_\Lambda(\theta_{\lambda,n}\eta(x)(0)) = \lambda \in \Lambda_0$. In particular, by the definition of $\theta_{\lambda,n}$, we get $\sigma(\rho(x), \psi_{\lambda,n}(x))(0) = \theta_{\lambda,n}\eta(x)(0) = (\lambda, n)$, so that $(\psi_{\lambda,n}(\rho(x)), \psi_{\lambda,n}(x)) \in \mathcal{S}$. It follows that for each $i \in F_0$, we have

$$\begin{aligned}&\sigma(\rho(x), \psi_{\lambda,n}(\rho(x)))(i)\\ &\quad= \sigma(\rho(x), \psi_{\lambda,n}(x))\sigma(\psi_{\lambda,n}(x), \psi_{\lambda,n}(\rho(x)))(i)\\ &\quad= \theta_{\lambda,n}\eta(x)\sigma(\psi_{\lambda,n}(x), \psi_{\lambda,n}(\rho(x)))(i) \qquad (\because\ (6.4) \text{ and } K\text{-invariance of } F_0).\end{aligned}$$

Since $\eta(x)\sigma(\psi_{\lambda,n}(x), \psi_{\lambda,n}(\rho(x))) \in K$, the result just obtained above indicates that the non-null subset $\mathrm{Im}(\rho)$ of X is included in the null set $(X_{F_0})^c$. This is a contradiction. Therefore, we complete the proof. $\square$

Definition 6.30. We call the pair $(\mathscr{G}(\sigma), K(\sigma))$ a Schlichting completion of the Hecke pair $(\mathcal{R}, \mathcal{S})$. As we proved in this section, $\mathscr{G}(\sigma)$ is a locally compact Hausdorff totally disconnected group, and $K(\sigma)$ is an open and compact subgroup of $\mathscr{G}(\sigma)$. Hence $(\mathscr{G}(\sigma), K(\sigma))$ is a Hecke pair of groups.

7. Dependency of the construction of the pair $(\mathscr{G}(\sigma), K(\sigma))$.

7.1. Dependency of the construction.

In the preceding section, we saw that every set of *preferable* choice functions for a Hecke pair $(\mathcal{R}, \mathcal{S})$ of ergodic equivalence relations produces a Hecke pair $(\mathscr{G}(\sigma), K(\sigma))$ of groups in $\mathrm{Per}(I)$. The purpose of this section is to clarify how the construction of $(\mathscr{G}(\sigma), K(\sigma))$ depends on the choice of preferable choice functions.

Let $\sigma \colon \mathcal{R} \to \mathrm{Per}(I)$ be the index cocycle that arises from the preferable choice functions $\{\psi_{\lambda,n}\}$ of the Hecke pair $(\mathcal{R}, \mathcal{S})$ which we have been considering so far. Thus $c := \sigma|_{\mathcal{S}}$ is a minimal cocycle into the compact subgroup $\prod_{\lambda \in \Lambda} P(\lambda)$ of $\mathrm{Per}(I)$. As before, we set $K := K(\sigma)$, the closed subgroup of $\prod_{\lambda \in \Lambda} P(\lambda)$ generated by $c(\mathcal{S})$.

Let $\sigma' \colon \mathcal{R} \to \mathrm{Per}(I)$ be another index cocycle derived from another family of preferable choice functions $\{\psi'_{\lambda,n}\}_{(\lambda,n)\in I}$ of $\mathcal{S} \subseteq \mathcal{R}$. Since σ' is cohomologous to σ, there is a Borel function $\phi \colon X \to \mathrm{Per}(I)$ such that

$$\sigma'(x,y) = \phi(x)\sigma(x,y)\phi(y)^{-1} \tag{7.1}$$

for a.e. $(x,y) \in \mathcal{R}$. Thus there is an $\mathcal{R}$-invariant Borel conull subset X' of X such that (7.1) holds true for all $(x,y) \in \mathcal{R} \cap (X' \times X')$. Since $\{\psi'_{\lambda,n}\}_{(\lambda,n)\in I}$ is preferable, the restriction $c' := \sigma'|_{\mathcal{S}}$ of σ' to $\mathcal{S}$ is a minimal 1-cocycle from $\mathcal{S}$ into the compact group $\prod_{\lambda\in\Lambda} P(\lambda)$. Let H be the closed subgroup of $\prod_{\lambda\in\Lambda} P(\lambda)$ generated by $c'(\mathcal{S})$. By definition, we have

$$c'(x,y) = \phi(x)c(x,y)\phi(y)^{-1} \quad \text{that is,} \quad c'(x,y)\phi(y) = \phi(x)c(x,y) \tag{7.2}$$

for a.e. $(x,y) \in \mathcal{S}$. It follows that there exists an $\mathcal{S}$-invariant Borel conull subset $X(0)$ of X, contained in X' defined above, such that with $\mathcal{S}(0) := \mathcal{S} \cap (X(0) \times X(0))$, which is a conull subset of $\mathcal{S}$, we have

$$c'(x,y) = \phi(x)c(x,y)\phi(y)^{-1} \qquad (\forall (x,y) \in \mathcal{S}(0)). \tag{7.3}$$

Multiplying both sides of the second identity of (7.2) by H from the left and by K from the right, we obtain

$$H\phi(y)K = H\phi(x)K. \tag{7.4}$$

It follows that the assignment $x \in X \longmapsto H\phi(x)K \in H\backslash\mathrm{Per}(I)/K$ from X into the double coset space $H\backslash\mathrm{Per}(I)/K$ is a Borel $\mathcal{S}$-invariant function. Since $H\backslash\mathrm{Per}(I)/K$ is a standard Borel space, we find that this function is constant up to a null set. So there exist an element $\tau \in \mathrm{Per}(I)$ and an $\mathcal{S}$-invariant conull subset $X(1)$ of X, contained in $X(0)$ introduced above, such that

$$H\phi(x)K = H\tau K \tag{7.5}$$

for all $x \in X(1)$. Define a Borel function $\phi_0 \colon X \to \mathrm{Per}(I)$ by

$$\phi_0(x) = \begin{cases} \phi(x) & (x \in X(1)), \\ \tau & (x \in X \setminus X(1)). \end{cases}$$

Then ϕ_0 obviously satisfies (7.5) for all $x \in X$. We consider the set

$$A := \{(x, h) \in X \times H \colon h^{-1}\phi_0(x)K = \tau K\}.$$

With two Borel functions $f_1 \colon X \to \mathrm{Per}(I)/K$ and $f_2 \colon H \to \mathrm{Per}(I)/K$ given by $f_1(x) = \phi_0(x)K$ and $f_2(h) = h\tau K$, we have $A = \{(x, h) \colon f_1(x) = f_2(h)\}$. It follows that A is a Borel subset of $X \times H$. For each $x \in X$, the section $A_x = \{h \in H \colon (x, h) \in A\}$ of A at x is nonempty, thanks to (7.5). Moreover, since $A_x = \{h \in H \colon h\tau K = \phi_0(x)K\}$, we easily see that A_x is closed, hence compact. By [**19**, Theorem 5.12.1], there is a Borel function $\ell \colon X \to H$ such that $(x, \ell(x)) \in A$ for all $x \in X$. This means that

$$\phi_0(x)K = \ell(x)\tau K \qquad (\forall x \in X). \tag{7.6}$$

By (7.6), we obtain a Borel function $r \colon X \to K$ satisfying

$$\phi_0(x) = \ell(x)\tau r(x) \qquad (\forall x \in X). \tag{7.7}$$

Let $\mathcal{S}_1 := \mathcal{S} \cap (X(1) \times X(1))$, which is a conull subset of $\mathcal{S}$. By (7.3), we have

$$c'(x, y) = \ell(x)\tau r(x)c(x, y)r(y)^{-1}\tau^{-1}\ell(y)^{-1} \qquad (\forall (x, y) \in \mathcal{S}_1) \qquad \text{or,}$$

$$\ell(x)^{-1}c'(x, y)\ell(y) = \tau \cdot r(x)c(x, y)r(y)^{-1} \cdot \tau^{-1} \qquad (\forall (x, y) \in \mathcal{S}_1). \tag{7.8}$$

Note that the function $c_1(x, y) = r(x)c(x, y)r(y)^{-1}$ $((x, y) \in \mathcal{S})$ is a Borel 1-cocycle, cohomologous to c, whose image is contained in K. Because c is minimal, it follows that the closed subgroup K_{c_1} generated by $c_1(\mathcal{S})$ equals K. Likewise, the closed subgroup K_{c_2} generated by the 1-cocycle c_2 defined by $c_2(x, y) = \ell(x)^{-1}c''(x, y)\ell(y)$ is equal to H. From this and (7.8), we find that

$$H = \tau K \tau^{-1}. \tag{7.9}$$

Let N_σ be as before. From Lemma 6.8, we see that there is an $\mathcal{R}$-invariant Borel conull subset Z_σ of X such that we have $\sigma(x, y) \in \mathscr{G}(\sigma)$ for all $(x, y) \in \mathcal{R} \cap (Z_\sigma \times Z_\sigma)$. Set $Z'_\sigma := \left(\bigcup_{i \in I}(\psi'_i)^{-1}((Z_\sigma)^c)\right)^c$, which is again an $\mathcal{R}$-invariant conull subset.

We also know that there is a null set $N_{\sigma'}$ of X such that $\mathscr{A}(\sigma', \lambda) = \mathscr{A}(x, \sigma', \lambda)$ for all $x \in (N_{\sigma'})^c$. Set

$$X(2) = X(1) \cap Z_\sigma \cap \left(\bigcup_{g \in G}\bigcup_{i \in I} g(\psi'_i)^{-1}(N_{\sigma'})\right)^c.$$

By definition, $X(2)$ is an $\mathcal{R}$-invariant conull set. Let $x \in X(2)$.

$$\begin{aligned} H\sigma'(x, \psi'_{\lambda,n}(x))H &= H\phi(x)\sigma(x, \psi'_{\lambda,n}(x))\phi(\psi'_{\lambda,n}(x))^{-1}H \\ &= H\ell(x)\tau r(x)\sigma(x, \psi'_{\lambda,n}(x))r(\psi'_{\lambda,n}(x))^{-1}\tau^{-1}\ell(\psi'_{\lambda,n}(x))^{-1}H \end{aligned}$$

$$
\begin{aligned}
&= H\tau r(x)\sigma(x,\psi'_{\lambda,n}(x))r(\psi'_{\lambda,n}(x))^{-1}\tau^{-1}H \\
&= \tau\cdot\tau^{-1}H\tau\cdot r(x)\sigma(x,\psi'_{\lambda,n}(x))r(\psi'_{\lambda,n}(x))^{-1}\cdot\tau^{-1}H\tau\cdot\tau^{-1} \\
&= \tau\cdot K\cdot r(x)\sigma(x,\psi'_{\lambda,n}(x))r(\psi'_{\lambda,n}(x))^{-1}\cdot K\cdot\tau^{-1} \\
&= \tau\cdot K\sigma(x,\psi'_{\lambda,n}(x))K\cdot\tau^{-1} \\
&\subseteq \tau K\mathscr{G}(\sigma)K\tau^{-1} \\
&= \tau\mathscr{G}(\sigma)\tau^{-1}.
\end{aligned}
$$

From this, we obtain

$$\mathscr{A}(x,\sigma',\lambda) = \bigcup_{n=1}^{n_\lambda} H\sigma'(x,\psi'_{\lambda,n}(x))H \subseteq \tau\mathscr{G}(\sigma)\tau^{-1}.$$

It follows that $\mathscr{G}(\sigma') \subseteq \tau\mathscr{G}(\sigma)\tau^{-1}$. Reversing the roles of σ and σ', we get $\mathscr{G}(\sigma) \subseteq \tau^{-1}\mathscr{G}(\sigma')\tau$. Therefore, we conclude that

$$\mathscr{G}(\sigma') = \tau\mathscr{G}(\sigma)\tau^{-1}. \tag{7.10}$$

We now summarize what we have obtained so far in the theorems that follow.

