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Forcing-theoretic aspects of Hindman's Theorem

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Abstract. We investigate the partial order $(\mathrm{FIN})^\omega$ of infinite block sequences, ordered by almost condensation, from the forcing-theoretic point of view. This order bears the same relationship to Hindman's Theorem as $\mathcal{P}(\omega)/\mathrm{fin}$ does to Ramsey's Theorem. While $(\mathcal{P}(\omega)/\mathrm{fin})^2$ completely embeds into $(\mathrm{FIN})^\omega$, we show this is consistently false for higher powers of $\mathcal{P}(\omega)/\mathrm{fin}$, by proving that the distributivity number $\mathfrak{h}_3$ of $(\mathcal{P}(\omega)/\mathrm{fin})^3$ may be strictly smaller than the distributivity number $\mathfrak{h}_{\mathrm{FIN}}$ of $(\mathrm{FIN})^\omega$. We also investigate infinite maximal antichains in $(\mathrm{FIN})^\omega$ and show that the least cardinality $\mathfrak{a}_{\mathrm{FIN}}$ of such a maximal antichain is at least the smallest size of a nonmeager set of reals. As a consequence, we obtain that $\mathfrak{a}_{\mathrm{FIN}}$ is consistently larger than $\mathfrak{a}$, the least cardinality of an infinite maximal antichain in $\mathcal{P}(\omega)/\mathrm{fin}$.

Introduction.

The forcing notion $\mathcal{P}(\omega)/\mathrm{fin}$ of infinite subsets of the natural numbers ω modulo the finite sets plays an important role in set theory of the reals. It is ordered by $[A] \leq [B]$ if $A \subseteq^* B$ where $A, B \in [\omega]^\omega$ are infinite subsets of ω and $\subseteq^*$ denotes *almost inclusion* as usual. $\mathcal{P}(\omega)/\mathrm{fin}$ is σ-closed (a straightforward diagonal argument) and thus does not add new real numbers. It generically adjoins a Ramsey ultrafilter on ω [**Ma**]. Recall that an ultrafilter $\mathcal{U}$ on ω is *Ramsey* if for all functions $f : \omega \to \omega$ there is $U \in \mathcal{U}$ such that $f \restriction U$ is either one-to-one or constant or, equivalently, if witnesses for the conclusion of *Ramsey's Theorem* can be found within $\mathcal{U}$; that is, if for all partitions $\pi : [\omega]^n \to k$, $n, k \in \omega$, there is $U \in \mathcal{U}$ such that $\pi \restriction [U]^n$ is constant.

In recent decades, a number of close relatives of $\mathcal{P}(\omega)/\mathrm{fin}$ have been investigated. One way to obtain such relatives goes by replacing the ideal fin of finite sets by an analytic ideal on ω. If $\mathcal{I}$ is an F_σ-ideal, then $\mathcal{P}(\omega)/\mathcal{I}$ is still σ-closed [**JK**] (and thus quite similar to $\mathcal{P}(\omega)/\mathrm{fin}$) while for non-$F_\sigma$-ideals the quotient may add real numbers. For example, Farah [**Fa**] proved that $\mathcal{P}(\omega)/\mathcal{Z}$ is forcing equivalent to the product $\mathcal{P}(\omega)/\mathrm{fin} \times \mathbb{B}_{\mathfrak{c}}$ where

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$\mathcal{Z}$ is the density zero ideal and $\mathbb{B}_{\mathfrak{c}}$ is the measure algebra adding $\mathfrak{c}$ random reals. Another way to obtain relatives of $\mathcal{P}(\omega)/\mathrm{fin}$ goes by considering structures which are naturally related to a Ramsey-theoretic statement in the same way $\mathcal{P}(\omega)/\mathrm{fin}$ is related to Ramsey's Theorem. For example, several authors (see, e.g., [**CKMW**], [**Ha**], [**Sp2**], [**Br1**]) have considered the forcing notion related to the Carlson–Simpson Theorem [**CS**] (see also [**To**, Theorem 5.70]), namely, the collection of partitions $(\omega)^\omega$ of ω into infinitely many blocks, ordered by almost coarsening.

In this work, we study the $\mathcal{P}(\omega)/\mathrm{fin}$-like forcing notion underlying Hindman's Theorem [**Hi**] (see also [**To**, Theorem 2.41]). Let FIN denote the finite non-empty subsets of ω. For $k \leq \omega$, a sequence $(d_i : i < k)$ of elements of FIN is called a *block sequence* if for all $i < k-1$, $d_i < d_{i+1}$, that is, $\max(d_i) < \min(d_{i+1})$. $(\mathrm{FIN})^\omega$ denotes the collection of infinite block sequences and $(\mathrm{FIN})^{<\omega}$ stands for the finite block sequences. If $k \leq \omega$ and $D = (d_i : i < k)$ is a block sequence, $\mathrm{FU}(D) = \{e \in \mathrm{FIN} : \exists \Gamma \subseteq k$ finite non-empty such that $e = \bigcup_{i \in \Gamma} d_i\}$ is the collection of finite unions from D. With this notation, Hindman's Theorem reads:

HINDMAN'S THEOREM. *For every partition $\pi : \mathrm{FIN} \to k$ there is $D \in (\mathrm{FIN})^\omega$ such that $\pi \restriction \mathrm{FU}(D)$ is constant.*

We equip $(\mathrm{FIN})^\omega$ with an order relation, as follows. An infinite block sequence $E = (e_i : i \in \omega)$ is a *condensation* of $D = (d_i : i \in \omega) \in (\mathrm{FIN})^\omega$, $E \sqsubseteq D$ in symbols, if $E \subseteq \mathrm{FU}(D)$ (or, equivalently, $\mathrm{FU}(E) \subseteq \mathrm{FU}(D)$). E is an *almost condensation* of D, $E \sqsubseteq^* D$ in symbols, if there is $n \in \omega$ such that $(e_i : i \geq n) \subseteq \mathrm{FU}(D)$. It is easy to see that $((\mathrm{FIN})^\omega, \sqsubseteq^*)$ is a σ-closed forcing notion and thus similar to $\mathcal{P}(\omega)/\mathrm{fin}$. It generically adjoins a stable ordered-union ultrafilter on FIN [**Ei**, Proposition 3.2]. Recall that an ultrafilter $\mathcal{U}$ on FIN is an *ordered-union ultrafilter* if witnesses for the conclusion of Hindman's Theorem can be found within $\mathcal{U}$; that is, if for all partitions $\pi : \mathrm{FIN} \to k$, $k \in \omega$, there is $D \in (\mathrm{FIN})^\omega$ such that $\mathrm{FU}(D) \in \mathcal{U}$ and $\pi \restriction \mathrm{FU}(D)$ is constant (or, equivalently, if $\mathcal{U}$ has a basis of sets of the form $\mathrm{FU}(D)$). An ultrafilter $\mathcal{U}$ on FIN is *stable* if given $(D_n : n \in \omega) \subseteq (\mathrm{FIN})^\omega$ such that $D_{n+1} \sqsubseteq^* D_n$ and $\mathrm{FU}(D_n) \in \mathcal{U}$ for all n, there is $E \in (\mathrm{FIN})^\omega$ such that $\mathrm{FU}(E) \in \mathcal{U}$ and $E \sqsubseteq^* D_n$ for all n.

A basic question one may ask about such relatives of $\mathcal{P}(\omega)/\mathrm{fin}$ is whether they are forcing equivalent or, at least, whether one can be completely embedded into another. Recall that two p.o.'s $\mathbb{P}$ and $\mathbb{Q}$ are *forcing equivalent* if they have the same completions, i.e., $\mathrm{r.o.}(\mathbb{P}) = \mathrm{r.o.}(\mathbb{Q})$. $\mathbb{P}$ *completely embeds into* $\mathbb{Q}$, $\mathbb{P} \lessdot \mathbb{Q}$ in symbols, if there is an embedding $e : \mathrm{r.o.}(\mathbb{P}) \to \mathrm{r.o.}(\mathbb{Q})$ preserving ordering, incompatibility and maximal antichains or, equivalently, if there is a *projection* $\pi : \mathrm{r.o.}(\mathbb{Q}) \to \mathrm{r.o.}(\mathbb{P})$ preserving the ordering such that for all $q \in \mathrm{r.o.}(\mathbb{Q})$ and all $p \leq \pi(q)$ in $\mathrm{r.o.}(\mathbb{P})$ there is $q' \leq q$ in $\mathrm{r.o.}(\mathbb{Q})$ with $\pi(q') \leq p$. This implies that (but is not equivalent to) forcing with $\mathbb{Q}$ adds a generic for $\mathbb{P}$.

In many cases, this basic question has a trivial answer under CH for then all σ-closed forcing notions of size $\mathfrak{c} = \aleph_1$ are forcing equivalent. Hence the real question is whether complete embeddability is provable in ZFC or whether it consistently fails. $\mathcal{P}(\omega)/\mathrm{fin}$ and even its square $(\mathcal{P}(\omega)/\mathrm{fin})^2$ are easily seen to completely embed into $(\mathrm{FIN})^\omega$ (see Proposition 5 in Section 1 below for the argument) and this embedding is "very definable" in the sense that the projection $\pi : (\mathrm{FIN})^\omega \to ([\omega]^\omega)^2$ is a continuous function. Indeed, there are mappings $\varphi^0 : \mathrm{FIN} \to \omega : e \mapsto \min(e)$ and $\varphi^1 : \mathrm{FIN} \to \omega : e \mapsto \max(e)$ such that for all $D \in (\mathrm{FIN})^\omega$ and all $(B^0, B^1) \subseteq^* (\varphi^0[D], \varphi^1[D])$ there is $E \sqsubseteq^* D$ such that $(\varphi^0[E], \varphi^1[E]) \subseteq^* (B^0, B^1)$. (Here we write $(B^0, B^1) \subseteq^* (A^0, A^1)$ for $B^0 \subseteq^* A^0$ and $B^1 \subseteq^* A^1$.) If 2 is replaced by 3 a strong version of the non-existence of such mappings is an easy consequence of Taylor's Canonization Theorem [**Ta**] (see also [**To**, Theorem 5.28]).

Taylor's Theorem. *For every $D \in (\mathrm{FIN})^\omega$ and every function $\varphi : \mathrm{FU}(D) \to \omega$, there is $E \sqsubseteq D$ such that $\varphi{\restriction}\mathrm{FU}(E)$ is one of the following five canonical functions*:

- *a constant function,*
- *a function of* min-*type, i.e.,* $\varphi(x) = \varphi(y) \iff \min(x) = \min(y)$ *for all* $x, y \in \mathrm{FU}(E)$,
- *a function of* max-*type, i.e.,* $\varphi(x) = \varphi(y) \iff \max(x) = \max(y)$ *for all* $x, y \in \mathrm{FU}(E)$,
- *a function of* min-max-*type, i.e.,* $\varphi(x) = \varphi(y) \iff \min(x) = \min(y) \wedge \max(x) = \max(y)$ *for all* $x, y \in \mathrm{FU}(E)$,
- *a one-to-one function.*

Corollary to Taylor's Theorem. *Assume $\varphi^j : \mathrm{FU}(D) \to \omega$, $j < 3$. Then there is $E \sqsubseteq D$ such that for all $(A^0, A^1, A^2) \in ([\omega]^\omega)^3$ there is $(B^0, B^1, B^2) \subseteq^* (A^0, A^1, A^2)$ with* $\mathrm{FU}(E) \cap (\varphi^0)^{-1}(B^0) \cap (\varphi^1)^{-1}(B^1) \cap (\varphi^2)^{-1}(B^2) = \emptyset$.

(The proof of the corollary from the theorem is implicit in the second part of the proof of Main Lemma 33 below. Here we only give a brief sketch: let E be such that all three functions φ^j are canonical on $\mathrm{FU}(E)$ in the sense of Taylor's Theorem. If at least one function is constant, the conclusion is immediate. If this is not the case, split into two cases, according to whether or not at least two functions are of min-type.)

This corollary suggests that there is no definable—and even no ZFC-provable—complete embedding of $(\mathcal{P}(\omega)/\mathrm{fin})^3$ into $(\mathrm{FIN})^\omega$. In fact, in all cases in which complete embeddability has been established, the witness is "very definable" in the above sense.

Closely connected with the problem of complete embeddability is the relationship between the distributivity numbers of the forcing notions involved. Given a p.o. $\mathbb{P}$, its *distributivity number* $\mathfrak{h}(\mathbb{P})$ is the minimal κ such that there is a family $(D_\alpha : \alpha < \kappa)$ of dense open subsets of $\mathbb{P}$ whose intersection is not dense. Equivalently, $\mathfrak{h}(\mathbb{P})$ is the minimal κ such that there is $p \in \mathbb{P}$ forcing that a new function from κ to V is adjoined. Using this reformulation, we immediately see that $\mathbb{P} \lessdot \mathbb{Q}$ implies $\mathfrak{h}(\mathbb{P}) \geq \mathfrak{h}(\mathbb{Q})$. So, for example, $\mathfrak{h}_{\mathrm{FIN}} \leq \mathfrak{h}_2$ where $\mathfrak{h}_{\mathrm{FIN}} := \mathfrak{h}((\mathrm{FIN})^\omega)$ and $\mathfrak{h}_n := \mathfrak{h}((\mathcal{P}(\omega)/\mathrm{fin})^n)$ for $n \in \omega$. Clearly, the $\mathfrak{h}_n$ form a decreasing chain, and Shelah and Spinas (see [**SS1**], [**SS2**]) proved that $\mathfrak{h}_{n+1} < \mathfrak{h}_n$ is consistent for every n. In fact, for a number of relatives $\mathbb{P}, \mathbb{Q}$ of $\mathcal{P}(\omega)/\mathrm{fin}$, the consistency of $\mathbb{P} \not\lessdot \mathbb{Q}$ has been established by showing the consistency of $\mathfrak{h}(\mathbb{P}) < \mathfrak{h}(\mathbb{Q})$. Along these lines we prove:

Theorem 1. *$\mathfrak{h}_3 < \mathfrak{h}_{\mathrm{FIN}}$ is consistent.*

In particular, consistently $(\mathcal{P}(\omega)/\mathrm{fin})^3 \not\lessdot (\mathrm{FIN})^\omega$. In view of the preceding discussion, this result also strengthens the consistency of $\mathfrak{h}_3 < \mathfrak{h}_2$ obtained by Shelah and Spinas [**SS2**]. Also:

Theorem 2. *It is consistent that $\mathfrak{h}_{\mathrm{FIN}} < \mathfrak{h}_n$ holds for all n.*

The consistency of $(\mathrm{FIN})^\omega \not\lessdot (\mathcal{P}(\omega)/\mathrm{fin})^n$ for all n follows.

There is another way to look at $\mathfrak{h}_{\mathrm{FIN}}$: by the general base tree theorem [**BDH**, Theorem 2.1], whose assumptions are satisfied for the partial order $((\mathrm{FIN})^\omega, \sqsubseteq^*)$, there are maximal antichains $\mathcal{D}_\alpha \subseteq (\mathrm{FIN})^\omega$, $\alpha < \mathfrak{h}_{\mathrm{FIN}}$, such that $\mathcal{D} = \bigcup_{\alpha<\mathfrak{h}_{\mathrm{FIN}}} \mathcal{D}_\alpha$ is a tree of height $\mathfrak{h}_{\mathrm{FIN}}$ with α-th level $\mathcal{D}_\alpha$, each $D \in \mathcal{D}$ has $\mathfrak{c}$ many immediate successors, and $\mathcal{D}$ is dense in $(\mathrm{FIN})^\omega$. This was originally established for $\mathcal{P}(\omega)/\mathrm{fin}$ in [**BPS**]. As a consequence, one obtains that forcing with $(\mathrm{FIN})^\omega$ collapses the continuum exactly to $\mathfrak{h}_{\mathrm{FIN}}$.

Another interesting problem concerns possible cardinalities of maximal antichains in such relatives of $\mathcal{P}(\omega)/\mathrm{fin}$. Note that $\mathcal{A} \subseteq [\omega]^\omega$ is a maximal antichain in $\mathcal{P}(\omega)/\mathrm{fin}$ iff it is a *MAD family*, that is, iff any two distinct members A and B of $\mathcal{A}$ are *almost disjoint* (i.e., $A \cap B$ is finite) and $\mathcal{A}$ is maximal with this property (i.e., for all $C \in [\omega]^\omega$ there is $A \in \mathcal{A}$ such that $C \cap A$ is infinite). In analogy, say that D and E in $(\mathrm{FIN})^\omega$ are *almost disjoint* FIN if there is no $F \in (\mathrm{FIN})^\omega$ with $F \sqsubseteq^* D, E$ or, equivalently, if $\mathrm{FU}(D) \cap \mathrm{FU}(E)$ is finite. $\mathcal{D} \subseteq (\mathrm{FIN})^\omega$ is an *AD_{FIN} family* if its members are pairwise almost disjoint FIN. $\mathcal{D}$ is a *MAD_{FIN} family* if it is AD_{FIN} and maximal with this property, that is, for all $E \in (\mathrm{FIN})^\omega$ there is $D \in \mathcal{D}$ such that $\mathrm{FU}(D) \cap \mathrm{FU}(E)$ is infinite. Clearly, the MAD_{FIN} families are exactly the maximal antichains in $(\mathrm{FIN})^\omega$.

The *almost disjointness number* $\mathfrak{a}$ is the minimal size of an infinite MAD family. Similarly, $\mathfrak{a}_{\mathrm{FIN}}$ is the minimal size of a MAD_{FIN} family of size at least 2. The reason for the restriction on size in these definitions is that finite partitions of ω into infinite sets clearly are finite MAD families, and that the block sequence $(\{n\} : n \in \omega)$ is a MAD_{FIN} family of size 1. A classical result says that $\mathfrak{a} \geq \mathfrak{b}$ where $\mathfrak{b}$ is the unbounding number [**Bl2**, Proposition 8.4]. (This is still true if $\mathfrak{a}$ is replaced by $\mathfrak{a}_n$, the minimal size of an infinite maximal antichain in $(\mathcal{P}(\omega)/\mathrm{fin})^n$ (or, equivalently, an infinite MAD family of n-dimensional cuboids) [**Sp1**]. It is clear that the $\mathfrak{a}_n$ form a decreasing chain like the $\mathfrak{h}_n$, but the consistency of $\mathfrak{a}_{n+1} < \mathfrak{a}_n$ is an open problem.)

For $\mathfrak{a}_{\mathrm{FIN}}$ we get a better lower bound:

Theorem 3. *$\mathfrak{a}_{\mathrm{FIN}} \geq \mathrm{non}(\mathcal{M})$.*

Here $\mathrm{non}(\mathcal{M})$ is the *uniformity* of the meager ideal $\mathcal{M}$, that is, the least size of a nonmeager set of reals. As a consequence we obtain the consistency of $\mathfrak{a}_{\mathrm{FIN}} > \mathfrak{a}$ (see Corollary 20 in Section 3). We also note that the consistency of $\mathfrak{a}_{\mathrm{FIN}} > \mathrm{non}(\mathcal{M})$ can be easily established by the methods of [**Br2**, Section 4]. While we do not know of an upper bound for $\mathfrak{a}_{\mathrm{FIN}}$ in terms of classical cardinal invariants of the continuum, we show:

Theorem 4. *$\mathfrak{a}_{\mathrm{FIN}} < \mathfrak{c}$ is consistent. In fact, $\mathfrak{a}_{\mathrm{FIN}} = \aleph_1$ in the Cohen model.*

While there is a natural way to produce an AD family from an AD_{FIN} family (see Proposition 21 in Section 3), the former never is maximal. In particular, we do not know the answer to the following:

QUESTION. *Is $\mathfrak{a}_{\mathrm{FIN}} < \mathfrak{a}$ consistent? Or does $\mathfrak{a} \le \mathfrak{a}_{\mathrm{FIN}}$ hold in ZFC?*

We have touched upon two cardinal invariants related to $(\mathrm{FIN})^\omega$, namely, $\mathfrak{h}_{\mathrm{FIN}}$ and $\mathfrak{a}_{\mathrm{FIN}}$, but it is clear that, as this has been done for example for $(\omega)^\omega$ [**CKMW**], other cardinals like $\mathfrak{s}_{\mathrm{FIN}}$ or $\mathfrak{r}_{\mathrm{FIN}}$ should be studied as well.

