J. Math. Soc. Japan
Vol. 68, No. 3 (2016) pp. 961–974
doi: 10.2969/jmsj/06830961

On an orbifold Hamiltonian structure for the first Painlevé equation

By Katsunori IWASAKI and Shu OKADA

(Received Sep. 9, 2013)
(Revised Sep. 19, 2014)

Abstract. For the first Painlevé equation we establish an orbifold polynomial Hamiltonian structure on the fibration of Okamoto's spaces and show that this geometric structure uniquely recovers the original Painlevé equation, thereby solving a problem posed by K. Takano.

1. Introduction.

For each of the six Painlevé equations P_J, Okamoto [**5**] constructed what he called the space of initial conditions. It is a fiber E_t of a fibration $\pi : E \to T$ on which P_J defines a foliation that is uniform and transversal to each fiber. For its construction, he first had a compact surface $\overline{E}_t$ as an eight-time blowup of a Hirzebruch surface and then obtained $E_t = \overline{E}_t \setminus V_t$ by removing a divisor V_t called the vertical leaves. Afterwards, Takano et al. [**3**], [**4**], [**9**] constructed a symplectic atlas of E_t, on each chart of which P_J enjoys a polynomial Hamiltonian structure, and they went on to show that such a structure uniquely recovers P_J. More precisely, they were able to do so for $J = \mathrm{II}, \mathrm{III}, \mathrm{IV}, \mathrm{V}, \mathrm{VI}$, but left open the case $J = \mathrm{I}$. We settle this last case in this article. Independently, Chiba [**1**] solved the problem based on his framework of Painlevé equations on weighted projective spaces. Our approach is more classical, along the lines of Okamoto and Takano, where what is new for $J = \mathrm{I}$ is the consideration of an orbifold Hamiltonian structure.

The first Painlevé equation P_I is a nonlinear ordinary differential equation

$$\frac{d^2x}{dt^2} = 6x^2 + t,$$

for an unknown function $x = x(t)$ with a time variable $t \in T := \mathbb{C}_t$. If we put $y := dx/dt$ then this equation can be represented as a time-dependent Hamiltonian system

$$\frac{dx}{dt} = \frac{\partial H_\mathrm{I}}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial H_\mathrm{I}}{\partial x}, \qquad H_\mathrm{I}(x, y, t) = \frac{1}{2}y^2 - 2x^3 - tx. \tag{1}$$

In order to construct the space E_t for system (1), it is sufficient to carry out an eight-time blowup of a Hirzebruch surface as in Okamoto [**5**], or alternatively a nine-time

---

2010 *Mathematics Subject Classification.* Primary 33E17; Secondary 34M55.

Key Words and Phrases. the first Painlevé equation, Hamiltonian system, orbifold.

This work is supported by Grant-in-aid for Scientific Research, JSPS, No. 25400102 (C), 16K05165 (C) and 23224001 (S).

blowup of $\mathbb{P}^2$ as in Duistermaat and Joshi [**2**], followed by removing vertical leaves. But this is not sufficient for the purpose of providing a symplectic atlas with E_t. Indeed, for $J = \mathrm{II}, \mathrm{III}, \mathrm{IV}, \mathrm{V}, \mathrm{VI}$, Takano et al. [**3**], [**9**] had to do some extra work in the course of successive blowups. We are in an even more intricate situation that is specific for $J = \mathrm{I}$. For this we shall carry out the following.

Construction of Okamoto's space. Start with the Hirzebruch surface Σ of degree 2. Take a two-time blowup of Σ to get a compact surface S, which contains a (-2)-curve C. Choose an open neighborhood U of C in S. Consider a branched double cover $(U, C) \leftarrow (V, D) \circlearrowleft \sigma$ ramifying along D, the fixed curve of the deck involution σ. Along the (-1)-curve D, take a blowdown $(V, D) \to (W, p)$ and let $\sigma : (W, p) \circlearrowleft$ be the induced involution. The result is the unique σ-fixed point $p \in W$, together with a pair of σ-equivalent singular points $p_\pm \in W$ of the foliation. To resolve the singularities $p_\pm$, carry out a pair of σ-equivariant six-time blowups $(W, p, p_\pm) \leftarrow (X, p, E_\pm)$. Take a quotient X/σ, which identifies E_+ and E_-, and make a gluing $F = (S \setminus C) \cup (X/\sigma)$ in accordance with the union $S = (S \setminus C) \cup U$. Then F is a compact surface with an A_1-singularity $p \in F$ arising from the σ-fixed point $p \in X$. Take a minimal resolution of $p \in F$ to obtain a smooth compact surface $\overline{E}_t$, which contains an $E_8^{(1)}$-type configuration V_t of (-2)-curves that are the vertical leaves, where the black-filled node in Figure 1 corresponds to the exceptional curve for the last resolution. Finally we get $E_t = \overline{E}_t \setminus V_t$ by removing the vertical leaves V_t. Details of these processes are described in Section 2 (see Figure 3–Figure 9).

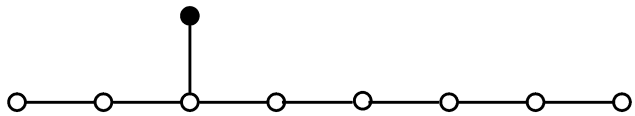

Figure 1. Dynkin diagram of type $E_8^{(1)}$.

Recipe for producing local charts. We look at how a blowup produces two new local charts from an old one. Start with an (x, y)-plane and blow up a point $(a, 0) \in \mathbb{C}^2_{(x,y)}$ on the x-axis $\{y = 0\}$. The ensuing morphism $\mathbb{C}^2_{(x,y)} \leftarrow \mathbb{C}^2_{(q,p)} \cup \mathbb{C}^2_{(Q,P)}$ is represented by

$$x - a = qp, \quad y = p; \qquad x - a = Q, \quad y = QP,$$

where the exceptional curve is $\{p = 0\} \cup \{Q = 0\} \cong \mathbb{P}^1$ while the strict transform of $\{y = 0\}$ is $\{P = 0\}$, respectively. This procedure leads to a creation of two local charts (see Figure 2):

$$(x, y) \rightsquigarrow (q, p), \ (Q, P). \tag{2}$$

Beginning with the local charts of the Hirzebruch surface Σ, we make a repeated application of recipe (2) to produce new local charts in the course of successive blowups. Recall that there is one step of blowdown $(V, D) \to (W, p)$, at which we apply (2) in the opposite direction.

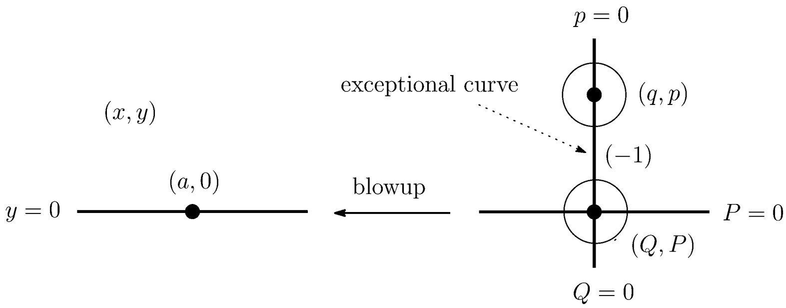

Figure 2. A blowup produces two new charts from an old one.