THEOREM 7.1. *Let $\{\psi_{\lambda,n}\}_{(\lambda,n)\in I}$ and $\{\psi'_{\lambda,n}\}_{(\lambda,n)\in I}$ be two preferable choice functions of the Hecke pair $(\mathcal{R},\mathcal{S})$. Let σ and σ' be the corresponding index cocycles respectively. Then the associated pairs $(\mathscr{G}(\sigma),K(\sigma))$ and $(\mathscr{G}(\sigma'),K(\sigma'))$ are conjugate in* $\mathrm{Per}(I)$. *To be precise, there exists an element $\tau\in\mathrm{Per}(I)$ such that*

$$\mathscr{G}(\sigma') = \tau\mathscr{G}(\sigma)\tau^{-1}, \qquad K(\sigma') = \tau K(\sigma)\tau^{-1}. \tag{7.11}$$

Hence the conjugacy class of the pair $(\mathscr{G}(\sigma),K(\sigma))$ in $\mathrm{Per}(I)$ *is independent of the choice of preferable choice functions of the Hecke pair $(\mathcal{R},\mathcal{S})$.*

THEOREM 7.2. *Let $\{\psi_{\lambda,n}\}_{(\lambda,n)\in I}$ be preferable choice functions of the Hecke pair $(\mathcal{R},\mathcal{S})$ and $\sigma\colon\mathcal{R}\to\mathrm{Per}(I)$ be the corresponding index cocycle.*

(1) *Let $r\colon X\to K(\sigma)$ be a Borel function, $\{\rho_i\}_{i\in I}$ be a family of maps in $[\mathcal{S}]_*$ with* $\mathrm{Dom}(\rho_i) = X$, *and $\tau\in\mathrm{Per}(I)$ be such that it satisfies $\tau K(\sigma)\tau^{-1}\subseteq\prod_{\lambda\in\Lambda}P(\lambda)$. For each $i\in I$, define a function $\psi'_i\colon X\to X$ by $\psi'_i(x) := \rho_i(\psi_{r(x)^{-1}\tau^{-1}(i)}(x))$. Then $\{\psi'_i\}$ are also preferable choice functions of $(\mathcal{R},\mathcal{S})$ whose associated index cocycle σ' satisfies $(\mathscr{G}(\sigma'),K(\sigma')) = (\tau\mathscr{G}(\sigma)\tau^{-1},\tau K(\sigma)\tau^{-1})$.*

(2) *Every set of preferable choice functions of $(\mathcal{R},\mathcal{S})$ arises in the manner described in* (1).

PROOF. (1) First, we prove that every ψ'_i is Borel. It suffices to show that the map $\gamma_i(x) := \psi_{r(x)^{-1}\tau^{-1}(i)}(x)$ is Borel for each $i\in I$. Fix an arbitrary $(\lambda,n)\in I$ and write $\tau^{-1}(\lambda,n) = (\lambda',n')$. Note that $r(x)^{-1}$ has the form $r(x)^{-1}(\lambda',n') = (\lambda',f(x)(n'))$, where $f(x)\in P(\lambda')$. Since $f\colon X\to P(\lambda')$ is Borel, $\{f^{-1}(\{\xi\})\colon \xi\in P(\lambda')\}$ is a Borel partition of X. Take any $E\in\mathfrak{B}$ and $\xi\in P(\lambda')$. For $x\in f^{-1}(\{\xi\})$, we have

$$\begin{aligned} x \in (\gamma_{\lambda,n})^{-1}(E) &\iff \psi_{r(x)^{-1}(\lambda',n')}(x) \in E \\ &\iff \psi_{\lambda',f(x)(n')}(x) \in E \\ &\iff x \in (\psi_{\lambda',\xi(n')})^{-1}(E). \end{aligned}$$

From this, it follows that one has

$$(\gamma_{\lambda,n})^{-1}(E) = \bigsqcup_{\xi \in P(\lambda')} (\psi'_{\lambda,n})^{-1}(E) \cap f^{-1}(\{\xi\}) = \bigsqcup_{\xi \in P(\lambda)} (\psi_{\lambda',\xi(n')})^{-1}(E) \cap f^{-1}(\{\xi\}).$$

This shows that $(\gamma_{\lambda,n})^{-1}(E)$ is a Borel subset. Therefore, $\psi'_{\lambda,n}$ is Borel.

The fact that $\{\psi'_{\lambda,n}\}$ are choice functions can be easily verified, so we leave the verification to the readers.

Let σ' be the index cocycle associated with $\{\psi'_{\lambda,n}\}$. Suppose that $\sigma'(x,y)(i) = j$. This means that $(\psi'_j(x), \psi'_i(y)) \in \mathcal{S}$, i.e., $(\psi_{r(x)^{-1}\tau^{-1}(j)}(x), \psi_{r(y)^{-1}\tau^{-1}(i)}(y)) \in \mathcal{S}$, because all ρ_i are in $[\mathcal{S}]_*$. So

$$\sigma(x,y)(r(y)^{-1}\tau^{-1}(i)) = r(x)^{-1}\tau^{-1}(j) = r(x)^{-1}\tau^{-1}(\sigma'(x,y)(i)).$$

It follows that

$$\sigma'(x,y) = \tau r(x)\sigma(x,y)r(y)^{-1}\tau^{-1} \qquad ((x,y) \in \mathcal{R}). \tag{7.12}$$

Define $c := \sigma|_{\mathcal{S}}$. By assumption, c is a minimal cocycle into $\prod_{\lambda\in\Lambda} P(\lambda)$ whose image generates $K(\sigma)$ as a closed subgroup.

Claim. *The restriction c' of σ' to $\mathcal{S}$ is a minimal cocycle in the compact group $\prod_{\lambda\in\Lambda} P(\lambda)$. The closed subgroup generated by $c'(\mathcal{S})$ is $\tau K(\sigma)\tau^{-1}$. (Note that since τ satisfies $\tau K(\sigma)\tau^{-1} \subseteq \prod_{\lambda\in\Lambda} P(\lambda)$, $\tau K(\sigma)\tau^{-1}$ is a compact group.)*

($\because$) The function $c_1(x,y) = r(x)c(x,y)r(y)^{-1}$ $((x,y) \in \mathcal{S})$ is a Borel 1-cocycle, cohomologous to c, whose image is contained in $K(\sigma)$. From the minimality of c, it follows that the closed subgroup K_{c_1} generated by $c_1(\mathcal{S})$ equals $K(\sigma)$. By (7.11), we find that the closed subgroup generated by $c'(\mathcal{S})$ is $\tau K(\sigma)\tau^{-1}$. So c' is a 1-cocycle of $\mathcal{S}$ into the compact group $\tau K(\sigma)\tau^{-1}$, and the closed subgroup generated by $c'(\mathcal{S})$ is $\tau K(\sigma)\tau^{-1}$.

Take any Borel function $q\colon X \to \tau K(\sigma)\tau^{-1}$, and set $c''(x,y) = q(x)c'(x,y)q(y)^{-1}$ $((x,y) \in \mathcal{S})$. Then $t(x) := \tau^{-1}q(x)\tau \cdot r(x)$ $(x \in X)$ is a Borel function from X into $K(\sigma)$ and satisfies

$$c_2(x,y) := t(x)c(x,y)t(y)^{-1} \in K(\sigma) \qquad ((x,y) \in \mathcal{S}).$$

Thus c_2 is a Borel 1-cocycle of $\mathcal{S}$ into $K(\sigma)$, cohomologous to c. From the minimality of c, it follows that the closed subgroup of $K(\sigma)$ generated by c_2 equals $K(\sigma)$. This in turn implies that

$$c''(x,y) = q(x)c'(c,y)q(y)^{-1} = \tau \cdot t(x)c(x,y)t(y)^{-1} \cdot \tau^{-1} = \tau c_2(x,y)\tau^{-1}$$

generates, as a closed subgroup, $\tau K(\sigma)\tau^{-1}$. This proves that every 1-cocycle into

$\tau K(\sigma)\tau^{-1}$ cohomologous to c' engenders as a closed subgroup $\tau K(\sigma)\tau^{-1}$. Hence c' is a minimal cocycle into $\tau K(\sigma)\tau^{-1}$. By [**23**, Theorem 3.9], c' is still a minimal cocycle into the compact group $\prod_{\lambda\in\Lambda} P(\lambda)$.

By *Claim* above, we see that $\{\psi'_{\lambda,n}\}$ are also preferable choice functions.

(2) Let $\{\psi'_{\lambda,n}\}_{(\lambda,n)\in I}$ be another family of preferable choice functions of $(\mathcal{R},\mathcal{S})$ and $\sigma'\colon \mathcal{R}\to \mathrm{Per}(I)$ be the associated index cocycle. Since σ' is cohomologous to σ, there is a Borel function $\phi\colon X\to \mathrm{Per}(I)$ such that

$$\sigma'(x,y)=\phi(x)\sigma(x,y)\phi(y)^{-1} \tag{7.13}$$

for a.e. $(x,y)\in\mathcal{R}$. In fact, due to [**10**, Lemma 1.2 (b)], the function ϕ can be characterized by the identity

$$\mathcal{S}(\psi'_i(x))=\mathcal{S}(\psi_{\phi(x)^{-1}(i)}(x)) \qquad (\forall i\in I,\ \forall x\in X). \tag{7.14}$$

Thus there is a family $\{\rho_i\}_{i\in I}$ of maps in $[\mathcal{S}]_*$ with $\mathrm{Dom}(\rho_i)=X$ such that

$$\psi'_i(x)=\rho_i(\psi_{\phi(x)^{-1}(i)}(x)) \qquad (\forall x\in X). \tag{7.15}$$

The restriction $c':=\sigma'|_{\mathcal{S}}$ of σ' to $\mathcal{S}$ is a minimal 1-cocycle from $\mathcal{S}$ into the compact group $\prod_{\lambda\in\Lambda} P(\lambda)$. Let H be the closed subgroup of $\prod_{\lambda\in\Lambda} P(\lambda)$ generated by $c'(\mathcal{S})$ (that is, $H=K(\sigma')$). As we have proved in this section, there is $\tau\in \mathrm{Per}(I)$ satisfying (7.5), where $K:=K(\sigma)$. We know that $\tau K\tau^{-1}=H\subseteq \prod_{\lambda\in\Lambda} P(\lambda)$. In particular, we have $H\tau=\tau K$. Thus

$$H\phi(x)K=H\tau K=\tau K.$$

Hence there exists a Borel function $r\colon X\to K$ such that $\phi(x)=\tau r(x)\,(x\in X)$. Substituting this into (7.15), we completes the proof. □

7.2. Relation to the pair $(r^*(\sigma), r^*(\sigma)_0)$.

As in the previous subsection, we fix preferable choice functions $\{\psi_{\lambda,n}\}_{(\lambda,n)\in I}$ of a Hecke pair $(\mathcal{R},\mathcal{S})$, and denote by σ the corresponding index cocycle, where $I=\{(\lambda,n)\colon \lambda\in\Lambda,\ n=1,\dots,n_\lambda\}$.