This paper is organized as follows: in Section 1 we collect a couple of basic results concerning complete embeddings and discuss Mathias-like forcing notions related to $(\mathrm{FIN})^\omega$ [**G2**], to $\mathcal{P}(\omega)/\mathrm{fin}$ [**Ma**] and to its products [**SS2**], as well as their connection with MAD families and with distributivity numbers. In Section 2 we prove Theorem 4, in Section 3, Theorem 3, and in Section 4, Theorem 1. Section 5 gives an outline of the proof of Theorem 2. Since many arguments are similar to the proof of Theorem 1, we only present the proofs whose combinatorics is substantially different.

Our notation is standard. $\forall^\infty n$ denotes "for all but finitely many $n \in \omega$", and $\exists^\infty n$ stands for "there are infinitely many $n \in \omega$". For basic results on cardinal invariants, we refer to [**Bl2**] or [**BJ**], for forcing theory, to [**Ku**] or [**BJ**], and for Ramsey theory, to [**To**].

1. Preliminaries.

For $D = (d_i : i \in \omega) \in (\mathrm{FIN})^\omega$ let $\min(D) = \{\min(d_i) : i \in \omega\}$ and $\max(D) = \{\max(d_i) : i \in \omega\}$. A similar definition applies to finite block sequences. We begin with:

PROPOSITION 5. *The product $\mathcal{P}(\omega)/\mathrm{fin} \times \mathcal{P}(\omega)/\mathrm{fin}$ completely embeds into $(\mathrm{FIN})^\omega$. In particular, $\mathfrak{h}_{\mathrm{FIN}} \le \mathfrak{h}_2$.*

PROOF. Define the projection $\pi : (\mathrm{FIN})^\omega \to ([\omega]^\omega)^2$ as follows: for $D = (d_i : i \in \omega) \in (\mathrm{FIN})^\omega$, let

$$\pi(D) := (\min(D), \max(D)).$$

It is straightforward to see that π is order-preserving: if $D \sqsubseteq^* D'$, then $\min(D) \subseteq^* \min(D')$ and $\max(D) \subseteq^* \max(D')$, and therefore $\pi(D) \le \pi(D')$. Clearly $\mathrm{ran}(\pi)$ is dense in $([\omega]^\omega)^2$.

Now let $D = (d_i : i \in \omega) \in (\mathrm{FIN})^\omega$ and assume $(A, B) \le \pi(D)$ for some $(A, B) \in ([\omega]^\omega)^2$. We need to find $D' \sqsubseteq^* D$ such that $\pi(D') \le (A, B)$. To this end, first choose $A' = \{a'_i : i \in \omega\} \subseteq A \cap \min(D)$ and $B' = \{b'_i : i \in \omega\} \subseteq B \cap \max(D)$ such that $a'_i < b'_i < a'_{i+1}$ for all $i \in \omega$. Since $A' \subseteq \{\min d_i : i \in \omega\}$ and $B' \subseteq \{\max d_i : i \in \omega\}$, $a'_i = \min d_{j_i}$ and $b'_i = \max d_{l_i}$ for some $j_i \le l_i < j_{i+1}$, for all $i \in \omega$. Letting $d'_i = d_{j_i} \cup d_{l_i}$ for all $i \in \omega$ and $D' = (d'_i : i \in \omega)$, we see that $D' \sqsubseteq D$ and $\pi(D') = (A', B') \le (A, B)$. Hence $(\mathcal{P}(\omega)/\mathrm{fin})^2$ is a projection of $(\mathrm{FIN})^\omega$. $\square$

There are other ways to look at the mapping π giving rise to this projection. Assume $\mathcal{U}$ is a stable ordered-union ultrafilter on FIN. Letting

$$\mathcal{U}_{\min} = \{A \subseteq \omega : \exists D \in (\mathrm{FIN})^\omega \text{ such that } \mathrm{FU}(D) \in \mathcal{U} \text{ and } \min(D) \subseteq A\}$$

and, similarly,

$$\mathcal{U}_{\max} = \{A \subseteq \omega : \exists D \in (\mathrm{FIN})^\omega \text{ such that } \mathrm{FU}(D) \in \mathcal{U} \text{ and } \max(D) \subseteq A\},$$

$\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$ are two Ramsey ultrafilters on ω [**Bl1**, Theorem 4.2 and Corollary 4.3], and they may be thought of as constructed from $\mathcal{U}$ via the mapping π: $\pi(\mathcal{U}) = (\mathcal{U}_{\min}, \mathcal{U}_{\max})$. Now, if $\dot{\mathcal{U}}$ is the generic stable ordered-union ultrafilter added by $(\mathrm{FIN})^\omega$, then $\pi(\dot{\mathcal{U}})$ is the pair of generic Ramsey ultrafilters added by $\mathcal{P}(\omega)/\mathrm{fin} \times \mathcal{P}(\omega)/\mathrm{fin}$.

Next recall that *Mathias forcing* $\mathbb{M}$ [**Ma**] consists of pairs (s, A) where $s \in [\omega]^{<\omega}$, $A \in [\omega]^\omega$, and $\max(s) < \min(A)$, ordered by $(t, B) \leq (s, A)$ if $s \subseteq t$, $B \subseteq A$, and $t \setminus s \subseteq A$. If $\mathcal{F}$ is a filter on ω, *Mathias forcing* $\mathbb{M}(\mathcal{F})$ *with* $\mathcal{F}$ consists of all $(s, A) \in \mathbb{M}$ with $A \in \mathcal{F}$, with the same ordering. $\mathbb{M}$ is a proper non-ccc forcing while $\mathbb{M}(\mathcal{F})$ is σ-centered and thus ccc. It is well-known that $\mathbb{M}$ decomposes as a two-step iteration $\mathbb{M} \cong \mathcal{P}(\omega)/\mathrm{fin} \star \mathbb{M}(\dot{\mathcal{U}})$ where $\dot{\mathcal{U}}$ is the generic Ramsey ultrafilter added by $\mathcal{P}(\omega)/\mathrm{fin}$ [**Ma**]. Furthermore, a real $C \in [\omega]^\omega$ is $\mathbb{M}$-generic over V iff for all MAD families $\mathcal{A}$ in V, there is $A \in \mathcal{A}$ such that $C \subseteq^* A$ [**Ma**]. Iterating $\mathbb{M}$ with countable support for ω_2 stages produces a model for $\mathfrak{h} = \aleph_2$.

In [**G2**] (see also [**G1**]), the second author defined an analogue of $\mathbb{M}$ for $(\mathrm{FIN})^\omega$: the forcing $\mathbb{P}_{\mathrm{FIN}}$ consists of pairs (σ, D) where $\sigma = (\sigma_i : i < |\sigma|) \in (\mathrm{FIN})^{<\omega}$ is a finite block sequence and $D = (d_i : i \in \omega) \in (\mathrm{FIN})^\omega$ with $\max(\sigma_{|\sigma|-1}) < \min(d_0)$, ordered by $(\tau, E) \leq (\sigma, D)$ if $\sigma \subseteq \tau$, $E \sqsubseteq D$, and $\tau_i \in \mathrm{FU}(D)$ for all i with $|\sigma| \leq i < |\tau|$. If $\mathcal{F}$ is a filter on FIN, $\mathbb{P}_{\mathcal{F}}$ consists of $(\sigma, D) \in \mathbb{P}_{\mathrm{FIN}}$ such that $\mathrm{FU}(D) \in \mathcal{F}$, with the same ordering. Again, $\mathbb{P}_{\mathrm{FIN}}$ is proper non-ccc while $\mathbb{P}_{\mathcal{F}}$ is σ-centered and thus ccc. Also, $\mathbb{P}_{\mathrm{FIN}}$ decomposes as a two-step iteration $\mathbb{P}_{\mathrm{FIN}} \cong (\mathrm{FIN})^\omega \star \mathbb{P}_{\dot{\mathcal{U}}}$ where $\dot{\mathcal{U}}$ is the generic stable ordered-union ultrafilter added by $(\mathrm{FIN})^\omega$ [**G2**, Lemma 4.16]. Furthermore, a block sequence $E \in (\mathrm{FIN})^\omega$ is $\mathbb{P}_{\mathrm{FIN}}$-generic over V iff for all MAD_{FIN} families $\mathcal{D} \in V$, there is $D \in \mathcal{D}$ such that $E \sqsubseteq^* D$ [**G3**, Theorem 3.9]. The ω_2-stage countable support iteration of $\mathbb{P}_{\mathrm{FIN}}$ forces $\mathfrak{h}_{\mathrm{FIN}} = \aleph_2$.

Since $(\mathcal{P}(\omega)/\mathrm{fin})^2$ completely embeds into $(\mathrm{FIN})^\omega$, it would be natural if a product of two copies of Mathias forcing also embedded into $\mathbb{P}_{\mathrm{FIN}}$. This is true—as long as "product" is interpreted in the right way. $\mathbb{M}_2$ consists of all pairs $((s, A), (t, B)) \in \mathbb{M} \times \mathbb{M}$ such that $|s| = |t|$ and if $s = \{s_i : i < n\}$ and $t = \{t_i : i < n\}$ are the increasing enumerations of s and t, respectively, then $s_i \leq t_i < s_{i+1}$ for all $i < n$. $\mathbb{M}_2$ is equipped with the product ordering: $((s', A'), (t', B')) \leq ((s, A), (t, B))$ if $(s', A') \leq (s, A)$ and $(t', B') \leq (t, B)$. This is the special case $n = 2$ of a forcing originally introduced by Shelah and Spinas [**SS2**, Definition 1.3], albeit with slightly different notation.

PROPOSITION 6. *$\mathbb{M}_2$ completely embeds into $\mathbb{P}_{\mathrm{FIN}}$.*

PROOF. Define the projection $\pi : \mathbb{P}_{\mathrm{FIN}} \to \mathbb{M}_2$ as follows: given $(\sigma, D) \in \mathbb{P}_{\mathrm{FIN}}$, let

$$\pi((\sigma, D)) := ((\min(\sigma), \min(D)), (\max(\sigma), \max(D))).$$

It is easy to see that π preserves order and that $\mathrm{ran}(\pi)$ is dense in $\mathbb{M}_2$. We need to show that for all $((s,A),(t,B)) \in \mathbb{M}_2$ and for all $(\sigma,D) \in \mathbb{P}_{\mathrm{FIN}}$ such that $((s,A),(t,B)) \leq \pi((\sigma,D))$, there is $(\sigma',D') \leq (\sigma,D)$ such that $\pi((\sigma',D')) \leq ((s,A),(t,B))$.

Assume $\sigma = (\sigma_0,\dots,\sigma_{m-1})$ and $D = (d_i : i \in \omega)$. There are $A' \subseteq A$ and $B' \subseteq B$ such that if $A' = \{a'_i : i \in \omega\}$ and $B' = \{b'_i : i \in \omega\}$ are their increasing enumerations, then $a'_i < b'_i < a'_{i+1}$ for all $i \in \omega$. Since $A' \subseteq A \subseteq \min(D)$ and $B' \subseteq B \subseteq \max(D)$, we can define $D' = (d'_l : l \in \omega)$ as follows: for $l \in \omega$, $d'_l = d_{i_l} \cup d_{j_l}$, where $a'_l = \min d_{i_l}$ and $b'_l = \max d_{j_l}$. It is clear that $D' \sqsubseteq D$.

Let $l := |s| = |t|$. Since $((s,A'),(t,B')) \in \mathbb{M}_2$, we have $s_0 \leq t_0 < s_1 \leq t_1 < \cdots < s_{l-1} \leq t_{l-1}$. Notice that $s \supseteq \min(\sigma)$ and if $x \in s \setminus \min(\sigma)$, then $x \in \min(D)$. Similarly, for all $y \in t \setminus \max(\sigma)$, $y \in \max(D)$. Thus, for $m \leq k < l$, we have $s_k = \min d_{i_k}$ and $t_k = \max d_{j_k}$ where $i_k \leq j_k < i_{k+1}$. Define $\sigma_k := d_{i_k} \cup d_{j_k}$ for such k. Put $\sigma' = \sigma\hat{\ }(\sigma_m,\dots,\sigma_{l-1})$. Now $(\sigma',D') \leq (\sigma,D)$ and $\pi((\sigma',D')) = ((s,A'),(t,B')) \leq ((s,A),(t,B))$.

Thus $\mathbb{P}_{\mathrm{FIN}}$ adds a generic for $\mathbb{M}_2$: if D^* is a $\mathbb{P}_{\mathrm{FIN}}$-generic block sequence over V, then $\pi(D^*) = (\min(D^*),\max(D^*))$ is an $\mathbb{M}_2$-generic over V. □

$\mathbb{M}_2$ is different from the usual product of two copies of Mathias forcing because of the following well-known fact whose proof we include for completeness' sake.

Observation 7. *$\mathbb{M}\times\mathbb{M}$ adds a Cohen real (in fact, $\mathbb{C} <\!\circ\, \mathbb{M}\times\mathbb{M}$ where $\mathbb{C}$ denotes Cohen forcing).*

Proof. Let (m_0,m_1) be a pair of Mathias reals $\mathbb{M}\times\mathbb{M}$-generic over V. We define $c \in 2^\omega$ as follows:

$$c(i) = \begin{cases} 0 & \text{if } m_0(i) \leq m_1(i) \\ 1 & \text{if } m_0(i) > m_1(i) \end{cases}$$

We shall show that c is a Cohen real over V. Let $D \subseteq \mathbb{C}$ be a dense subset.

Given $(a,b) \in [\omega]^{<\omega} \times [\omega]^{<\omega}$ such that $|a| = |b| = k$, and $a = \{a_i : i < k\}$ and $b = \{b_i : i < k\}$ are increasing enumerations of a and b, respectively, we define a function $q_{(a,b)} : k \to 2$ as follows:

$$q_{(a,b)}(i) = \begin{cases} 0 & \text{if } a_i \leq b_i \\ 1 & \text{if } a_i > b_i \end{cases}$$

Define

$$D' := \{((a,A),(b,B)) \in \mathbb{M}\times\mathbb{M} : |a| = |b| \text{ and } \exists q \in D \text{ such that } q = q_{(a,b)}\}.$$

Claim 8. *D' is a dense subset of $\mathbb{M}\times\mathbb{M}$.*

Proof. Let $((a,A),(b,B)) \in \mathbb{M}\times\mathbb{M}$. Assume without loss of generality that $|a| \leq |b|$. There exists a condition $(a',A') \in \mathbb{M}$ such that $(a',A') \leq (a,A)$ and $|a'| = |b| = k$. Consider $q_{(a',b)}$ defined as before. Since D is dense in $\mathbb{C}$, there is $q \in D$ such that $q \leq q_{(a',b)}$, i.e., $q \restriction k = q_{(a',b)}$. Assume $|q| = \ell$. Find increasing sequences $(a_i : k \leq i < \ell) \subseteq A'$ and $(b_i : k \leq i < \ell) \subseteq B$ such that, for all i with $k \leq i < \ell$,

$a_i \leq b_i$ iff $q(i) = 0$. Let $a'' = a' \cup \{a_i : k \leq i < \ell\}$, $b' = b \cup \{b_i : k \leq i < \ell\}$, $A'' = A' \setminus (a_{\ell-1}+1)$, and $B' = B \setminus (b_{\ell-1}+1)$. Then $((a'', A''), (b', B')) \leq ((a, A), (b, B))$ and $((a'', A''), (b', B')) \in D'$. So D' is a dense subset of $\mathbb{M} \times \mathbb{M}$. □

Since the generic filter G for $\mathbb{M} \times \mathbb{M}$ meets all such D', the real $c \in 2^\omega$ extends a $q \in D$ for all dense $D \subseteq \mathbb{C}$. Hence c is a Cohen real. In fact, the map π sending $((a, A), (b, B)) \in \mathbb{M} \times \mathbb{M}$ to $q_{(a,b)}$ is a projection establishing $\mathbb{C} <\!\!\circ\ \mathbb{M} \times \mathbb{M}$. □

By [**G2**, Theorem 4.26 and Corollary 4.28], $\mathbb{P}_{\mathrm{FIN}}$ has the Laver property and thus does not add Cohen reals, and the same is true for $\mathbb{M}_2$ by Proposition 6 (the latter was already proved by Shelah and Spinas [**SS2**, Lemma 1.17]). In particular, $\mathbb{M} \times \mathbb{M}$ cannot completely embed into either $\mathbb{M}_2$ or $\mathbb{P}_{\mathrm{FIN}}$.

On the other hand, it is easy to see that $\mathbb{M}_2$ completely embeds into $\mathbb{M} \times \mathbb{M}$: if $(A, A') \in ([\omega]^\omega)^2$ is a generic for $\mathbb{M} \times \mathbb{M}$, then define $B^0 = \{b_i^0 : i \in \omega\} \subseteq A^0$ and $B^1 = \{b_i^1 : i \in \omega\} \subseteq A^1$ recursively as follows:

$$\begin{aligned} b_0^0 &= \min(A^0), \\ b_0^1 &= \min(A^1 \setminus b_0^0), \\ b_{n+1}^0 &= \min\{x \in A^0 : x > b_n^1\}, \\ b_{n+1}^1 &= \min\{x \in A^1 : x \geq b_{n+1}^0\}. \end{aligned}$$

We leave it to the reader to verify that $(B^0, B^1) \in ([\omega]^\omega)^2$ is generic for $\mathbb{M}_2$.

2. Proof of Theorem 4.

This is similar to the proof that $\mathfrak{a} = \aleph_1$ in the Cohen model [**Ku**, Chapter VIII, Theorem 2.3].

Let V be a model of CH. In V, we shall construct a MAD_{FIN} family $\mathcal{A}$ of size ω_1 such that for all $I \in V$, $\mathcal{A}$ remains maximal in $V[G]$ whenever G is $\mathrm{Fn}(I, 2)$-generic over V. By [**Ku**, Chapter VIII, Lemma 2.2], it is sufficient to verify maximality of $\mathcal{A}$ in the extension via I_0 which are countable in V, because any $X \in (\mathrm{FIN})^\omega$ in $V[G]$ is in $V[G \cap \mathrm{Fn}(I_0, 2)]$, for some such I_0. When $|I_0| = \omega$ in V, $\mathrm{Fn}(I_0, 2)$ is isomorphic to $\mathrm{Fn}(\omega, 2)$ in V. It is therefore sufficient to define $\mathcal{A}$ such that whenever G is $\mathrm{Fn}(\omega, 2)$-generic over V, no infinite block sequence in $V[G]$ is AD_{FIN} from all elements in $\mathcal{A}$.