THEOREM 1.1. *The construction mentioned above leads to the following description of E_t:*

$$E_t = \mathbb{C}^2_{(x,y)} \cup (\mathbb{C}^2_{(z,w)} \cup \mathbb{C}^2_{(u,v)})/\sigma, \tag{3}$$

where $\mathbb{C}^2_{(z,w)}$ and $\mathbb{C}^2_{(u,v)}$ are glued together along the subset $\{w \neq 0\} = \{v \neq 0\}$ via

$$u = z - 2tw^{-2} - 8w^{-6}, \qquad v = w, \tag{4}$$

with $\sigma : \mathbb{C}^2_{(z,w)} \cup \mathbb{C}^2_{(u,v)} \circlearrowleft$ being a holomorphic involution that restricts to

$$\begin{aligned} &\sigma : \mathbb{C}^2_{(z,w)} \to \mathbb{C}^2_{(u,v)}, \quad (z,w) \mapsto (u,v) = (-z,-w), \\ &\sigma : \mathbb{C}^2_{(z,w)} \to \mathbb{C}^2_{(z,w)}, \quad (z,w) \mapsto (-z + 2tw^{-2} + 8w^{-6},\, -w), \\ &\sigma : \mathbb{C}^2_{(u,v)} \to \mathbb{C}^2_{(u,v)}, \quad (u,v) \mapsto (-u - 2tv^{-2} - 8v^{-6},\, -v), \end{aligned} \tag{5}$$

while $\mathbb{C}^2_{(x,y)}$ and the quotient space $(\mathbb{C}^2_{(z,w)} \cup \mathbb{C}^2_{(u,v)})/\sigma$ are glued together via

$$x = \frac{1}{w^2}, \qquad y = -\frac{2}{w^3} - \frac{tw}{2} - \frac{w^2}{2} + \frac{zw^3}{2}, \tag{6}$$

$$x = \frac{1}{v^2}, \qquad y = \frac{2}{v^3} + \frac{tv}{2} - \frac{v^2}{2} + \frac{uv^3}{2}, \tag{7}$$

along the subset $\{x \neq 0\} = \{w \neq 0\}/\sigma = \{v \neq 0\}/\sigma$.

We remark that formulas (6) and (7) were already known to Painlevé [**7**] in a different context, that is, through the Laurent expansion around a pole of a solution, where any pole must be of order two so that it is converted into a simple zero via the transformations $x = w^{-2} = v^{-2}$.

The total space E of the fibration $\pi : E \to T$ is made up of three (orbifold) charts $\mathbb{C}^3_{(x,y,t)}$, $\mathbb{C}^3_{(z,w,t)}$ and $\mathbb{C}^3_{(u,v,t)}$ patched together through the symplectic mappings (4), (6) and (7), where by *symplectic* we mean $\delta x \wedge \delta y = \delta z \wedge \delta w = \delta u \wedge \delta v$ with δ being the relative exterior differentiation on the fibration $\pi : E \to T$ so that t is thought of as a constant. In this situation one can speak of a time-dependent Hamiltonian structure on the fibration, which can be represented by a triple of Hamiltonians $H = H(x, y, t)$, $K = K(z, w, t)$ and $L = L(u, v, t)$ that should share a fundamental 2-form Ω in common, to the effect that

$$\Omega = dy \wedge dx - dH \wedge dt = dw \wedge dz - dK \wedge dt = dv \wedge du - dL \wedge dt,$$

where d is the exterior differentiation on the total space E so that t is regarded as a variable. Under transformation rules (4), (6) and (7), this last condition can be written

$$H = K + 1/w = L - 1/v, \qquad K = L - 2/v. \tag{8}$$

In order to speak of *an orbifold Hamiltonian structure* we should also take into account the σ-invariance of Ω. In view of formulas (5) the condition $\sigma^*\Omega = \Omega$ can be written

$$K \circ \sigma = K + 2/w, \qquad L \circ \sigma = L - 2/v. \tag{9}$$

The first Painlevé equation $\mathrm{P_I}$ admits an orbifold Hamiltonian structure. Indeed, its Hamiltonian triple $\{H_{\mathrm{I}}, K_{\mathrm{I}}, L_{\mathrm{I}}\}$ is given by formula (1) together with

$$\begin{aligned} K_{\mathrm{I}}(z, w, t) &= \frac{1}{8} w^6 z^2 - \frac{1}{4}(4 + tw^4 + w^5) z + \frac{1}{8} w^2 (t + w)^2, \\ L_{\mathrm{I}}(u, v, t) &= \frac{1}{8} v^6 u^2 + \frac{1}{4}(4 + tv^4 - v^5) u + \frac{1}{8} v^2 (t - v)^2. \end{aligned} \tag{10}$$

Note that H_{I}, K_{I} and L_{I} are polynomials of their respective variables. Suppose that

H, K, L are entire holomorphic in their respective variables
and meromorphic on $\overline{E}$, (11)

where $\pi : \overline{E} \to T$ is the fibration with compactified fibers $\overline{E}_t$ $(t \in T)$. The following theorem asserts that such an orbifold Hamiltonian structure is unique and just coming from $\mathrm{P_I}$.

Theorem 1.2. *If a function triple* $\{H, K, L\}$ *satisfies conditions* (8), (9) *and* (11), *then* $H = H_{\mathrm{I}}$, $K = K_{\mathrm{I}}$ *and* $L = L_{\mathrm{I}}$ *modulo functions of* $t \in T$.

Here we remark that a Hamiltonian makes sense only up to addition of a function of $t \in T$. Theorem 1.2 is an easy consequence of the following function-theoretic property of E_t and $\overline{E}_t$.

Theorem 1.3. *Any function holomorphic on* E_t *and meromorphic on* $\overline{E}_t$ *must be constant.*

Indeed, take the differences $\mathcal{H} = H - H_{\mathrm{I}}$, $\mathcal{K} = K - K_{\mathrm{I}}$ and $\mathcal{L} = L - L_{\mathrm{I}}$. Since both $\{H, K, L\}$ and $\{H_{\mathrm{I}}, K_{\mathrm{I}}, L_{\mathrm{I}}\}$ satisfy conditions (8), (9) and (11), one has $\mathcal{H} = \mathcal{K} = \mathcal{L}$,

$\mathcal{K} \circ \sigma = \mathcal{K}$ and $\mathcal{L} \circ \sigma = \mathcal{L}$ so that $\mathcal{H} = \mathcal{K} = \mathcal{L}$ defines a function h holomorphic on E and meromorphic on $\overline{E}$. Theorem 1.3 then implies that h is only a function of $t \in T$. This proves Theorem 1.2. Theorem 1.1 and Theorem 1.3 will be proved in Section 2 and Section 3, respectively.

2. Construction of Okamoto's Space.

Our construction of E_t and thus a proof of Theorem 1.1 consist of the following twelve steps.

