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Fractional order error estimates for the renewal density

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Dedicated to Professor Shinzo Watanabe on the occasion of his eightieth birthday

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Abstract. We study the rate of convergence for the renewal density with the interarrival times that are absolutely continuous, not necessarily positive, and has finite moment of αth order with $\alpha > 3/2$. We obtain an error estimate that is better than known results. Our method is based on modification of functions that have the same tails as the original ones and have integrable Fourier transform.

1. Introduction and the main result.

Let F be a probability measure on $\mathbb{R}$ such that $0 < \mu = \int_{-\infty}^{\infty} tF(dt) < \infty$ and F^{n*} be the n-fold convolution (as a measure) of F with itself. The measure U is defined via $U(I) = \sum_{n=1}^{\infty} F^{n*}(I)$ for an interval I and is called the renewal measure.

The well-known renewal theorem states that, intuitively, U is close to the Lebesgue measure multiplied by $1/\mu$ near $+\infty$; See Feller (1966) [**4**] for its developments. The aim of the present note is to study the rate of convergence in the absolutely continuous case. We assume F has the density $f(t)$ and denote that of F^{n*} by $f_n(t)$. In view of the monotone convergence theorem, $u(t) := \sum_{n=1}^{\infty} f_n(t)$ is a version of the density of U and is called the renewal density. It may be natural that one expects $u(t) \to 1/\mu$ as $t \to +\infty$.

We denote $\int_{-\infty}^{\infty} |t|^{\alpha} f(t)dt \in (0, \infty]$ by $M(\alpha)$ for $\alpha \in (1, \infty)$. To state the main result we introduce several functions: Let $q(t) = (1/\mu) \int_t^{\infty} f(s)ds$ and $r(t) = (1/\mu) \int_t^{\infty} q(s)ds$ for $t \geq 0$; let $q(t) = -(1/\mu) \int_{-\infty}^{t} f(s)ds$ and $r(t) = -(1/\mu) \int_{-\infty}^{t} q(s)ds$ for $t < 0$; and let $r_1(t) = r(t) - q * r(t)$ for $t \in \mathbb{R}$, where this convolution is well-defined as is seen in Lemma 2.5. We denote by $\varphi(\theta)$ the Fourier transform of $f(t)$: $\varphi(\theta) = \mathcal{F}f(\theta) = \int_{-\infty}^{\infty} e^{i\theta t} f(t)dt$.

We state our result for the case $3/2 < \alpha < 2$ and the case $\alpha \geq 2$ separately.

Theorem 1.1. *Assume $0 < \mu < \infty$, $M(\alpha) < \infty$ for some $\alpha \in (3/2, 2)$, and $\varphi(\theta) \in L^p$ for some $p \in [1, \infty)$. Let $N = \lceil p \rceil + 1$ where $\lceil p \rceil$ is the smallest integer that is greater than or equal to p. Then we have*

$$u(t) = \frac{1}{\mu} 1_{[0,\infty)}(t) + \sum_{n=1}^{N} f_n(t) + r(t) + r_1(t) + o(|t|^{-\alpha}) \quad \text{as } t \to \pm\infty. \tag{1.1}$$

Theorem 1.2. *Assume $0 < \mu < \infty$, $M(\alpha) < \infty$ for some $\alpha \in [2, \infty)$ and $\varphi(\theta) \in L^p$ for some $p \in [1, \infty)$. Let $N = \lceil p \rceil + \lfloor \alpha \rfloor$ where $\lfloor \alpha \rfloor$ is the greatest integer that is less than*

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or equal to α. *Then*

$$u(t) = \frac{1}{\mu} 1_{[0,\infty)}(t) + \sum_{n=1}^{N} f_n(t) + r(t) + o(|t|^{-\alpha}) \quad \text{as } t \to \pm\infty. \tag{1.2}$$

The rate of convergence of $u(t)$ have been studied by several authors ([**3**], [**13**], [**11**], [**14**], [**6**], [**9**]). Among them, Grubel [**6**] and Stone [**11**] are especially remarkable.

Grubel [**6**] is based on a method from the Banach algebra. If we adapt his Satz 1(i) to the case $\alpha(t)$, a function in [**6**], is given by $(|t| \vee 2)^{\alpha-1}$ we have the following.

Theorem A (Grubel). *Assume that the density* $f(t)$ *of* F *is an absolutely continuous function,* $0 < \mu < \infty$, $M(\alpha) < \infty$, *and* $\int_{-\infty}^{\infty} |f_k'(t)|(|t| \vee 2)^{\alpha-1} dt < \infty$ *for some* $\alpha \in (1, \infty)$ *and* $k \in \mathbb{N}$. *Then* $u(t) = (1/\mu) 1_{[0,\infty)}(t) + \sum_{n=1}^{k-1} f_n(t) + o(|t|^{-(\alpha-1)})$ *as* $t \to \pm\infty$.

Note that this theorem is applicable both to the case with and without finite second moment. Compared with Theorem A, Theorems 1.1 and 1.2 have additional terms, $r(t)$ and $r_1(t)$, and a better error estimate. In fact, with some effort we can derive the estimate in Theorem A from those of Theorems 1.1 and 1.2 since the condition in Theorem A implies $\varphi(\theta)^k = o(1/|\theta|)$ as $\theta \to \pm\infty$ and $f_k(t) = o(|t|^{-(\alpha-1)})$ as $t \to \pm\infty$.

Stone [**11**] studies the case F is not necessarily absolutely continuous using the Fourier analysis. To state his result we adapt the definition of $r(t)$ to this case: Let $r(t) = (1/\mu^2) \int_t^{\infty} ds \int_s^{\infty} F(dy)$ for $t \geq 0$ and let $r(t) = (1/\mu^2) \int_{-\infty}^{t} ds \int_{-\infty}^{s} F(dy)$ for $t < 0$.

Theorem B (Stone). *Assume* $0 < \mu < \infty$, $M(\alpha) < \infty$ *for some* $\alpha \in [2, \infty)$, *and* F^{n*} *has a non-trivial absolutely continuous component for some* $n \in \mathbb{N}$. *Then there exist an absolutely continuous measure* $\nu'(dt)$ *and a finite measure* $\nu''(dt)$ *such that* $U(dt) = \nu'(dt) + \nu''(dt)$, $\int_{-\infty}^{\infty} |t|^{\lfloor \alpha \rfloor} \nu''(dt) < \infty$, *and the density* $u'(t)$ *of* $\nu'(dt)$ *is continuous and satisfies* $u'(t) = (1/\mu) 1_{[0,\infty)}(t) + r(t) + o(|t|^{-\lfloor \alpha \rfloor})$ *as* $t \to \pm\infty$.