From now on, all forcing terminology refers to the p.o. $\mathrm{Fn}(\omega, 2)$. Within V, do the following: by CH, let (p_ξ, τ_ξ) for $\omega \leq \xi < \omega_1$ enumerate all pairs (p, τ) such that $p \in \mathrm{Fn}(\omega, 2)$ and τ is a nice name for an infinite block sequence. By recursion, pick infinite $A_\xi \in (\mathrm{FIN})^\omega$ as follows. Let A_n, for $n < \omega$, be any disjoint-FIN infinite block sequences. If $\omega \leq \xi < \omega_1$, and we have A_η for $\eta < \xi$, choose A_ξ such that:

(1) $\forall \eta < \xi\ (|\ \mathrm{FU}(A_\eta) \cap \mathrm{FU}(A_\xi)\ | < \omega)$, and

(2) if $p_\xi \Vdash$ "τ_ξ is an element of $(\mathrm{FIN})^\omega$" and

$$\forall \eta < \xi \quad (p_\xi \Vdash |\ \mathrm{FU}(\tau_\xi) \cap \mathrm{FU}(A_\eta)\ | < \omega), \quad (*)$$

then for all n and all $q \leq p_\xi$ there are $r \leq q$ and $s \in \mathrm{FU}(A_\xi)$ such that $\min(s) > n$ and $r \Vdash s \in \mathrm{FU}(\tau_\xi)$.

If $(*)$ fails then only (1) needs to be considered and we simply apply the fact that there are no MAD_{FIN} families of size ω.

Assume that $p_\xi \Vdash \tau_\xi \in (\mathrm{FIN})^\omega$ and $p_\xi \Vdash | \mathrm{FU}(\tau_\xi) \cap \mathrm{FU}(A_\eta) | < \omega$ for all $\eta < \xi$. Let B_i, $i \in \omega$, enumerate $\{A_\eta : \eta < \xi\}$ and let (n_i, q_i), $i \in \omega$, enumerate $\omega \times \{q : q \leq p_\xi\}$. By $(*)$, for each i, $q_i \Vdash | \mathrm{FU}(\tau_\xi) \setminus (\mathrm{FU}(B_0) \cup \cdots \cup \mathrm{FU}(B_i)) | = \omega$. We recursively produce $A_\xi = (d_i : i \in \omega)$ as follows.

Assume that we have elements $d_0, \ldots, d_j$ of FIN such that $d_i < d_{i+1}$ for all $i < j$, $\min d_i > n_i$ for all $i \leq j$, $\bigcup_{l \in I} d_l \cup d_j \notin \mathrm{FU}(B_0) \cup \cdots \cup \mathrm{FU}(B_j)$ for all $I \subseteq j$, and such that there is $r_i \leq q_i$ with $r_i \Vdash d_i \in \mathrm{FU}(\tau_\xi)$ for all $i \leq j$. Let $k = \max\{\max b^l_m : l \in \{0, \ldots, j+1\}\}$, where $B_l = (b^l_i : i \in \omega)$ and m is the least number such that $b^l_m > d_j$ for all $l \in \{0, \ldots, j+1\}$. Since $q_{j+1} \Vdash | \mathrm{FU}(\tau_\xi) \setminus \mathrm{FU}(B_0) \cup \cdots \cup \mathrm{FU}(B_{j+1}) | = \omega$, there exist $r_{j+1} \leq q_{j+1}$ and $d_{j+1} \in \mathrm{FIN}$ such that $\min d_{j+1} > \max\{k, n_{j+1}\}$, $d_{j+1} \notin \mathrm{FU}(B_0) \cup \cdots \cup \mathrm{FU}(B_{j+1})$ and $r_{j+1} \Vdash d_{j+1} \in \mathrm{FU}(\tau_\xi)$.

Claim 9. *$\bigcup_{l \in I} d_l \cup d_{j+1} \notin \mathrm{FU}(B_0) \cup \cdots \cup \mathrm{FU}(B_{j+1})$ for all $I \subseteq j+1$.*

Proof. Assume that there exists $I' \subseteq j+1$ such that $\bigcup_{l \in I'} d_l \cup d_{j+1} \in \mathrm{FU}(B_i)$ for some $i \leq j+1$. Since $d_j < b^i_m < d_{j+1}$ this implies that $d_{j+1} \in \mathrm{FU}(B_i)$, which is a contradiction. □

Let $A_\xi = (d_i : i \in \omega)$. Clearly A_ξ satisfies (1) and (2). This completes the recursive construction.

Consider $\mathcal{A} = \{A_\xi : \xi < \omega_1\}$. By (1), $\mathcal{A}$ is an AD_{FIN}-family. Let G be a $\mathrm{Fn}(\omega, 2)$-generic filter over V.

Claim 10. *$\mathcal{A}$ is a MAD_{FIN} family in $V[G]$.*

Proof. If $\mathcal{A}$ is not maximal in $V[G]$, there exists $\xi < \omega_1$ such that $p_\xi \in G$, $p_\xi \Vdash \tau_\xi \in (\mathrm{FIN})^\omega$ and $p_\xi \Vdash \forall \eta < \omega_1\ (| \mathrm{FU}(\tau_\xi) \cap \mathrm{FU}(A_\eta) | < \omega)$. Thus $(*)$ holds at ξ and also $p_\xi \Vdash | \mathrm{FU}(\tau_\xi) \cap \mathrm{FU}(A_\xi) | < \omega$. Therefore there are $q \leq p_\xi$ and $n \in \omega$ such that $q \Vdash \bigcup(\mathrm{FU}(\tau_\xi) \cap \mathrm{FU}(A_\xi)) \subseteq n$. But by (2), there are $r \leq q$ and $s \in \mathrm{FU}(A_\xi)$ such that $\min(s) > n$ and $r \Vdash s \in \mathrm{FU}(\tau_\xi)$, a contradiction. □

This completes the proof of Theorem 4.

3. Proof of Theorem 3.

As a preliminary step towards proving Theorem 3, we first show the weaker inequality $\mathfrak{b} \leq \mathfrak{a}_{\mathrm{FIN}}$ (Proposition 12 below).

We say that an infinite block sequence $A = (a_n : n \in \omega)$ has *almost no holes* if

- $\forall^\infty n\ \ (a_n = [\min(a_n), \max(a_n)])$,
- $\forall^\infty n\ \ (\max(a_n) + 1 = \min(a_{n+1}))$.

Otherwise A has *infinitely many holes.* Note that A has almost no holes is equivalent to $\bigcup A$ is cofinite.

Lemma 11. *Let A and A' be two almost disjoint infinite block sequences such that A and A' have almost no holes. Define $B := \min(A)$ and $B' := \min(A')$, then $|B \cap B'| < \omega$.*

Proof. Let $A = (a_n : n \in \omega)$ and $A' = (a'_n : n \in \omega)$. Assume that $|B \cap B'| = \omega$. Let $B'' = B \cap B' = \{b_i : i \in \omega\}$ such that $b_i < b_{i+1}$ for all $i \in \omega$. We have $b_i = \min a_{n_i} = \min a'_{m_i}$ for some strictly increasing sequences $(n_i : i \in \omega)$ and $(m_i : i \in \omega)$. Since A and A' have almost no holes, we have $\bigcup_{j=n_i}^{n_{i+1}-1} a_j = \bigcup_{j=m_i}^{m_{i+1}-1} a'_j$ for almost all $i \in \omega$. Therefore, $|\mathrm{FU}(A) \cap \mathrm{FU}(A')| = \omega$, which is a contradiction. Hence $|B \cap B'| < \omega$. □

Proposition 12. $\mathfrak{b} \leq \mathfrak{a}_{\mathrm{FIN}}$.

Proof. Let $2 \leq \kappa < \mathfrak{b}$ and assume $\{A^\alpha : \alpha < \kappa\}$ is an almost disjoint FIN family. Say $A^\alpha = (a_n^\alpha : n \in \omega)$. For every $\alpha < \kappa$ such that A^α has infinitely many holes we define a function $f_\alpha : \omega \to \omega$ as follows:

$$f_\alpha(n) = \min \left\{ k : k \notin \bigcup A^\alpha \text{ and } k > n \right\}.$$

Note that $f_\alpha(n) > n$ for all $n \in \omega$.

Also define $B^\alpha = \min(A^\alpha)$ for all $\alpha < \kappa$. If there is an α such that A^α has almost no holes, fix such an α and let $B = B^\alpha$. B^α is coinfinite because $\kappa \geq 2$. Otherwise, let B be an arbitrary infinite coinfinite set.

Since $\kappa < \mathfrak{b}$, there is $f : \omega \to \omega$ such that $f_\beta <^* f$ for all $\beta < \kappa$. Define $A = (a_n : n \in \omega)$ as follows: for all $n \in \omega$

- a_n is an interval, i.e., $a_n = [\min a_n, \max a_n]$
- $\min a_n \in B$
- $\max a_n + 1 \notin B$
- $f(\min a_n) \leq \max a_n$

We shall prove that $|\mathrm{FU}(A) \cap \mathrm{FU}(A^\beta)| < \omega$ for all $\beta < \kappa$.

First assume that A^β has infinitely many holes.

Claim 13. *For all $\beta < \kappa$ such that A^β has infinitely many holes, $a_n \not\subseteq \bigcup A^\beta$ for almost all n.*

Proof. Let $\beta < \kappa$ such that A^β has infinitely many holes. There exists $m \in \omega$ such that $f_\beta(k) < f(k)$ for all $k \geq m$. Let $n \in \omega$ such that $\min a_n \geq m$. Then $f_\beta(\min a_n) \in a_n$ because

$$\min a_n < f_\beta(\min a_n) < f(\min a_n) \leq \max a_n.$$

Thus $a_n \not\subseteq \bigcup A^\beta$ follows. □

Therefore $|\mathrm{FU}(A^\beta) \cap \mathrm{FU}(A)| < \omega$ follows for β such that A^β has infinitely many holes.

So assume A^β has almost no holes. First suppose $\beta = \alpha$. Then $B = B^\alpha = \{\min a_i^\alpha : i \in \omega\}$. Note that $\forall^\infty i, j \in \omega$, $\max a_i \neq \max a_j^\alpha$, because $\max a_i + 1 \notin B$ while $\max a_j^\alpha + 1 = \min a_{j+1}^\alpha \in B^\alpha = B$. Assume towards a contradiction that $|\mathrm{FU}(A) \cap \mathrm{FU}(A^\alpha)| = \omega$. Then there is $C \in (\mathrm{FIN})^\omega$ with $C \sqsubseteq A$ and $C \sqsubseteq A^\alpha$. Therefore, if $b \in C$ then $b \in \mathrm{FU}(A) \cap \mathrm{FU}(A^\alpha)$, so $\max b = \max a_i$ for some i and $\max b = \max a_j^\alpha$ for some j, which is a contradiction. Hence $|\mathrm{FU}(A) \cap \mathrm{FU}(A^\alpha)| < \omega$.

Next suppose that $\beta \neq \alpha$. By Lemma 11, $|B^\alpha \cap B^\beta| < \omega$. Assume that $|\mathrm{FU}(A) \cap \mathrm{FU}(A^\beta)| = \omega$. Then $|B \cap B^\beta| = \omega$, contradicting $B = B^\alpha$. Therefore $|\mathrm{FU}(A) \cap \mathrm{FU}(A^\beta)| < \omega$. □

⋆⋆⋆

For completing the proof of Theorem 3, recall the following cardinal invariants:

$$\begin{aligned}\mathfrak{b}(p \neq^*) = \min\{\, |\mathcal{F}| : {} & \mathcal{F} \subseteq \omega^\omega \text{ and for all partial functions } g : \omega \to \omega \\ & \text{with infinite domain } \exists f \in \mathcal{F}\ \exists^\infty n \in \mathrm{dom}(g)\ (f(n) = g(n))\}\end{aligned}$$

For $h : \omega \to \omega$ with $h(n) \to \infty$ as $n \to \infty$, let

$$\begin{aligned}\mathfrak{b}_h(p \neq^*) = \min\{\, |\mathcal{F}| : {} & \mathcal{F} \subseteq \omega^\omega \text{ and for all partial functions } g : \omega \to \omega \\ & \text{with infinite domain bounded by } h \\ & \exists f \in \mathcal{F}\ \exists^\infty n \in \mathrm{dom}(g)\ (f(n) = g(n))\}\end{aligned}$$

It is well-known that this cardinal does not depend on the function h. We include a proof for completeness' sake.

Lemma 14. *$\mathfrak{b}_h(p \neq^*) = \mathfrak{b}_{h'}(p \neq^*)$ for $h, h' \in \omega^\omega$ with $h(n), h'(n) \to \infty$ as $n \to \infty$.*

Proof. For $k \in \omega$ let n_k be such that $h'(k) < h(n_k)$ and $n_k < n_{k+1}$ and $k < n_k$.

Let $\mathcal{F}$ be a witness of $\mathfrak{b}_h(p \neq^*)$. Given $f \in \mathcal{F}$, define $f' : \omega \to \omega$ by $f'(k) = f(n_k)$. Consider $\mathcal{F}' = \{f' : f \in \mathcal{F}\}$. Note that $|\mathcal{F}'| \leq |\mathcal{F}|$. We claim that $\mathcal{F}'$ is a witness for $\mathfrak{b}_{h'}(p \neq^*)$.

Let $g : \omega \to \omega$ be partial such that $|\mathrm{dom}(g)| = \omega$ and $g(k) < h'(k)$ for all $k \in \mathrm{dom}(g)$. Define a partial function $g' : \omega \to \omega$ as follows: $\mathrm{dom}(g') = \{n_k : k \in \mathrm{dom}(g)\}$ and $g'(n_k) = g(k)$ for $k \in \mathrm{dom}(g)$. For every $k \in \mathrm{dom}(g)$, $g'(n_k) = g(k) < h'(k) < h(n_k)$. Hence there is $f \in \mathcal{F}$ such that $f(n_k) = g'(n_k)$ for infinitely many $n_k \in \mathrm{dom}(g')$. Thus $f'(k) = f(n_k) = g'(n_k) = g(k)$ for infinitely many $k \in \mathrm{dom}(g)$, and $\mathcal{F}'$ is as required.

This shows $\mathfrak{b}_{h'}(p \neq^*) \leq \mathfrak{b}_h(p \neq^*)$ and, by symmetry, $b_{h'}(p \neq^*) = b_h(p \neq^*)$ follows. □

Let $\mathfrak{b}(pbd \neq^*)$ be the common value of the $\mathfrak{b}_h(p \neq^*)$. The following is also a folklore result whose proof we include.

Lemma 15. $\mathfrak{b}(p \neq^*) = \max\{\mathfrak{b}, \mathfrak{b}(pbd \neq^*)\}$.

Proof. If $\mathcal{F}$ is a witness for $\mathfrak{b}(p \neq^*)$, then clearly it is a witness for both $\mathfrak{b}$ and $\mathfrak{b}(pbd \neq^*)$. On the other hand, assume $\mathcal{H}$ is an unbounded family consisting of increasing functions and for each $h \in \mathcal{H}$, $\mathcal{F}_h$ is a witness for $\mathfrak{b}_h(p \neq^*)$. We claim that $\mathcal{F} = \bigcup\{\mathcal{F}_h : h \in \mathcal{H}\}$ is a witness for $\mathfrak{b}(p \neq^*)$.

Indeed, if $g : \omega \to \omega$ is a partial function, $|\mathrm{dom}(g)| = \omega$, then there is $h \in \mathcal{H}$ such that $h(n) > g(n)$ for infinitely many $n \in \mathrm{dom}(g)$. Let $g' = g \restriction \{n \in \mathrm{dom}(g) : h(n) > g(n)\}$. Since $\mathcal{F}_h$ is a witness for $\mathfrak{b}_h(p \neq^*)$, there is $f \in \mathcal{F}_h$ such that $f(n) = g'(n) = g(n)$ for infinitely many $n \in \mathrm{dom}(g')$, as required. □

An old result of Miller [**Mi**] (see also [**BJ**, Lemma 2.4.8]) says:

MILLER'S THEOREM. $\mathrm{non}(\mathcal{M}) = \mathfrak{b}(p \neq^*)$.

We note that, by a result of Bartoszyński [**Ba**] (see also [**BJ**, Theorem 2.4.7 and Lemma 2.4.8]), this is also true when only total functions are considered. However, the two preceding lemmata are false for total functions, and this is why we work with partial functions here. By Lemma 15 and Proposition 12, the proof of Theorem 3 is complete if we show:

THEOREM 16. $\mathfrak{b}(pbd \neq^*) \leq \mathfrak{a}_{\mathrm{FIN}}$.

PROOF. Fix $h : \omega \to \omega$ such that $h(n) \to \infty$ when $n \to \infty$. By Lemma 14 and the discussion after its proof, $\mathfrak{b}(pbd \neq^*) = \mathfrak{b}_h(p \neq^*)$. Let $\mathcal{A}$ be an AD_{FIN} family of size $2 \leq \kappa < \mathfrak{b}_h(p \neq^*)$. We shall prove that $\mathcal{A}$ is not maximal.

For $A = (a_n : n \in \omega)$ in $\mathcal{A}$, let $E_A = \bigcup\{a_n \in A : a_n \text{ singleton}\}$. Note that E_A is coinfinite, because otherwise A is not almost disjoint from other elements in $\mathcal{A}$ (here we use $\kappa \geq 2$). Also, if A, A' are distinct elements of $\mathcal{A}$, then $|E_A \cap E_{A'}| < \omega$, because $|E_A \cap E_{A'}| = \omega$ implies $|FU(A) \cap FU(A')| = \omega$, which is a contradiction.

We will define finite sets c_n and d_n, where $c_n = \{c_n^0, \ldots, c_n^{h(n)+1}\}$, $d_n = \{d_n^0, \ldots, d_n^{h(n)+1}\}$, $c_n < d_n < c_{n+1} < d_{n+1}$ for all $n \in \omega$ and $|c_n| = |d_n| = h(n) + 2$.

Assume that there is $A_0 = (a_n : n \in \omega)$ in $\mathcal{A}$ such that A_0 contains infinitely many singletons, i.e., E_{A_0} is infinite. In this case we choose c_n and d_n such that $\bigcup\{c_n : n \in \omega\} \subseteq E_{A_0}$ and $\bigcup\{d_n : n \in \omega\} \cap E_{A_0} = \emptyset$. If there is no A_0 such that E_{A_0} is infinite then we choose c_n and d_n arbitrarily.

For $n \in \omega$ and $k < h(n)$, define

$$\begin{aligned} b_n^k &= \{c_n^0, \ldots, c_n^k, c_n^{k+2}, \ldots, c_n^{h(n)+1}, d_n^0, \ldots, d_n^k, d_n^{k+2}, \ldots, d_n^{h(n)+1}\} \\ &= (c_n \cup d_n) \setminus \{c_n^{k+1}, d_n^{k+1}\}. \end{aligned}$$

Let $g : \omega \to \omega$ be partial with $|\mathrm{dom}(g)| = \omega$ and $g(n) < h(n)$ for all $n \in \mathrm{dom}(g)$. Define an infinite block sequence $B^g = (b_n^g : n \in \mathrm{dom}(g))$ by $b_n^g = b_n^{g(n)}$ for $n \in \mathrm{dom}(g)$.

For $a, b \in \mathrm{FIN}$, b is an *interval inside* a if $b \subseteq a$ and $a \setminus b \subseteq \min(b) \cup \omega \setminus (\max(b) + 1)$. Let A be an element in $\mathcal{A}$. We say that b_n^k is *compatible with* A if there is $a \in \mathrm{FU}(A)$ such that b_n^k is an interval inside a.

CLAIM 17. *Assume $k \neq k'$ and b_n^k, $b_n^{k'}$ are compatible with A, then $\{c_n^{k+1}\}$, $\{c_n^{k'+1}\}$, $\{d_n^{k+1}\}, \{d_n^{k'+1}\} \in A$.*

Proof. Without loss of generality assume $k < k' < h(n)$. Since b_n^k and $b_n^{k'}$ are compatible with A, there exist $a, a' \in \mathrm{FU}(A)$ such that b_n^k is an interval inside a and $b_n^{k'}$ is an interval inside a'.

Note that c_n^{k+1} and d_n^{k+1} do not belong to b_n^k and thus not to a, but $c_n^{k+1}, d_n^{k+1} \in b_n^{k'}$.