First, let us note that by Proposition 4.4, the stabilizer $r^*(\sigma)_0$ at 0 of the asymptotic range $r^*(\sigma)$ of σ is nothing but $r^*(c)$, where $c:=\sigma|_{\mathcal{S}}$. Recall that we explained at the end of Section 5 that $r^*(c)$ actually coincides with $K(\sigma)$. As mentioned at the beginning of Section 5, this property of the cocycle c is usually referred to as regularity of c. We explained in Section 5 how the regularity of c follows from the results in [**23**] and [**17**]. Let us indicate that one can also show the regularity of c by using [**2**, Proposition 7.4]. In any case, this fact together with Theorem 7.1 implies the following

Proposition 7.3. *The stabilizer $r^*(\sigma)_0$ at 0 of the asymptotic range $r^*(\sigma)$ of σ equals $K(\sigma)$.*

Proposition 7.4. *The asymptotic range $r^*(\sigma)$ coincides with $\mathscr{G}(\sigma)$.*

Proof. It suffices to show that $\theta_{\lambda,n}$ is in $r^*(\sigma)$ for each $(\lambda,n)\in I$. For this, we shall show that $\theta_{\lambda,n}$ is in the essential range of $\sigma|_{\mathcal{R}\cap X'\times X'}$ for each nonnull Borel subset X' of X.

Let the notation be as in the proof of Proposition 6.29.

We proved there that $E=\{x\in X\colon \sigma(x,\psi_{\lambda,n}(x))\in\theta_{\lambda,n}K\}$ is a null set. Recall that we have $E=\bigcup_{F\in\mathcal{F}(I)}X_F$. Hence X_F is null for any $F\in\mathcal{F}(I)$. Fix a K-invariant $F_0\in\mathcal{F}(I)$ and set $\Lambda_0:=q_\Lambda(F_0)$. Put $X_0=X'\cap X_{F_0}$, which is a conull Bore subset of X'. By the definition of X_{F_0}, we see that for every $x\in X_0$, there exists $k_x\in K$ such that the equation $\sigma(x,\psi_{\lambda,n}(x))(i)=\theta_{\lambda,n}k_x(i)$ holds for all $i\in F_0$. Suggested by this, we define a subset B of $X_0\times K$ by

$$B:=\{(x,k)\in X_0\times K\colon \theta_{\lambda,n}^{-1}\sigma(x,\psi_{\lambda,n}(x))(i)=k(i)\ (\forall i\in F_0)\}.$$

So $(x,k_x)\in B$ for all $x\in X_0$. Remark that with the map $p_a\colon \operatorname{Map}(I)\to I$ introduced in Section 4 and the continuous map $f\colon \operatorname{Per}(I)\times K\to\operatorname{Per}(I)$ given by $f(\phi,k)=k^{-1}\phi$, we have

$$B=\Big((\theta_{\lambda,n}^{-1}\circ\sigma\circ(id_{X_0}\times\psi_{\lambda,n}|_{X_0})\times id_K\Big)^{-1}\circ f^{-1}\Bigg(\bigcap_{i\in F}p_i^{-1}(\{i\})\Bigg).$$

This proves that B is Borel. Let $pr_1\colon X_0\times K\to X_0$ be the projection onto the first coordinate : $pr_1(x,k)=x$. Then, since $(x,k_x)\in B$, the map pr_1 is onto. By von Neumann selection theorem (see [**7**, Theorem I.14] for example), there exist a conull Borel subset Y_0 in X_0 (hence conull in X' in particular) and a Borel map $\xi\colon Y_0\to K$ such that $(y,\xi(y))\in B$ for all $y\in Y_0$. By using the same arguments as in the proof of Proposition 6.29, there are non-null Borel subset Z_0 of Y_0 and an element g in the finite group $\pi_{\Lambda_0}(K)$ such that $\pi_{\Lambda_0}(\xi(z))=g$ for all $z\in Z_0$. Since $K=r^*(\sigma|_{\mathcal{S}})\subset r^*(\sigma)$ and $\mathcal{P}_{\Lambda_0}$ $(\subset\mathcal{S})$ is ergodic, there exists $\rho\in[\mathcal{S}]_*$ which satisfies $\operatorname{Dom}(\rho)\subset\psi_{\lambda,n}(Z_0)$, $\operatorname{Im}(\rho)\subset X'$ and $\pi_{\Lambda_0}(\sigma(\psi_{\lambda,n}(x),\rho(\psi_{\lambda,n}(x))))=g^{-1}$. It then follows that for any $z\in Z_0$ and any $i\in F_0$, we have

$$\sigma(z,\rho(\psi_{\lambda,n}(z)))(i)=\sigma(z,\psi_{\lambda,n}(z))\sigma(\psi_{\lambda,n}(z),\rho(\psi_{\lambda,n}(z)))(i)=\theta_{\lambda,n}g\cdot g^{-1}(i)=\theta_{\lambda,n}(i).$$

Since F_0 is arbitrary, we conclude that $\theta_{\lambda,n}$ is in the essential range of $\sigma|_{\mathcal{R}\cap X'\times X'}$. Therefore, we complete the proof. □

Proposition 7.5. *The Schlichting completion $(\mathscr{G}(\sigma),K(\sigma))$ of the Hecke pair $(\mathcal{R},\mathcal{S})$ is reduced in the sense that $K(\sigma)$ contains no nontrivial normal subgroup of $\mathscr{G}(\sigma)$.*

Proof. Suppose that H is a subgroup of $K(\sigma)$ which is normal in $\mathscr{G}(\sigma)$. We shall show that H is trivial. Put $\mathcal{P}:=\sigma^{-1}(H)$. By Proposition 7.4, we have that H is contained in $r^*(\sigma)$. So it suffices to show that $\mathcal{P}$ coincides with $\operatorname{Ker}(\sigma)$. By Lemma 6.27, we have that $\mathscr{G}(\sigma)$ is equal to $\bigcup_{i\in I}\theta_iK(\sigma)$ and $\sigma(x,\psi_i(x))\in\theta_iK(\sigma)$ for a.e. $x\in X$. So, for a.e. $(x,y)\in\mathcal{P}$ and $i\in I$, we have

$$\sigma(\psi_i(x),\psi_i(y))=\sigma(\psi_i(x),x)\sigma(x,y)\sigma(y,\psi_i(y))\in K(\sigma)\theta_i^{-1}H\theta_iK(\sigma)=K(\sigma).$$

It means that $\sigma(x,y)(i) = i$. This completes the proof. $\square$

8. On the Hecke von Neumann algebra $\mathcal{H}^*(\mathcal{R},\mathcal{S})$.

We briefly review the construction of the Hecke von Neumann algebra $\mathcal{H}^*(\mathcal{R},\mathcal{S})$. Consider the set $L^\infty(I\times X)^{\mathcal{S}}$ of all functions F in $L^\infty(I\times X,\mu_c\times\mu)$ satisfying

$$F(i,x) = F(\sigma(y,x)(i),y) \qquad \text{for a.e.}\,(x,y)\in\mathcal{S} \text{ and all } i\in I,$$

where μ_c is the counting measure on I. By [**4**, Lemma 9.7], the map $\Xi\colon I\times X\to\Lambda$ defined by $\Xi((\lambda,n),x)=\lambda$ induces a $*$-isomorphism Ξ^* from $\ell^\infty(\Lambda)$ onto $L^\infty(I\times X)^{\mathcal{S}}$ given by $\Xi^*(f) := f\circ\Xi$, where $f\in\ell^\infty(\Lambda)$. Set $F^\lambda := \Xi^*(\delta_\lambda)$ for all $\lambda\in\Lambda$. Refer to [**4**, Sections 8 and 9] for basic properties of $\{F^\lambda\}_{\lambda\in\Lambda}$. We define a faithful normal semifinite trace Ω on $L^\infty(I\times X)^{\mathcal{S}}$ given by

$$\Omega(F) = \sum_{\lambda\in\Lambda}|f(\lambda)|n_\lambda,$$

where $F=\Xi^*(f)$.

We denote by $\mathcal{I}(\mathcal{R},\mathcal{S})$ the set of all functions F in $L^\infty(I\times X)^{\mathcal{S}}$ satisfying, with $F=\Xi^*(f)$,

$$\|F\|_{1,\ell} := \sum_{\lambda\in\Lambda}|f(\lambda)|n_\lambda<\infty \quad\text{and}\quad \|F\|_{1,r} := \sum_{\lambda\in\Lambda}|f(\lambda^{-1})|n_\lambda<\infty.$$

For any $F\in L^\infty(I\times X)^{\mathcal{S}}$, define a Borel function $F^\sharp$ on $I\times X$ by

$$F^\sharp(i,x) := \overline{F(\sigma(\psi_i(x),x)(0),\psi_i(x))} \qquad ((i,x)\in I\times X),$$

which can be proved to belong to $\mathcal{I}(\mathcal{R},\mathcal{S})$ as well (see [**4**, p. 643]). Thus $\mathcal{I}(\mathcal{R},\mathcal{S})$ is a subspace of $L^\infty(I\times X)^{\mathcal{S}}$ which is closed under the $\sharp$-operation. Since $\Xi^*(\delta_\lambda)=F^\lambda$ for any $\lambda\in\Lambda$, the linear span $\mathcal{I}_0(\mathcal{R},\mathcal{S})$ of $\{F^\lambda\colon\lambda\in\Lambda\}$ is contained in $\mathcal{I}(\mathcal{R},\mathcal{S})$. Because $\mathcal{I}_0(\mathcal{R},\mathcal{S})$ is σ-strongly* dense in $L^\infty(I\times X)^{\mathcal{S}}$, so is $\mathcal{I}(\mathcal{R},\mathcal{S})$. If $F\in\mathfrak{n}_\Omega$ and $\Xi^*(f)=F$, then

$$\|\Lambda_\Omega(F)\|^2 = \sum_{\lambda\in\Lambda}|f(\lambda)|^2 n_\lambda.$$

From this, we see that $\mathcal{I}(\mathcal{R},\mathcal{S})$ is contained in $\mathfrak{n}_\Omega$. Since $\mathcal{I}_0(\mathcal{R},\mathcal{S})$ is σ-strongly* dense $*$-subalgebra contained in $\mathfrak{n}_\Omega$, it follows that $\Lambda_\Omega(\mathcal{I}_0(\mathcal{R},\mathcal{S}))$ is dense in the GNS Hilbert space H_Ω. In particular, $\Lambda_\Omega(\mathcal{I}(\mathcal{R},\mathcal{S}))$ is total in H_Ω.