If α is not an integer, F is absolutely continuous, and $\varphi(\theta) \in L^p$, Theorem 1.2 (combined with Lemma 2.3 (ii) below) gives a refinement of Theorem B with $\nu''(dt) = \sum_{n=1}^{k-1} f_n(t) dt$.

In Section 2 we will give an explanation of the origin of $r(t)$ and $r_1(t)$ using (2.6). The importance of these terms is suggested in Stone–Wainger [**12**, Theorem 2] that studies the lattice case, i.e. the case F is supported on $\{dm | m \in \mathbb{Z}\}$ with some $d > 0$.

Carlsson [**1**] studies $U((-\infty, t])$, called the renewal function, in the non-lattice case with finite second moment and without absolute continuity. His result also suggests $r(t)$ and $r_1(t)$ in the sense that indefinite integrals of $r(t)$ and $r_1(t)$ appear in the estimates of $U((-\infty, t])$, which coincides with $\int_{-\infty}^{t} u(t) dt$ in the absolutely continuous case.

We prove Theorems 1.1 and 1.2 by the Fourier analysis. The Fourier transforms of $r(t)$ and $r_1(t)$ in our Theorems will be seen to be only locally integrable and we cannot invert the transform easily. This fact also blocks us to generalize the known estimates in the lattice case to the absolutely continuous case. We will modify the terms so that they have the same behavior as $t \to \pm\infty$ and that their Fourier transform decrease rapidly enough as $\theta \to \pm\infty$, where θ is the variable for the Fourier transform.

REMARK 1.1. Our method is also applicable to the case $M(\alpha) < \infty$ with $\alpha \in (1, 3/2]$. For instance, if $\alpha \in (4/3, 3/2]$, we define $r_2(t) = r_1(t) - q * r_1(t)$ and have

$$u(t) = \frac{1}{\mu} 1_{[0,\infty)}(t) + \sum_{n=1}^{N} f_n(t) + r(t) + r_1(t) + r_2(t) + o(|t|^{-\alpha})$$

as $t \to \pm\infty$ under some technical conditions. The origin of $r_2(t)$ can be explained in the same way as $r(t)$ and $r_1(t)$ using (2.6). In the case $\alpha \in (1, 4/3]$ we would need more terms such as $r_3(t) = r_2(t) - q * r_2(t)$ but we do not proceed in this direction.

REMARK 1.2. The estimates (1.1) and (1.2) hold for all sufficiently large N since we easily deduce that $f_n(t) = o(|t|^{-\alpha})$ as $|t| \to \infty$ if $n \ge \lceil p \rceil + \lfloor \alpha \rfloor + 1$ by the method appearing around (3.12) below.

This paper is organized as follows. In Section 2 we prove preliminary lemmas that are mainly devoted to studying properties of $r(t)$ and $r_1(t)$. In Section 3 we complete the proof of Theorem 1.1 and give a sketch of proof of Theorem 1.2.

2. Preliminary lemmas.

Recall that $M(\alpha) = \int_{-\infty}^{\infty} |t|^\alpha f(t) dt$, $\varphi(\theta) = \mathcal{F}f(\theta)$, and that $f_n(t)$ is the n-fold convolution of $f(t)$ with itself. We introduce some notations for functions that may vary from place to place. Let $a \in \mathbb{R}$. By $O_{\theta\to\pm\infty}(|\theta|^a)$ and $O_{\theta\to 0}(|\theta|^a)$ we denote a function $g(\theta)$ such that $\sup_{|\theta|\ge 1} g(\theta)/|\theta|^a < \infty$ and $\sup_{0<|\theta|\le 1} g(\theta)/|\theta|^a < \infty$ respectively. The notations $O_{\varepsilon\to+0}(\varepsilon^a)$, $O_{t\to\pm\infty}(|t|^a)$ and $O_{t\to 0}(|t|^a)$ are defined similarly. We denote by $o_{\varepsilon\to+0}(\varepsilon^a)$ a function $g(\varepsilon)$ such that $\lim_{\varepsilon\to+0} g(\varepsilon)/\varepsilon^a = 0$ and $g(\varepsilon) = O_{\varepsilon\to+0}(\varepsilon^a)$. The notations $o_{t\to\pm\infty}(|t|^a)$, $o_{t\to 0}(|t|^a)$, $o_{\theta\to\pm\infty}(|\theta|^a)$ and $o_{\theta\to 0}(|\theta|^a)$ are defined similarly.

Such notations for functions with two variables will appear in Lemma 3.1.

LEMMA 2.1. *If $\alpha \in (1, 2)$ and $M(\alpha) < \infty$ we have*

$$\sup_{\theta\in\mathbb{R}} |\varphi'(\theta)| \le M(1) < \infty, \tag{2.1}$$

$$\sup_{\theta_0\in\mathbb{R}} |\varphi(\theta_0 + \theta) - \varphi(\theta_0) - \varphi'(\theta_0)\theta| = o_{\theta\to 0}(|\theta|^\alpha), \tag{2.2}$$

$$\sup_{\theta_0\in\mathbb{R}} |\varphi'(\theta_0 + \theta) - \varphi'(\theta_0)| = o_{\theta\to 0}(|\theta|^{\alpha-1}). \tag{2.3}$$

PROOF. The inequality (2.1) follows from $\varphi'(\theta) = \int_{-\infty}^{\infty} e^{i\theta t} itf(t) dt$.

For any $x \in \mathbb{R}$, it is obvious that $\min(|x|, |x|^2) \le |x|^\alpha$, $|e^{ix} - 1| \le |x|$, $|e^{ix} - 1 - ix| \le |x|^2/2$ and hence $|e^{ix} - 1 - ix| \le \min(2|x|, |x|^2/2) \le 2|x|^\alpha$.