Since $c_n^{k+1} \in b_n^{k'}$ and $b_n^{k'} \subseteq a'$, there is $a_i \in A$ such that $c_n^{k+1} \in a_i$. Clearly a_i is an interval inside a' and $a_i \cap a = \emptyset$. If $|a_i| \geq 2$, then, since a_i and $b_n^{k'}$ both are intervals inside a', we must have either $\{c_n^k, c_n^{k+1}\} \subseteq a_i$ or $\{c_n^{k+1}, c_n^{k+2}\} \subseteq a_i$ or (in case $k' = k+1$) $\{c_n^{k+1}, c_n^{k+3}\} \subseteq a_i$. In either case we obtain a contradiction because $c_n^k, c_n^{k+2}, c_n^{k+3}$ all belong to a and $a_i \cap a = \emptyset$. Hence $|a_i| = 1$, so $\{c_n^{k+1}\} = a_i$ and $\{c_n^{k+1}\} \in A$.

In a similar way we prove that $\{c_n^{k'+1}\}$, $\{d_n^{k+1}\}$ and $\{d_n^{k'+1}\}$ belong to A. $\square$

Claim 18. *For $A \in \mathcal{A}$ there exists $f_A : \omega \to \omega$ such that for almost all $n \in \omega$, if $k \neq f_A(n)$ then b_n^k is not compatible with A.*

Proof. Assume that there are $A \in \mathcal{A}$ and $M \subseteq \omega$ such that $|M| = \omega$ and for all $n \in M$ there are $k'_n \neq k_n$ such that $b_n^{k_n}$ and $b_n^{k'_n}$ are compatible with A. Then $\{c_n^{k_n+1}\}$, $\{c_n^{k'_n+1}\}$, $\{d_n^{k_n+1}\}$ and $\{d_n^{k'_n+1}\}$ are elements of A by the previous claim.

If $A \neq A_0$, then $|E_A \cap E_{A_0}| = \omega$, which is a contradiction. If $A = A_0$, then $E_A \cap \bigcup_{n\in\omega} d_n \neq \emptyset$, which is a contradiction. In the case that there is no A_0 which contains infinitely many singletons, c_n and d_n are arbitrary and A has infinitely many singletons, which is a contradiction.

Hence for all $A \in \mathcal{A}$ and almost all n, there is at most one k such that b_n^k is compatible with A. Thus we define

$$f_A(n) = \begin{cases} k & \text{if } b_n^k \text{ is compatible with } A \\ 0 & \text{otherwise} \end{cases}$$

We now have that there exists an m such that for all $n \geq m$, if $k \neq f_A(n)$, then b_n^k is not compatible with A. $\square$

Consider $\mathcal{F} = \{f_A : A \in \mathcal{A}\}$. Since $|\mathcal{F}| < \mathfrak{b}_h(p \neq^*)$, there is g such that $|\mathrm{dom}(g)| = \omega$, $g(n) < h(n)$ for all $n \in \mathrm{dom}(g)$ and for all $A \in \mathcal{A}$ and almost all $n \in \mathrm{dom}(g)$, $g(n) \neq f_A(n)$. In view of the preceding claim, this means in particular that for all $A \in \mathcal{A}$ and almost all $n \in \mathrm{dom}(g)$, $b_n^{g(n)}$ is not compatible with A.

Claim 19. *For all $A \in \mathcal{A}$, B^g is incompatible with A.*

Proof. Fix $A \in \mathcal{A}$. There is $l \in \omega$ such that for all $n \geq l$ with $n \in \mathrm{dom}(g)$, $b_n^{g(n)}$ is not compatible with A. Assume there is $a \in \mathrm{FU}(B^g) \cap \mathrm{FU}(A)$ with $\max(a) \geq \max(d_l)$. Then $a = \bigcup_{i\in I} b_{n_i}^{g(n_i)}$ for some finite $I \subseteq \mathrm{dom}(g)$ with $n_i \geq l$ for some $i \in I$. Since all $b_{n_i}^{g(n_i)}$ are intervals inside a and thus compatible with A, this is a contradiction. Hence $\mathrm{FU}(B^g) \cap \mathrm{FU}(A)$ is finite, and it follows that B^g is incompatible with A. $\square$

This completes the proof of Theorem 16—and also of Theorem 3. $\square$

Corollary 20. $\mathfrak{a} < \mathfrak{a}_{\mathrm{FIN}}$ *is consistent.*

Proof. In fact, this holds in the random model, that is, the model obtained by adding (at least) ω_2 many random reals to a model of CH with the measure algebra. Then $\mathfrak{a} = \aleph_1$ by [**Bl2**, Section 11.4]. On the other hand, $\mathrm{non}(\mathcal{M}) = \mathfrak{c} = \aleph_2$ [**BJ**, Model 7.6.8], and $\mathfrak{a}_{\mathrm{FIN}} = \mathfrak{c} = \aleph_2$ follows by Theorem 3. □

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A natural way to construct an almost disjoint family from an almost disjoint FIN family is as follows. Let $\mathcal{A}$ be an almost disjoint FIN family. Put $\mathcal{B}_{\mathcal{A}} = \{\mathrm{FU}(A) : A \in \mathcal{A}\}$. Then $\mathcal{B}_{\mathcal{A}}$ is an almost disjoint family of subsets of FIN. However, it is never maximal.

Proposition 21. *Assume* $|\mathcal{A}| \geq 2$. *Then* $\mathcal{B}_{\mathcal{A}}$ *is not maximal.*

Proof. Let $B^0 = (b^0_i : i \in \omega)$ and $B^1 = (b^1_i : i \in \omega)$ in $\mathcal{A}$. Define $C = (c_i : i \in \omega) \subseteq \mathrm{FIN}$ by $c_0 := b^1_0 \cup b^0_{i_0}$ where $b^0_{i_0} > b^1_0$, $c_1 := b^1_0 \cup b^1_{j_1} \cup b^0_{i_1}$ where $b^0_{i_1} > b^1_{j_1} > b^0_{i_0}$, and $c_n := b^1_0 \cup b^1_{j_1} \cup \cdots \cup b^1_{j_n} \cup b^0_{i_n}$ such that $b^0_{i_n} > b^1_{j_n} > b^0_{i_{n-1}}$. It is clear that $C \subseteq \mathrm{FIN}$ and $|C| = \omega$.

Claim 22. $|C \cap \mathrm{FU}(A)| < \omega$ *for all* $A \in \mathcal{A}$.

Proof. Since $b^0_{i_n} \in B^0$ and $c_n \setminus b^0_{i_n} \in \mathrm{FU}(B^1)$, we have $c_n \setminus b^0_{i_n} \notin \mathrm{FU}(B^0)$ and $b^0_{i_n} \notin \mathrm{FU}(B^1)$ for almost all n. Thus $|C \cap \mathrm{FU}(B^0)| < \omega$ and $|C \cap \mathrm{FU}(B^1)| < \omega$ follow.

Let $A \in \mathcal{A}$ be different from B^0 and B^1. There is ℓ such that for all $n \geq \ell$, $c_n \setminus b^0_{i_n} \notin \mathrm{FU}(A)$. If $c_n \in \mathrm{FU}(A)$ for some such n, then there must be $a \in A$ such that $a \cap b^1_{j_n}$ and $a \cap b^0_{i_n}$ are both non-empty. Then, however, we have that for all $m > n$, $a \cap c_m \neq \emptyset$ and $a \setminus c_m \neq \emptyset$. Thus $c_m \notin \mathrm{FU}(A)$. Again, $|C \cap \mathrm{FU}(A)| < \omega$ follows. □

This completes the proof of the proposition. □

4. Proof of Theorem 1.

The framework for the proof of Theorem 1 is similar to the arguments of [**Br3**, Theorem 3.1] and [**Br4**, Theorems 1 and 2], but the combinatorial details are rather different. We therefore do not assume knowledge of either [**Br3**] or [**Br4**] and present the whole argument.

We assume CH and $\diamondsuit_{S^2_1}$. Recall that the latter means that there is a sequence $\{S_\alpha : \mathrm{cf}(\alpha) = \omega_1 \text{ and } \alpha < \omega_2\}$ such that for all $S \subseteq \omega_2$, the set $\{\alpha < \omega_2 : \mathrm{cf}(\alpha) = \omega_1$ and $S \cap \alpha = S_\alpha\}$ is stationary. We perform a finite support iteration $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \omega_2)$ of ccc forcing such that

(A) if $\mathrm{cf}(\alpha) = \omega_1$ then $\dot{\mathbb{Q}}_\alpha$ is Laver forcing $\mathbb{L}_{\dot{\mathcal{U}}_\alpha}$ with a stable ordered-union ultrafilter $\dot{\mathcal{U}}_\alpha$ in $(\mathrm{FIN})^\omega$,

(B) if $cf(\alpha) \neq \omega_1$ then $\dot{\mathbb{Q}}_\alpha$ is Hechler forcing $\dot{\mathbb{D}}$.

(The definition and description of these forcing notions will be given below, after Lemma 26.) The idea is that at limit stages α of cofinality ω_1 we use (A) to kill potential witnesses for $\mathfrak{h}_{\mathrm{FIN}} = \aleph_1$ by building the $\mathcal{U}_\alpha$ accordingly. More explicitly, $\mathcal{U}_\alpha$ will diagonalize (an initial segment of) a witness for $\mathfrak{h}_{\mathrm{FIN}}$ handed down by $\diamondsuit_{S^2_1}$, see below, Lemma 35 and the discussion preceding its proof. Since every witness for $\mathfrak{h}_{\mathrm{FIN}} = \aleph_1$ will have initial segments guessed by the diamond sequence stationarily often on S^2_1, this will complete the argument for $\mathfrak{h}_{\mathrm{FIN}} = \aleph_2$.

The main point of the argument is the proof of $\mathfrak{h}_3 = \aleph_1$. We shall build, along the iteration, families $\{\mathcal{A}_\beta : \beta < \omega_1\}$ such that

(a) all $\mathcal{A}_\beta$ are MAD families of cuboids in ω^3 (that is, all $\mathcal{A}_\beta$ are maximal antichains in $(\mathcal{P}(\omega)/\mathrm{fin})^3$),

(b) $\beta < \beta'$ implies that $\mathcal{A}_{\beta'}$ refines $\mathcal{A}_\beta$ (that is, for all $(A', B', C') \in \mathcal{A}_{\beta'}$ there is $(A, B, C) \in \mathcal{A}_\beta$ with $A' \subseteq^* A$, $B' \subseteq^* B$, and $C' \subseteq^* C$).

Put $\mathcal{A}^{\leq\alpha}_\beta = \mathcal{A}_\beta \cap V_\alpha$. This set will always be a member of V_α. For simplicity let $\mathcal{A}^{\leq 0}_\beta = \emptyset$ for all $\beta < \omega_1$ (though this choice does not really matter). A book-keeping argument will hand us down an ordinal $\alpha = \alpha_{(X,Y,Z),\beta} \in \omega_2$ for $(X, Y, Z) \in ([\omega]^\omega)^3$ and $\beta < \omega_1$ such that $(X, Y, Z) \in V_\alpha$ and the function sending the pair $(X, Y, Z), \beta$ to α is one-to-one and onto ordinals of cofinality less than ω_1. For α with $\mathrm{cf}(\alpha) \neq \omega_1$, let $(X, Y, Z)_\alpha$ and β_α be unique such that $\alpha = \alpha_{(X,Y,Z)_\alpha,\beta_\alpha}$. The idea is that at each successor stage $\alpha + 1$ with $\mathrm{cf}(\alpha) \neq \omega_1$, countably many $\mathcal{A}_\beta$ will get exactly one new element. More explicitly, we stipulate

(c) $\mathcal{A}^{\leq\alpha}_\beta = \bigcup_{\gamma<\alpha} \mathcal{A}^{\leq\gamma}_\beta$ for limit ordinals α, and $\mathcal{A}^{\leq\alpha+1}_\beta = \mathcal{A}^{\leq\alpha}_\beta$ if $\mathrm{cf}(\alpha) = \omega_1$ or $\beta \geq \beta_\alpha$,

(d) if $\beta < \beta_\alpha$ and if $\mathcal{A}^{\leq\alpha}_\beta$ is not predense below $(X, Y, Z)_\alpha$, then there is $(A, B, C) \subseteq^* (X, Y, Z)_\alpha$ belonging to $\mathcal{A}^{\leq\alpha+1}_\beta \setminus \mathcal{A}^{\leq\alpha}_\beta$,

(e) for all α and for $\beta < \beta' < \omega_1$, $\mathcal{A}^{\leq\alpha}_{\beta'}$ refines $\mathcal{A}^{\leq\alpha}_\beta$,

(f) whenever $\beta < \beta_\alpha$ and $(A, B, C) \in \mathcal{A}^{\leq\alpha+1}_\beta \setminus \mathcal{A}^{\leq\alpha}_\beta$, $A = \{a_n : n \in \omega\}$, $B = \{b_n : n \in \omega\}$, and $C = \{c_n : n \in \omega\}$ are their increasing enumerations, then

$$d(d(a_n)) < b_n < d(d(b_n)) < c_n < d(d(c_n)) < a_{n+1}$$

for all n, where $d = d_\alpha$ is the dominating real over V_α added by $\mathbb{Q}_\alpha = \mathbb{D}$.

This last condition should be seen as meaning that A, B, C grow fast with respect to reals in V_α. Let us verify that we can indeed extend the AD families so as to guarantee these conditions.

Observation 23. *If* $\mathrm{cf}(\alpha) < \omega_1$, $(A, B, C) \in \mathcal{A}^{\leq\alpha+1}_\beta \setminus \mathcal{A}^{\leq\alpha}_\beta$ *can be chosen so as to satisfy* (c), (d), (e), *and* (f).

Proof. Let β^0 be minimal such that $\mathcal{A}^{\leq\alpha}_{\beta^0}$ is not predense below $(X, Y, Z)_\alpha$ (if there is such a β^0). Then, by (e) for α, $\mathcal{A}^{\leq\alpha}_\beta$ is not predense below $(X, Y, Z)_\alpha$ for all $\beta \geq \beta^0$. If there is no such β^0 or if $\beta^0 \geq \beta_\alpha$, we will have $\mathcal{A}^{\leq\alpha+1}_\beta = \mathcal{A}^{\leq\alpha}_\beta$ for all β, and there is nothing to prove.

So assume $\beta^0 < \beta_\alpha$. Let $(A^0, B^0, C^0) \subseteq^* (X, Y, Z)_\alpha$ be incompatible with all of $\mathcal{A}_{\beta^0}^{\leq\alpha}$. Since $\mathcal{A}_\beta^{\leq\alpha}$ is predense below (A^0, B^0, C^0) for all $\beta < \beta^0$, we can find a decreasing chain $(A_\beta, B_\beta, C_\beta) \in \mathcal{A}_\beta^{\leq\alpha}$, $\beta < \beta^0$, all compatible with (A^0, B^0, C^0). Let (A^1, B^1, C^1) be a common extension of (A^0, B^0, C^0) and this chain. Finally let (A, B, C) be an extension of (A^1, B^1, C^1) such that, letting $A = \{a_n : n \in \omega\}$, $B = \{b_n : n \in \omega\}$, and $C = \{c_n : n \in \omega\}$ be their respective increasing enumerations,

$$d(d(a_n)) < b_n < d(d(b_n)) < c_n < d(d(c_n)) < a_{n+1}$$

holds for all n. Adding (A, B, C) to $\mathcal{A}_\beta^{\leq\alpha+1}$ for $\beta^0 \leq \beta < \beta_\alpha$, we see that (d) and (e) are satisfied. $\square$

Corollary 24. *The* $\mathcal{A}_\beta = \bigcup_{\alpha<\omega_2} \mathcal{A}_\beta^{\leq\alpha}$ *satisfy* (*a*) *and* (*b*).

Proof. (b) follows from (e), and (a) follows from (d) together with the bookkeeping. $\square$

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To argue that $\{\mathcal{A}_\beta : \beta < \omega_1\}$ witnesses $\mathfrak{h}_3 = \aleph_1$, we need to show that for all $\alpha < \omega_2$,

$$\forall (X, Y, Z) \in ([\omega]^\omega)^3 \cap V_\alpha \; \exists \beta < \omega_1 \; \forall (A, B, C) \in \mathcal{A}_\beta (X \not\subseteq^* A \text{ or } Y \not\subseteq^* B \text{ or } Z \not\subseteq^* C)$$

By (f), it suffices to show by induction on $\alpha < \omega_2$ that

$$(*_\alpha) \begin{cases} \forall (X, Y, Z) \in ([\omega]^\omega)^3 \cap V_\alpha \; \exists \beta < \omega_1 \; \forall (A, B, C) \in \mathcal{A}_\beta^{\leq\alpha} \\ (X \not\subseteq^* A \text{ or } Y \not\subseteq^* B \text{ or } Z \not\subseteq^* C) \end{cases}$$

For technical reasons we shall need a slightly stronger property:

$$(\star_\alpha) \begin{cases} \forall k \; \forall X^{j,i} = \{x_n^{j,i} : n \in \omega\} \in [\omega]^\omega \cap V_\alpha \; (j < 3 \text{ and } i < k) \\ \text{listed in increasing enumeration } \exists \beta < \omega_1 \; \forall (A^0, A^1, A^2) \in \mathcal{A}_\beta^{\leq\alpha} \\ \exists \text{ countable } D \subseteq \omega \; \exists j < 3 \; \forall i < k \; \forall n \in D \; (x_n^{j,i} \notin A^j) \end{cases}$$

Note that $(*_\alpha)$ is $(\star_\alpha)$ for the case $k = 1$. In the proof of Main Lemma 33, we will need a reformulation of $(\star_\alpha)$ which we now explain. Say that $X = \{x_s : s \in \text{FIN}\} \in [\omega]^\omega$ is listed in *canonical enumeration* if

- the listing is of one of the four canonical types (in Taylor's Theorem) min, max, min-max, or one-to-one,
- if the listing is of min-type, then $\min(s) < \min(s')$ implies $x_s < x_{s'}$,
- if the listing is of one of the three other types, then $\max(s) < \max(s')$ implies $x_s < x_{s'}$.

Clearly, these clauses completely determine the listing for the min- and max-types, but not for the other two types. Now consider the following property:

$$(\dagger_\alpha)\begin{cases}\forall k\ \forall X^{j,i}=\{x^{j,i}_s : s\in \mathrm{FIN}\}\in[\omega]^\omega\cap V_\alpha\ (j<3 \text{ and } i<k)\\ \text{listed in canonical enumeration } \exists\beta<\omega_1\ \forall(A^0,A^1,A^2)\in\mathcal{A}^{\leq\alpha}_\beta\\ \exists \text{ countable } D\subseteq\omega\ \exists j<3\ \forall i<k\ \forall s\subseteq D\ (x^{j,i}_s\notin A^j)\end{cases}$$

Again $(\dagger_\alpha)$ is obviously stronger than $(\star_\alpha)$. However, the two properties are in fact equivalent.

Observation 25. *$(\star_\alpha)$ implies $(\dagger_\alpha)$.*

Proof. Assume k and $X^{j,i}=\{x^{j,i}_s : s\in \mathrm{FIN}\}$, $j<3$ and $i<k$, are given as required. Given j,i and $s\in\mathrm{FIN}$, form new sets $\bar X^{j,i,s}=\{x^{j,i,s}_n : n>\max(s)\}$ as follows:

- if $X^{j,i}$ is of min-type, then $x^{j,i,s}_n=x^{j,i}_{\{n\}}$,
- if $X^{j,i}$ is not of min-type, then $x^{j,i,s}_n=x^{j,i}_{s\cup\{n\}}$.