1. The Hirzebruch surface Σ of degree 2 is made up of four local charts $\mathbb{C}^2_{(q_i,p_i)}$, $i = 1, 2, 3, 4$, glued together according to the relations:

$$q_1 q_2 = 1, \quad p_1 = -q_2^2 p_2; \qquad q_3 = q_1, \quad p_1 p_3 = 1; \qquad q_4 = q_2, \quad p_2 p_4 = 1, \tag{12}$$

where $(q_3, p_3) = (x, y)$ is the original chart for system (1). Consider the Pfaffian system on $\mathbb{C}^2_{(x,y)} \times T$ defined by formula (1) and extend it to the entire space $\Sigma \times T$. For each $t \in T$ the associated foliation has two vertical leaves $\{p_1 = 0\} \cup \{p_2 = 0\} \cong \mathbb{P}^1$ and $\{q_2 = 0\} \cup \{q_4 = 0\} \cong \mathbb{P}^1$, together with an accessible singular point $a_t^{(0)} = \{(q_4, p_4) = (0, 0)\}$ (see Figure 3). In what follows by a singularity we always mean an accessible singularity.

2. Blowup at $a_t^{(0)}$ produces two new charts $(q^{(1)}, p^{(1)})$ and $(Q^{(1)}, P^{(1)})$ such that

$$q_4 = q^{(1)} p^{(1)}, \quad p_4 = p^{(1)}; \qquad q_4 = Q^{(1)}, \quad p_4 = Q^{(1)} P^{(1)}.$$

Rewrite the Pfaffian system in terms of the new charts. The ensuing foliation has two vertical leaves; the exceptional curve $\{p^{(1)} = 0\} \cup \{Q^{(1)} = 0\}$ of the blowup and the proper

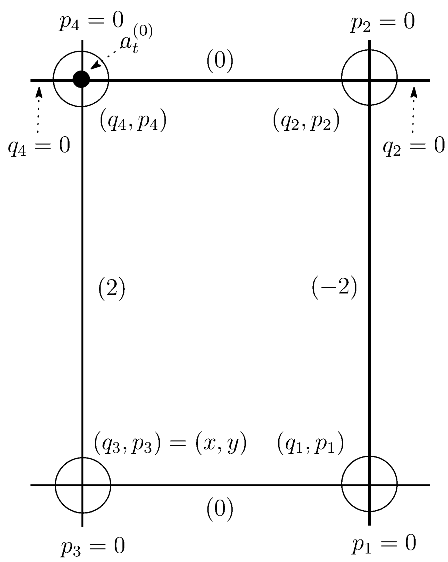

Figure 3. Start with Σ.

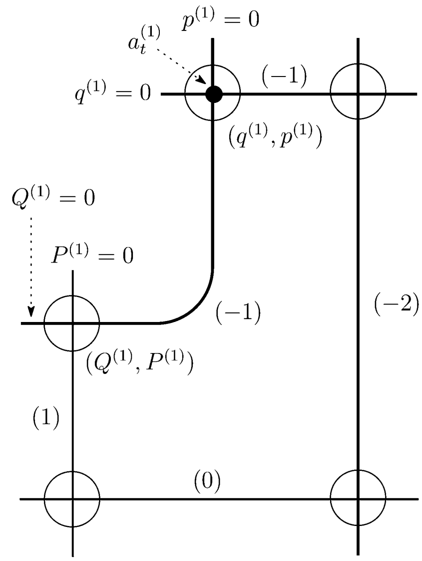

Figure 4. Blowup at $a_t^{(0)}$.

image $\{q^{(1)}=0\}$ of $\{q_4=0\}$, together with a singular point $a_t^{(1)}=\{(q^{(1)},p^{(1)})=(0,0)\}$ (see Figure 4).

3. Blowup at $a_t^{(1)}$ produces two new charts $(q^{(2)},p^{(2)})$ and $(Q^{(2)},P^{(2)})$ such that

$$q^{(1)}=q^{(2)}p^{(2)},\quad p^{(1)}=p^{(2)};\qquad q^{(1)}=Q^{(2)},\quad p^{(1)}=Q^{(2)}P^{(2)}.$$

In terms of the new charts there are three vertical leaves; the exceptional curve $\{p^{(2)}=0\}\cup\{Q^{(2)}=0\}$ of the blowup, the proper images $\{q^{(2)}=0\}$ of $\{q^{(1)}=0\}$ and $\{P^{(2)}=0\}$ of $\{p^{(1)}=0\}$, together with a singular point $a_t^{(2)}=\{(Q^{(2)},P^{(2)})=(0,4)\}$ (see Figure 5).

4. Consider the (-2)-curve $C=\{Q^{(1)}=0\}\cup\{P^{(2)}=0\}$ and its tubular neighborhood $U=\mathbb{C}^2_{(Q^{(1)},P^{(1)})}\cup\mathbb{C}^2_{(Q^{(2)},P^{(2)})}$. Let $(U,C)\leftarrow(V,D)$ with $V=\mathbb{C}^2_{(r,s)}\cup\mathbb{C}^2_{(R,S)}$ be the branched double covering ramifying along $D=\{s=0\}\cup\{R=0\}$, which is defined by

$$Q^{(1)}=s^2,\quad P^{(1)}=r;\qquad Q^{(2)}=S,\quad P^{(2)}=R^2.$$

The deck involution $\sigma: V \circlearrowleft$ maps $(r,s)\mapsto(r,-s)$ on $\mathbb{C}^2_{(r,s)}$ and $(R,S)\mapsto(-R,S)$ on $\mathbb{C}^2_{(R,S)}$, respectively. Three vertical leaves mentioned in step 3 become $\{S=0\}$, $\{q^{(2)}=0\}$ and $\{R=0\}$, respectively, while the singular point $a_t^{(2)}\in U$ lifts up to a pair of σ-equivalent points in V:

$$\tilde{a}_t^{(2)}=\{(R,S)=(2,0)\}\oplus\tilde{b}_t^{(2)}=\{(R,S)=(-2,0)\},\tag{13}$$

where $(*)\oplus(**)$ indicates that $(*)$ and $(**)$ are permuted by the involution σ (see Figure 6). In what follows $(*)\oplus(**)$ will be thought of as a single (that is, not a dual) object.

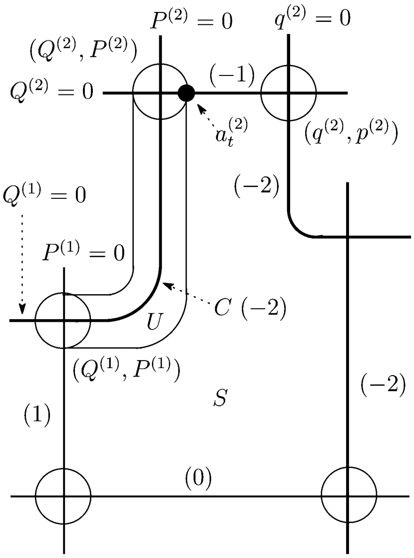

Figure 5. Blowup at $a_t^{(1)}$.

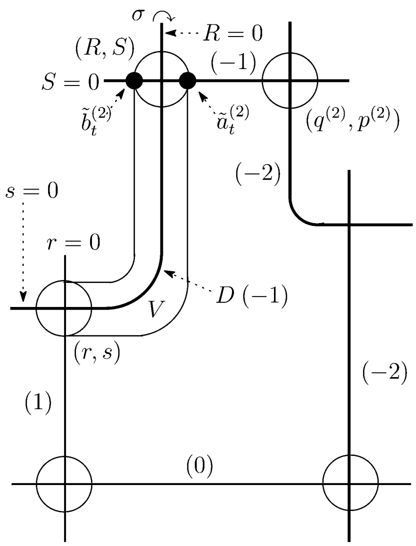

Figure 6. Double cover.