Let F_1 and F_2 be in $\mathcal{I}(\mathcal{R},\mathcal{S})$. Define a Borel function F_1*F_2 on $I\times X$ by

$$(F_1*F_2)(i,x) := \sum_{j\in I}F_1(\sigma(\psi_j(x),x)(i),\psi_j(x))F_2(j,x). \tag{8.1}$$

This defines an associative product on $\mathcal{I}(\mathcal{R},\mathcal{S})$ which makes it a $\sharp$-algebra. $\mathcal{I}_0(\mathcal{R},\mathcal{S})$ is a $\sharp$-subalgebra of $\mathcal{I}(\mathcal{R},\mathcal{S})$, which we call the *algebraic Hecke algebra associated with*

$(\mathcal{R},\mathcal{S})$. It turns out that $\Lambda_\Omega(\mathcal{I}(\mathcal{R},\mathcal{S}))$ is a left Hilbert algebra in H_Ω. We call the left von Neumann algebra of $\Lambda_\Omega(\mathcal{I}(\mathcal{R},\mathcal{S}))$ the *Hecke von Neumann algebra associated with* $(\mathcal{R},\mathcal{S})$, and denote it by $\mathcal{H}^*(\mathcal{R},\mathcal{S})$. Let π_l be the left multiplication of the left Hilbert algebra $\Lambda_\Omega(\mathcal{I}(\mathcal{R},\mathcal{S}))$. So, by definition, we have $\pi_l(\Lambda_\Omega(\mathcal{I}(\mathcal{R},\mathcal{S})))'' = \mathcal{H}^*(\mathcal{R},\mathcal{S})$. Thus

$$F \in \mathcal{I}(\mathcal{R},\mathcal{S}) \longmapsto \pi_l(\Lambda_\Omega(F))$$

defines a $*$-representation of the involutive algebra $\mathcal{I}(\mathcal{R},\mathcal{S})$, which we still denote by π_l. Hence

$$\overline{\pi_l(\mathcal{I}(\mathcal{R},\mathcal{S}))}^{\sigma\text{-strong}*} = \mathcal{H}^*(\mathcal{R},\mathcal{S}).$$

It is also true that

$$\overline{\pi_l(\mathcal{I}_0(\mathcal{R},\mathcal{S}))}^{\sigma\text{-strong}*} = \mathcal{H}^*(\mathcal{R},\mathcal{S}).$$

It is proved in [**4**] among other things that if $W^*(\mathcal{S}) \subseteq W^*(\mathcal{R}) \subseteq A_1 \subseteq A_2$ is the basic extension of the inclusion $W^*(\mathcal{S}) \subseteq W^*(\mathcal{R})$, then $A_2 \cap A'$ is $*$-isomorphic to $\mathcal{H}^*(\mathcal{R},\mathcal{S})$. This particularly shows that $\mathcal{H}^*(\mathcal{R},\mathcal{S})$ is independent of the choice of choice functions for $\mathcal{S} \subseteq \mathcal{R}$.

Define $\ell_1^\infty(\Lambda) := (\Xi^*)^{-1}(\mathcal{I}(\mathcal{R},\mathcal{S}))$ and $\ell_0^\infty(\Lambda) := (\Xi^*)^{-1}(\mathcal{I}_0(\mathcal{R},\mathcal{S}))$. By definition, $\ell_0^\infty(\Lambda)$ is the linear span of the set $\{\delta_\lambda \colon \lambda \in \Lambda\}$. Hence $\ell_0^\infty(\Lambda)$ consists of functions on Λ with finite support. One can transport the $\sharp$-algebraic structure of $\mathcal{I}(\mathcal{R},\mathcal{S})$ to $\ell_1^\infty(\Lambda)$. We still denote by $*$ and $\sharp$ the corresponding convolution and involution on $\ell_1^\infty(\Lambda)$. Then Lemma 9.8 in [**4**] tells us that

$$f^\sharp(\lambda) = \overline{f(\lambda^{-1})} \tag{8.2}$$

holds for any $f \in \ell_1^\infty(\Lambda)$. To see how the convolution on $\ell_0^\infty(\Lambda)$ is defined, we pay attention to the equality obtained on page 648 in [**4**], which is

$$(F^{\lambda_1} * F^{\lambda_2})((\lambda,n),x) = \sum_{k=1}^{n_{\lambda_2}} \chi_{\mathcal{C}_{\lambda_1}}(\psi_{\lambda_2,k}(x),\psi_{\lambda,n}(x)). \tag{8.3}$$

Throughout the remainder of this section, we consider the situation taken up in Section 6. Let K be the open and compact subgroup $K(\sigma)$. Fix arbitrary $\lambda_1,\lambda_2 \in \Lambda$ in what follows. By Lemma 6.26 (3), there exists a conull Borel subset $X(\lambda_2)$ of X such that $\sigma(x,\psi_{\lambda_2,m}(x)) \in \bigcup_{n=1}^{n_{\lambda_2}} \theta_{\lambda_2,n}K$ for all $x \in X(\lambda_2)$ and all $m \in \{1,\dots,n_{\lambda_2}\}$. Fix any $k_1 \in \{1,\dots,n_{\lambda_1}\}$ and any $k_2 \in \{1,\dots,n_{\lambda_2}\}$. Define $\phi_{k_2} \colon X \to K$ by

$$\phi_{k_2}(x) := \begin{cases} \theta_{\lambda_2,k_2}^{-1}\sigma(x,\psi_{\lambda_2,k_2}(x)) & \text{if } x \in X(\lambda_2), \\ id & \text{if } x \in X(\lambda_2)^c. \end{cases}$$

Since $\sigma(x,\psi_{\lambda_2,k_2}(x)) \in \theta_{\lambda_2,k_2}K$ for all $x \in X(\lambda_2)$, the image of ϕ_{k_2} is indeed contained in K.

For the next lemma, recall that for each $\lambda_0 \in \Lambda$, the Borel subset $E_{\lambda_0}^{k_1,k_2}$ defined by

$$E_{\lambda_0}^{k_1,k_2} := \{x \in X \colon (x, \psi_{\lambda_1,k_1}(\psi_{\lambda_2,k_2}(x))) \in \mathcal{C}_{\lambda_0}\}$$

was introduced in Section 9 of [**4**] (see the discussion on page 649 there).

LEMMA 8.1. *Let $\lambda_0 \in \Lambda$. For $x \in X(\lambda_2)$ to belong to $E_{\lambda_0}^{k_1,k_2}$, it is necessary and sufficient that it should satisfy $\theta_{\lambda_2,k_2}\theta_{\phi_{k_2}(x)(\lambda_1,k_1)} \in K\theta_{\lambda_0,1}K$.*

PROOF. Let $x \in X(\lambda_2)$.

Suppose first that $x \in E_{\lambda_0}^{k_1,k_2}$. So $(x, \psi_{\lambda_1,k_1}(\psi_{\lambda_2,k_2}(x))) \in \mathcal{C}_{\lambda_0}$. By the definition of $\mathcal{C}_{\lambda_0}$, there exists $z \in X$ such that $(x,z) \in \mathcal{S}$ and $(\psi_{\lambda_0,1}(z), \psi_{\lambda_1,k_1}(\psi_{\lambda_2,k_2}(x))) \in \mathcal{S}$. Thus

$$\sigma(x,z) \in K \quad \text{and} \quad \sigma(z, \psi_{\lambda_2,k_2}(x))(\lambda_1,k_1) = (\lambda_0, 1).$$

This yields

$$\sigma(x, \psi_{\lambda_2,k_2}(x))(\lambda_1,k_1) = \sigma(x,z)(\lambda_0,1).$$

By the definition of ϕ_{k_2}, we have $\sigma(x, \psi_{\lambda_2,k_2}(x)) = \theta_{\lambda_2,k_2}\phi_{k_2}(x)$. Hence the identity above can be written as

$$\theta_{\lambda_2,k_2}\phi_{k_2}(x)(\lambda_1,k_1) = \sigma(x,z)(\lambda_0,1).$$

Because $\phi_{k_2}(x)(\lambda_1,k_1) = \theta_{\phi_{k_2}(x)(\lambda_1,k_1)}(0)$ and $(\lambda_0,1) = \theta_{\lambda_0,1}(0)$, it follows that

$$\theta_{\lambda_2,k_2}\theta_{\phi_{k_2}(x)(\lambda_1,k_1)}(0) = \sigma(x,z)\theta_{\lambda_0,1}(0).$$

Namely,

$$(\theta_{\lambda_0,1})^{-1}\sigma(z,x)\theta_{\lambda_2,k_2}\theta_{\phi_{k_2}(x)(\lambda_1,k_1)}(0) = 0.$$

This means that $(\theta_{\lambda_0,1})^{-1}\sigma(z,x)\theta_{\lambda_2,k_2}\theta_{\phi_{k_2}(x)(\lambda_1,k_1)} \in K$, because the stabilizer subgroup of G at $0 \in I$ coincides with K due to the results obtained before. Hence

$$\theta_{\lambda_2,k_2}\theta_{\phi_{k_2}(x)(\lambda_1,k_1)} \in \sigma(x,z)\theta_{\lambda_0,1}K \subseteq K\theta_{\lambda_0,1}K.$$

Assume conversely that x satisfies $\theta_{\lambda_2,k_2}\theta_{\phi_{k_2}(x)(\lambda_1,k_1)} \in K\theta_{\lambda_0,1}K$. By Lemma 6.26 (2), there exist $h \in K$ and $m \in \{1, \dots, n_{\lambda_0}\}$ such that $\theta_{\lambda_2,k_2}\theta_{\phi_{k_2}(x)(\lambda_1,k_1)} = \theta_{\lambda_0,m}h$. Hence we have

$$\begin{aligned}
\theta_{\lambda_2,k_2}\theta_{\phi_{k_2}(x)(\lambda_1,k_1)}(0) &= \theta_{\lambda_0,m}h(0),\\
\theta_{\lambda_2,k_2}\phi_{k_2}(x)(\lambda_1,k_1) &= \theta_{\lambda_0,m}(0),\\
\sigma(x, \psi_{\lambda_2,k_2}(x))(\lambda_1,k_1) &= (\lambda_0,m),\\
(\psi_{\lambda_0,m}(x), \psi_{\lambda_1,k_1}(\psi_{\lambda_2,k_2}(x))) &\in \mathcal{S},\\
(x, \psi_{\lambda_1,k_1}(\psi_{\lambda_2,k_2}(x))) &\in \mathcal{C}_{\lambda_0,m} = \mathcal{C}_{\lambda_0}.
\end{aligned}$$

Therefore, $x \in E_{\lambda_0}^{k_1,k_2}$. □

According to the arguments on page 649 in [**4**], there corresponds to each pair

(k_1, k_2), where $k_1 \in \{1, \ldots, n_{\lambda_1}\}$ and $k_2 \in \{1, \ldots, n_{\lambda_2}\}$, an index $\lambda_{k_1,k_2} \in \Lambda$ such that $E := \bigcap_{k_1=1}^{n_{\lambda_1}} \bigcap_{k_2=1}^{n_{\lambda_2}} E^{k_1,k_2}_{\lambda_{k_1,k_2}}$ is a non-null Borel subset of X. Put $\Lambda_0 := \{\lambda_{k_1,k_2} \colon k_1 \in \{1, \ldots, n_{\lambda_1}\},\ k_2 \in \{1, \ldots, n_{\lambda_2}\}\}$.