Note that, by our definition of canonical enumeration, these are indeed increasing enumerations. By $(\star_\alpha)$, for each n_0, there is a β_{n_0} satisfying the conclusion for the family of sets $\bar X^{j,i,s}$ where $s\subseteq n_0$ and $i<k$. Let β be the supremum of the β_{n_0}, $n_0\in\omega$. We claim that β witnesses $(\dagger_\alpha)$. Fix $(A^0,A^1,A^2)\in\mathcal{A}^{\leq\alpha}_\beta$. It clearly suffices to verify the following:

$$(*)\quad \forall n_0\ \exists n\geq n_0\ \exists j\ \forall s\subseteq n_0\ \forall i<k\ (x^{j,i,s}_n\notin A^j)$$

For indeed, $(*)$ easily implies that for some j there is an infinite $D\subseteq\omega$ such that for all $n\in D$, $s\subseteq D$ with $\max(s)<n$, and $i<k$, $x^{j,i,s}_n$ does not belong to A^j. This clearly implies that for $s\subseteq D$ and $i<k$, $x^{j,i}_s$ does not belong to A^j.

To show $(*)$, fix n_0. By $(\star_\alpha)$ there are infinite $D\subseteq\omega$ and $j<3$ such that for all $i<k$, $s\subseteq n_0$, and $n\in D$, $x^{j,i,s}_n$ does not belong to A^j. □

Preservation of $(\star_\alpha)$ is the main issue of the proof of $\mathfrak{h}_3=\aleph_1$. We start with preservation in limit steps.

Lemma 26. *$(\star_\alpha)$ is preserved in limit steps.*

Proof. Clearly it suffices to consider limit steps of cofinality ω. Assume $\mathrm{cf}(\alpha)=\omega$, α is the increasing union of the α_m, and $(\star_\gamma)$ holds for all $\gamma<\alpha$. We need to show that $(\star_\alpha)$ holds.

Fix k, and let $\dot X^{j,i}=\{\dot x^{j,i}_n : n\in\omega\}$ be $\mathbb{P}_\alpha$-names for subsets of ω, $j<3$ and $i<k$. Fix m, and let G_m be $\mathbb{P}_{\alpha_m}$-generic. In $V_{\alpha_m}=V[G_m]$ let $\{p^\ell_m : \ell\in\omega\}$ be a decreasing sequence of conditions in the remainder forcing $\mathbb{P}_{[\alpha_m,\alpha)}$ such that p^ℓ_m decides the values of $\dot x^{j,i}_n$ for $n\leq\ell$ and all j and i. Say $p^\ell_m\Vdash \dot x^{j,i}_n=x^{j,i}_{m,n}$. By $(\star_{\alpha_m})$ we find β_m such that for all $(A^0,A^1,A^2)\in\mathcal{A}^{\leq\alpha_m}_{\beta_m}$, there are countable $D_m\subseteq\omega$ and $j<3$ with $x^{j,i}_{m,n}\notin A^j$ for all $i<k$ and all $n\in D_m$.

Now step back into the ground model V. By ccc-ness there is a $\beta<\omega_1$ such that the trivial condition forces that all β_m are smaller than β. We show that this β witnesses $(\star_\alpha)$. To this end fix $(A^0,A^1,A^2)\in\mathcal{A}^{\leq\alpha}_\beta$ in the generic extension V_α. By construction there is some m_0 such that (A^0,A^1,A^2) belongs to $\mathcal{A}^{\leq\alpha_{m_0}}_\beta$. That is, $(\dot A_0,\dot A_1,\dot A_2)$ is a

$\mathbb{P}_{\alpha_{m_0}}$-name. Next fix $p \in \mathbb{P}_\alpha$ and $m_1 \in \omega$. By increasing m_1, if necessary, we may assume that $m_1 \geq m_0$ and $p \in \mathbb{P}_{\alpha_{m_1}}$. It suffices to show that there are $q \leq p$, $n \geq m_1$, and $j < 3$ such that

$$(**) \quad q \Vdash \dot{x}_n^{j,i} \notin \dot{A}^j \text{ for all } i < k.$$

Then a genericity argument will yield $(\star_\alpha)$ in the extension.

To see $(**)$ step into $V_{\alpha_{m_1}}$ such that $p \in G_{m_1}$. Let $n \in D_{m_1}$ with $n \geq m_1$ and $j < 3$ such that $x_{m_1,n}^{j,i} \notin A^j$ for all $i < k$. Thus $p_{m_1}^n$ forces $\dot{x}_n^{j,i} \notin A^j$ for all $i < k$. In G_{m_1} there is $\bar{p} \leq p$ deciding n, j and $p_{m_1}^n$. Then q given by $q{\restriction}\alpha_{m_1} = \bar{p}$ and $q{\restriction}[\alpha_{m_1}, \alpha) = p_{m_1}^n$ will force $\dot{x}_n^{j,i} \notin \dot{A}^j$ for all $i < k$, as required. □

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We next come to preservation of $(\star_\alpha)$ under Hechler forcing $\mathbb{D}$. We start with a discussion of rank arguments for Laver-style forcing notions. Such rank arguments for forcing go back to [**BD**] and have been used often since.

Let $\mathcal{F}$ be a filter on a countable set Ω extending the filter of cofinite sets. *Laver forcing* $\mathbb{L}_\mathcal{F}$ with the filter $\mathcal{F}$ consists of all trees $T \subseteq \Omega^{<\omega}$ such that for any node $\sigma \in T$ beyond the stem, the set $\{a \in \Omega : \sigma\hat{\ }a \in T\}$ belongs to $\mathcal{F}$. $\mathbb{L}_\mathcal{F}$ is ordered by inclusion. It is easy to see that $\mathbb{L}_\mathcal{F}$ is a σ-centered forcing notion which adds a dominating real $\ell_\mathcal{F} := \bigcup\{\text{stem}(T) : T \in G\} = \bigcap\{[T] : T \in G\} \in \Omega^\omega$, where G is the generic filter. Furthermore, the range of $\ell_\mathcal{F}$ is a pseudointersection of the filter $\mathcal{F}$, i.e., $\text{ran}(\ell_\mathcal{F}) \subseteq^* F$ for all $F \in \mathcal{F}$.

Let ψ be a sentence of the $\mathbb{L}_\mathcal{F}$-forcing language. Say that $\sigma \in \Omega^{<\omega}$ *forces* ψ if there is $T \in \mathbb{L}_\mathcal{F}$ with stem σ such that $T \Vdash \psi$. Say that σ *favors* ψ if σ does not force $\neg\psi$. Clearly, any σ can force at most one of ψ and $\neg\psi$ and favors at least one them. Alternatively, "σ favors ψ" can be defined using a rank function. Since we shall use rank arguments later on in various places, it is instructive to explain this rank here. Define by recursion on α:

$$\begin{aligned} &\text{rk}_\psi(\sigma) = 0 \Longleftrightarrow \sigma \text{ forces } \psi \\ \alpha > 0: \quad &\text{rk}_\psi(\sigma) = \alpha \Longleftrightarrow \neg(\text{rk}_\psi(\sigma) < \alpha) \wedge \{a \in \Omega :\ \text{rk}_\psi(\sigma\hat{\ }a) < \alpha\} \in \mathcal{F}^+ \end{aligned}$$

where $\mathcal{F}^+$ denotes the collection of *$\mathcal{F}$-positive sets*, that is, $\mathcal{F}^+ = \{A \subseteq \Omega :\ \forall B \in \mathcal{F}\ (A \cap B \neq \emptyset)\}$. Say $\text{rk}_\psi(\sigma) = \infty$ if the rank is undefined. Clearly, either $\text{rk}_\psi(\sigma) < \omega_1$ or $\text{rk}_\psi(\sigma) = \infty$. Furthermore, an easy argument left to the reader shows that $\text{rk}_\psi(\sigma) < \omega_1$ iff σ favors ψ.

Now assume $k \in \omega$ and $\mathbb{L}_\mathcal{F}$-names $\dot{X}^{j,i} = \{\dot{x}_n^{j,i} : n \in \omega\}$, $j < 3$ and $i < k$, as in $(\star_\alpha)$ are given. By thinning out these sets, we may assume, without loss of generality, that there is a total order R on $3 \times k$ such that

$$(***) \quad (j,i)R(j',i') \Longleftrightarrow \forall n\ \Vdash \dot{x}_n^{j,i} \leq \dot{x}_n^{j',i'}$$

We can also assume that the function $n \mapsto \dot{x}_n^{j,i}$ dominates the Laver generic real for all j and i.

Fix $n \in \omega$. Let ψ be a sentence of the $\mathbb{L}_{\mathcal{F}}$-forcing language. For any pair j, i define the *rank* $\rho^{\psi}_{j,i,n}$ *of the name* $\dot{x}^{j,i}_n$ *relative to* ψ by recursion on α as follows.

$$\begin{aligned} &\rho^{\psi}_{j,i,n}(\sigma) = 0 \Longleftrightarrow \text{ for some } \ell \in \omega, \sigma \text{ favors the statement "}\psi \wedge \dot{x}^{j,i}_n = \ell\text{"} \\ \alpha > 0 : \quad &\rho^{\psi}_{j,i,n}(\sigma) = \alpha \Longleftrightarrow \neg(\rho^{\psi}_{j,i,n}(\sigma) < \alpha) \wedge \{a \in \Omega : \rho^{\psi}_{j,i,n}(\sigma\hat{\ }a) < \alpha\} \in \mathcal{F}^+ \end{aligned}$$

Note that $\rho^{\psi}_{j,i,n}(\sigma) > 0$ means that for all $\ell \in \omega$, σ forces either $\neg\psi$ or $\dot{x}^{j,i}_n \neq \ell$. In particular, $\rho^{\psi}_{j,i,n}(\sigma) > 0$ does not imply that σ forces $\neg\psi$. Notice that the definition for $\alpha > 0$ is exactly the same as for the general rank above. If ψ is trivial (i.e., any tautology), we omit it.

Claim 27. *Assume σ favors ψ. Then $\rho^{\psi}_{j,i,n}(\sigma) < \omega_1$.*

Proof. Assume that $\rho^{\psi}_{j,i,n}(\sigma) = \infty$. Then there is $T \in \mathbb{L}_{\mathcal{F}}$ with stem σ such that for all $\tau \in T$ extending the stem, $\rho^{\psi}_{j,i,n}(\tau) = \infty$. On the other hand, since σ favors ψ, there is $T' \leq T$ forcing ψ. Now let $T'' \leq T'$ be such that T'' decides the value of $\dot{x}^{j,i}_n$, and let τ be the stem of T''. By construction of T, τ has rank ∞, while by definition of the rank, τ has rank 0, a contradiction. $\square$

Claim 28. *If $(j', i')R(j, i)$ then $\rho^{\psi}_{j',i',n}(\sigma) \leq \rho^{\psi}_{j,i,n}(\sigma)$.*

Proof. By induction on rank $\rho^{\psi}_{j,i,n}(\sigma)$. If this rank is 0, there is ℓ such that σ favors $\psi \wedge \dot{x}^{j,i}_n = \ell$. We claim that for some $\ell' \leq \ell$, σ favors $\psi \wedge \dot{x}^{j',i'}_n = \ell'$, thus showing $\rho^{\psi}_{j',i',n}(\sigma) = 0$. For if this were not the case, for every $\ell' \leq \ell$, σ forces $\neg\psi \vee \dot{x}^{j',i'}_n \neq \ell'$. Hence there is a condition T with stem σ forcing this. Since σ favors $\psi \wedge \dot{x}^{j,i}_n = \ell$, there is $T' \leq T$ forcing this. Thus T' must force $\dot{x}^{j',i'}_n > \ell$. This contradicts $(***)$.

The case rank > 0 is straightforward and left to the reader. $\square$

Let $\Omega = \omega$ and let $\mathcal{F}$ be the filter of cofinite sets. Define *Hechler forcing* $\mathbb{D}$ to be the forcing $\mathbb{L}_{\mathcal{F}}$. Notice that this is *not* the standard definition of Hechler forcing. Our reason for using $\mathbb{L}_{\mathcal{F}}$ is that this simplifies rank arguments. It is known that $\mathbb{D} = \mathbb{L}_{\mathcal{F}}$ completely embeds into standard Hechler forcing, and vice-versa – and, therefore, they have the same effect in iterated forcing constructions – but they are not forcing equivalent [**Pa**].

Main Lemma 29. *Hechler forcing preserves $(\star_\alpha)$. More explicitly, if* $\mathrm{cf}(\alpha) \neq \omega_1$ *and $(\star_\alpha)$ holds in V_α, then $(\star_{\alpha+1})$ holds in $V_{\alpha+1}$.*

Proof. Assume $k \in \omega$ and $\mathbb{D}$-names $\dot{X}^{j,i} = \{\dot{x}^{j,i}_n : n \in \omega\}$, $j < 3$ and $i < k$, are given as in the discussion in the paragraphs preceding the main lemma.

Fix $n \in \omega$. Let I be a (possibly empty) R-initial segment of $3 \times k$. Let (j_0, i_0) be the R-minimal element not belonging to I. Let J be a non-empty R-interval with $\min J = (j_0, i_0)$. Finally, let $L = (\ell^{j,i} : (j, i) \in I)$ be a sequence of natural numbers and let ψ be the statement "$\dot{x}^{j,i}_n = \ell^{j,i}$ for all $(j, i) \in I$". Let $\sigma \in \omega^{<\omega}$. Say that σ is *n-good for the triple* (I, J, L) if

- σ favors ψ,

- $\rho^{\psi}_{j,i,n}(\sigma) = 1$ for all $(j,i) \in J$,
- $\rho^{\psi}_{j,i,n}(\sigma) > 1$ for all (j,i) R-larger than $\max J$.

Using the previous claim, for such n-good σ, we can find an infinite set $V^{\sigma,n} = \{v^{\sigma,n}_m : m \in \omega\} \subseteq \omega$ and numbers $\ell^{j,i,\sigma,n}_m$, $(j,i) \in J$ and $m \in \omega$, such that for all m,

- $\rho^{\psi}_{j,i,n}(\sigma\hat{\ }v^{\sigma,n}_m) = 0$ and, in fact, $\sigma\hat{\ }v^{\sigma,n}_m$ favors "ψ and $\dot{x}^{j,i}_n = \ell^{j,i,\sigma,n}_m$ for $(j,i) \in J$",
- $\rho^{\psi}_{j,i,n}(\sigma\hat{\ }v^{\sigma,n}_m) \geq 1$ for all (j,i) R-larger than $\max J$.

Note that the function $m \mapsto \ell^{j,i,\sigma,n}_m$ must be finite-to-one for all $(j,i) \in J$. For otherwise we would have $\rho^{\psi}_{j,i,n}(\sigma) = 0$, a contradiction. Therefore we may assume that all functions $m \mapsto \ell^{j,i,\sigma,n}_m$ are in fact one-to-one and that the finitely many sets $L^{j,i,\sigma,n} = \{\ell^{j,i,\sigma,n}_m : m \in \omega\}$, $(j,i) \in J$, are listed in increasing enumeration. For $(j,i) \notin J$, let $L^{j,i,\sigma,n} = \{\ell^{j,i,\sigma,n}_m : m \in \omega\}$ be arbitrary sets listed in increasing enumeration.

Unfix n. For each $n \in \omega$, each finite set $\Sigma \subseteq \omega^{<\omega}$ such that all $\sigma \in \Sigma$ are n-good for some triple (I, J, L), apply $(\star_\alpha)$ to the family $\{L^{j,i,\sigma,n} : j < 3, i < k \text{ and } \sigma \in \Sigma\}$ to obtain a $\beta_{n,\Sigma}$ depending on n and Σ. Let β be the supremum of the countably many $\beta_{n,\Sigma}$ so obtained. We may assume $\beta \geq \beta_\alpha$.

Now let $T \in \mathbb{D}$ be arbitrary. Fix $n_0 \in \omega$. Also fix $(A^0, A^1, A^2) \in \mathcal{A}^{\leq\alpha}_{\beta} = \mathcal{A}^{\leq\alpha+1}_{\beta}$. To complete the proof of the main lemma, it clearly suffices to show the following:

CLAIM 30. *For some $n \geq n_0$, some $T' \leq T$, and some $j < 3$, T' forces $\dot{x}^{j,i}_n \notin A^j$ for all $i < k$.*

For then an easy genericity argument yields the conclusion of $(\star_{\alpha+1})$.

PROOF. Let $n \geq n_0$ be such that $\rho_{j,i,n}(\sigma) \geq 1$ for any $(j,i) \in 3 \times k$ where σ is the stem of T. Such an n must exist because the functions $n \mapsto \dot{x}^{j,i}_n$ dominate the generic real. By extending σ, if necessary, we may assume, without loss of generality, that $\rho_{j,i,n}(\sigma) = 1$ for the R-minimal (j,i).

Since for each finite Σ consisting of n-good τ we can find j such that the conclusion of $(\star_\alpha)$ holds for the $L^{j,i,\tau,n}$ ($i < k$ and $\tau \in \Sigma$) and for (A^0, A^1, A^2), the directedness of the Σ yields that there is a single j_0 that works for all Σ. Fix such j_0.

Recursively build $\sigma = \sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_q$ all in T, $I_0, I_1, \ldots, I_{q-1}$, and $\ell^{j,i}$, $j < 3$ and $i < k$, such that

- the I_r form an interval partition of the total order $(3 \times k, R)$,
- $\ell^{j_0,i} \notin A^{j_0}$,
- σ_r is n-good for $(I_{<r}, I_r, L_r)$ where $I_{<r} := \bigcup_{t<r} I_t$ and $L_r := \{\ell^{j,i} : (j,i) \in I_{<r}\}$; in particular, σ_r favors $\psi_{<r}$ and $\rho^{\psi_{<r}}_{j,i,n}(\sigma_r) = 1$ for all $(j,i) \in I_r$ where $\psi_{<r}$ is "$\dot{x}^{j,i}_n = \ell^{j,i}$ for all $(j,i) \in I_{<r}$".

To carry out the recursive construction, assume σ_r (as well as all items with lower index) has been obtained for some $r \geq 0$ such that $\rho^{\psi_{<r}}_{j,i,n}(\sigma_r) = 1$ for the R-minimal $(j,i) \notin I_{<r}$ (in particular, $I_{<r}$ is a strict R-initial segment of $3 \times k$). By Claim 28, the (j,i) such that

$\rho^{\psi_{<r}}_{j,i,n}(\sigma_r) = 1$ form a (non-trivial) R-initial segment I_r of $(3 \times k) \setminus I_{<r}$, and $\rho^{\psi_{<r}}_{j,i,n}(\sigma_r) > 1$ for all (j,i) beyond this initial segment. By assumption $(\star_\alpha)$ and choice of j_0, we find $m \in \omega$ such that $\sigma_r \hat{}\, v^{\sigma_r,n}_m \in T$ and $\ell^{j_0,i,\sigma_r,n}_m \notin A^{j_0}$ for all i. Put $\ell^{j,i} = \ell^{j,i,\sigma_r,n}_m$ for $(j,i) \in I_r$. Since $\rho^{\psi_{<r}}_{j,i,n}(\sigma_r \hat{}\, v^{\sigma_r,n}_m) = 0$ for all $(j,i) \in I_r$, $\sigma_r \hat{}\, v^{\sigma_r,n}_m$ in fact favors $\psi_{\le r}$. We also know that $\rho^{\psi_{<r}}_{j,i,n}(\sigma_r \hat{}\, v^{\sigma_r,n}_m) \ge 1$ for all $(j,i) \notin I_{\le r}$. A fortiori, $\rho^{\psi_{\le r}}_{j,i,n}(\sigma_r \hat{}\, v^{\sigma_r,n}_m) \ge 1$ for all $(j,i) \notin I_{\le r}$. In case $I_{\le r} \subset 3 \times k$, we can extend $\sigma_r \hat{}\, v^{\sigma_r,n}_m$ to $\sigma_{r+1} \in T$ such that $\rho^{\psi_{\le r}}_{j,i,n}(\sigma_{r+1}) = 1$ for the R-minimal $(j,i) \notin I_{\le r}$. In case $I_{\le r} = 3 \times k$, let $q = r+1$ and $\sigma_q = \sigma_r \hat{}\, v^{\sigma_r,n}_m$.