5. Since the branching locus C downstairs is a (-2)-curve, the ramifying locus D upstairs is a (-1)-curve that can be blown down into a smooth point p. Blowdown of D into p induces a morphism $(V, D) \to (W, p)$ with $W = \mathbb{C}^2_{(r^{(2)}, s^{(2)})}$ and $p = \{(r^{(2)}, s^{(2)}) = (0,0)\}$ such that

$$r^{(2)} = rs, \quad s^{(2)} = s; \qquad r^{(2)} = R, \quad s^{(2)} = RS.$$

The induced involution $\sigma : (W, p) \circlearrowleft$ maps $(r^{(2)}, s^{(2)}) \mapsto (-r^{(2)}, -s^{(2)})$. Through the blowdown morphism the vertical leaf $\{S = 0\}$ descends to $\{s^{(2)} = 0\}$, while the singular point (13) to

$$p_+ = \{(r^{(2)}, s^{(2)}) = (2, 0)\} \oplus p_- = \{(r^{(2)}, s^{(2)}) = (-2, 0)\}. \qquad \text{(see Figure 7)}$$

6. Blowup at $p_+ \oplus p_-$ produces new charts $(z^{(3)}, w^{(3)}) \oplus (u^{(3)}, v^{(3)})$ and $(Z^{(3)}, W^{(3)}) \oplus (U^{(3)}, V^{(3)})$ such that

$$\begin{aligned} r^{(2)} &= 2 + z^{(3)} w^{(3)}, \quad s^{(2)} = w^{(3)}; \qquad & r^{(2)} &= 2 + Z^{(3)}, \quad s^{(2)} = Z^{(3)} W^{(3)}, \\ r^{(2)} &= -2 + u^{(3)} v^{(3)}, \quad s^{(2)} = v^{(3)}; \qquad & r^{(2)} &= -2 + U^{(3)}, \quad s^{(2)} = U^{(3)} V^{(3)}, \end{aligned}$$

where the induced involution σ maps $(z^{(3)}, w^{(3)}) \mapsto (z^{(3)} + 4/w^{(3)}, \; -w^{(3)})$ on $\mathbb{C}^2_{(z^{(3)}, w^{(3)})}$ and $(u^{(3)}, v^{(3)}) \mapsto (u^{(3)} - 4/v^{(3)}, \; -v^{(3)})$ on $\mathbb{C}^2_{(u^{(3)}, v^{(3)})}$, respectively. In terms of the new charts there are two vertical leaves; the exceptional curve $\{w^{(3)} = 0\} \cup \{Z^{(3)} = 0\} \oplus \{v^{(3)} = 0\} \cup \{U^{(3)} = 0\}$ and the proper image $\{W^{(3)} = 0\} = \{V^{(3)} = 0\}$ of $\{s^{(2)} = 0\}$, together with a singular point $a_t^{(3)} = \{(z^{(3)}, w^{(3)}) = (0, 0)\} \oplus b_t^{(3)} = \{(u^{(3)}, v^{(3)}) = (0, 0)\}$ (see Figure 8).

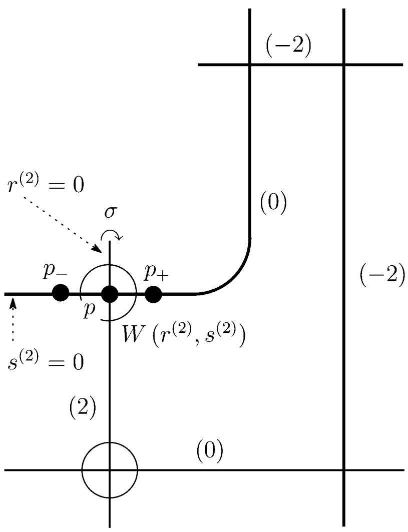

Figure 7. Blowdown of D.

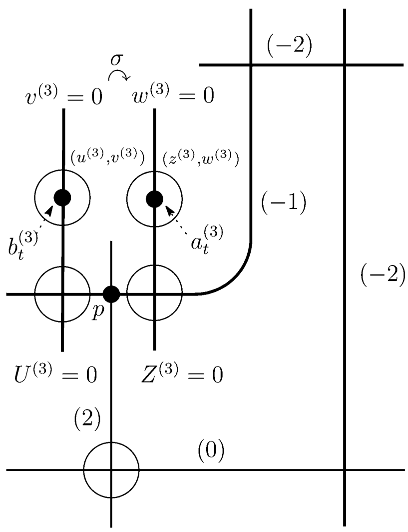

Figure 8. Blowup at $p_+ \oplus p_-$.

7. Blowup at $a_t^{(3)} \oplus b_t^{(3)}$ produces new charts $(z^{(4)}, w^{(4)}) \oplus (u^{(4)}, v^{(4)})$ and $(Z^{(4)}, W^{(4)}) \oplus (U^{(4)}, V^{(4)})$ such that

$$\begin{aligned} z^{(3)} &= z^{(4)}w^{(4)}, \quad w^{(3)} = w^{(4)}, \qquad z^{(3)} = Z^{(4)}, \quad w^{(3)} = Z^{(4)}W^{(4)}, \\ u^{(3)} &= u^{(4)}v^{(4)}, \quad v^{(3)} = v^{(4)}, \qquad u^{(3)} = U^{(4)}, \quad v^{(3)} = U^{(4)}V^{(4)}, \end{aligned}$$

where the induced involution σ maps $(z^{(4)}, w^{(4)}) \mapsto (-z^{(4)} - 4(w^{(4)})^{-2}, -w^{(4)})$ on $\mathbb{C}^2_{(z^{(4)},w^{(4)})}$ and $(u^{(4)}, v^{(4)}) \mapsto (-u^{(4)} + 4(v^{(4)})^{-2}, -v^{(4)})$ on $\mathbb{C}^2_{(u^{(4)},v^{(4)})}$, respectively. In terms of the new charts there are two vertical leaves; the exceptional curve $\{w^{(4)} = 0\} \cup \{Z^{(4)} = 0\} \oplus \{v^{(4)} = 0\} \cup \{U^{(4)} = 0\}$ and the proper image $\{W^{(4)} = 0\} \oplus \{V^{(4)} = 0\}$ of $\{w^{(3)} = 0\} \oplus \{v^{(3)} = 0\}$, as well as a singular point $a_t^{(4)} = \{(z^{(4)}, w^{(4)}) = (0,0)\} \oplus b_t^{(4)} = \{(u^{(4)}, v^{(4)}) = (0,0)\}$.