Lemma 8.2. *In the notation introduced above, we have*

$$\Lambda_0 = \{\lambda \in \Lambda \colon\ \exists k_1 \in \{1, \ldots, n_{\lambda_1}\} \text{ and } \exists k_2 \in \{1, \ldots, n_{\lambda_2}\} \text{ such that } \theta_{\lambda_2,k_2}\theta_{\lambda_1,k_1} \in K\theta_{\lambda,1}K\}.$$

Proof. Take any $\lambda_0 \in \Lambda_0$. By definition, there exist $k_1 \in \{1, \ldots, n_{\lambda_1}\}$ and $k_2 \in \{1, \ldots, n_{\lambda_2}\}$ such that $\lambda_0 = \lambda_{k_1,k_2}$. Choose any $x \in E = \bigcap_{k_1=1}^{n_{\lambda_1}} \bigcap_{k_2=1}^{n_{\lambda_2}} E^{k_1,k_2}_{\lambda_{k_1,k_2}}$. Since $x \in E^{k_1,k_2}_{\lambda_{k_1,k_2}}$, it follows from Lemma 8.1 that $\theta_{\lambda_2,k_2}\theta_{\phi_{k_2}(x)(\lambda_1,k_1)} \in K\theta_{\lambda_{k_1,k_2},1}K = K\theta_{\lambda_0,1}K$. Because $\phi_{k_2}(x)$ is in K, $\phi_{k_2}(x)(\lambda_1, k_1)$ has the form $(\lambda_1, m) = \phi_{k_2}(x)(\lambda_1, k_1)$ for some $m \in \{1, \ldots, n_{\lambda_1}\}$. Then $\theta_{\lambda_2,k_2}\theta_{\lambda_1,m} \in K\theta_{\lambda_0,1}K$.

Next let $\lambda \in \Lambda$ have the property that there exist $k_1 \in \{1, \ldots, n_{\lambda_1}\}$ and $k_2 \in \{1, \ldots, n_{\lambda_2}\}$ such that $\theta_{\lambda_2,k_2}\theta_{\lambda_1,k_1} \in K\theta_{\lambda,1}K$. Again, choose one $x \in E$. Then there is a unique $m \in \{1, \ldots, n_{\lambda_1}\}$ such that $\phi_{k_2}(x)(\lambda_1, m) = (\lambda_1, k_1)$. By Lemma 8.1, we have

$$\begin{aligned} K\theta_{\lambda,1}K &\ni \theta_{\lambda_2,k_2}\theta_{\lambda_1,k_1} \\ &= \theta_{\lambda_2,k_2}\theta_{\phi_{k_2}(x)(\lambda_1,m)} \\ &\in K\theta_{\lambda_{m,k_2},1}K \qquad (\because\ x \in E^{m,k_2}_{\lambda_{m,k_2}}). \end{aligned}$$

It follows that $K\theta_{\lambda,1}K = K\theta_{\lambda_{m,k_2},1}K$. Hence we have to have $\lambda = \lambda_{m,k_2} \in \Lambda_0$. $\square$

As in the arguments in Section 9 of [**4**] (see page 648 there), for each $\lambda \in \Lambda$, let us look at the Borel subset $K^{\lambda_1}_{\lambda_2,\lambda}$ of X defined by

$$K^{\lambda_1}_{\lambda_2,\lambda} := \bigcup_{k=1}^{n_\lambda} \bigcup_{k_2=1}^{n_{\lambda_2}} \{x \in X \colon (\psi_{\lambda_2,k_2}(x), \psi_{\lambda,k}(x)) \in \mathcal{C}_{\lambda_1}\}.$$

It was observed that

- $K^{\lambda_1}_{\lambda_2,\lambda}$ is either null or conull;
- For any $\lambda \in \Lambda \setminus \Lambda_0$, $K^{\lambda_1}_{\lambda_2,\lambda}$ is contained in E, so it is null.

Lemma 8.3. *For $\lambda \in \Lambda$ to belong to Λ_0, it is necessary and sufficient that $K^{\lambda_1}_{\lambda_2,\lambda}$ is conull. Therefore,*

$$\Lambda_0 = \{\lambda \in \Lambda \colon K^{\lambda_1}_{\lambda_2,\lambda} \textit{ is conull}\}.$$

Proof. Let $\lambda \in \Lambda$.

As noted just before this lemma, λ belongs to Λ_0 if $K^{\lambda_1}_{\lambda_2,\lambda}$ is conull.

Suppose that $\lambda \in \Lambda_0$. By definition, there are $k_1 \in \{1, \ldots, n_{\lambda_1}\}$ and $k_2 \in \{1, \ldots, n_{\lambda_2}\}$ such that $\lambda = \lambda_{k_1,k_2}$. Let us take an arbitrary $x \in E$. Since x particularly belongs to $E^{k_1,k_2}_\lambda = E^{k_1,k_2}_{\lambda_{k_1,k_2}}$, it follows that $(x, \psi_{\lambda_1,k_1}(\psi_{\lambda_2,k_2}(x))) \in \mathcal{C}_\lambda$. Thus there exists $z \in X$ such that $(x, z) \in \mathcal{S}$ and $(\psi_{\lambda,1}(z), \psi_{\lambda_1,k_1}(\psi_{\lambda_2,k_2}(x))) \in \mathcal{S}$. The second fact implies that

$\sigma(z,\psi_{\lambda_2,k_2}(x))(\lambda_1,k_1)=(\lambda,1)$. Applying $\sigma(x,z)$ to both sides yields

$$\sigma(x,\psi_{\lambda_2,k_2}(x))(\lambda_1,k_1)=\sigma(x,z)(\lambda,1).$$

Since $\sigma(x,z)\in K$, there is some $m\in\{1,\ldots,n_\lambda\}$ such that $\sigma(x,z)(\lambda,1)=(\lambda,m)$. Hence we obtain

$$\sigma(x,\psi_{\lambda_2,k_2}(x))(\lambda_1,k_1)=(\lambda,m).$$

Due to the proof of Lemma 8.5 of [**4**] (or one of the results in Section 3.1), we see that $(\psi_{\lambda_2,k_2}(x),\psi_{\lambda,m}(x))\in\mathcal{C}_{\lambda_1}$. Hence x belongs to $K^{\lambda_1}_{\lambda_2,\lambda}$. We have proved that the non-null subset E is contained in $K^{\lambda_1}_{\lambda_2,\lambda}$. Because $K^{\lambda_1}_{\lambda_2,\lambda}$ is either null or conull, it must be conull. □

Set $X_{\lambda_1,\lambda_2}=\left(\bigcap_{\lambda\in\Lambda_0}K^{\lambda_1}_{\lambda_2,\lambda}\right)\cap\left(\bigcap_{\lambda\in\Lambda\setminus\Lambda_0}\left(K^{\lambda_1}_{\lambda_2,\lambda}\right)^c\right)\cap X(\lambda_2)$. Thanks to the results obtained so far, X_{λ_1,λ_2} is a conull Borel subset of X. In [**4**] (refer to one of the equations on page 648), we obtained the identity

$$(F^{\lambda_1}*F^{\lambda_2})((\lambda,n),x)=\sum_{k_2=1}^{n_{\lambda_2}}\chi_{\mathcal{C}_{\lambda_1}}(\psi_{\lambda_2,k_2}(x),\psi_{\lambda,n}(x)). \tag{8.4}$$

As in the arguments on page 650 in [**4**], one can easily show by using the identity above that $(F^{\lambda_1}*F^{\lambda_2})((\lambda,n),x)=0$ whenever $\lambda\in\Lambda\setminus\Lambda_0$ and $x\in X_{\lambda_1,\lambda_2}$.

Suppose now that $\lambda\in\Lambda_0$ and $x\in X_{\lambda_1,\lambda_2}$. Due to the proof of Lemma 8.5 of [**4**] (or one of the results in Section 3.1), we have

$$\begin{aligned}\chi_{\mathcal{C}_{\lambda_1}}(\psi_{\lambda_2,k_2}(x),\psi_{\lambda,n}(x))=1 \quad&\Longleftrightarrow\quad (\psi_{\lambda_2,k_2}(x),\psi_{\lambda,n}(x))\in\mathcal{C}_{\lambda_1}\\ &\Longleftrightarrow\quad \exists k_1\in\{1,\ldots,n_{\lambda_1}\}\text{ s.t. }(\lambda,n)=\sigma(x,\psi_{\lambda_2,k_2}(x))(\lambda_1,k_1).\end{aligned}$$

Owing to this equivalence, (8.4) can be reduced to the form

$$(F^{\lambda_1}*F^{\lambda_2})((\lambda,n),x)=\sum_{k_1=1}^{n_{\lambda_1}}\sum_{k_2=1}^{n_{\lambda_2}}\delta_{(\lambda,n),\,\sigma(x,\psi_{\lambda_2,k_2}(x))(\lambda_1,k_1)}. \tag{8.5}$$

Since $x\in X(\lambda_2)$, it follows that $\sigma(x,\psi_{\lambda_2,k_2}(x))(\lambda_1,k_1)=\theta_{\lambda_2,k_2}\phi_{k_2}(x)(\lambda_1,k_1)$. For a fixed k_2, we have

$$\{\phi_{k_2}(x)(\lambda_1,k_1)\colon k_1\in\{1,\ldots,n_{\lambda_1}\}\}=\{(\lambda_1,k_1)\colon k_1\in\{1,\ldots,n_{\lambda_1}\}\},$$

because $\phi_{k_2}(x)\in K$. Hence (8.5) can be further reduced to the form

$$(F^{\lambda_1}*F^{\lambda_2})((\lambda,n),x)=\sum_{k_1=1}^{n_{\lambda_1}}\sum_{k_2=1}^{n_{\lambda_2}}\delta_{(\lambda,n),\theta_{\lambda_2,k_2}(\lambda_1,k_1)}. \tag{8.6}$$

We observe

$$\begin{aligned}(\lambda,n)=\theta_{\lambda_2,k_2}(\lambda_1,k_1) \quad &\iff \quad \theta_{\lambda,n}(0)=\theta_{\lambda_2,k_2}\theta_{\lambda_1,k_1}(0)\\ &\iff \quad (\theta_{\lambda,n})^{-1}\theta_{\lambda_2,k_2}\theta_{\lambda_1,k_1}(0)=0\\ &\iff \quad (\theta_{\lambda,n})^{-1}\theta_{\lambda_2,k_2}\theta_{\lambda_1,k_1}\in K\\ &\iff \quad \theta_{\lambda_2,k_2}\theta_{\lambda_1,k_1}\in\theta_{\lambda,n}K.\end{aligned}$$