Now, σ_q favors "$\dot{x}^{j,i}_n = \ell^{j,i}$ for all $(j,i) \in 3 \times k$" and $\ell^{j_0,i} \notin A^{j_0}$ for all $i < k$. Hence we can find $T' \le T$ with stem extending σ_q such that T' forces that $\dot{x}^{j_0,i}_n \notin A^{j_0}$ for all $i < k$, as required. □

This completes the proof of the main lemma. □

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To prove the preservation of $(\star_\alpha)$ under forcing of the type $\mathbb{L}_{\mathcal{U}}$ is harder, and, in fact, we need a property stronger than $(\star_\alpha)$ to be able to build a stable ordered-union ultrafilter $\mathcal{U}$ preserving $(\star_\alpha)$:

$$(\clubsuit_\alpha)\begin{cases} \forall k\ \forall D \in (\mathrm{FIN})^\omega\ \forall \varphi^{j,i} : \mathrm{FU}(D) \to \omega\ (j < 3 \text{ and } i < k) \text{ in } V_\alpha \\ \text{of one of the four types min, max, min-max, or one-to-one} \\ \exists E \sqsubseteq D\ \exists \beta < \omega_1\ \forall (A^0, A^1, A^2) \in \mathcal{A}^{\le\alpha}_\beta\ \forall E' \sqsubseteq E\ \exists E'' \sqsubseteq E' \\ \exists j < 3\ \forall i < k\ (\mathrm{FU}(E'') \cap (\varphi^{j,i})^{-1}(A^j) = \emptyset) \end{cases}$$

For technical reasons (see below, Lemma 35 and Main Lemma 36) we stated this principle for arbitrary k, but it turns out that this is equivalent to the simpler case where $k = 1$. We shall use this observation below in the proof of Main Lemma 33.

OBSERVATION 31. *$(\clubsuit_\alpha)$ is the same if restricted to $k = 1$.*

PROOF. Assume $(\clubsuit_\alpha)$ for 1 and let k be arbitrary. Let D and $\varphi^{j,i}$ be given. Let $\{f_\ell : \ell \in k^3\}$ list all functions from 3 to k. Recursively construct sets E_ℓ and ordinals β_ℓ such that

- $E_0 \sqsubseteq D$, $E_{\ell+1} \sqsubseteq E_\ell$, and $\beta_{\ell+1} \ge \beta_\ell$,
- E_ℓ and β_ℓ witness $(\clubsuit_\alpha)$ for the triple of functions $\varphi^{j,f_\ell(j)}$, $j < 3$.

Let $\beta = \beta_{k^3-1}$ and $E = E_{k^3-1}$. We shall see that β and E witness $(\clubsuit_\alpha)$ for the given $\varphi^{j,i}$. To this end, let $(A^0, A^1, A^2) \in \mathcal{A}^{\le\alpha}_\beta$ and $E' \sqsubseteq E$. By repeatedly applying Hindman's Theorem we find $E'' \sqsubseteq E'$ such that

- either $\exists j < 3\ \forall i < k\ (\{\varphi^{j,i}(s) : s \in \mathrm{FU}(E'')\} \cap A^j = \emptyset)$,
- or $\forall j < 3\ \exists i = f(j)\ (\{\varphi^{j,i}(s) : s \in \mathrm{FU}(E'')\} \subseteq A^j)$.

Letting ℓ be such that $f = f_\ell$, we see that the second alternative contradicts step ℓ of the construction. Hence the first alternative must hold and we are done. □

Also, $(\clubsuit_\alpha)$ is clearly stronger than $(\star_\alpha)$. (This is not really needed in the proof.)

OBSERVATION 32. *$(\clubsuit_\alpha)$ implies $(\star_\alpha)$.*

PROOF. Let $X^{j,i} = \{x^{j,i}_n : n \in \omega\}$ be given and let $\varphi^{j,i} : \mathrm{FIN} \to X^{j,i}$ be canonical bijections. Let β and E be as given by $(\clubsuit_\alpha)$. Next let $(A^0, A^1, A^2) \in \mathcal{A}^{\leq\alpha}_\beta$ and fix $E'' \sqsubseteq E$ and $j < 3$ such that $\{\varphi^{j,i}(s) : s \in \mathrm{FU}(E'')\} \cap A^j = \emptyset$ for all $i < k$. Then clearly $D = \{n \in \omega : x^{j,i}_n = \varphi^{j,i}(s)$ for some $s \in \mathrm{FU}(E'')\}$ is as required (note that the definition of D is independent of the choice of j, i). $\square$

On the other hand, in limit stages of cofinality ω_1, not only $(\star_\alpha)$ but even the stronger $(\clubsuit_\alpha)$ will hold – and this is the heart of the proof.

MAIN LEMMA 33. *Assume $\mathrm{cf}(\alpha) = \omega_1$ and $(\star_\gamma)$ holds for all $\gamma < \alpha$. Then $(\clubsuit_\alpha)$ holds.*

PROOF. By Observation 31, it suffices to consider the case $k = 1$. Let $D \in (\mathrm{FIN})^\omega$ and $\varphi^j : \mathrm{FU}(D) \to \omega$ in V_α be given as required. Since $\mathrm{cf}(\alpha) = \omega_1$, there is $\gamma < \alpha$ such that D and the φ^j belong to V_γ.

First step. We show that there are $\beta < \omega_1$ and $E \sqsubseteq D$ in $V_{\gamma+1}$ such that the conclusion of $(\clubsuit_\alpha)$ holds for all $(A^0, A^1, A^2) \in \mathcal{A}^{\leq\gamma+1}_\beta$.

For the moment work in the model V_γ. Say that a set $F \subseteq \mathrm{FIN}$ is *non-trivial* if there is a non-zero condition in the forcing $(\mathrm{FIN})^\omega$ below it, that is, if there is $E \in (\mathrm{FIN})^\omega$ such that $\mathrm{FU}(E) \subseteq F$. Otherwise F is trivial. For $A^j \subseteq \omega$ ($j < 3$), consider $F_{\bar{A}} = \mathrm{FU}(D) \cap (\varphi^0)^{-1}(A^0) \cap (\varphi^1)^{-1}(A^1) \cap (\varphi^2)^{-1}(A^2)$. For $\beta < \omega_1$, let $\mathcal{E}_\beta = \{F_{\bar{A}} : \bar{A} = (A^0, A^1, A^2) \in \mathcal{A}^{\leq\gamma}_\beta$ and $F_{\bar{A}}$ is non-trivial$\}$. Notice that if $\bar{A}$ and $\bar{B}$ are two distinct elements of $\mathcal{A}^{\leq\gamma}_\beta$, then $F_{\bar{A}} \cap F_{\bar{B}}$ is necessarily trivial. Thus the $\mathcal{E}_\beta$ are antichains.

Suppose first that there is some β such that $\mathcal{E}_\beta$ is not a maximal antichain below $\mathrm{FU}(D)$, that is, there is $E \sqsubseteq D$ such that for all $\bar{A} \in \mathcal{A}^{\leq\gamma}_\beta$, $\mathrm{FU}(E) \cap F_{\bar{A}}$ is trivial. Without loss of generality assume $\beta \geq \beta_\gamma$. Then $E \in V_\gamma$ is as required. (For suppose that $\bar{A} \in \mathcal{A}^{\leq\gamma+1}_\beta = \mathcal{A}^{\leq\gamma}_\beta$ and $E' \sqsubseteq E$. Applying Hindman's Theorem three times we find $E'' \sqsubseteq E'$ such that for all $j < 3$ either $\mathrm{FU}(E'') \subseteq (\varphi^j)^{-1}(A^j)$ or $\mathrm{FU}(E'') \cap (\varphi^j)^{-1}(A^j) = \emptyset$. By triviality, the first alternative cannot always hold, so there must be j with $\mathrm{FU}(E'') \cap (\varphi^j)^{-1}(A^j) = \emptyset$.)

Hence we may assume that all $\mathcal{E}_\beta$ are maximal antichains below $\mathrm{FU}(D)$. The following argument is reminiscent of the proof of the base-tree lemma [**BPS**] (see also [**Bl2**, Theorem 6.20] or [**BDH**]). We build an increasing sequence β_n of ordinals, a binary tree $\bar{A}_\sigma = (A^0_\sigma, A^1_\sigma, A^2_\sigma)$ of conditions in $([\omega]^\omega)^3$, and a binary tree of $E_\sigma \sqsubseteq D$, $\sigma \in 2^{<\omega}$, such that

- $\bar{A}_\sigma \in \mathcal{A}^{\leq\gamma}_{\beta_n}$ where $n = |\sigma|$,
- $\bar{A}_{\sigma\hat{}0}$ and $\bar{A}_{\sigma\hat{}0}$ are incompatible extensions of $\bar{A}_\sigma$,
- $E_{\sigma\hat{}0}$ and $E_{\sigma\hat{}1}$ are incompatible extensions of E_σ,
- $\mathrm{FU}(E_\sigma) \subseteq F_{\bar{A}_\sigma}$.

To see that this can be done, first choose β_0 and $\bar{A}_\emptyset$ arbitrary with $F_{\bar{A}_\emptyset} \in \mathcal{E}_{\beta_0}$. Then find $E_\emptyset \sqsubseteq D$ with $\mathrm{FU}(E_\emptyset) \subseteq F_{\bar{A}_\emptyset}$. Assume β_n and $\bar{A}_\sigma$, E_σ for $\sigma \in 2^n$ have been constructed. Fix $\sigma \in 2^n$. Let $x^j_s = \varphi^j(s)$ for $s \in \mathrm{FU}(E_\sigma)$. Applying $(\dagger_\gamma)$ to the $X^j = \{x^j_s : s \in \mathrm{FU}(E_\sigma)\}$ we obtain $\beta_\sigma > \beta_n$ such that the conclusion of $(\dagger_\gamma)$ holds. Unfixing σ, let β_{n+1} be the maximum of the β_σ. Applying $(\dagger_\gamma)$ together with the maximality of the antichain $\mathcal{E}_{\beta_{n+1}}$, we see that there must be two distinct $\bar{A}_{\sigma\frown 0}$ and $\bar{A}_{\sigma\frown 1} \in \mathcal{A}^{\leq\gamma}_{\beta_{n+1}}$ such that $\mathrm{FU}(E_\sigma) \cap F_{\bar{A}_{\sigma\frown 0}}$ and $\mathrm{FU}(E_\sigma) \cap F_{\bar{A}_{\sigma\frown 1}}$ are both non-trivial. Hence we can find $E_{\sigma\frown i}$ with $\mathrm{FU}(E_{\sigma\frown i}) \subseteq \mathrm{FU}(E_\sigma) \cap F_{\bar{A}_{\sigma\frown i}}$ as required.

Let β be the supremum of the β_n. For $f \in 2^\omega$, choose $E_f \sqsubseteq^* E_{f\restriction n}$ for all n. Again by maximality, there must be $\bar{A}_f \in \mathcal{A}^{\leq\gamma}_\beta$ such that $\mathrm{FU}(E_f) \cap F_{\bar{A}_f}$ is non-trivial. Note that for such $\bar{A}_f$ we necessarily must have $A^j_f \subseteq^* A^j_{f\restriction n}$ for all n. (We know that $\bar{A}_f$ refines a member $\bar{A}_n$ of $\mathcal{A}^{\leq\gamma}_{\beta_n}$. If $\bar{A}_n$ were distinct from $\bar{A}_{f\restriction n}$, then $\mathrm{FU}(E_{f\restriction n}) \cap F_{\bar{A}_n}$, and thus $\mathrm{FU}(E_f) \cap F_{\bar{A}_f}$, would be trivial, a contradiction.)

Now step into $V_{\gamma+1}$. Without loss of generality assume $\beta \geq \beta_\gamma$. So $\mathcal{A}^{\leq\gamma+1}_\beta = \mathcal{A}^{\leq\gamma}_\beta$. Let $f \in 2^\omega \cap (V_{\gamma+1} \setminus V_\gamma)$ (note that each stage of the forcing adds a new real, so clearly there is such an f). Consider E_f. Suppose that $\mathrm{FU}(E_f) \cap F_{\bar{A}_f}$ is non-trivial for some $\bar{A}_f \in \mathcal{A}^{\leq\gamma+1}_\beta$. By the discussion in the preceding paragraph, we see that $A^j_f \subseteq^* A^j_{f\restriction n}$ for all n. This means, however, that from $\bar{A}_f$ we can reconstruct the function f. This contradicts the fact that $\bar{A}_f \in V_\gamma$ and $f \notin V_\gamma$. Thus $\mathrm{FU}(E_f) \cap F_{\bar{A}}$ is trivial for all $\bar{A} \in \mathcal{A}^{\leq\gamma+1}_\beta$ and $E = E_f \in V_{\gamma+1}$ is as required, by the same argument as above (in the case one of the $\mathcal{E}_\beta$ is not a maximal antichain).

Second step. We show by induction on δ with $\gamma < \delta \leq \alpha$ that the conclusion of $(\clubsuit_\alpha)$ holds for all $(A^0, A^1, A^2) \in \mathcal{A}^{\leq\delta}_\beta$, for the β and E obtained in the first step.

The case $\delta = \gamma + 1$ is the first step. The limit step is trivial. So assume this has been proved for δ, and we show it for $\delta + 1$. Let $(A^0, A^1, A^2) \in \mathcal{A}^{\leq\delta+1}_\beta \setminus \mathcal{A}^{\leq\delta}_\beta$. Assume $A^j = \{a^j_n : n \in \omega\}$ are their respective increasing enumerations. We use condition (f).

Assume first two of the three functions φ^j are of min-type, say φ^0 and φ^1. Note that the functions $f^j : k \mapsto \min\{s : \varphi^j(s) = k\}$, $j = 0, 1$, are partial one-to-one functions from $V_\gamma \subseteq V_\delta$. In particular, we have $f^j, (f^j)^{-1} \leq^* d = d_\delta$ (on the respective domains of the functions). Let n_0 be large enough so that for all $n \geq n_0$ we have $f^j(a^j_n) < d(a^j_n)$ and $a^j_n = (f^j)^{-1}(f^j(a^j_n)) < d(f^j(a^j_n))$, $j = 0, 1$. Then we see that

$$f^0(a^0_n) < d(a^0_n) < f^1(a^1_n) < d(a^1_n) < f^0(a^0_{n+1})$$

where the second inequality follows from $d(d(a^0_n)) < a^1_n$ for $f^1(a^1_n) \leq d(a^0_n)$ would yield $a^1_n < d(f^1(a^1_n)) \leq d(d(a^0_n))$, a contradiction to (f). Similarly for the forth inequality. Hence, $(\varphi^0)^{-1}(A^0)$ and $(\varphi^1)^{-1}(A^1)$ have only finitely many mins in common. Thus clearly $\mathrm{FU}(E) \cap (\varphi^0)^{-1}(A^0) \cap (\varphi^1)^{-1}(A^1)$ is trivial.

Similarly, if two of the functions are not of min-type, say again, φ^0 and φ^1, then we see that $(\varphi^0)^{-1}(A^0)$ and $(\varphi^1)^{-1}(A^1)$ only have finitely many maxs in common, and we conclude as before. The only slight complication is that this time the functions $f^j : k \mapsto \max\{s : \varphi^j(s) = k\}$ are only finite-to-one and not necessarily one-to-one. We leave details to the reader. This ends the proof of the main lemma. $\square$

Note that the second step of this argument is the only place in the proof of Theorem 1 where we use that we work with *triples* (A^0, A^1, A^2).

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Now consider the following property of ultrafilters $\mathcal{U}$ on FIN in V_α:

$$(\spadesuit_\alpha)\begin{cases}\forall k\ \forall D\in(\mathrm{FIN})^\omega \text{ with } \mathrm{FU}(D)\in\mathcal{U}\ \forall\varphi^{j,i}:\mathrm{FU}(D)\to\omega\ (j<3 \text{ and } i<k)\\ \text{in } V_\alpha \text{ of one of the four types min, max, min-max, or one-to-one}\\ \exists\beta<\omega_1\ \forall(A^0,A^1,A^2)\in\mathcal{A}_\beta^{\leq\alpha}\ \exists E\sqsubseteq D \text{ with } \mathrm{FU}(E)\in\mathcal{U}\\ \exists j<3\ \forall i<k\ (\mathrm{FU}(E)\cap(\varphi^{j,i})^{-1}(A^j)=\emptyset)\end{cases}$$

Again this is equivalent to the special case $k=1$, but we will not need this.

OBSERVATION 34. *If $\mathcal{U}$ is an ordered-union ultrafilter, then $(\spadesuit_\alpha)$ is the same if restricted to $k=1$.*

PROOF. Like the proof of Observation 31. □

LEMMA 35. *Assume $cf(\alpha)=\omega_1$ and $(\clubsuit_\alpha)$ holds. Then, in V_α, there is a stable ordered-union ultrafilter $\mathcal{U}_\alpha$ satisfying $(\spadesuit_\alpha)$ and diagonalizing the witness handed down by $\diamondsuit_{S^2_1}$ at stage α.*

The latter means that if the α-th object of the diamond sequence is a $\mathbb{P}_\alpha$-name which interprets as a family $\{\mathcal{D}_\beta:\beta<\omega_1\}$ where each $\mathcal{D}_\beta$ is an open dense set in $(\mathrm{FIN})^\omega$, then, for all β, there is $D\in\mathcal{D}_\beta$ such that $\mathrm{FU}(D)\in\mathcal{U}_\alpha$. This implies that $\mathbb{L}_{\mathcal{U}_\alpha}$ will generically add an $E\in(\mathrm{FIN})^\omega$ belonging to all $\mathcal{D}_\beta$ so that $\{\mathcal{D}_\beta:\beta<\omega_1\}$ cannot be extended to a witness for $\mathfrak{h}_{\mathrm{FIN}}=\aleph_1$ in the final model, as required.

PROOF. Work in V_α. Note that CH still holds. $\{F_\delta:\delta<\omega_1\}$ lists $[\mathrm{FIN}]^\omega$, that is, the infinite subsets of FIN. Also let $\{\{\varphi_\delta^{j,i}:j<3 \text{ and } i<k_\delta\}:\delta<\omega_1\}$ list all finite sequences of functions from FIN to ω. Recursively construct $E_\delta\in(\mathrm{FIN})^\omega$ (the infinite block sequences) and ordinals $\beta_\delta<\omega_1$, $\delta<\omega_1$, such that

(i) the sequence E_δ is $\sqsubseteq^*$-decreasing,

(ii) $\mathrm{FU}(E_{\delta+1})\subseteq F_\delta$ or $\mathrm{FU}(E_{\delta+1})\subseteq\omega\setminus F_\delta$,

(iii) $E_{\delta+1}\in\mathcal{D}_\delta$ (see the paragraph preceding the proof),

(iv) if all $\varphi_\delta^{j,i}\restriction\mathrm{FU}(D)$ are one of the four types, for some $D\sqsubseteq E_\delta$, then $E_{\delta+1}\sqsubseteq D$ and $\beta_{\delta+1}$ satisfy the conclusion of $(\clubsuit_\alpha)$.