8. Blowup at $a_t^{(4)} \oplus b_t^{(4)}$ produces new charts $(z^{(5)}, w^{(5)}) \oplus (u^{(5)}, v^{(5)})$ and $(Z^{(5)}, W^{(5)}) \oplus (U^{(5)}, V^{(5)})$ such that

$$\begin{aligned} z^{(4)} &= z^{(5)}w^{(5)}, \quad w^{(4)} = w^{(5)}, \qquad z^{(4)} = Z^{(5)}, \quad w^{(4)} = Z^{(5)}W^{(5)}, \\ u^{(4)} &= u^{(5)}v^{(5)}, \quad v^{(4)} = v^{(5)}, \qquad u^{(4)} = U^{(5)}, \quad v^{(4)} = U^{(5)}V^{(5)}, \end{aligned}$$

where the induced involution σ maps $(z^{(5)}, w^{(5)}) \mapsto (z^{(5)} + 4(w^{(5)})^{-3}, -w^{(5)})$ on $\mathbb{C}^2_{(z^{(5)},w^{(5)})}$ and $(u^{(5)}, v^{(5)}) \mapsto (u^{(5)} - 4(v^{(5)})^{-3}, -v^{(5)})$ on $\mathbb{C}^2_{(u^{(5)},v^{(5)})}$, respectively. In terms of the new charts there are two vertical leaves; the exceptional curve $\{w^{(5)} = 0\} \cup \{Z^{(5)} = 0\} \oplus \{v^{(5)} = 0\} \cup \{U^{(5)} = 0\}$ and the proper image $\{W^{(5)} = 0\} \oplus \{V^{(5)} = 0\}$ of $\{w^{(4)} = 0\} \oplus \{v^{(4)} = 0\}$, as well as a singular point $a_t^{(5)} = \{(z^{(5)}, w^{(5)}) = (0,0)\} \oplus b_t^{(5)} = \{(u^{(5)}, v^{(5)}) = (0,0)\}$.

9. Blowup at $a_t^{(5)} \oplus b_t^{(5)}$ produces new charts $(z^{(6)}, w^{(6)}) \oplus (u^{(6)}, v^{(6)})$ and $(Z^{(6)}, W^{(6)}) \oplus (U^{(6)}, V^{(6)})$ such that

$$\begin{aligned} z^{(5)} &= z^{(6)}w^{(6)}, \quad w^{(5)} = w^{(6)}, \qquad z^{(5)} = Z^{(6)}, \quad w^{(5)} = Z^{(6)}W^{(6)}, \\ u^{(5)} &= u^{(6)}v^{(6)}, \quad v^{(5)} = v^{(6)}, \qquad u^{(5)} = U^{(6)}, \quad v^{(5)} = U^{(6)}V^{(6)}, \end{aligned}$$

where the induced involution σ maps $(z^{(6)}, w^{(6)}) \mapsto (-z^{(6)} - 4(w^{(6)})^{-4}, -w^{(6)})$ on $\mathbb{C}^2_{(z^{(6)},w^{(6)})}$ and $(u^{(6)}, v^{(6)}) \mapsto (-u^{(6)} + 4(v^{(6)})^{-4}, -v^{(6)})$ on $\mathbb{C}^2_{(u^{(6)},v^{(6)})}$, respectively. In terms of the new charts there are two vertical leaves; the exceptional curve $\{w^{(6)} = 0\} \cup \{Z^{(6)} = 0\} \oplus \{v^{(6)} = 0\} \cup \{U^{(6)} = 0\}$ and the proper image $\{W^{(6)} = 0\} \oplus \{V^{(6)} = 0\}$ of $\{w^{(5)} = 0\} \oplus \{v^{(5)} = 0\}$, as well as a singular point $a_t^{(6)} = \{(z^{(6)}, w^{(6)}) = (t/2, 0)\} \oplus b_t^{(6)} = \{(u^{(6)}, v^{(6)}) = (-t/2, 0)\}$.

10. Blowup at $a_t^{(6)} \oplus b_t^{(6)}$ produces new charts $(z^{(7)}, w^{(7)}) \oplus (u^{(7)}, v^{(7)})$ and $(Z^{(7)}, W^{(7)}) \oplus (U^{(7)}, V^{(7)})$ such that

$$\begin{aligned} z^{(6)} &= t/2 + z^{(7)}w^{(7)}, \quad w^{(6)} = w^{(7)}, \qquad z^{(6)} = t/2 + Z^{(7)}, \quad w^{(6)} = Z^{(7)}W^{(7)}, \\ u^{(6)} &= -t/2 + u^{(7)}v^{(7)}, \quad v^{(6)} = v^{(7)}, \qquad u^{(6)} = -t/2 + U^{(7)}, \quad v^{(6)} = U^{(7)}V^{(7)}, \end{aligned}$$

where the induced involution σ maps $(z^{(7)}, w^{(7)}) \mapsto (z^{(7)} + t/w^{(7)} + 4(w^{(7)})^{-5},\ -w^{(7)})$ on $\mathbb{C}^2_{(z^{(7)},w^{(7)})}$ and $(u^{(7)}, v^{(7)}) \mapsto (u^{(7)} - t/v^{(7)} - 4(v^{(7)})^{-5},\ -v^{(7)})$ on $\mathbb{C}^2_{(u^{(7)},v^{(7)})}$, respectively. In terms of the new charts there are two vertical leaves; the exceptional curve $\{w^{(7)} = 0\} \cup \{Z^{(7)} = 0\} \oplus \{v^{(7)} = 0\} \cup \{U^{(7)} = 0\}$ and the proper image $\{W^{(7)} = 0\} \oplus \{V^{(7)} = 0\}$ of $\{w^{(6)} = 0\} \oplus \{v^{(6)} = 0\}$, as well as a singular point $a_t^{(7)} = \{(z^{(7)}, w^{(7)}) = (1/2, 0)\} \oplus b_t^{(7)} = \{(u^{(7)}, v^{(7)}) = (1/2, 0)\}$.

11. Blowup at $a_t^{(7)} \oplus b_t^{(7)}$ produces new charts $(z^{(8)}, w^{(8)}) \oplus (u^{(8)}, v^{(8)})$ and $(Z^{(8)}, W^{(8)}) \oplus (U^{(8)}, V^{(8)})$ such that

$$z^{(7)} = 1/2 + z^{(8)}w^{(8)}, \quad w^{(7)} = w^{(8)}, \quad z^{(7)} = 1/2 + Z^{(8)}, \quad w^{(7)} = Z^{(8)}W^{(8)},$$
$$u^{(7)} = 1/2 + u^{(8)}v^{(8)}, \quad v^{(7)} = v^{(8)}, \quad u^{(7)} = 1/2 + U^{(8)}, \quad v^{(7)} = U^{(8)}V^{(8)},$$

where the induced involution σ maps $(z^{(8)}, w^{(8)}) \mapsto (-z^{(8)} - t(w^{(8)})^{-2} - 4(w^{(8)})^{-6},\ -w^{(8)})$ on $\mathbb{C}^2_{(z^{(8)},w^{8)})}$ and $(u^{(8)}, v^{(8)}) \mapsto (-u^{(8)} + t(v^{(8)})^{-2} + 4(v^{(8)})^{-6},\ -v^{(8)})$ on $\mathbb{C}^2_{(u^{(8)},v^{(8)})}$, respectively. In terms of the new charts there is only one vertical leaf; the proper image $\{W^{(8)} = 0\} \oplus \{V^{(8)} = 0\}$ of $\{w^{(7)} = 0\} \oplus \{v^{(7)} = 0\}$. Observe that the exceptional curve $\{Z^{(8)} = 0\} \cup \{w^{(8)} = 0\} \oplus \{U^{(8)} = 0\} \cup \{v^{(8)} = 0\}$ is *not* a vertical leaf and there is *no* singular point of the foliation.