From this, we obtain

$$(F^{\lambda_1}*F^{\lambda_2})((\lambda,n),x)=\left|\{(k_1,k_2)\in\{1,\dots,n_{\lambda_1}\}\times\{1,\dots,n_{\lambda_2}\}\colon \theta_{\lambda_2,k_2}\theta_{\lambda_1,k_1}\in\theta_{\lambda,n}K\}\right|. \quad (8.7)$$

So, for any $n\in\{1,\dots,n_\lambda\}$, we define

$$P(\lambda)_n=\{(k_1,k_2)\in\{1,\dots,n_{\lambda_1}\}\times\{1,\dots,n_{\lambda_2}\}\colon \theta_{\lambda_2,k_2}\theta_{\lambda_1,k_1}\in\theta_{\lambda,n}K\}.$$

Let $n\in\{1,\dots,n_\lambda\}$. Since $\theta_{\lambda,n}\in K\theta_{\lambda,1}K$, there are $h_1,h_2\in K$ such that $\theta_{\lambda,1}=h_1\theta_{\lambda,n}h_2$. Take any $(k_1,k_2)\in P_n$. By Lemma 6.26 (1), there is a unique $\psi(k_2)\in\{1,\dots,n_{\lambda_2}\}$ such that $h_1\theta_{\lambda_2,k_2}=\theta_{h_1(\lambda_2,k_2)}K=\theta_{\lambda_2,\phi(k_2)}K$. This in turn implies that there is a unique $h\in K$ such that $h_1\theta_{\lambda_2,k_2}=\theta_{\lambda_2,\phi(k_2)}h$. By Lemma 6.26 (1) again, there is a unique $\phi(k_1)\in\{1,\dots,n_{\lambda_1}\}$ such that $h\theta_{\lambda_1,k_1}\in\theta_{\lambda_1,\phi(k_1)}K$. Thus we obtain

$$\begin{aligned}\theta_{\lambda_2,\phi(k_2)}\theta_{\lambda_1,\phi(k_1)}&\in\theta_{\lambda_2,\psi(k_2)}\cdot h\theta_{\lambda_1,k_1}K\\ &=\theta_{\lambda_2,\psi(k_2)}h\cdot\theta_{\lambda_1,k_1}K\\ &=h_1\theta_{\lambda_2,k_2}\theta_{\lambda_1,k_1}K\\ &\subseteq h_1\cdot\theta_{\lambda,n}K\cdot K \qquad (\because\ \theta_{\lambda_2,k_2}\theta_{\lambda_1,k_1}\in\theta_{\lambda,n}K)\\ &=h_1\theta_{\lambda,n}h_2K\\ &=\theta_{\lambda,1}K.\end{aligned}$$

Hence $(\phi(k_1),\psi(k_2))\in P(\lambda)_1$. In this way, we obtain a map $F\colon P(\lambda)_n\to P(\lambda)_1$ given by $F(k_1,k_2)=(\phi(k_1),\psi(k_2))$.

Claim. *The map F is bijective.*

Proof of Claim. Suppose that $F(k_1,k_2)=F(k_1',k_2')$. First, by definition, $\psi(k_2)$ and $\psi(k_2')$ are determined by the equations

$$h_1(\lambda_2,k_2)=(\lambda_2,\psi(k_2)) \quad\text{and}\quad h_1(\lambda_2,k_2')=(\lambda_2,\psi(k_2')).$$

If $\psi(k_2)=\psi(k_2')$, then $k_2=k_2'$ by the bijectivity of h_1. From this, we see that $\phi(k_1)$ and $\phi(k_1')$ are determined by the equations

$$h(\lambda_1,k_1)=(\lambda_1,\phi(k_1)) \quad\text{and}\quad h(\lambda_1,k_1')=(\lambda_1,\psi(k_1')).$$

If $\phi(k_1)=\phi(k_1')$, then $k_1=k_1'$ by the bijectivity of h. Hence F is injective.

To prove that F is surjective, take any $(l_1,l_2)\in P(\lambda)_1$. By Lemma 6.26 (1) again, there is a unique $k_2\in\{1,\dots,n_{\lambda_2}\}$ such that $h_1^{-1}(\lambda_2,l_2)=(\lambda_2,k_2)$ and $(h_1)^{-1}\theta_{\lambda_2,l_2}\in\theta_{\lambda_2,k_2}K$. So there exists a unique $h'\in K$ such that $(h_1)^{-1}\theta_{\lambda_2,l_2}=\theta_{\lambda_2,k_2}h'$. By Lemma 6.26

(1) again, there is a unique $k_1 \in \{1, \dots, n_{\lambda_1}\}$ such that $h'(\lambda_1, l_1) = (\lambda_1, k_1)$ and $h'\theta_{\lambda_1, l_1} \in \theta_{\lambda_1, k_1} K$. Thus we obtain

$$\begin{aligned}
\theta_{\lambda_2,k_2}\theta_{\lambda_1,k_1} &\in (h_1)^{-1}\theta_{\lambda_2,l_2}(h')^{-1} \cdot h'\theta_{\lambda_1,l_1}K \\
&= (h_1)^{-1}\theta_{\lambda_2,l_2}\theta_{\lambda_1,l_1}K \\
&\subseteq (h_1)^{-1} \cdot \theta_{\lambda,1}K \cdot K \qquad (\because\ \theta_{\lambda_2,l_2}\theta_{\lambda_1,l_1} \in \theta_{\lambda,1}K) \\
&= \theta_{\lambda,n}h_2K \\
&= \theta_{\lambda,n}K.
\end{aligned}$$

This shows that (k_1, k_2) belongs to $P(\lambda)_n$. Because of how k_1 and k_2 are defined above, one can verify that $F(k_1, k_2) = (l_1, l_2)$. □

Thanks to Claim, it makes sense to define

$$C_\lambda^{\lambda_1,\lambda_2} = |P(\lambda)_1| \in \mathbb{N} \cup \{0\} \qquad (\lambda_1, \lambda_2, \lambda \in \Lambda).$$

Owing to Lemma 8.2, we find that $C_\lambda^{\lambda_1,\lambda_2} > 0$ for all $\lambda \in \Lambda_0$. Then from (8.7), we obtain

$$(F^{\lambda_1} * F^{\lambda_2})((\lambda, n), x) = C_\lambda^{\lambda_1,\lambda_2}. \tag{8.8}$$

Hence

$$F^{\lambda_1} * F^{\lambda_2} = \sum_{\lambda \in \Lambda_0} C_\lambda^{\lambda_1,\lambda_2} F^\lambda. \tag{8.9}$$

This proves that we have the identity

$$\delta_{\lambda_1} * \delta_{\lambda_2} = \sum_{\lambda \in \Lambda_0} C_\lambda^{\lambda_1,\lambda_2} \delta_\lambda \qquad (\lambda_1, \lambda_2 \in \Lambda). \tag{8.10}$$

This, together with (8.2), completely describes the involutive($\sharp$)-algebraic structure of $\ell_0^\infty(\Lambda)$. Note that we should write $\Lambda_0(\lambda_1, \lambda_2)$ for Λ_0, for it completely depends on the pair (λ_1, λ_2).

9. Review on the Hecke von Neumann algebra $W^*(G, K)$ associated with a Hecke pair (G, K) of groups.

Throughout this section, we fix a Hecke pair (G, K) of groups. Hence G is a locally compact Hausdorff totally connected group, and K is an open and compact subgroup of G. We freely identify

- the functions on $K\backslash G$ with the left K-invariant functions on G;
- the functions on G/K with the right K-invariant functions on G;
- the functions on $K\backslash G/K$ with the two-sided K-invariant functions on G.

Let $\mathcal{H}(G, K)$ be the vector space consisting of all functions on the discrete space $K\backslash G/K$, identified freely with the two-sided K-invariant functions on G, with finite support. If we fix a complete set of the representatives $\{t_q\}_{q\in K\backslash G}$ of the right coset space $K\backslash G$ satisfying $t_K = id$, then the equation

$$(f_1 * f_2)(g) = \sum_{q\in K\backslash G} f_1(gt_q^{-1})f_2(t_q) \qquad (f_1, f_2 \in \mathcal{H}(G,K)) \tag{9.1}$$

defines an associative product operation on $\mathcal{H}(G,K)$, which is independent of a choice of the representatives $\{t_q\}_{q\in K\backslash G}$. We also have an expression

$$(f_1 * f_2)(g) = \sum_{p\in G/K} f_1(s_p)f_2(s_p^{-1}g) \qquad (g \in G), \tag{9.2}$$

where $\{s_p\}_{p\in G/K}$ is a complete set of representatives of the left coset space G/K. We also let $\{q_c \in K\backslash G\colon c \in K\backslash G/K\}$ be a complete set of representatives of $K\backslash G/K$ satisfying $q_K = K \in K\backslash G$. We simply write t_c for t_{q_c} for each $c \in K\backslash G/K$. Consider the right K-action $K \curvearrowright K\backslash G$: $(q,k) \in K\backslash G \times K \to q\cdot k \in K\backslash G$. One can easily check that the stabilizer group K_{Kg} at the point $Kg \in K\backslash G$ equals $K \cap g^{-1}Kg$. Note that $|K_{Kg}\backslash K| = L(g)$, where $L(g)$ is in general defined by $L(g) = [K : K\cap g^{-1}Kg]$. If $Kg = q_c$ for some $c \in K\backslash G/K$, then we get $|K_{q_c}\backslash K| = |K_{Kt_c}\backslash K| = L(t_c)$. For any $c \in K\backslash G/K$, we choose a set $\{k_i^{(c)}\colon 1 \le i \le L(t_c)\} \subseteq K$ of representatives of the right coset space $K_{q_c}\backslash K$. Then, by construction, the points $q_ck_i^{(c)}$ $(c \in K\backslash G/K,\ 1 \le i \le L(t_c))$ are all distinct and $K\backslash G = \{q_ck_i^{(c)}\colon c \in K\backslash G/K,\ 1 \le i \le L(t_c)\}$.

Let $f \in \mathcal{H}(G,K)$ with $\operatorname{supp}(f) := \{c \in K\backslash G/K\colon f(c) \neq 0\}$. With the notation introduced above, we have,

$$\begin{aligned}\sum_{q\in K\backslash G} f(q) &= \sum_{c\in K\backslash G/K}\sum_{i=1}^{L(t_c)} f(q_ck_i^{(c)}) = \sum_{c\in K\backslash G/K}\sum_{i=1}^{L(t_c)} f(q_c) \\ &= \sum_{c\in K\backslash G/K} f(q_c)L(t_c) = \sum_{c\in\operatorname{supp}(f)} f(c)L(t_c).\end{aligned} \tag{9.3}$$

This particularly shows that f belongs to $\ell^r(K\backslash G)$ for any $r \in [1,\infty]$. Now take any $\xi \in \ell^2(K\backslash G)$. As in Equation (9.1), define a function $f * \xi$ on G by

$$(f * \xi)(g) = \sum_{q\in\Gamma\backslash G} f(gt_q^{-1})\xi(t_q) \qquad (g \in G). \tag{9.4}$$

As we have observed before, we see that $f * \xi$ is independent of a choice of the representatives $\{t_q\}$, and that it is right K-invariant. Thus we regard $f * \xi$ as a function on $K\backslash G$.