Item (i) is taken care of in limit steps, items (ii) through (iv) in successor steps. To get (ii) use Hindman's Theorem. The other conditions can be easily guaranteed using the assumptions. (i) and (ii) imply that the decreasing chain of the $\mathrm{FU}(E_\delta)$ generates a stable ordered-union ultrafilter $\mathcal{U}_\alpha$. We are left with showing $(\spadesuit_\alpha)$.

Let k, D, and $\varphi^{j,i}$, $j<3$ and $i<k$, be given as required. Let δ be such that $k=k_\delta$ and $\varphi^{j,i}=\varphi_\delta^{j,i}$. We may assume D extends E_δ. Hence $E_{\delta+1}$ and $\beta_{\delta+1}$ were constructed

so as to satisfy ($\clubsuit_\alpha$). Now take $(A^0, A^1, A^2) \in \mathcal{A}^{\leq\alpha}_{\beta_{\delta+1}}$. Since $\mathcal{U}_\alpha$ is an ordered-union ultrafilter, there is $E \sqsubseteq E_{\delta+1}$ with $\mathrm{FU}(E) \in \mathcal{U}_\alpha$ such that for all j and i,

- either $\mathrm{FU}(E) \cap (\varphi^{j,i})^{-1}(A^j) = \emptyset$,
- or $\mathrm{FU}(E) \subseteq (\varphi^{j,i})^{-1}(A^j)$.

If for all j there were i with $\mathrm{FU}(E) \subseteq (\varphi^{j,i})^{-1}(A^j)$, this would clearly contradict the conclusion of ($\clubsuit_\alpha$). Hence the conclusion of ($\spadesuit_\alpha$) holds. $\square$

Main Lemma 36. *Assume the stable ordered-union ultrafilter $\mathcal{U}_\alpha$ satisfies ($\spadesuit_\alpha$). Then the forcing $\mathbb{L}_{\mathcal{U}_\alpha}$ preserves ($\star_\alpha$). In particular, if $\mathrm{cf}(\alpha) = \omega_1$ and ($\clubsuit_\alpha$) holds in V_α, then ($\star_{\alpha+1}$) holds in $V_{\alpha+1}$.*

Proof. This proof is very similar to the proof of Main Lemma 29. Since the filter is an ultrafilter this time, "forces" and "favors" mean the same, and we do not need the relativized ranks. We do need, however, the relation R and the ranks $\rho_{j,i,n}$ as discussed in the paragraphs preceding Main Lemma 29.

Assume $k \in \omega$ and $\mathbb{L}_{\mathcal{U}_\alpha}$-names $\dot{X}^{j,i} = \{\dot{x}^{j,i}_n : n \in \omega\}$, $j < 3$ and $i < k$, are given. Fix n for the moment. As in the proof of Main Lemma 29, let J be a non-empty R-interval. Say that $\sigma \in (\mathrm{FIN})^{<\omega}$ is *n-good for J* if

- $\rho_{j,i,n}(\sigma) = 0$ for all (j,i) R-smaller than $\min J$,
- $\rho_{j,i,n}(\sigma) = 1$ for all $(j,i) \in J$,
- $\rho_{j,i,n}(\sigma) > 1$ for all (j,i) R-larger than $\max J$.

Using the definition of the rank, we can find a set $E \in (\mathrm{FIN})^\omega$ such that $\mathrm{FU}(E) \in \mathcal{U}_\alpha$ and functions $\varphi^{j,i,\sigma,n} : \mathrm{FU}(E) \to \omega$, $(j,i) \in J$, such that for all $e \in \mathrm{FU}(E)$,

- $\rho_{j,i,n}(\sigma\hat{\ }e) = 0$ and, in fact, $\sigma\hat{\ }e$ forces $\dot{x}^{j,i}_n = \varphi^{j,i,\sigma,n}(e)$ for $(j,i) \in J$,
- $\rho_{j,i,n}(\sigma\hat{\ }e) \geq 1$ for all (j,i) R-larger than $\max J$.

Since $\mathcal{U}_\alpha$ is a stable ordered-union ultrafilter and thus satisfies Taylor's Theorem by [**Bl1**, Theorem 4.2], we may assume that all the functions $\varphi^{j,i,\sigma,n}$, $(j,i) \in J$, are of one of the five canonical types. Furthermore, no such function can be of constant type, because $\varphi^{j,i,\sigma,n}$ being constant would imply $\rho_{j,i,n}(\sigma) = 0$, contradicting the assumption. Hence they are of one of the other four types. For $(j,i) \notin J$, let $\varphi^{j,i,\sigma,n} : \mathrm{FU}(E) \to \omega$ be an arbitrary function of one of the four types.

Unfix n. For each $n \in \omega$, each finite set $\Sigma \subseteq (\mathrm{FIN})^{<\omega}$ such that all $\sigma \in \Sigma$ are n-good for some J, apply ($\spadesuit_\alpha$) to the family $\{\varphi^{j,i,\sigma,n} : j < 3, i < k \text{ and } \sigma \in \Sigma\}$ to obtain $\beta_{n,\Sigma}$ depending on n and Σ and satisfying the conclusion of ($\spadesuit_\alpha$). Let β be the supremum of the $\beta_{n,\Sigma}$.

Now let $T \in \mathbb{L}_{\mathcal{U}_\alpha}$ be arbitrary. Fix $n_0 \in \omega$. Also fix $(A^0, A^1, A^2) \in \mathcal{A}^{\leq\alpha}_\beta = \mathcal{A}^{\leq\alpha+1}_\beta$. Again, a standard genericity argument shows that it suffices to prove the following:

Claim 37. *For some $n \geq n_0$, some $T' \leq T$, and some $j < 3$, $T' \Vdash \dot{x}^{j,i}_n \notin A^j$ for all $i < k$.*

Proof. The proof is similar to the proof of Claim 30. Let $n \geq n_0$ be such that $\rho_{j,i,n}(\sigma) \geq 1$ for any $(j,i) \in 3 \times k$ where σ is the stem of T. Again assume that $\rho_{j,i,n}(\sigma) = 1$ for the R-minimal (j,i).

Now, for each finite Σ consisting of n-good τ, we can find $j = j_\Sigma$ and $E_\Sigma \sqsubseteq D$ with $\mathrm{FU}(E_\Sigma) \in \mathcal{U}_\alpha$ such that the conclusion of $(\spadesuit_\alpha)$ holds for the $\varphi^{j,i,\tau,n}$, $i < k$ and $\tau \in \Sigma$, and (A^0, A^1, A^2). The directedness of the Σ yields that a single j_0 works for all Σ. Fix such j_0.

Recursively build $\sigma = \sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_q$ all in T, an interval partition $I_0, I_1, \ldots, I_{q-1}$ of $(3 \times k, R)$, and numbers $\ell^{j,i}$, $j < 3$ and $i < k$, such that

- $\ell^{j_0,i} \notin A^{j_0}$,
- σ_r is n-good for I_r; in particular, $\rho_{j,i,n}(\sigma_r) = 1$ for all $(j,i) \in I_r$,
- σ_r forces $\dot{x}_n^{j,i} = \ell^{j,i}$ for all $(j,i) \in I_{<r} := \bigcup_{t<r} I_t$.

To do the recursion, assume σ_r has been obtained for some $r \geq 0$ such that $\rho_{j,i,n}(\sigma_r) = 1$ for the R-minimal $(j,i) \notin I_{<r}$ (in particular, $I_{<r}$ is a proper R-initial segment of $3 \times k$). By Claim 28, the (j,i) with $\rho_{j,i,n}(\sigma_r) = 1$ form an R-initial segment I_r of $(3 \times k) \setminus I_{<r}$. By assumption $(\spadesuit_\alpha)$ and choice of j_0, we find $e \in \mathrm{FIN}$ such that $\sigma_r\hat{\ }e \in T$, $\sigma\hat{\ }e$ forces $\dot{x}_n^{j,i} = \varphi^{j,i,\sigma_r,n}(e)$ for all $(j,i) \in I_r$, $\varphi^{j_0,i,\sigma_r,n}(e) \notin A^{j_0}$ for all i, and $\rho_{j,i,n}(\sigma_r\hat{\ }e) \geq 1$ for all $(j,i) \notin I_{\leq r}$. Put $\ell^{j,i} = \varphi^{j,i,\sigma_r,n}(e)$ for $(j,i) \in I_r$. In case $I_{\leq r}$ is a proper subset of $3 \times k$, we can extend $\sigma_r\hat{\ }e$ to $\sigma_{r+1} \in T$ such that $\rho_{j,i,n}(\sigma_{r+1}) = 1$ for the R-minimal $(j,i) \notin I_{\leq r}$. If $I_{\leq r} = 3 \times k$, let $q = r + 1$ and $\sigma_q = \sigma_r\hat{\ }e$.

Now, σ_q forces "$\dot{x}_n^{j,i} = \ell^{j,i}$ for all $(j,i) \in 3 \times k$" and $\ell^{j_0,i} \notin A^{j_0}$ for all $i < k$. Hence we can find $T' \leq T$ with stem σ_q forcing $\dot{x}_n^{j_0,i} \notin A^{j_0}$ for all $i < k$, as required. □

This completes the proof of the main lemma. □

This also completes the proof of Theorem 1.

5. Proof of Theorem 2.

The proof of Theorem 2 is very similar to the proof of Theorem 1. Therefore, we will confine ourselves to stressing the main combinatorial differences and leave the proofs of a number of facts to the reader.

Before starting, however, it is instructive to review what we did in the proof of Theorem 1. We used $\diamondsuit_{S_1^2}$ and a Laver-style forcing in limit steps of cofinality ω_1 to get $\mathfrak{h}_{\mathrm{FIN}} = \aleph_2$ – and that was the easy part. We also built up a family of $\aleph_1$ many maximal antichains in $([\omega]^\omega)^3$ as a witness for $\mathfrak{h}_3 = \aleph_1$ along the iteration. For preserving this family we introduced property $(\star_\alpha)$ which was only slightly stronger than the obviously needed property. Preservation of $(\star_\alpha)$ in limit steps and successor steps $\alpha + 1$ with $\mathrm{cf}(\alpha) \neq \omega_1$ – the places where we used Hechler forcing – were straightforward albeit technical. Then main problem was the preservation of $(\star_\alpha)$ in successor steps $\alpha + 1$ with $\mathrm{cf}(\alpha) = \omega_1$ – and for this we needed that the ultrafilter $\mathcal{U}_\alpha$ of the Laver forcing $\mathbb{L}_{\mathcal{U}_\alpha}$ satisfied property $(\spadesuit_\alpha)$. To be able to build $\mathcal{U}_\alpha$ with $(\spadesuit_\alpha)$, property $(\clubsuit_\alpha)$ was necessary, and the heart of the whole argument was to prove $(\clubsuit_\alpha)$ for α with $\mathrm{cf}(\alpha) = \omega_1$, just assuming $(\star_\gamma)$ – more explicitly, its equivalent reformulation $(\dagger_\gamma)$ – for $\gamma < \alpha$. This central argument was based on a canonization theorem, Taylor's Theorem.

For the proof of CON($\mathfrak{h}_n > \mathfrak{h}_{\mathrm{FIN}}$ for all n), we shall again need a canonization result, the Erdős–Rado Theorem [**To**, Theorem 1.8] which is a direct consequence of Ramsey's Theorem. Fix $\ell \in \omega$. Let $A = (A^j : j < \ell)$ be infinite subsets of ω. By $\vec{A}$ we denote the set of all strictly increasing sequences in the product of the A^j, i.e., $\vec{A} = \{\bar{a} = (a^j : j < \ell) \in \prod_{j<\ell} A^j : a^j < a^{j+1}$ for all $j < \ell - 1\}$. If all A^j are equal to ω, write $\vec{\omega}$ – or $\vec{\omega}^\ell$ if we want to stress the number of coordinates – for $\vec{A}$. We say that $\vec{B} \leq \vec{A}$ if $B^j \subseteq A^j$ for all $j < \ell$. Similarly, we define $\leq^*$: $\vec{B} \leq^* \vec{A}$ if $B^j \subseteq^* A^j$ for all $j < \ell$.

Erdős–Rado Theorem. *For every $A = (A^j : j < \ell) \in ([\omega]^\omega)^\ell$ and every function $f : \vec{A} \to \omega$, there are $\vec{B} \leq \vec{A}$ and $T \subseteq \ell$ such that $f{\restriction}\vec{B}$ depends exactly on the coordinates in T, that is, for all $\bar{a}, \bar{b} \in \vec{B}$,*

$$f(\bar{a}) = f(\bar{b}) \Longleftrightarrow \forall j \in T\ (a^j = b^j)$$

Assume again CH and $\diamondsuit_{S^2_1}$. We make a finite support iteration $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \omega_2)$ such that

(A′) if $\mathrm{cf}(\alpha) = \omega_1$, then $\dot{\mathbb{Q}}_\alpha$ is Laver forcing $\mathbb{L}_{\dot{\mathcal{U}}_\alpha}$ where $\dot{\mathcal{U}}_\alpha$ is an Erdős–Rado ultrafilter on ω^{ℓ_α} with ℓ_α being a natural number handed down by the diamond sequence,

(B′) if $\mathrm{cf}(\alpha) = \omega_1$, then $\dot{\mathbb{Q}}_\alpha$ is Hechler forcing $\mathbb{D}$.

(Here an ultrafilter is *Erdős-Rado* if canonical sets in the sense of the Erdős–Rado Theorem can be found in the ultrafilter; see after Main Lemma 42 for details.) $\mathfrak{h}_\ell = \aleph_2$ for all ℓ will again follow as in the proof of Theorem 1, and the main point is the argument for $\mathfrak{h}_{\mathrm{FIN}} = \aleph_1$. Construct families $\{\mathcal{D}_\beta : \beta < \omega_1\}$ such that

(a′) the $\mathcal{D}_\beta$ are MAD_{FIN}-families,

(b′) $\beta < \beta'$ implies that $\mathcal{D}_{\beta'}$ refines $\mathcal{D}_\beta$.

Again we have $\mathcal{D}^{\leq\alpha}_\beta = \mathcal{D}_\beta \cap V_\alpha \in V_\alpha$, and a book-keeping argument gives $\alpha = \alpha_{E,\beta}$ for $E \in (\mathrm{FIN})^\omega$ and $\beta < \omega_1$ with $E \in V_\alpha$, such that the function $E, \beta \mapsto \alpha$ is one-to-one and onto ordinals of cofinality $< \omega_1$. For α with $\mathrm{cf}(\alpha) \neq \omega_1$, denote by E_α and β_α the unique E and β with $\alpha = \alpha_{E,\beta}$. Again we stipulate

(c′) $\mathcal{D}^{\leq\alpha}_\beta = \bigcup_{\gamma<\alpha} \mathcal{D}^{\leq\gamma}_\beta$ for limit α, and $\mathcal{D}^{\leq\alpha+1}_\beta = \mathcal{D}^{\leq\alpha}_\beta$ if $\mathrm{cf}(\alpha) = \omega_1$ or $\beta \geq \beta_\alpha$,

(d′) if $\beta < \beta_\alpha$ and if $\mathcal{D}^{\leq\alpha}_\beta$ is not predense below E_α, then there is $D \sqsubseteq^* E_\alpha$ belonging to $\mathcal{D}^{\leq\alpha+1}_\beta \setminus \mathcal{D}^{\leq\alpha}_\beta$,

(e′) for all α and for $\beta < \beta' < \omega_1$, $\mathcal{D}^{\leq\alpha}_{\beta'}$ refines $\mathcal{D}^{\leq\alpha}_\beta$,

(f′) whenever $\beta < \beta_\alpha$ and $D \in \mathcal{D}^{\leq\alpha+1}_\beta \setminus \mathcal{D}^{\leq\alpha}_\beta$, $D = (d_n : n \in \omega)$, then

$$d(\min(d_n)) < \max(d_n) \text{ and } d(\max(d_n)) < \min(d_{n+1})$$

for all n, where $d = d_\alpha$ is the dominating real over V_α added by $\mathbb{Q}_\alpha$.

It is easy to see that this can be done. See the discussion in Observation 23 and Corollary 24.

⋆⋆⋆

To see that $\{\mathcal{D}_\beta : \beta < \omega_1\}$ witnesses $\mathfrak{h}_{\mathrm{FIN}} = \aleph_1$, by (f′) it suffices to show by induction on $\alpha < \omega_2$ that

$$(**_\alpha) \quad \forall E \in (\mathrm{FIN})^\omega \cap V_\alpha \; \exists \beta < \omega_1 \; \forall D \in \mathcal{D}_\beta^{\leq\alpha} \; (E \not\sqsubseteq^* D)$$

We shall preserve the following slightly stronger property:

$$(\star\star_\alpha) \begin{cases} \forall k \; \forall \text{enumerations } E^i = \{e^i_n : n \in \omega\} \in [\mathrm{FIN}]^\omega \cap V_\alpha \; (i < k) \\ \exists \beta < \omega_1 \; \forall D \in \mathcal{D}_\beta^{\leq\alpha} \; \exists^\infty n \; \forall i < k \; (e^i_n \notin \mathrm{FU}(D)) \end{cases}$$

Clearly $(\star\star_\alpha)$ strengthens $(**_\alpha)$. Note that we allow the E^i to be arbitrary infinite subsets of FIN and not just block sequences. This is important for preservation purposes. Say that $E = \{e_{\bar{a}} : \bar{a} \in \vec{\omega}\} \subseteq \mathrm{FIN}$ is listed in *canonical enumeration* if the listing is one of the non-constant canonical types given by the Erdős–Rado Theorem. Now consider:

$$(\dagger\dagger_\alpha) \begin{cases} \forall \ell \; \forall k \; \forall E^i = \{e^i_{\bar{a}} : \bar{a} \in \vec{\omega}^\ell\} \in [\mathrm{FIN}]^\omega \cap V_\alpha \; (i < k) \\ \text{listed in canonical enumeration } \exists \beta < \omega_1 \; \forall D \in \mathcal{D}_\beta^{\leq\alpha} \\ \exists \vec{R} \leq \vec{\omega}^\ell \; \forall \bar{a} \in \vec{R} \; \forall i < k \; (e^i_{\bar{a}} \notin \mathrm{FU}(D)) \end{cases}$$

OBSERVATION 38. *$(\star\star_\alpha)$ and $(\dagger\dagger_\alpha)$ are equivalent.*

PROOF. This is similar to the proof of Observation 25. However, since the combinatorial structure is different, we include the argument.

Assume ℓ, k, and $E^i = \{e^i_{\bar{a}} : \bar{a} \in \vec{\omega}^\ell\}$, $i < k$, are given as required. Since all listings E^i are of non-constant canonical type, for each $i < k$ there is $T^i \subseteq \ell$, $T^i \neq \emptyset$, such that for all $\bar{a}, \bar{b} \in \vec{\omega}^\ell$,

$$e^i_{\bar{a}} = e^i_{\bar{b}} \iff \forall j \in T^i \; (a^j = b^j)$$

Let $j^i = \max(T^i)$. For $j < \ell$, $i < k$ such that $j^i = j$, and $\bar{c} \in \vec{\omega}^j$, define the sets $E^{i,\bar{c}} = \{e^{i,\bar{c}}_n : n > \max \bar{c}\}$ where $e^{i,\bar{c}}_n = e^i_{\bar{a}}$ with $\bar{a} \restriction j = \bar{c}$ and $a^j = n$. Note that for $\bar{a}, \bar{b} \in \vec{\omega}^\ell$ with $\bar{c} \subseteq \bar{a}, \bar{b}$,

$$e^i_{\bar{a}} = e^i_{\bar{b}} \iff a^j = b^j$$

so that $e^{i,\bar{c}}_n$ is welldefined and $E^{i,\bar{c}}$ is an enumeration of an infinite subset of FIN.