12. Composition of steps 6–11 leads to a proper modification $(W, p, p_\pm) \leftarrow (X, p, E_\pm)$. The rest is just as mentioned in Section 1. Make a gluing $F = (S \setminus C) \cup (X/\sigma)$ to have a compact surface F with an A_1-singularity $p \in F$; take its minimal resolution to get a smooth compact space $\overline{E}_t$; and finally remove the vertical leaves V_t to obtain E_t. All these procedures are symbolically represented by Figure 9. In order to make the final result exactly symplectic, we define the final charts (z, w) and (u, v) by $z^{(8)} = -z/2$,

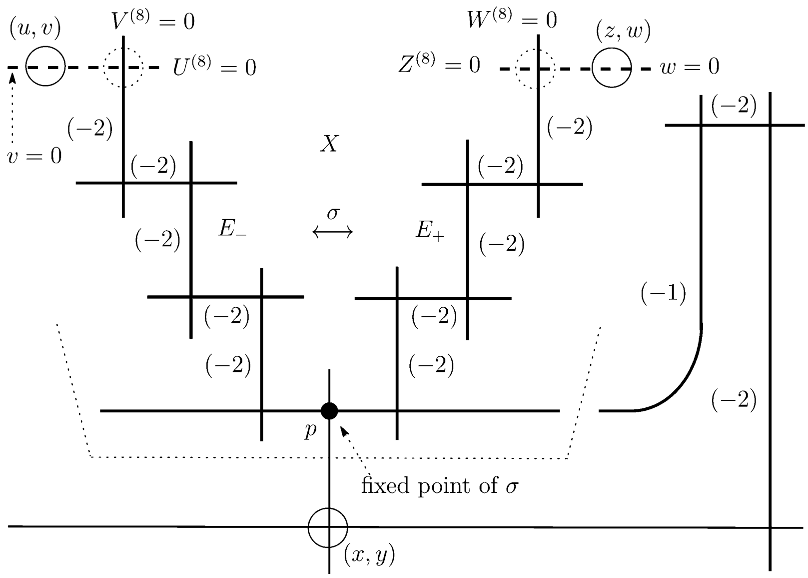

Figure 9. A "deer" whose dual "horns" $E_\pm$ are identified by σ.

$w^{(8)} = w$; $u^{(8)} = -u/2$, $v^{(8)} = v$. Among all charts of $\overline{E}_t$ we have constructed, those which are disjoint with V_t are exactly $\mathbb{C}^2_{(x,y)}$, $\mathbb{C}^2_{(z,w)}$ and $\mathbb{C}^2_{(u,v)}$. These three make an orbifold symplectic atlas of E_t. A careful check of steps 1–11 yields the desired relations (4), (6), (7) as well as formula (5) for the involution σ.

It might be fun to think of Figure 9 as a "deer" whose dual "horns" $E_{\pm}$ are identified by σ, and whose "nose" is just the fixed point $p \in X$ or the A_1-singularity $p \in X/\sigma$ arising from it.

3. Holomorphic functions.

We prove Theorem 1.3. Fixing $t \in T$ we do not refer to the dependence upon t. Any function holomorphic on E_t and meromorphic on $\overline{E}_t$ is represented by a triple $\{H, K, L\}$ of functions $H = H(x, y)$, $K = K(z, w)$ and $L = L(u, v)$ entire in their respective variables such that $H = K = L$ under transformations (4), (6) and (7), as well as $K \circ \sigma = K$ and $L \circ \sigma = L$.

Lemma 3.1. *We have $H \in \mathbb{C}[x, y]$, $K \in \mathbb{C}[z, w]$ and $L \in \mathbb{C}[u, v]$.*

Proof. As an entire holomorphic function of (x, y), H admits a Taylor expansion

$$H = \sum_{i,j=0}^{\infty} c_{ij}\, x^i y^j, \qquad c_{ij} \in \mathbb{C}.$$

Using relation (12) we can rewrite it in terms of (q_2, p_2) to have

$$H = \sum_{i,j=0}^{\infty} (-1)^j c_{ij}\, q_2^{-(i+2j)} p_2^{-j}.$$

In order for this to be meromorphic on the vertical leaf $\{q_2 = 0\}$, there must be a nonnegative integer N such that $c_{ij} = 0$ for every $i + 2j > N$, which forces $H \in \mathbb{C}[x, y]$. Next we show that $K \in \mathbb{C}[z, w]$. Under transformation (6) we have $K = H \in \mathbb{C}[x, y]$ so that $K \in \mathbb{C}[z, w, w^{-1}]$. On the other hand, $K \in \mathbb{C}\{z, w\}$ (the convergent power series ring), since it is an entire function of (z, w). Thus we have $K \in \mathbb{C}[z, w, w^{-1}] \cap \mathbb{C}\{z, w\} = \mathbb{C}[z, w]$. Similarly, $L \in \mathbb{C}[u, v]$. $\square$

Let us discuss the problem of finding a function K such that

$$K \circ \sigma = K, \qquad K = K(z, w) \in \mathbb{C}[z, w], \tag{14}$$

where $\sigma : \mathbb{C}^2_{(z,w)} \circlearrowleft$ is the involution defined by formula (5). We begin by a simple reduction. Consider the decomposition of K into even and odd components with respect to w:

$$K = K^{+} + K^{-}, \qquad K^{\pm} := (K \pm \check{K})/2, \qquad \check{K}(z, w) := K(z, -w). \tag{15}$$

Lemma 3.2. *If K is a solution to problem* (14), *then so are $K^{\pm}$.*

Proof. It suffices to show that if K is a solution to problem (14) then so is $\check{K}$. First, it is obvious that $K \in \mathbb{C}[z,w]$ implies $\check{K} \in \mathbb{C}[z,w]$. Next, observe that

$$\begin{aligned}(\check{K} \circ \sigma)(z,w) &= \check{K}(8w^{-6} + 2tw^{-2} - z,\, -w) = K(8w^{-6} + 2tw^{-2} - z,\, w) \\ &= K(8(-w)^{-6} + 2t(-w)^{-2} - z,\, -(-w)) = (K \circ \sigma)(z,-w) \\ &= K(z,-w) = \check{K}(z,w),\end{aligned}$$

where $K \circ \sigma = K$ is used in the fifth equality. Thus $\check{K}$ is also a solution to problem (14). $\square$

Let K be a solution to problem (14) and put $\xi = w^2$. The even component of K can be written $K^{+}(z,w) = F(z,w^2)$ with $F = F(z,\xi)$ being a solution to the problem

$$F \circ \tau = F, \qquad F = F(z,\xi) \in \mathbb{C}[z,\xi], \tag{16}$$

where $\tau : \mathbb{C}^2_{(z,\xi)} \circlearrowleft$ is an involution $(z,\xi) \mapsto (8\xi^{-3} + 2t\xi^{-1} - z,\, \xi)$. There is a particular solution $E(z,\xi) := z(\xi^3 z - 2t\xi^2 - 8)$ to problem (16), which plays an important role in the following.

Lemma 3.3. *Any nontrivial solution to problem* (16) *must be of the form*

$$F(z,\xi) = \sum_{m=0}^{M} f_m(\xi) E^m(z,\xi), \qquad f_m(\xi) \in \mathbb{C}[\xi], \tag{17}$$

where $M \geq 0$ and $f_M(\xi)$ is a nonzero polynomial of ξ.