It can be shown that $f * \xi$ belongs to $\ell^2(K\backslash G)$ and $\|f * \xi\|_2 \le \sqrt{\|f\|_{1,l}\|f\|_{1,r}}\,\|\xi\|_2$, where

$$\|f\|_{1,l} = \sum_{c\in\operatorname{supp}(f)} |f(c)|L(t_c^{-1}), \qquad \|f\|_{1,r} := \sum_{c\in\operatorname{supp}(f)} |f(c)|L(t_c). \tag{9.5}$$

It follows that, for each $f \in \mathcal{H}(G,K)$, the equation

$$\mathcal{L}(f)\xi := f * \xi \qquad (\xi \in \ell^2(K\backslash G))$$

defines a bounded operator $\mathcal{L}(f)$ on $\ell^2(K\backslash G)$ satisfying $\|\mathcal{L}(f)\| \le \sqrt{\|f\|_{1,l}\|f\|_{1,r}}$. It is easy to check that $\mathcal{L}\colon \mathcal{H}(G,K) \to B(\ell^2(K\backslash G))$ is a representation. We call $\mathcal{L}$ the regular representation of the Hecke algebra $\mathcal{H}(G,K)$. One can check also that $\mathcal{L}(\chi_e)$ is the identity operator. Thus $\mathcal{L}$ is a unital map.

For any $f \in \mathcal{H}(G,K)$, define a function $f^\sharp$ on G by $f^\sharp(g) := \overline{f(g^{-1})}$. Clearly, $f^\sharp$ is two-sided K-invariant, so it is regarded as a function on $K\backslash G/K$. With the notation introduced before, we have $f^\sharp(c) = \overline{f(t_c^{-1})}$ for all $c \in K\backslash G/K$. Since $\mathrm{supp}(f^\sharp) = \{Kt_c^{-1}K \in K\backslash G/K\colon c \in \mathrm{supp}(f)\}$, $f^\sharp$ too has finite support. Hence it belongs to $\mathcal{H}(G,K)$. One can verify that the operation $\sharp$ defined above makes $\mathcal{H}(G,K)$ an involutive($\sharp$) algebra. One can easily verify that $\mathcal{L}(f)^* = \mathcal{L}(f^\sharp)$. Therefore, $\mathcal{L}$ is a $*$-representation. Remark that $\mathcal{L}$ is faithful, because $\mathcal{L}(f)\chi_K = f$ for any $f \in \mathcal{H}(G,K)$.

Definition 9.1. We denote by $W^*(G,K)$ the von Neumann algebra generated by the unital $*$-subalgebra $\mathcal{L}(\mathcal{H}(G,K))$ and call it the Hecke von Neumann algebra associated with the Hecke pair (G,K).

It is known that $\chi_K \in \ell^2(K\backslash G)$ is a separating vector for $W^*(G,K)$. Thus the equation

$$\omega_0(x) := (x\chi_K \mid \chi_K) \qquad (x \in W^*(G,K))$$

defines a faithful normal state on $W^*(G,K)$. It is easy to verify that

$$\omega_0(\mathcal{L}(f)) = f(e), \qquad \omega_0(\mathcal{L}(g)^*\mathcal{L}(f)) = \sum_{c \in K\backslash G/K} f(t_c)\overline{g(t_c)}L(t_d). \tag{9.6}$$

In what follows, we consider the situation taken up in Section 6. We let $G := \mathscr{G}(\sigma)$ and $K := K(\sigma)$. From

$$K\theta_{\lambda,n}K = (K\theta_{\lambda^{-1},1}K)^{-1} = \left(\bigcup_{n=1}^{n_{\lambda^{-1}}} \theta_{\lambda^{-1},n}K\right)^{-1} = \bigcup_{n=1}^{n_{\lambda^{-1}}} K(\theta_{\lambda^{-1},n})^{-1}, \tag{9.7}$$

we may choose $\{K(\theta_{\lambda^{-1},n})^{-1}\colon \lambda \in \Lambda,\, n \in \{1,\dots,n_{\lambda^{-1}}\}\}$ for a complete set of representatives of $K\backslash G$. Namely, with the previous notation, we have $t_{K(\theta_{\lambda^{-1},n})^{-1}} = (\theta_{\lambda^{-1},n})^{-1}$. The double coset space $K\backslash G/K$ equals $\{K\theta_{\lambda,1}K\}_{\lambda\in\Lambda}$, so it is parametrized by the set Λ. Hence one can choose $\{K(\theta_{\lambda^{-1},1})^{-1}\}_{\lambda\in\Lambda}$ for the complete set $\{q_c\}$ of representatives of $K\backslash G/K$. More precisely, with the previous notation, we have $q_{K\theta_{\lambda,1}K} = K(\theta_{\lambda^{-1},1})^{-1}$ with $c = K\theta_{\lambda,1}K$, since $K(\theta_{\lambda^{-1},1})^{-1}K = K\theta_{\lambda,1}K$ for all $\lambda \in \Lambda$. In particular, $t_{K\theta_{\lambda,1}K} = (\theta_{\lambda^{-1},1})^{-1}$. In this case, the convolution (9.1) becomes

$$(f_1 * f_2)(g) = \sum_{\lambda\in\Lambda} \sum_{n=1}^{n_{\lambda^{-1}}} f_1(g\theta_{\lambda^{-1},n}) f_2((\theta_{\lambda^{-1},n})^{-1}). \tag{9.8}$$

Because $\{\theta_{\lambda,n}K\colon (\lambda,n) \in I\}$ is a complete set of representatives of the left coset space

G/K, it follows from (9.2) that the convolution (9.1) can be also written in the form

$$(f_1 * f_2)(g) = \sum_{(\lambda,n)\in I} f_1(\theta_{\lambda,n}) f_2((\theta_{\lambda,n})^{-1} g). \tag{9.9}$$

The equation (9.3) has the form

$$\sum_{\lambda\in\Lambda} \sum_{n=1}^{n_{\lambda^{-1}}} f(K(\theta_{\lambda^{-1},n})^{-1}) = \sum_{\lambda\in\Lambda} f(K\theta_{\lambda,1}K) L((\theta_{\lambda^{-1},1})^{-1}) = \sum_{\lambda\in\Lambda} f(K\theta_{\lambda,1}K) n_{\lambda^{-1}}. \tag{9.10}$$

Also note that we have

$$\|f\|_{1,l} = \sum_{\lambda\in\Lambda} |f(K\theta_{\lambda,1}K)| n_\lambda, \qquad \|f\|_{1,r} = \sum_{\lambda\in\Lambda} |f(K\theta_{\lambda,1}K)| n_{\lambda^{-1}}. \tag{9.11}$$

Fix any $c_1 = K\theta_{\lambda_1,1}K$ and $c_2 = K\theta_{\lambda_2,1}K$ in $K\backslash G/K$. Let $c = KgK \in K\backslash G/K$ be arbitrary. By (9.9), we have

$$(\delta_{c_1} * \delta_{c_2})(c) = \sum_{(\lambda,n)\in I} \delta_{K\theta_{\lambda_1,1}K}(\theta_{\lambda,n}) \delta_{K\theta_{\lambda_2,1}K}((\theta_{\lambda,n})^{-1} g).$$

Note that

$$\delta_{K\theta_{\lambda_1,1}K}(\theta_{\lambda,n}) = 1 \quad \Longleftrightarrow \quad \lambda = \lambda_1.$$

From this, we get

$$(\delta_{c_1} * \delta_{c_2})(c) = \sum_{n=1}^{n_{\lambda_1}} \delta_{K\theta_{\lambda_2,1}K}((\theta_{(\lambda_1),n})^{-1} g).$$

Now we observe that

$$\begin{aligned} \delta_{K\theta_{\lambda_2,1}K}((\theta_{\lambda_1,n})^{-1} g) = 1 \quad &\Longleftrightarrow \quad (\theta_{\lambda_1,n})^{-1} g \in K\theta_{\lambda_2,1}K \\ &\Longleftrightarrow \quad g \in \theta_{\lambda_1,n}\theta_{\lambda_2,k_2}K \quad \text{for some } k_2 \in \{1,\dots,n_{\lambda_2}\} \\ &\Longleftrightarrow \quad KgK \subseteq K\theta_{\lambda_1,1}K\theta_{\lambda_2,1}K. \end{aligned}$$

Suggested by this, we introduce the number $D_c^{c_1,c_2}$ by

$$D_c^{c_1,c_2} = \left|\{(k_1,k_2)\colon \theta_{\lambda_1,k_1}\theta_{\lambda_2,k_2} \in gK\}\right| \in \mathbb{N}\cup\{0\}, \tag{9.12}$$

where $c_1 = K\theta_{\lambda_1,1}K$, $c_2 = K\theta_{\lambda_2,1}K$ and $c = KgK$. Then, from the results above, we obtain

$$(\delta_{c_1} * \delta_{c_2})(c) = D_c^{c_1,c_2}. \tag{9.13}$$

Therefore, we obtain

$$\delta_{c_1} * \delta_{c_2} = \sum_{\substack{c \in K\backslash G/K \\ c \subseteq K\theta_{\lambda_1,1}K\theta_{\lambda_2,1}K}} D_c^{c_1,c_2}\delta_c, \tag{9.14}$$

where $c_1 = K\theta_{\lambda_1,1}K$, $c_2 = K\theta_{\lambda_2,1}K$, as before.

10. Relation between $\mathcal{H}^*(\mathcal{R},\mathcal{S})$ and $W^*(G,K)$.

Again, we consider the situation taken up in Section 6. We let $G := \mathscr{G}(\sigma)$ and $K := K(\sigma)$.

We define a map Φ from $\ell^\infty(\Lambda)$ into $\ell^\infty(K\backslash G/K)$ by

$$\Phi(f) = \sum_{\lambda\in\Lambda} f(\lambda^{-1})\, \delta_{K\theta_{\lambda,1}K} \qquad (f \in \ell^\infty(\Lambda)). \tag{10.1}$$

It is easy to see that Φ is indeed a $*$-isomorphism between abelian von Neumann algebras $\ell^\infty(\Lambda)$ and $\ell^\infty(K\backslash G/K)$. The restriction of Φ to $\ell_0^\infty(\Lambda)$ will be still denoted by Φ, which is a vector space isomorphism from $\ell_0^\infty(\Lambda)$ onto $\mathcal{H}(G,K)$. Let $f \in \ell_0^\infty(\Lambda)$. Then

$$\begin{aligned}
\Phi(f^\sharp) &= \sum_{\lambda\in\Lambda} f^\sharp(\lambda^{-1})\, \delta_{K\theta_{\lambda,1}K} \\
&= \sum_{\lambda\in\Lambda} \overline{f(\lambda)}\, \delta_{K\theta_{\lambda,1}K} \\
&= \overline{\sum_{\lambda\in\Lambda} f(\lambda)\, \delta_{K\theta_{\lambda,1}K}} \\
&= \Phi(f)^\sharp.
\end{aligned}$$

Thus Φ is involutive (i.e., preserves the $\sharp$-operation).