By $(\star\star_\alpha)$, for each $m \in \omega$, there is a β_m satisfying the conclusion for the family of sets $E^{i,\bar{c}}$ where $\max \bar{c} \leq m$ and $i < k$. Let β be the supremum of the β_m, $m \in \omega$. We claim that β witnesses $(\dagger\dagger_\alpha)$. To see this, fix $D \in \mathcal{D}_\beta^{\leq\alpha}$. It is easy to see that

$$(**) \quad \forall j \; \forall m \; \exists^\infty n > m \; \forall i \text{ with } j^i = j \; \forall \bar{c} \in \vec{\omega}^j \text{ with } \max \bar{c} \leq m \; (e^{i,\bar{c}}_n \notin \mathrm{FU}(D))$$

In fact, this is a straightforward consequence of the conclusion of $(\star\star_\alpha)$ and the choice of β.

$(**)$, however, allows us to build infinite sets $R^j \subseteq \omega$, $j < \ell$, such that for all i with $j^i = j$, all $\bar{c} \in \vec{R}\restriction j$, and all $n \in R^j$ with $n > \max \bar{c}$, $e_n^{i;\bar{c}} \notin \mathrm{FU}(D)$. This means that for $\bar{a} \in \vec{R}$ and $i < k$, $e^i_{\bar{a}}$ does not belong to $\mathrm{FU}(D)$. □

LEMMA 39. *$(\star\star_\alpha)$ is preserved in limit steps of finite support iterations.*

PROOF. Like the proof of Lemma 26. □

LEMMA 40. *$(\star\star_\alpha)$ is preserved by Hechler forcing.*

PROOF. Like the proof of Main Lemma 29, but simpler because we only have one set E^i for each i and not three sets $X^{j,i}$, $j < 3$. □

To prove the preservation of $(\star\star_\alpha)$ under forcing of type $\mathbb{L}_{\mathcal{U}}$, we need again a stronger property:

$$(\clubsuit\clubsuit_\alpha)\begin{cases}\forall \ell\ \forall k\ \forall A = (A^0,\dots,A^{\ell-1})\ \forall f^i : \vec{A} \to \mathrm{FIN}\ (i<k)\ \text{ in } V_\alpha\\ \text{of non-constant canonical type}\\ \exists \vec{B} \le \vec{A}\ \exists \beta < \omega_1\ \forall D \in \mathcal{D}_\beta^{\le\alpha}\ \forall \vec{B}' \le \vec{B}\ \exists \vec{B}'' \le \vec{B}'\\ \forall i < k\ (\vec{B}'' \cap (f^i)^{-1}(\mathrm{FU}(D)) = \emptyset)\end{cases}$$

OBSERVATION 41. *$(\clubsuit\clubsuit_\alpha)$ is the same if restricted to $k = 1$.*

PROOF. Like the proof of Observation 31 but easier. □

Again, it is easy to see that $(\clubsuit\clubsuit_\alpha)$ implies $(\star\star_\alpha)$ (cf. Observation 32).

MAIN LEMMA 42. *Assume* $\mathrm{cf}(\alpha) = \omega_1$ *and* $(\star\star_\gamma)$ *holds for all* $\gamma < \alpha$. *Then* $(\clubsuit\clubsuit_\alpha)$ *holds.*

PROOF. The proof proceeds like the proof of Main Lemma 33, in two steps. The first step is similar, and we shall confine ourselves to only sketching the argument, while the second step involves the combinatorics of the Erdős–Rado Theorem and will be presented in detail.

By Observation 41, it suffices to consider the case $k = 1$. Let $\ell \in \omega$, $A = (A^0,\dots,A^{\ell-1})$, and $f : \vec{A} \to \mathrm{FIN}$ in V_α be given. Let $\gamma < \alpha$ be such that $\vec{A}, f \in V_\gamma$.

First step. Show that there are $\beta < \omega_1$ and $\vec{B} \le \vec{A}$ in $V_{\gamma+1}$ satisfying the conclusion of $(\clubsuit\clubsuit_\alpha)$ for all $D \in \mathcal{D}_\beta^{\le\gamma+1}$.

Work in V_γ. Say $C \subseteq \vec{A}$ is *non-trivial* if there is $\vec{B} \subseteq C$. For $D \in (\mathrm{FIN})^\omega$, consider $C_D = \vec{A} \cap f^{-1}(\mathrm{FU}(D))$. For $\beta < \omega_1$, let $\mathcal{C}_\beta = \{C_D : D \in \mathcal{D}_\beta^{\le\gamma}$ and C_D is non-trivial$\}$. Note that if D and E are two distinct elements of $\mathcal{D}_\beta^{\le\gamma}$, then $C_D \cap C_E$ is trivial (this is so because, if D and E are incompatible, then $\mathrm{FU}(D) \cap \mathrm{FU}(E)$ is finite; since f is not a constant function, it is one-to-one in at least one coordinate, and then $C_D \cap C_E$ is finite in this coordinate).

If, for some $\beta \geq \beta_\gamma$, $\mathcal{C}_\beta$ is not maximal below $\vec{A}$, then we find $\vec{B} \leq \vec{A}$ in V_γ incompatible with everything from $\mathcal{C}_\beta$, and this $\vec{B}$ is easily seen to be as required.

If, for all β, $\mathcal{C}_\beta$ is maximal below $\vec{A}$, we build a tree as in the proof of Main Lemma 33, using the assumption $(\dagger\dagger_\gamma)$ which is equivalent to $(\star\star_\gamma)$ by Observation 38, and argue that a new branch from $V_{\gamma+1} \setminus V_\gamma$ gives us $\vec{B} \leq \vec{A}$ incompatible with everything from $\mathcal{C}_\beta$ for some $\beta \geq \beta_\gamma$. Again, this $\vec{B}$ is as required.

Second step. Show by induction on δ with $\gamma < \delta \leq \alpha$ that the conclusion of $(\clubsuit\clubsuit_\alpha)$ holds for all $D \in \mathcal{D}^{\leq\delta}_{\vec{\beta}}$, for the β and $\vec{B}$ obtained in the first part.

The case $\delta = \gamma + 1$ was done above, and the limit step is trivial. So assume this is true for δ, and we shall show this for $\delta + 1$. Let $D = (d_n : n \in \omega) \in \mathcal{D}^{\leq\delta+1}_{\vec{\beta}} \setminus \mathcal{D}^{\leq\delta}_{\vec{\beta}}$.

Recall that $T \subseteq \ell$ is the set of coordinates on which the function f depends, that is, for all $\bar{a}, \bar{b} \in \vec{B}$, $f(\bar{a}) = f(\bar{b})$ iff $a^j = b^j$ for all $j \in T$. Since f is of non-constant type, $T \neq \emptyset$. Since it suffices to show that for all $\vec{B}' \leq \vec{B}$ there is $\vec{B}'' \leq \vec{B}'$ such that $\vec{B}'' \cap f^{-1}(\mathrm{FU}(D))$ is trivial, we may assume that $\vec{B}$ has further canonization properties. So suppose that $\vec{B}$ is canonical with respect to the function $\min f$, that is, there is a set $T_{\min} \subseteq T$ such that for all $\bar{a}, \bar{b} \in \vec{B}$, $\min(f(\bar{a})) = \min(f(\bar{b}))$ iff $a^j = b^j$ for all $j \in T_{\min}$. Put $j_0 := \max T$.

Case 1. $j_0 \in T_{\min}$. Let $\{b^{j_0}_n : n \in \omega\}$ be the increasing enumeration of B^{j_0}. Since f does not depend on coordinates beyond j_0, for each n there are only finitely many values $f(\bar{a}_n)$ where $\bar{a}_n \in \vec{B}$ is arbitrary with $a^{j_0}_n = b^{j_0}_n$. Also, $\min\{\min(f(\bar{a}_n)) : \bar{a}_n \in \vec{B}$ and $a^{j_0}_n = b^{j_0}_n\}$ goes to ∞ as $n \to \infty$. Since $d = d_\delta$ is dominating over V_δ, we conclude that, for almost all n, the following holds: for all $\bar{a}_n \in \vec{B}$ with $a^{j_0}_n = b^{j_0}_n$, $d(\min(f(\bar{a}_n))) > \max(f(\bar{a}_n))$. For such $\bar{a}_n$, $f(\bar{a}_n)$ cannot belong to $\mathrm{FU}(D)$, for if some d_m was an initial segment of $f(\bar{a}_n)$, then by (f′)

$$\max(d_m) > d(\min(d_m)) = d(\min(f(\bar{a}_n))) > \max(f(\bar{a}_n))$$

a contradiction. Hence $\vec{B} \cap f^{-1}(\mathrm{FU}(D))$ is trivial.

Case 2. $j_0 \notin T_{\min}$. Fix strictly increasing $b^j \in B^j$ for $j \in T \setminus \{j_0\}$. Let $\bar{b} := (b^j : j \in T \setminus \{j_0\})$. Let $\bar{a}_n = \bar{a}_{\bar{b},n} \in \vec{B}$ be arbitrary with $a^j_n = b^j$ for $j \in T \setminus \{j_0\}$ and $a^{j_0}_n = b^{j_0}_n$. Note that $f(\bar{a}_n)$ then depends only on $a^{j_0}_n = b^{j_0}_n$. Think of $f(\bar{a}_n) \in \mathrm{FIN}$ as an element of the Cantor space 2^ω. By thinning out the set B^{j_0}, if necessary, we may assume that there is a real $x = x_{\bar{b}} \in 2^\omega$ such that the sequence $f(\bar{a}_n)$, $n \in \omega$, converges to x. Again, identify x with a subset of ω. (Note that x is non-empty because $j_0 \notin T_{\min}$ means that $n \mapsto \min(f(\bar{a}_n))$ is constant and therefore $\min(f(\bar{a}_n)) \in x$.) Unfixing the b^j we may assume that for each $\bar{b}$, the sequence $f(\bar{a}_{\bar{b},n})$, $n \in \omega$, converges to $x_{\bar{b}}$. To see that $\vec{B} \cap f^{-1}(\mathrm{FU}(D))$ is trivial, it suffices to show that for all $\bar{b}$, for almost all n, $f(\bar{a}_{\bar{b},n})$ does not belong to $\mathrm{FU}(D)$. Fix $\bar{b}$. We consider two cases.

Case 2a. $x = x_{\bar{b}}$ is finite. Note that for large enough n, $f(\bar{a}_{\bar{b},n}) \in \mathrm{FU}(D)$ is equivalent to $f(\bar{a}_{\bar{b},n}) \setminus x_{\bar{b}} \in \mathrm{FU}(D)$. Also, for large enough n, $d(\min(f(\bar{a}_{\bar{b},n}) \setminus x_{\bar{b}})) > \max(f(\bar{a}_{\bar{b},n}))$ holds because d is a dominating real. Thus, as in Case 1, if d_m was an initial segment of $f(\bar{a}_{\bar{b},n}) \setminus x_{\bar{b}}$, then by (f′)

$$\max(d_m) > d(\min(d_m)) = d(\min(f(\bar{a}_{\bar{b},n}) \setminus x_{\bar{b}})) > \max(f(\bar{a}_{\bar{b},n}))$$

a contradiction. Hence $f(\bar{a}_{\bar{b},n})$ does not belong to $\mathrm{FU}(D)$ for almost all n, as required.

Case 2b. $x = x_{\bar{b}}$ is infinite. Say $x = \{y_p : p \in \omega\}$ is its increasing enumeration. Note that each y_p must belong to almost all $f(\bar{a}_{\bar{b},n})$. Also, for large enough p, $d(y_p) > y_{p+1}$ because d is a dominating real. Thus, for large enough n, if $f(\bar{a}_{\bar{b},n})$ belonged to $\mathrm{FU}(D)$, we would have that for some p and m, $y_p = \max(d_m)$, $y_p, y_{p+1} \in f(\bar{a}_{\bar{b},n})$, and $d(y_p) > y_{p+1}$. However, by (f′),

$$\min(d_{m+1}) > d(\max(d_m)) = d(y_p) > y_{p+1}$$

then contradicts $f(\bar{a}_{\bar{b},n}) \in \mathrm{FU}(D)$, a contradiction. □

⋆⋆⋆

To be able to complete the proof of Theorem 2, we introduce the ultrafilters which we use for the Laver forcing in limit steps of uncountable cofinality, and discuss their basic properties.

Fix $\ell \in \omega$, $\ell \geq 1$. An ultrafilter $\mathcal{U}$ on ω^ℓ is *Erdős–Rado* if for all $A = (A^j : j < \ell) \in ([\omega]^\omega)^\ell$ with $\vec{A} \in \mathcal{U}$ and all $f : \vec{A} \to \omega$, there is $\vec{B} \leq \vec{A}$ such that $\vec{B} \in \mathcal{U}$ and $f \restriction \vec{B}$ is canonical (in the sense of the Erdős–Rado Theorem). We collect a couple of basic properties of such ultrafilters.

Observation 43. *Assume $\mathcal{U}$ is an Erdős–Rado ultrafilter on ω^ℓ, $\ell \geq 1$. Then:*

(i) *$\mathcal{U}$ has a basis of sets of the form $\vec{A}$.*

(ii) *If $\vec{A}_n \in \mathcal{U}$, $\vec{A}_{n+1} \leq^* \vec{A}_n$ for all n, then there is $\vec{B} \in \mathcal{U}$ with $\vec{B} \leq^* \vec{A}_n$ for all n.*

(iii) *The projection of $\mathcal{U}$ on the j-th coordinate, $\mathcal{U}^j = \{X \subseteq \omega : \{\bar{a} \in \omega^\ell : a^j \in X\} \in \mathcal{U}\}$, $j < \ell$, is a Ramsey ultrafilter on ω.*

Furthermore, for $\ell = 1$, $\mathcal{U}$ is Erdős-Rado iff $\mathcal{U}$ is Ramsey.

Proof. (i) Assume $X \subseteq \omega^\ell$, $X \in \mathcal{U}$. Define $f : \vec{\omega}^\ell \to 2$ by

$$f(\bar{a}) = \begin{cases} 0 & \text{if } \bar{a} \notin X \\ 1 & \text{if } \bar{a} \in X \end{cases}$$

Let $\vec{A} \in \mathcal{U}$ be such that $f \restriction \vec{A}$ is canonical. Since f is two-valued, $f \restriction \vec{A}$ must be constant. Since $\vec{A} \cap X \neq \emptyset$, the constant value must be 1, and $\vec{A} \subseteq X$ follows.

(ii) Let $\vec{A}_n \in \mathcal{U}$ be given as required. For simplicity assume $\vec{A}_{n+1} \leq \vec{A}_n$ and $\bigcap_n \vec{A}_n = \emptyset$. Define $f : \vec{A}_0 \to \omega$ by

$$f(\bar{a}) = \min\{n : \bar{a} \notin \vec{A}_{n+1}\}$$

Assume $\vec{B} \in \mathcal{U}$ canonizes f. We must show $\vec{B} \leq^* \vec{A}_n$ for all n. Suppose this was false, let n be minimal such that $\vec{B} \not\leq^* \vec{A}_{n+1}$ and let j be such that $B^j \not\subseteq^* A^j_{n+1}$. Since $\vec{A}_{n+1}$ and $\vec{B}$ both belong to $\mathcal{U}$ and $\mathcal{U}$ is Erdős-Rado, $(B^j \cap A^j_n) \setminus A^j_{n+1}$ and $B^j \cap A^j_{n+1}$ are both infinite. This means we can find distinct $i_0, i_1 \in (B^j \cap A^j_n) \setminus A^j_{n+1}$, $i_2 \in B^j \cap A^j_{n+1}$, and

$\bar{a}_2 \in \vec{B} \cap \vec{A}_{n+1}$ such that $a_2^j = i_2$ and $\bar{a}_0, \bar{a}_1$ given by $\bar{a}_0 \restriction \ell \setminus \{j\} = \bar{a}_1 \restriction \ell \setminus \{j\} = \bar{a}_2 \restriction \ell \setminus \{j\}$, $a_0^j = i_0$, and $a_1^j = i_1$ also belong to $\vec{B}$. But then $f(\bar{a}_2) > n$ while $f(\bar{a}_0) = f(\bar{a}_1) = n$, contradicting the assumption that f is canonical on $\vec{B}$. Hence $\vec{B}$ is as required.

(iii) Given $f : \omega \to \omega$, let $g : \omega^\ell \to \omega$ be defined by $g(\bar{a}) = f(a^j)$. Let $\vec{A} \in \mathcal{U}$ be canonical for g. Then f is either constant or one-to-one on $A^j \in \mathcal{U}^j$. Hence $\mathcal{U}^j$ is a Ramsey ultrafilter.

The final statement is immediate from (iii) and the definition of Ramseyness. $\square$

Now consider the following property of ultrafilters $\mathcal{U}$ on ω^ℓ in V_α:

$$(\spadesuit\spadesuit_\alpha) \begin{cases} \forall k \; \forall A = (A^0, \ldots, A^{\ell-1}) \text{ with } \vec{A} \in \mathcal{U} \; \forall f^i : \vec{A} \to \mathrm{FIN} \; (i < k) \text{ in } V_\alpha \\ \text{of non-constant canonical type} \\ \exists \beta < \omega_1 \; \forall D \in \mathcal{D}_\beta^{\leq \alpha} \; \exists \vec{B} \leq \vec{A} \text{ with } \vec{B} \in \mathcal{U} \\ \forall i < k \; (\vec{B} \cap (f^i)^{-1}(\mathrm{FU}(D)) = \emptyset) \end{cases}$$

LEMMA 44. *Assume $cf(\alpha) = \omega_1$ and $(\clubsuit\clubsuit_\alpha)$ holds. Then, in V_α, there is an Erdős–Rado ultrafilter $\mathcal{U}_\alpha$ satisfying $(\spadesuit\spadesuit_\alpha)$ such that $\mathcal{U}_\alpha$ diagonalizes the witness for $\mathfrak{h}_\ell$ handed down at stage α where $\ell = \ell_\alpha$.*

Again, the last part of this statement means that if the α-th object of the $\diamondsuit_{S_1^2}$-sequence is a $\mathbb{P}_\alpha$-name which interprets as a family $\{\mathcal{A}_\beta : \beta < \omega_1\}$ where each $\mathcal{A}_\beta$ is an open dense set in $([\omega]^\omega)^\ell$ with $\ell = \ell_\alpha$, then for all $\beta < \omega_1$ there is $A = (A^0, \ldots, A^{\ell-1}) \in \mathcal{A}_\beta$ such that $\vec{A} \in \mathcal{U}_\alpha$. This entails that $\mathbb{L}_{\mathcal{U}_\alpha}$ adds a set $C \subseteq \omega^\ell$ such that, letting $B^j = \{b : \exists \bar{c} \in C \; (c^j = b)\}$, a standard genericity argument shows that all B^j are infinite and that $B = (B^0, \ldots, B^{\ell-1})$ belongs to all $\mathcal{A}_\beta$ so that $\{\mathcal{A}_\beta : \beta < \omega_1\}$ cannot be an initial segment of a witness for $\mathfrak{h}_\ell = \aleph_1$ in the final model.

PROOF. This is a standard recursive construction, like the construction of the ultrafilter in Lemma 35. We leave the details to the reader. $\square$

LEMMA 45. *Assume the Erdős-Rado ultrafilter $\mathcal{U}_\alpha$ satisfies $(\spadesuit\spadesuit_\alpha)$. Then the forcing $\mathbb{L}_{\mathcal{U}_\alpha}$ preserves $(\star\star_\alpha)$. In particular, if $cf(\alpha) = \omega_1$ and $(\clubsuit\clubsuit_\alpha)$ holds in V_α, then $(\star\star_{\alpha+1})$ holds in $V_{\alpha+1}$.*

PROOF. This is similar to, but easier than, the proof of Main Lemma 36. $\square$

This completes our outline of the proof of Theorem 2.

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