To prove (2.2) note that $|\theta|^{-\alpha}|(e^{i\theta t} - 1 - i\theta t)f(t)| \le 2|t|^\alpha f(t)$, where the left hand side converges to 0 as $\theta \to 0$ and the right hand side is integrable. Since $\varphi(\theta_0 + \theta) -$

$\varphi(\theta_0) - \varphi'(\theta_0)\theta = \int_{-\infty}^{\infty} e^{i\theta_0 t}(e^{i\theta t} - 1 - i\theta t)f(t)dt$ we have as $\theta \to 0$

$$
\begin{aligned}
&|\theta|^{-\alpha} \sup_{\theta_0 \in \mathbb{R}} |\varphi(\theta_0 + \theta) - \varphi(\theta_0) - \varphi'(\theta_0)\theta| \\
&\le \int_{-\infty}^{\infty} \sup_{\theta_0 \in \mathbb{R}} \left| \frac{e^{i\theta_0 t}(e^{i\theta t} - 1 - i\theta t)f(t)}{|\theta|^{\alpha}} \right| dt \\
&= \int_{-\infty}^{\infty} \left| \frac{(e^{i\theta t} - 1 - i\theta t)f(t)}{|\theta|^{\alpha}} \right| dt \to 0.
\end{aligned}
$$

A similar argument based on $|e^{ix} - 1| \le \min(2, |x|) \le 2|x|^{\alpha-1}$ and $|\theta|^{-(\alpha-1)}|(e^{i\theta t} - 1)it\, f(t)| \le 2|t|^{\alpha} f(t)$ leads us to (2.3). □

We introduce a family of finite measures $U_s = \sum_{n=1}^{\infty} s^n F^{n*}$ where $s \in [0, 1)$. If we set $u_s(t) = \sum_{n=1}^{\infty} s^n f_n(t)$, it is a version of the density of U_s and satisfies $\lim_{s\to 1-0} u_s(t) = u(t)$ for a.e. t by the monotone convergence theorem.

Lemma 2.2. *Assume $\varphi(\theta) \in L^p$ for some $p \in [1, \infty)$. Let N be any integer such that $N \ge p$.*

- (i) *For any $n \ge \lceil p \rceil$, $\mathcal{F} f_n(\theta) = \varphi(\theta)^n$ is integrable and its inverse Fourier transform $(1/2\pi)\int_{-\infty}^{\infty} e^{-it\theta}\varphi(\theta)^n d\theta$ is continuous in t and coincides a.e. with $f_n(t)$.*
- (ii) *Fix $s \in [0, 1)$. For $n \ge N$ we identify $f_n(t)$ with the continuous version in* (i)*. Then $\sum_{n=N}^{\infty} s^n f_n(t)$ is integrable and is the uniform limit of a sequence of continuous functions. Its Fourier transform $s^N\varphi(\theta)^N/(1 - s\varphi(\theta))$ is integrable in θ and it holds $\sum_{n=N}^{\infty} s^n f_n(t) = (1/2\pi)\int_{-\infty}^{\infty} e^{-it\theta} s^N \varphi(\theta)^N/(1 - s\varphi(\theta))d\theta$ for all $t \in \mathbb{R}$.*

Proof. (i) Since $N \ge p$ and $\|\varphi(\theta)\|_{\infty} = 1$, $\varphi(\theta)^n$ is integrable for any $n \ge N$. The other statements follow immediately.

(ii) It is trivial that $\|f_n(t)\|_1 = 1$ and hence $\sum_{n=N}^{\infty} s^n f_n(t)$ is integrable. For any $n \ge N$ it holds $\|\varphi(\theta)^n\|_1 \le \|\varphi(\theta)^N\|_1$ and $\|f_n(t)\|_{\infty} \le (1/2\pi)\|\varphi(\theta)^N\|_1$, which implies $\lim_{M\to\infty} \sum_{n=N}^{M} s^n f_n(t)$ converges uniformly in t. Its Fourier transform is obviously equal to $s^N\varphi(\theta)^N/(1 - s\varphi(\theta))$ and is integrable since $\|1/(1 - s\varphi(\theta))\|_{\infty} \le 1/(1 - s) < \infty$. The inverse Fourier transform coincides with $\sum_{n=N}^{\infty} s^n f_n(t)$ for at least a.e. t. Their continuity implies the equality for all t. □

Lemma 2.3. *Assume $\varphi(\theta) \in L^p$ for some $p \in [1, \infty)$ and $\mu = \int_{-\infty}^{\infty} tf(t)dt \in (0, \infty)$. Let $N = \lceil p \rceil + 1$. Then the following hold.*

- (i) *For all $t \in \mathbb{R}$, $\lim_{s\to 1-0} \int_{-\infty}^{\infty} e^{-it\theta}(s^{N+1}\varphi(\theta)^{N+1}/(1 - s\varphi(\theta)))d\theta = \pi/\mu + \int_{-\infty}^{\infty} \Re(e^{-it\theta}(\varphi(\theta)^{N+1}/(1 - \varphi(\theta))))d\theta$.*
- (ii) *The function $u(t) - \sum_{n=1}^{N} f_n(t)$ has a continuous version $1/2\mu + (1/2\pi)\int_{-\infty}^{\infty} \Re(e^{-it\theta}(\varphi(\theta)^{N+1}/(1 - \varphi(\theta))))d\theta$.*

Proof. (i) Since $\sum_{n=N+1}^{\infty} s^n f_n(t)$ is real we have by Lemma 2.2 (ii)

$$2\pi \sum_{n=N+1}^{\infty} s^n f_n(t) = \int_{-\infty}^{\infty} \Re\left(e^{-it\theta} \frac{s^{N+1}\varphi(\theta)^{N+1}}{1-s\varphi(\theta)}\right) d\theta$$

$$= \int_{-\infty}^{\infty} \left(\cos(t\theta) \Re \frac{s^{N+1}\varphi(\theta)^{N+1}}{1-s\varphi(\theta)} + \sin(t\theta) \Im \frac{s^{N+1}\varphi(\theta)^{N+1}}{1-s\varphi(\theta)} \right) d\theta.$$

We divide this into two integrals over $[-1,1]$ and $[-1,1]^c$ and will prove their convergence as $s \to 1-0$.