Proof. We prove the lemma by induction on $\deg_z F(z,\xi)$. If $\deg_z F(z,\xi) = 0$ then formula (17) obviously holds with $M = 0$. Suppose that $\deg_z F(z,\xi) \geq 1$ and put $f_0(\xi) := F(0,\xi) \in \mathbb{C}[\xi]$. Notice that $F_0(z,\xi) := F(z,\xi) - f_0(\xi) \in \mathbb{C}[z,\xi]$ is also a solution to problem (16). It is divisible by z, that is, $F_0(z,\xi) = zF_1(z,\xi)$ for some $F_1(z,\xi) \in \mathbb{C}[z,\xi]$. The τ-invariance of $F_0(z,\xi)$ implies $zF_1(z,\xi) = (8\xi^{-3} + 2t\xi^{-1} - z)F_2(z,\xi)$, where $F_2(z,\xi) := F_1(8\xi^{-3} + 2t\xi^{-1} - z, \xi) \in \mathbb{C}(\xi)[z]$. Writing $F_2(z,\xi) = -F_3(z,\xi)/a(\xi)$ with $F_3(z,\xi) \in \mathbb{C}[z,\xi]$ and $a(\xi) \in \mathbb{C}[\xi]$, we obtain $a(\xi)\xi^3 zF_1(z,\xi) = (\xi^3 z - 8 - 2t\xi^2)F_3(z,\xi)$ in $\mathbb{C}[z,\xi]$. Since the right-hand side is divisible by $\xi^3 z - 8 - 2t\xi^2$, so must be the left-hand side, but $a(\xi)\xi^3 z$ and $\xi^3 z - 8 - 2t\xi^2$ have no common factor, so that $F_1(z,\xi)$ must be divisible by $\xi^3 z - 8 - 2t\xi^2$, that is, $F_1(z,\xi) = (\xi^3 z - 8 - 2t\xi^2)F_4(z,\xi)$ for some $F_4(z,\xi) \in \mathbb{C}[z,\xi]$. Thus we have $F_0(z,\xi) = E(z,\xi)F_4(z,\xi)$. Since $F_0(z,\xi)$ and $E(z,\xi)$ are τ-invariant, $F_4(z,\xi)$ is also τ-invariant and hence yields a solution to problem (16) with $\deg_z F_4(z,\xi) = \deg_z F_0(z,\xi) - 2 = \deg_z F(z,\xi) - 2$. By induction hypothesis we can write $F_4(z,\xi) = \sum_{m=1}^{M} f_m(\xi)E^{m-1}(z,\xi)$ for some $f_m(\xi) \in \mathbb{C}[\xi]$. Substituting this into $F(z,\xi) = f_0(\xi) + E(z,\xi)F_4(z,\xi)$ yields formula (17). The induction is complete. $\square$

On the other hand, the odd component of K can be written $K^-(z,w) = wG(z,w^2)$ with $G = G(z,\xi)$ being a solution to the problem

$$G \circ \tau = -G, \qquad G = G(z,\xi) \in \mathbb{C}[z,\xi]. \tag{18}$$

Notice that $\mathit{\Delta}(z,\xi) := \xi^3 z - t\xi^2 - 4$ is a particular solution to problem (18).

Lemma 3.4. *Any nontrivial solution to problem* (18) *must be of the form*

$$G(z,\xi) = \mathit{\Delta}(z,\xi)\sum_{n=0}^{N} g_n(\xi)E^n(z,\xi), \qquad g_n(\xi) \in \mathbb{C}[\xi], \tag{19}$$

where $N \geq 0$ and $g_N(\xi)$ is a nonzero polynomial of ξ.

Proof. Substituting $z = t\xi^{-1} + 4\xi^{-3}$ into the skew τ-invariance $G(8\xi^{-3} + 2t\xi^{-1} - z,\xi) = -G(z,\xi)$ yields $G(t\xi^{-1} + 4\xi^{-3},\xi) = -G(t\xi^{-1} + 4\xi^{-3},\xi)$, which forces $G(t\xi^{-1} + 4\xi^{-3},\xi) = 0$. Thus $G(z,\xi)$ is divisible by $z - t\xi^{-1} - 4\xi^{-3}$ in $\mathbb{C}(\xi)[z]$, that is, $G(z,\xi) = (z - t\xi^{-1} - 4\xi^{-3})G_1(z,\xi)$ for some $G_1(z,\xi) \in \mathbb{C}(\xi)[z]$. Writing $G_1(z,\xi) = G_2(z,\xi)/b(\xi)$ with $G_2(z,\xi) \in \mathbb{C}[z,\xi]$ and $b(\xi) \in \mathbb{C}[\xi]$, we obtain $b(\xi)\xi^3 G(z,\xi) = \mathit{\Delta}(z,\xi)G_2(z,\xi)$ in $\mathbb{C}[z,\xi]$. Since the right-hand side is divisible by $\mathit{\Delta}(z,\xi)$, so must be the left-hand side, but $b(\xi)\xi^3$ and $\mathit{\Delta}(z,\xi)$ have no common factor, so that $G(z,\xi)$ must be divisible by $\mathit{\Delta}(z,\xi)$, that is, $G(z,\xi) = \mathit{\Delta}(z,\xi)G_3(z,\xi)$ for some $G_3(z,\xi) \in \mathbb{C}[z,\xi]$. Since $G(z,\xi)$ and $\mathit{\Delta}(z,\xi)$ are skew τ-invariant, $G_3(z,\xi)$ is τ-invariant and so yields a solution to problem (16). Lemma 3.3 then allows us to write $G_3(z,\xi) = \sum_{n=0}^{N} g_n(\xi)E^n(z,\xi)$ for some $g_n(\xi) \in \mathbb{C}[\xi]$, which leads to representation (19). $\square$

Now a general solution to problem (14) can be written $K(z,w) = K^+(z,w) + K^-(z,w)$ with $K^+(z,w) = F(z,w^2)$, $K^-(z,w) = wG(z,w^2)$, where $F(z,\xi)$ and $G(z,\xi)$ are as in formulas (17) and (19) respectively. Recall that we have $H(x,y) = K(z,w)$ under transformation (6). Let $H(x,y) = H^+(x,y) + H^-(x,y)$ be the decomposition parallel to the one $K(z,w) = K^+(z,w) + K^-(z,w)$. Notice that $H^{\pm}(x,y) \in \mathbb{C}[x,x^{-1},y]$. Observe that

$$E(z,w^2) = 4y^2 + 4x^{-1}y + x^{-2} - x^{-1}(4x^2+t)^2, \qquad w\mathit{\Delta}(z,w^2) = 2x^{-2}y + x^{-3}, \tag{20}$$

under relation (6). Indeed, the second formula readily follows from (6), while the first formula is derived from the second one and the relation $E(t,w^2) = w^{-8}\{w\mathit{\Delta}(z,w^2)\}^2 - w^2\left(t + 4w^{-4}\right)^2$. Formulas (20) are substituted into formulas (17) and (19) to find

$$\begin{aligned} H^+(x,y) &= \sum_{m=0}^{M} f_m(x^{-1})\bigl\{4y^2 + 4x^{-1}y + x^{-2} - x^{-1}(4x^2+t)^2\bigr\}^m, \\ H^-(x,y) &= (2x^{-2}y + x^{-3})\sum_{n=0}^{N} g_n(x^{-1})\bigl\{4y^2 + 4x^{-1}y + x^{-2} - x^{-1}(4x^2+t)^2\bigr\}^n, \end{aligned} \tag{21}$$

with $f_m(\xi) \in \mathbb{C}[\xi]$ and $g_n(\xi) \in \mathbb{C}[\xi]$, where if $H^+(x,y)$ is nontrivial then $M \geq 0$ and $f_M(\xi)$ is a nonzero polynomial, while if $H^-(x,y)$ is nontrivial then $N \geq 0$ and $g_N(\xi)$ is a nonzero polynomial. By convention we put $M = -1$ resp. $N = -1$ if $H^+(x,y)$ resp. $H^-(x,y)$ is trivial. Suppose that $H(x,y)$ is nontrivial, so that at least one of $H^{\pm}(x,y)$ is nontrivial.