For the next proposition, let us introduce the notation: for any $f \in \ell_0^\infty(\Lambda)$, define $f^\vee$ by $f^\vee(\lambda) = f(\lambda^{-1})$. We use the same notation for the functions in $\mathcal{H}(G,K)$: $\eta^\vee(g) = \eta(g^{-1})$ ($\eta \in \mathcal{H}(G,K)$). Then by a straightforward computation, we can verify the following

Lemma 10.1. *For any $f_1, f_2 \in \ell_0^\infty(\Lambda)$ and $\lambda \in \Lambda$, we have $(f_1 * f_2)^\vee = f_2^\vee * f_1^\vee$ and $(\delta_\lambda)^\vee = \delta_{\lambda^{-1}}$. Also, we have $(\eta_1 * \eta_2)^\vee = \eta_2^\vee * \eta_1^\vee$ for $\eta_1, \eta_2 \in \mathcal{H}(G,K)$.*

Proposition 10.2. *Φ is a involutive($\sharp$)-isomorphism from $\ell_0^\infty(\Lambda)$ into $\mathcal{H}(G,K)$.*

Proof. It suffices to check that Φ satisfies $\Phi(\delta_{\lambda_1} * \delta_{\lambda_2}) = \Phi(\delta_{\lambda_1}) * \Phi(\delta_{\lambda_2})$. Take any $\lambda_1, \lambda_2 \in \Lambda$. As we saw in Section 8 (see (8.10)), we have

$$(\delta_{\lambda_1} * \delta_{\lambda_2})(\lambda^{-1}) = C_{\lambda^{-1}}^{\lambda_1,\lambda_2}, \tag{10.2}$$

which in turns equals $D_{K\theta_{\lambda^{-1},1}K}^{K\theta_{\lambda_2,1}K,K\theta_{\lambda_1,1}K}$ by (9.13). So we have

$$\Phi(\delta_{\lambda_1} * \delta_{\lambda_2}) = \sum_{\lambda\in\Lambda} (\delta_{\lambda_1} * \delta_{\lambda_2})(\lambda^{-1})\delta_{K\theta_{\lambda,1}K}$$

$$
\begin{aligned}
&= \sum_{\lambda\in\Lambda} C_{\lambda^{-1}}^{\lambda_1,\lambda_2}\delta_{K\theta_{\lambda,1}K} \\
&= \sum_{\lambda\in\Lambda} D_{K\theta_{\lambda^{-1},1}K}^{K\theta_{\lambda_2,1}K,K\theta_{\lambda_1,1}K}\delta_{K\theta_{\lambda,1}K} \\
&= \sum_{\lambda\in\Lambda}(\delta_{K\theta_{\lambda_2,1}K} * \delta_{K\theta_{\lambda_1,1}K})(K\theta_{\lambda^{-1},1}K)\delta_{K\theta_{\lambda,1}K} \\
&= \sum_{\lambda\in\Lambda}(\delta_{K\theta_{\lambda_2,1}K} * \delta_{K\theta_{\lambda_1,1}K})^\vee(K\theta_{\lambda,1}K)\delta_{K\theta_{\lambda,1}K} \\
&= \sum_{\lambda\in\Lambda}\left(\delta_{K\theta_{\lambda_1^{-1},1}K} * \delta_{K\theta_{\lambda_2^{-1},1}K}\right)(K\theta_{\lambda,1}K)\delta_{K\theta_{\lambda,1}K} \qquad \text{(by Lemma 10.1)} \\
&= \sum_{\lambda\in\Lambda}(\Phi(\delta_{\lambda_1}) * \Phi(\delta_{\lambda_2}))(K\theta_{\lambda,1}K)\delta_{K\theta_{\lambda,1}K} \\
&= \Phi(\delta_{\lambda_1}) * \Phi(\delta_{\lambda_2}).
\end{aligned}
$$

Thus we are done. □

Recall (see [**4**, Equations (9.4) and (9.5)]) that we considered a faithful normal semifinite trace Ω on $L^\infty(I\times X)^{\mathcal{S}}$ given by

$$\Omega(F) = \sum_{\lambda\in\Lambda} f(\lambda)n_\lambda, \tag{10.3}$$

where $F = \Xi^*(f) \in \left(L^\infty(I\times X)^{\mathcal{S}}\right)_+$. Set $\Omega_1 := \Omega\circ\Xi^*$, a faithful normal semifinite trace on $\ell^\infty(\Lambda)$ given by $\Omega_1(f) = \sum_{\lambda\in\Lambda} f(\lambda)n_\lambda\,(f\in\ell^\infty(\Lambda)_+)$. Then set $\Omega_2 := \Omega_1\circ\Phi^{-1}$, a faithful normal semifinite trace on $\ell^\infty(K\backslash G/K)$ given by

$$\Omega_2(f) = \sum_{\lambda\in\Lambda} f(K\theta_{\lambda^{-1},1}K)n_\lambda \qquad (f\in\ell^\infty(K\backslash G/K)_+). \tag{10.4}$$

In comparison with (9.11), we obtain the next identities

$$\|f\|_{1,l} = \Omega_2(|f^\vee|), \qquad \|f\|_{1,r} = \Omega_2(|f|). \tag{10.5}$$

By definition, all the GNS Hilbert spaces H_Ω, H_{Ω_1}, H_{Ω_2} are naturally isomorphic. The subspaces

$$\Lambda_\Omega(\mathcal{I}_0(\mathcal{R},\mathcal{S})), \quad \Lambda_{\Omega_1}(\ell_0^\infty(\Lambda)), \quad \Lambda_{\Omega_2}(\mathcal{H}(G,K))$$

are dense in the relevant Hilbert spaces. For example, the equation

$$\mathcal{V}\Lambda_\Omega(F) = \Lambda_{\Omega_2}(\Phi\circ(\Xi^*)^{-1}(F)) \qquad (F\in\mathcal{I}_0(\mathcal{R},\mathcal{S})) \tag{10.6}$$

defines a unitary from H_Ω onto H_{Ω_2}. Recall (see [**4**, p. 653]) that the $\sharp$-algebra $\mathcal{I}_0(\mathcal{R},\mathcal{S})$ admits an involutive representation π_l on $B(H_\Omega)$ defined by

$$\pi_l(F_1)\Lambda_\Omega(F_2) = \Lambda_\Omega(F_1 * F_2) \qquad (F_1, F_2\in\mathcal{I}_0(\mathcal{R},\mathcal{S})).$$

Moreover, due to [**4**, Proposition 9.22], the Hecke von Neumann algebra $\mathcal{H}^*(\mathcal{R},\mathcal{S})$ associated with the Hecke pair $(\mathcal{R},\mathcal{S})$ equals $\pi_l(\mathcal{I}_0(\mathcal{R},\mathcal{S}))'' = \overline{\pi_l(\mathcal{I}_0(\mathcal{R},\mathcal{S}))}^{\sigma\text{-strong}^*}$.

Fix any $(\lambda,n)\in I$. Due to condition (2) in Lemma 6.26, $K\theta_{\lambda,n}K$ equals $\bigcup_{m=1}^{n_\lambda}\theta_{\lambda,m}K$. This, together with Corollary 6.25, implies that $L((\theta_{\lambda,n})^{-1})=n_\lambda$. Moreover, by Lemma 6.26 (2) again, we have $K\theta_{\lambda^{-1},m}K = \bigcup_{n=1}^{n_{\lambda^{-1}}}\theta_{\lambda^{-1},n}K$. In the meantime, by (9.7), we have $L(\theta_{\lambda,n}) = n_{\lambda^{-1}}$.

By definition, for any $f_1, f_2 \in \mathcal{H}(G,K)$, we have

$$(\Lambda_{\Omega_2}(f_1)\mid\Lambda_{\Omega_2}(f_2)) = \Omega_2(f_2^* f_1) = \sum_{\lambda\in\Lambda} f_1(K\theta_{\lambda^{-1},1}K)\overline{f_2(K\theta_{\lambda^{-1},1}K)}n_\lambda. \tag{10.7}$$

With the notation just before (9.8), we have $t_{K\theta_{\lambda^{-1},1}K}=(\theta_{\lambda,1})^{-1}$. So we have $L(t_{K\theta_{\lambda^{-1},1}K}) = L((\theta_{\lambda,1})^{-1}) = n_\lambda$. Hence (10.7) becomes

$$(\Lambda_{\Omega_2}(f_1)\mid\Lambda_{\Omega_2}(f_2)) = \sum_{c\in K\backslash G/K} f_1(t_c)\overline{f_2(t_c)}L(t_c). \tag{10.8}$$

By (9.6), we have

$$\sum_{c\in K\backslash G/K} f_1(t_c)\overline{f_2(t_c)}L(t_c) = \omega_0(\mathcal{L}(f_2)^*\mathcal{L}(f_1)) = (\Lambda_{\omega_0}(\mathcal{L}(f_1))\mid\Lambda_{\omega_0}(\mathcal{L}(f_2))).$$

Owing to this, the equation

$$\mathcal{W}\Lambda_{\Omega_2}(f) = \Lambda_{\omega_0}(\mathcal{L}(f)) \qquad (f\in\mathcal{H}(G,K))$$

defines a unitary $\mathcal{W}$ from H_{Ω_2} onto H_{ω_0}. If $F_1, F_2\in\mathcal{I}_0(\mathcal{R},\mathcal{S})$, then

$$\begin{aligned}
\mathcal{W}\mathcal{V}\pi_l(F_1)\Lambda_\Omega(F_2) &= \mathcal{W}\mathcal{V}\Lambda_\Omega(F_1 * F_2)\\
&= \mathcal{W}\Lambda_{\Omega_2}(\Phi\circ(\Xi^*)^{-1}(F_1) * \Phi\circ(\Xi^*)^{-1}(F_2))\\
&= \Omega_{\omega_0}(\mathcal{L}(\Phi\circ(\Xi^*)^{-1}(F_1) * \Phi\circ(\Xi^*)^{-1}(F_2)))\\
&= \mathcal{L}(\Phi\circ(\Xi^*)^{-1}(F_1))\Lambda_{\omega_0}(\Phi\circ(\Xi^*)^{-1}(F_2))\\
&= \mathcal{L}(\Phi\circ(\Xi^*)^{-1}(F_1))\mathcal{W}\mathcal{V}\Lambda_\Omega(F_2).
\end{aligned}$$

This shows that

$$(\mathcal{W}\mathcal{V})\pi_l(F)(\mathcal{W}\mathcal{V})^* = \mathcal{L}(\Phi\circ(\Xi^*)^{-1}(F)) \tag{10.9}$$

holds for all $F\in\mathcal{I}_0(\mathcal{R},\mathcal{S})$, which implies that

$$(\mathcal{W}\mathcal{V})\mathcal{H}^*(\mathcal{R},\mathcal{S})(\mathcal{W}\mathcal{V})^* = W^*(G,K).$$

Therefore, we have proved

THEOREM 10.3. *$(\mathcal{H}^*(\mathcal{R},\mathcal{S}), H_\Omega)$ is spatially isomorphic to $(W^*(G,K), H_{\omega_0})$. Namely, the Hecke von Neumann algebra of a Hecke pair of ergodic measured equiv-*

alence relations is ∗-isomorphic to the Hecke von Neumann algebra of its Schlichting completion.

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