Since $\varphi(\theta) \neq 1$ for all $\theta \neq 0$, $|\varphi(\theta)| \leq 1$ for all $\theta \in \mathbb{R}$, and $\varphi(\theta) = 1 + i\mu\theta + o_{\theta\to 0}(|\theta|)$ with $\mu > 0$, it is elementary to deduce that, for some $K > 0$, $\sup_{s\in[0,1)} |1/(1-s\varphi(\theta))| < K/|\theta|$ for all θ with $0 < |\theta| \leq 1$. Hence we have

$$\left| \sin(t\theta) \Im \frac{s^{N+1}\varphi(\theta)^{N+1}}{1-s\varphi(\theta)} \right| \leq \frac{|t|\,|\theta|}{|1-s\varphi(\theta)|} \leq K|t|$$

if $0 < |\theta| \leq 1$.

We quote a result from [**2**] (see also [**5**, p. 62]): The measure $1_{[-1,1]}(\theta)\Re(1/(1-s\varphi(\theta)))d\theta$ converges weakly to $(\pi/\mu)\delta_0(d\theta) + 1_{[-1,1]}(\theta)\Re(1/(1-\varphi(\theta)))d\theta$ as $s \to 1-0$. Note that

$$\Re \frac{s^{N+1}\varphi(\theta)^{N+1}}{1-s\varphi(\theta)} = \Re \frac{1}{1-s\varphi(\theta)} - \Re \frac{1-s^{N+1}\varphi(\theta)^{N+1}}{1-s\varphi(\theta)}$$

$$= \Re \frac{1}{1-s\varphi(\theta)} - \Re \sum_{n=0}^{N} s^n \varphi(\theta)^n$$

and the absolute value of $\sum_{n=0}^{N} s^n \varphi(\theta)^n$ is bounded by $N+1$.

Hence we have, as $s \to 1-0$,

$$\int_{[-1,1]} \left(\cos(t\theta) \Re \frac{s^{N+1}\varphi(\theta)^{N+1}}{1-s\varphi(\theta)} + \sin(t\theta) \Im \frac{s^{N+1}\varphi(\theta)^{N+1}}{1-s\varphi(\theta)} \right) d\theta$$

$$\to \int_{[-1,1]} \cos(t\theta) \left(\frac{\pi}{\mu}\delta_0(d\theta) + \Re \frac{1}{1-\varphi(\theta)} d\theta \right)$$

$$+ \int_{[-1,1]} \left(\cos(t\theta) \Re \frac{\varphi(\theta)^{N+1} - 1}{1-\varphi(\theta)} + \sin(t\theta) \Im \frac{\varphi(\theta)^{N+1}}{1-\varphi(\theta)} \right) d\theta$$

$$= \frac{\pi}{\mu} + \int_{[-1,1]} \left(\cos(t\theta) \Re \frac{\varphi(\theta)^{N+1}}{1-\varphi(\theta)} + \sin(t\theta) \Im \frac{\varphi(\theta)^{N+1}}{1-\varphi(\theta)} \right) d\theta. \tag{2.4}$$

A bound for $\theta \in [-1,1]^c$ can be obtained by using $\inf_{|\theta|>1} |1-\varphi(\theta)| > 0$:

$$\sup_{s\in[0,1)} \left| e^{-it\theta} \frac{s^{N+1}\varphi(\theta)^{N+1}}{1-s\varphi(\theta)} \right| \leq \mathrm{const}\, \varphi(\theta)^{N+1} \quad \text{for } \theta \in [-1,1]^c.$$

Hence

$$\int_{[-1,1]^c} \Re\left(e^{-it\theta}\frac{s^{N+1}\varphi(\theta)^{N+1}}{1-s\varphi(\theta)}\right)d\theta \to \int_{[-1,1]^c} \Re\left(e^{-it\theta}\frac{\varphi(\theta)^{N+1}}{1-\varphi(\theta)}\right)d\theta. \tag{2.5}$$

By (2.4) and (2.5) we have the statement of (i).

(ii) The integral in (2.5) is the sum of the cosine and the sine transform of integrable functions and thus continuous in t.

The proof of (i) also reveals $1_{[-1,1]}(\theta)\Re(\varphi(\theta)^{N+1}/(1-\varphi(\theta)))$ is integrable and its cosine transform is continuous.

Set $I(t) := \int_{[-1,1]} \sin(t\theta)\Im(\varphi(\theta)^{N+1}/(1-\varphi(\theta)))d\theta$. Note that $\varphi(\theta)^{N+1}/(1-\varphi(\theta)) = O_{\theta\to 0}(|\theta|^{-1})$. Hence for any $t, t' \in \mathbb{R}$, it follows from $|\sin(t\theta)-\sin(t'\theta)| \le |t-t'|\cdot|\theta|$ that $|I(t)-I(t')| \le \text{const}\,|t-t'|$ and the proof of continuity of the integral appearing in (ii) is completed. The rest of (ii) follows from

$$u(t) - \sum_{n=1}^{N} f_n(t) = \lim_{s\to 1-0}\left(u_s(t) - \sum_{n=1}^{N} s^n f_n(t)\right) = \lim_{s\to 1-0}\sum_{n=N+1}^{\infty} s^n f_n(t)$$

and Lemma 2.2 (ii). $\square$

Intuitively $u(t)$ is related in Lemma 2.3 (ii) to "the inverse Fourier transform" of $\varphi(\theta)^{N+1}/(1-\varphi(\theta))$ if it could be justified. When we are interested in the asymptotic behavior as $t\to\infty$, the behavior of $\varphi(\theta)^{N+1}/(1-\varphi(\theta))$ near $\theta = 0$ is important. Since $\varphi(\theta)^{N+1}$ will be shown later to be well approximated by 1 as $\theta\to 0$, we here focus on $1/(1-\varphi(\theta))$. By (2.2) we have $a := (\varphi(\theta)-1-i\mu\theta)/-i\mu\theta = o_{\theta\to 0}(|\theta|^{\alpha-1})$ and hence, if $|\theta|$ is sufficiently small, $1/(1-a) = 1+a+a^2+\cdots$ is convergent. We divide the both sides by $(-i\mu\theta)$ and obtain

$$\frac{1}{1-\varphi(\theta)} = \frac{1}{-i\mu\theta} + \frac{\varphi(\theta)-1-i\mu\theta}{(-i\mu\theta)^2} + \frac{(\varphi(\theta)-1-i\mu\theta)^2}{(-i\mu\theta)^3} + \cdots. \tag{2.6}$$