Lemma 3.5. *We have $M > N$ and $H^+(x,y)$ must be nontrivial.*

Proof. Expanding formulas (21) into powers of y yields

$$\begin{aligned} H^+(x,y) &= 2^{2M} f_M(x^{-1})\, y^{2M} + 2^{2M} M x^{-1} f_M(x^{-1})\, y^{2M-1} + \cdots, \\ H^-(x,y) &= 2^{2N+1} x^{-2} g_N(x^{-1})\, y^{2N+1} + \cdots, \end{aligned} \tag{22}$$

where $\cdots$ denotes lower-degree terms with respect to y. If $M \leq N$ then $2M < 2N+1$ in formulas (22) so that $H(x,y) = 2^{2N+1}x^{-2}g_N(x^{-1})y^{2N+1} + \cdots \in \mathbb{C}[x,y]$ and hence $x^{-2}g_N(x^{-1}) \in \mathbb{C}[x]$. This is possible only if $g_N(\xi)$ is the zero polynomial, in which case $H^-(x,y)$ is trivial with $N = -1$; then $M = -1$ and so $H^+(x,y)$ is also trivial. This contradiction shows that $M > N \geq -1$. Since M is nonnegative, $H^+(x,y)$ must be nontrivial. □

Lemma 3.6. *We have $M = 0$, $N = -1$ and $H(x,y) = c \in \mathbb{C}^{\times}$.*

Proof. By Lemma 3.5 we have $M > N$ and hence $2M > 2N+1$, so that formulas (22) yield $H(x,y) = 2^{2M} f_M(x^{-1})\, y^{2M} + \cdots \in \mathbb{C}[x,y]$, which implies $f_M(x^{-1}) \in \mathbb{C}[x]$. On the other hand we have $f_M(x^{-1}) \in \mathbb{C}[x^{-1}]$. Thus $f_M(\xi)$ must be a constant, say, $c \in \mathbb{C}$. Since $H^+(x,y)$ is nontrivial, $f_M(\xi) = c \in \mathbb{C}^{\times}$ must be a nonzero constant. To show that $M = 0$, suppose the contrary that $M \geq 1$. In the first case where $N = M-1$, formulas (22) imply

$$H(x,y) = 2^{2M} c\, y^{2M} + \{(M \cdot 2^{2M} \cdot c)x^{-1} + 2^{2M-1}x^{-2}g_{M-1}(x^{-1})\}y^{2M-1} + \cdots \in \mathbb{C}[x,y],$$

and so $(M \cdot 2^{2M} \cdot c)x^{-1} + 2^{2M-1}x^{-2}g_{M-1}(x^{-1}) \in \mathbb{C}[x]$, which is impossible. In the second case where $N < M-1$, formulas (22) imply $H(x,y) = 2^{2M} c\, y^{2M} + (M \cdot 2^{2M} \cdot c)x^{-1}y^{2M-1} + \cdots \in \mathbb{C}[x,y]$, and so $(M \cdot 2^{2M} \cdot c)x^{-1} \in \mathbb{C}[x]$, which is also impossible. Thus $M = 0$ and $H^+(x,y) = c$. Since $N < M = 0$, we have $N = -1$ and $H^-(x,y) = 0$ so that $H(x,y) = c$. □

With the proof of Lemma 3.6 above, Theorem 1.3 has also been established completely.

Formulas (21) can be used to construct a meromorphic function on E_t that is holomorphic on $E_t \setminus \{x = 0\}$ with poles only along $\{x = 0\}$. There is a connection of this formula with a proof of the Painlevé property. In a qualitative proof of it, which does not use isomonodromic deformations, it is crucial to deal with a kind of Lyapunov function that can control the trajectories near the vertical leaves. As a Lyapunov function for $\mathrm{P_I}$ we usually employ

$$U(x,y,t)=2H_{\mathrm{I}}(x,y,t)+\frac{y}{x}=y^2-4x^3-2tx+\frac{y}{x}$$

as in [**6**, formula (5)] or [**8**, formula (3.8)]. This function is just obtained by putting $M=1$, $f_1(\xi)=1/4$, $f_0(\xi)=\xi(t^2-\xi)/4$ and $N=-1$, i.e., $H^-(x,y,t)=0$ in formulas (21).

A quite different proof, but still of a qualitative nature, for the Painlevé property has been proposed by H. Chiba in his framework of Painlevé equations on weighted projective spaces [**1**].

References

[1] H. Chiba, The first, second and fourth Painlevé equations on weighted projective spaces, preprint, April 16, 2014.

[2] J. J. Duistermaat and N. Joshi, Okamoto's space for the first Painlevé equation in Boutroux coordinates, Arch. Rational Mech. Anal., **202** (2011), 707–785.

[3] T. Matano, A. Matumiya and K. Takano, On some Hamiltonian structures of Painlevé systems, Ⅱ, J. Math. Soc. Japan, **51** (1999), 843–866.

[4] A. Matumiya, On some Hamiltonian structures of Painlevé systems, Ⅲ, Kumamoto J. Math., **10** (1997), 45–73.

[5] K. Okamoto, Sur les feuilletages associés aux équations du second ordre à points critiques fixes de P. Painlevé, Espaces des conditions initiales, Japan. J. Math., **5** (1979), 1–79.

[6] K. Okamoto and K. Takano, The proof of the Painlevé property by Masuo Hukuhara, Funkcial. Ekvac., **44** (2001), 201–217.

[7] P. Painlevé, Mémoire sur les équations différentielles dont l'integrale generale est uniforme, Bull. Soc. Math. France, **28** (1900), 206–261.

[8] S. Shimomura, Proofs of the Painlevé property for all Painlevé equations, Japan. J. Math., **29** (2003), 159–180.

[9] T. Shioda and K. Takano, On some Hamiltonian structures of Painlevé systems, I, Funckcial. Ekvac., **40** (1997), 271–291.

Katsunori Iwasaki
Department of Mathematics
Hokkaido University
Kita 10, Nishi 8, Kita-ku
Sapporo 060-0810, Japan
E-mail: iwasaki@math.sci.hokudai.ac.jp

Shu Okada
Nishiwaki-Kita High School
669-32 Gonose-cho
Nishiwaki
Hyogo 677-0014, Japan