Although we will take advantage of the inverse Fourier transform of integrable functions, we first use the inverse Fourier transform in the sense of distributions to foretell a couple of leading terms in the approximation of $u(t)$. Namely what we will prove in the sequel can be summarized that the quantity in Lemma 2.3 (ii) is close to

$$\begin{aligned}
&\frac{1}{2\mu} + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-it\theta}\frac{1}{1-\varphi(\theta)}d\theta \\
&\quad = \frac{1}{2\mu} + \mathcal{F}^{-1}\frac{1}{1-\varphi(\theta)} \\
&\quad = \frac{1}{2\mu} + \mathcal{F}^{-1}\frac{1}{-i\mu\theta} + \mathcal{F}^{-1}\frac{\varphi(\theta)-1-i\mu\theta}{(-i\mu\theta)^2} + \mathcal{F}^{-1}\frac{(\varphi(\theta)-1-i\mu\theta)^2}{(-i\mu\theta)^3} + \cdots.
\end{aligned}$$

Note first that $1/-i\mu\theta = \mathcal{F}((1/2\mu)\,\mathrm{sgn}(t))$ where $\mathrm{sgn}(t) = t/|t|$ for $t \ne 0$ and $\mathrm{sgn}(0) = 0$.

Hence $1/2\mu + \mathcal{F}^{-1}(1/{-i\mu\theta}) = (1/2\mu)(1+\operatorname{sgn}(t)) = (1/2\mu)1_{\{0\}}(t) + (1/\mu)1_{(0,\infty)}(t)$ and this density yields the Lebesgue measure on $(0,\infty)$ multiplied by $1/\mu$, which is plausible as the first approximation of the renewal measure.

Since the second term in the right hand side of (2.6) includes $\varphi(\theta)/\theta^2$ it should be the Fourier transform of an iterated integral of $f(t)$, which will be shown to be $r(t)$ defined in Section 1.

Being the square of the second term multiplied by $(-i\mu\theta)$, the third term in the right hand side of (2.6) should be the Fourier transform of $\mu(r*r)'$. We justify this formal treatment in Lemmas 2.4 and 2.5.

We introduce a function $v(t) = r(t) + ((t-1)/\mu)1_{[0,1]}(t)$. We set

$$\gamma(\theta) = \mathcal{F}(r-v)(\theta) = \int_0^1 e^{i\theta t}\frac{1-t}{\mu}dt = \frac{-1}{i\mu\theta} + \frac{\mu(e^{i\theta}-1)}{(-i\mu\theta)^2}. \tag{2.7}$$

The importance of $v(t)$ lies in the fact that it has the same asymptotic behavior as $r(t)$ as $|t|\to\infty$ and $\mathcal{F}v(\theta)$ has a thin tail as $|\theta|\to\infty$, see Lemma 2.4 (iii) below.

LEMMA 2.4. *Assume $\mu = \int_{-\infty}^{\infty} tf(t)dt \in (0,\infty)$, $\alpha \in (1,\infty)$, and $M(\alpha) < \infty$.*

(i) *The function $q(t)$ belongs to $L^1 \cap L^\infty$, is absolutely continuous, and is monotone nonincreasing both on $(-\infty,0)$ and on $[0,\infty)$. It holds $q(t) = o_{t\to\pm\infty}(|t|^{-\alpha})$, $q(0-) \le 0 \le q(0) = q(0-) + 1/\mu$, $\mathcal{F}q(\theta) = (\varphi(\theta)-1)/i\mu\theta$ for any $\theta \ne 0$, and $\mathcal{F}q(0) = 1$.*

(ii) *The function $r(t)$ is convex and of the class C^1 both on $(-\infty,0)$ and on $[0,\infty)$. Moreover, it is nonnegative, bounded, and satisfies $r(0) = r(0-) + 1/\mu$ and $r(t) = o_{t\to\pm\infty}(|t|^{1-\alpha})$. For any $\theta \ne 0$, the Fourier transform $\mathcal{F}r(\theta) = \int_{-\infty}^{\infty} e^{i\theta t} r(t)dt$ converges as an improper Riemann integral and equals to $(\mathcal{F}q(\theta)-1)/i\mu\theta = (\varphi(\theta)-1-i\mu\theta)/(-i\mu\theta)^2$. It satisfies $\mathcal{F}r(\theta) = -1/i\mu\theta + O_{\theta\to\pm\infty}(|\theta|^{-2})$; $\mathcal{F}r(\theta) = o_{\theta\to 0}(|\theta|^{\alpha-2})$ if $1<\alpha<2$; $\mathcal{F}r(\theta) = O_{\theta\to 0}(1)$ if $\alpha \ge 2$.*

(iii) *The function $v(t)$ is bounded and continuous on $\mathbb{R}$. For any $\theta \ne 0$, the Fourier transform $\mathcal{F}v(\theta) = \int_{-\infty}^{\infty} e^{i\theta t} v(t)dt$ converges as an improper Riemann integral and equals to $\mathcal{F}r(\theta) - \gamma(\theta) = (\varphi(\theta) - 1 - \mu(e^{i\theta}-1))/(-i\mu\theta)^2$. It satisfies $\mathcal{F}v(\theta) = O_{\theta\to\pm\infty}(|\theta|^{-2})$; $\mathcal{F}v(\theta) = o_{\theta\to 0}(|\theta|^{\alpha-2})$ if $1<\alpha<2$; $\mathcal{F}v(\theta) = O_{\theta\to 0}(1)$ if $\alpha \ge 2$. Consequently, $\mathcal{F}v(\theta)$ is integrable.*

(iv) *If $\alpha > 3/2$ then $r(t) \in L^2$, $v(t) \in L^2$, and $\mathcal{F}v(\theta) \in L^1 \cap L^2$. For all $t \in \mathbb{R}$ it holds*

$$v(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-it\theta}\mathcal{F}v(\theta)d\theta.$$

PROOF. (i) Since $\int_0^\infty t^\alpha f(t)dt < \infty$,

$$\lim_{A\to\infty} A^\alpha q(A) = \lim_{A\to\infty}\frac{1}{\mu}\int_A^\infty A^\alpha f(t)dt \le \lim_{A\to\infty}\frac{1}{\mu}\int_A^\infty t^\alpha f(t)dt = 0.$$

Similarly, we have $\lim_{A\to-\infty} A^\alpha q(A) = 0$.

Let $\theta \ne 0$. Since $\lim_{|A|\to\infty} q(A) = 0$, by integration by parts we have $\int_0^\infty e^{i\theta t}q(t)dt =$

$[(e^{i\theta t}/i\theta)q(t)]_0^\infty + (1/\mu)\int_0^\infty (e^{i\theta t}/i\theta)f(t)dt = -q(0)/i\theta + (1/\mu)\int_0^\infty (e^{i\theta t}/i\theta)f(t)dt$ and $\int_{-\infty}^0 e^{i\theta t}q(t)dt = q(0-)/i\theta + (1/\mu)\int_{-\infty}^0 (e^{i\theta t}/i\theta)f(t)dt$. The sum of these integrals yields $\mathcal{F}q(\theta) = -(q(0)-q(0-))/i\theta + (1/\mu)\int_{-\infty}^\infty (e^{i\theta t}/i\theta)f(t)dt = -1/i\mu\theta + \varphi(\theta)/i\mu\theta$.

Other statements in (i) are easily verified using $\int_0^\infty q(t)dt = (1/\mu)\int_0^\infty tf(t)dt$ and $\int_{-\infty}^0 q(t)dt = (-1/\mu)\int_{-\infty}^0 |t|f(t)dt$.

(ii) On $(0,\infty)$, it holds $r'(t) = -q(t)/\mu$ and q is monotonously nonincreasing and continuous, so that r is convex and of the class C^1. Similar result holds for $(-\infty,0)$.

By the definition we have $r(0) - r(0-) = (1/\mu)\int_{-\infty}^\infty q(s)ds = 1/\mu$.

By integration by parts and a result in (i), we have $\int_0^\infty \alpha t^{\alpha-1}\mu q(t)dt = \int_0^\infty t^\alpha f(t)dt < \infty$. An argument similar to the proof of (i) leads us to $\alpha A^{\alpha-1}\mu^2 r(A) \le \int_A^\infty \alpha t^{\alpha-1}\mu q(t)dt \to 0$ as $A\to\infty$. We also have $\lim_{A\to-\infty}|A|^{\alpha-1}r(A) = 0$.

Let $\theta\neq 0$. Since $\lim_{|t|\to\infty} r(t) = 0$, by integration by parts we have $\int_0^\infty e^{i\theta t}r(t)dt = [(e^{i\theta t}/i\theta)r(t)]_0^\infty + (1/\mu)\int_0^\infty (e^{i\theta t}/i\theta)q(t)dt = -r(0)/i\theta + (1/\mu)\int_0^\infty (e^{i\theta t}/i\theta)q(t)dt$ and $\int_{-\infty}^0 e^{i\theta t}r(t)dt = r(0-)/i\theta + (1/\mu)\int_{-\infty}^0 (e^{i\theta t}/i\theta)q(t)dt$. The sum of these integrals yields $\mathcal{F}r(\theta) = -(r(0)-r(0-))/i\theta + (1/\mu)\int_{-\infty}^\infty (e^{i\theta t}/i\theta)q(t)dt = -1/i\mu\theta + \mathcal{F}q(\theta)/i\mu\theta = -1/i\mu\theta + (\varphi(\theta)-1)/(-i\mu\theta)^2$. It follows from $|\varphi(\theta)|\le 1$ that $\mathcal{F}r(\theta) = -1/i\mu\theta + O_{\theta\to\pm\infty}(|\theta|^{-2})$.

If $1<\alpha<2$ the estimate $\mathcal{F}r(\theta) = o_{\theta\to 0}(|\theta|^{\alpha-2})$ follows from (2.2) and $\varphi'(0) = i\mu$. If $\alpha\ge 2$ we have $r(t)\in L^1$, $\mathcal{F}r(\theta)\in \mathrm{C}(\mathbb{R})$ and hence $\mathcal{F}r(\theta) = O_{\theta\to 0}(1)$. Other statements can be proved immediately.

(iii) By (ii), $r(t)$ is continuous except the jump by $1/\mu$ at $t=0$. The modification in the definition of $v(t)$ gets rid of this jump and makes it a continuous function. Since $\gamma(\theta) = -1/i\mu\theta + O_{\theta\to\pm\infty}(|\theta|^{-2})$, other statements are proved using $\mathcal{F}v(\theta) = \mathcal{F}r(\theta) - \gamma(\theta)$ and (ii).

(iv) If $\alpha > 3/2$, the statements in (ii) and (iii) imply that $r(t)\in L^2$, $v(t)\in L^2$, $\mathcal{F}v(\theta)\in L^2$, and that $v_A(t) := (1/2\pi)\int_{-A}^A e^{-it\theta}\mathcal{F}v(\theta)d\theta$ converges to $v(t)$ in L^2 as $A\to\infty$. There exists a sequence $A_k\to\infty$ such that $v_{A_k}(t)$ converges to $v(t)$ almost everywhere. On the other hand, since $\mathcal{F}v(\theta)$ is integrable, $\lim_{A\to\infty} v_A(t)$ converges for all t and yields a continuous function. Since $v(t)$ is also continuous, they coincide for all t. $\square$

We next relate the third term in the right side of (2.6) to the Fourier transform of $\mu r'(t) * r(t)$, as is predicted just before Lemma 2.4. Since formally $r'(t) = (1/\mu)\delta(t) - (1/\mu)q(t)$ where $\delta(t)$ is the Dirac delta at $t=0$, $\mu r'(t) * r(t)$ should be replaced by $r_1(t) = r(t) - q * r(t)$. We also introduce the function $w(t)$ that behaves in the same way as $r_1(t)$ as $|t|\to\infty$ and has the Fourier transform with a thin tail (see Lemma 2.5 (ii) below) as $|\theta|\to\infty$: $w(t) = r_1(t) + ((t-1)/\mu)1_{[0,1]}(t) = v(t) - q * r(t)$.

Lemma 2.5. *Assume* $\mu = \int_{-\infty}^\infty tf(t)dt \in (0,\infty)$ *and* $M(\alpha) < \infty$ *for some* $\alpha\in(1,\infty)$. *Then the following statements hold.*

(i) *The functions* $q * r(t)$ *and* $w(t)$ *are bounded and continuous.*
(ii) *Assume* $1<\alpha<2$. *Then it holds* $w(t) = o_{t\to\pm\infty}(|t|^{2-2\alpha})$ *and* $r_1(t) = o_{t\to\pm\infty}(|t|^{2-2\alpha})$.
(iii) *Assume* $3/2<\alpha$. *Then* $r_1(t)\in L^1\cap L^\infty$, $\mathcal{F}r_1(\theta) = (\varphi(\theta)-1-i\mu\theta)^2/(-i\mu\theta)^3$ *for*

any $\theta \neq 0$, *and* $\mathcal{F}r_1(0) = 0$.

(iv) *Assume* $3/2 < \alpha$. *Then* $w(t) \in L^1 \cap L^\infty$, $\mathcal{F}w(\theta) = (\varphi(\theta) - 1 - i\mu\theta)^2/(-i\mu\theta)^3 - 1/-i\mu\theta - \mu(e^{i\theta} - 1)/(-i\mu\theta)^2$ *for any* $\theta \neq 0$, $\mathcal{F}w(\theta) = O_{\theta\to\pm\infty}(|\theta|^{-2})$, *and* $\mathcal{F}w(\theta) \in L^1 \cap L^\infty$. *It holds* $w(t) = (1/2\pi)\int_{-\infty}^{\infty} e^{-it\theta}\mathcal{F}w(\theta)d\theta$ *for all* $t \in \mathbb{R}$.

Proof. (i) We can prove boundedness and continuity of $q * r(t)$ immediately using $q(t) \in L^1$, $r(t) \in L^\infty$, and Young's inequality. The statement for $w(t)$ follows from $w(t) = v(t) - q * r(t)$ and Lemma 2.4 (iii).

(ii) We only prove here the tail estimate when $t \to +\infty$. Since $\int_{-\infty}^{\infty} q(s)ds = 1$, it holds $r_1(t) = \int_{-\infty}^{\infty}(r(t) - r(t - s))q(s)ds$. We divide this integral into four parts and denote each of them by I_i ($i \in \{1, 2, 3, 4\}$) so that

$$r_1(t) = \left(\int_{-\infty}^{-t/2} + \int_{-t/2}^{t/2} + \int_{t/2}^{3t/2} + \int_{3t/2}^{\infty}\right)(r(t) - r(t - s))q(s)ds = I_1 + I_2 + I_3 + I_4.$$

By Lemma 2.4 (ii), $|I_1| \leq o_{t\to\pm\infty}(|t|^{1-\alpha})\int_{-\infty}^{-t/2} q(s)ds = o_{t\to\pm\infty}(|t|^{1-\alpha}r(-t/2)) = o_{t\to\pm\infty}(|t|^{2-2\alpha})$ and similarly $|I_4| \leq o_{t\to\pm\infty}(|t|^{1-\alpha}r(3t/2)) = o_{t\to\pm\infty}(|t|^{2-2\alpha})$. We have $|r(t) - r(u)| \leq |q(t/2)|\,|t - u|/\mu = o_{t\to\pm\infty}(|t|^{-\alpha})|t - u|$ for any $u \in [t/2, 3t/2]$ by convexity of $r(u)$. Therefore $|I_2| \leq o_{t\to\pm\infty}(|t|^{-\alpha})\int_{t/2}^{-t/2} |s|q(s)ds = o_{t\to\pm\infty}(|t|^{2-2\alpha})$. Finally, $|I_3| \leq r(t)\int_{t/2}^{3t/2} q(s)ds + q(t/2)\int_{t/2}^{3t/2} r(t - s)ds = o_{t\to\pm\infty}(|t|^{2-2\alpha})$. The proof of the rest of (ii) can be done similarly.

(iii) If $3/2 < \alpha$, the estimates in (ii) and the (forthcoming) Remark 2.1 imply $r_1(t) \in L^1 \cap L^\infty$. Since $q(t) \in L^1$ and $r(t) \in L^2$ we have $\mathcal{F}(q * r) = (\mathcal{F}q)(\mathcal{F}r)$ a.e. by [**7**, p. 122]. Hence $\mathcal{F}(r_1)(\theta) = (1 - \mathcal{F}q(\theta))\mathcal{F}r(\theta) = (1 - \mathcal{F}q(\theta))(\mathcal{F}q(\theta) - 1)/(i\mu\theta) = (\varphi(\theta) - 1 - i\mu\theta)^2/(-i\mu\theta)^3$ for a.e. $\theta \neq 0$. By continuity the equality holds for all $\theta \neq 0$.

(iv) The difference between $w(t)$ and $r_1(t)$ is a bounded function supported on a compact set and hence $w(t) \in L^1 \cap L^\infty$. By (2.7) we have $\mathcal{F}w(\theta) = \mathcal{F}r_1(\theta) - \gamma(\theta)$. It is easily shown that $\mathcal{F}r_1(\theta) = -1/i\mu\theta + O_{\theta\to\pm\infty}(|\theta|^{-2})$ and $\gamma(\theta) = -1/i\mu\theta + O_{\theta\to\pm\infty}(|\theta|^{-2})$ by the proof of Lemma 4 (iii), which yields $\mathcal{F}w(\theta) = O_{\theta\to\pm\infty}(|\theta|^{-2})$. Since $\mathcal{F}w(\theta)$ is bounded, it is also integrable. Hence its inverse transform coincides with $w(t)$. □

Remark 2.1. In the proof of Lemma 2a in Carlsson [**1**], it is assumed that $\alpha \geq 2$ and hence the tail estimate in Lemma 2.5 (ii) is modified. Indeed, since $|t|q(t)$ is integrable it holds $r(t) \in L^1$, $q * r(t) \in L^1$, and $\int_{-t/2}^{t/2} |s|q(s)ds = O_{t\to\pm\infty}(1)$, which is greater than $|t|^{2-\alpha}$. Consequently one has $I_2 = o_{t\to\pm\infty}(|t|^{-\alpha})$ in [**1**] so that $r_1(t) = o_{t\to\pm\infty}(|t|^{-\alpha})$ if $\alpha \geq 2$.

3. Proof of the main theorem.

We recall the three terms appearing in the right hand side of (2.6) and will approximate them by the functions $e^{-\theta^2}/-i\mu\theta$, $\mathcal{F}v(\theta)$, and $\mathcal{F}w(\theta)$ respectively that behave similarly to the original ones as $|\theta| \to 0$, have thin tails as $|\theta| \to \infty$, and admit the inverse Fourier transform. By a formula in Oberhettinger [**8**, p. 126] we have

$$\frac{1}{2\mu}\operatorname{Erf}(t/2)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\sin(t\theta)\frac{1}{\mu\theta}e^{-\theta^2}d\theta=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Re\Big(e^{-it\theta}\frac{e^{-\theta^2}}{-i\mu\theta}\Big)d\theta, \tag{3.1}$$

where $\operatorname{Erf}(x)=(2/\sqrt{\pi})\int_0^x e^{-t^2}dt$ for $x\in\mathbb{R}$. Subtracting the both sides of (3.1) from the result of Lemma 2.3 (ii) we have

$$u(t)-\frac{1+\operatorname{Erf}(t/2)}{2\mu}-\sum_{n=1}^{N}f_n(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Re\Big(e^{-it\theta}\Big(\frac{\varphi(\theta)^{N+1}}{1-\varphi(\theta)}-\frac{e^{-\theta^2}}{-i\mu\theta}\Big)\Big)d\theta. \tag{3.2}$$

We set

$$u_1(t)=u(t)-\frac{1+\operatorname{Erf}(t/2)}{2\mu}-\sum_{n=1}^{N}f_n(t)-v(t)-w(t). \tag{3.3}$$

If $\alpha>3/2$ it follows from (3.2), Lemma 2.4 (iv), and Lemma 2.5 (iv) that

$$u_1(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Re\Big(e^{-it\theta}\Big(\frac{\varphi(\theta)^{N+1}}{1-\varphi(\theta)}-\frac{e^{-\theta^2}}{-i\mu\theta}-\mathcal{F}v(\theta)-\mathcal{F}w(\theta)\Big)\Big)d\theta. \tag{3.4}$$

We define a function $\Psi_1(\theta)$ by

$$\Psi_1(\theta)=\frac{\varphi(\theta)^{N+1}}{1-\varphi(\theta)}-\frac{e^{-\theta^2}}{-i\mu\theta}-\mathcal{F}v(\theta)-\mathcal{F}w(\theta) \tag{3.5}$$

for $\theta\neq 0$ and $\Psi_1(0)=-N-1+2\gamma(0)$.

Lemma 3.1. *Let $3/2<\alpha<2$, $M(\alpha)<\infty$, and $0<\varepsilon<1/2$.*

(i) *For $\theta\neq 0$, $\Psi_1(\theta)$ is differentiable and satisfies*

$$\Psi_1(\theta)=\frac{(\varphi(\theta)-1-i\mu\theta)^3}{(1-\varphi(\theta))(-i\mu\theta)^3}-\sum_{k=0}^{N}\varphi(\theta)^k-\frac{e^{-\theta^2}-1}{-i\mu\theta}+2\gamma(\theta). \tag{3.6}$$

$\Psi_1(\theta)$ is continuous on $\mathbb{R}$. It holds $(\varphi(\theta)-1-i\mu\theta)^3/(1-\varphi(\theta))(-i\mu\theta)^3=o_{\theta\to 0}(|\theta|^{3\alpha-4})$.

(ii) *$\Psi_1(\theta)$ is integrable on $\mathbb{R}$ and satisfies $\lim_{\theta\to\pm\infty}\Psi_1(\theta)=0$.*

(iii) *It holds $(d/d\theta)(\varphi(\theta)-1-i\mu\theta)^3/(1-\varphi(\theta))(-i\mu\theta)^3=o_{\theta\to 0}(|\theta|^{3\alpha-5})$. On $\{\theta\neq 0\}$, $\Psi_1'(\theta)$ is integrable, continuous and satisfies $\lim_{\theta\to\pm\infty}\Psi_1'(\theta)=0$.*

(iv) *As $\varepsilon\to+0$, $\int_{[-1,1]^c}|\Psi_1'(\theta)-\Psi_1'(\theta+\varepsilon)|d\theta=o_{\varepsilon\to+0}(\varepsilon^{\alpha-1})$.*

(v) *As $\varepsilon\to+0$, $\int_{\{2\varepsilon<|\theta|<1\}}|\Psi_1'(\theta)-\Psi_1'(\theta+\varepsilon)|d\theta=o_{\varepsilon\to+0}(\varepsilon^{\alpha-1})$.*

Proof. (i) Since $\mathcal{F}v=\mathcal{F}r-\gamma$ and $\mathcal{F}w=\mathcal{F}r_1-\gamma$, (3.6) can be derived from (3.5), Lemma 2.4 (ii), and Lemma 2.5 (iii) by elementary manipulations. Other statements are proved easily using (2.2) and $\gamma(\theta)=-1/i\mu\theta+\mu(e^{i\theta}-1)/(-i\mu\theta)^2$.

(ii) The integrability of $\Psi_1(\theta)$ on $[-1,1]$ follows from (i). Since $\varphi^{N+1} \in L^1$, it follows from $\lim_{\theta\to\pm\infty} \varphi(\theta) = 0$ and $\inf_{|\theta|\geq 1} |1-\varphi(\theta)| > 0$ that $\varphi(\theta)^{N+1}/(1-\varphi(\theta))$ has a vanishing tail and is integrable on $[-1,1]^c$. Other terms in (3.5) also have vanishing tails and are integrable on $[-1,1]^c$ by Lemmas 2.4 (iii) and 2.5 (iv).

(iii) We can verify the first estimate in (iii) using Lemma 2.1 and

$$\begin{aligned}\frac{d}{d\theta}\frac{(\varphi(\theta)-1-i\mu\theta)^3}{(1-\varphi(\theta))(-i\mu\theta)^3} &= 3\frac{(\varphi(\theta)-1-i\mu\theta)^2(\varphi'(\theta)-i\mu)}{(1-\varphi(\theta))(-i\mu\theta)^3} \\ &\quad + \frac{(\varphi(\theta)-1-i\mu\theta)^3\varphi'(\theta)}{(1-\varphi(\theta))^2(-i\mu\theta)^3} - 3\frac{(\varphi(\theta)-1-i\mu\theta)^3}{(1-\varphi(\theta))(-i\mu)^3\theta^4}. \qquad (3.7)\end{aligned}$$

Continuity of $\Psi_1'(\theta)$ on $\{\theta \neq 0\}$ follows from (3.6), (3.7), (2.3), and smoothness of $(e^{-\theta^2}-1)/-i\mu\theta$ and $\gamma(\theta)$. Since $|\theta|^{3\alpha-5}$ is locally integrable and $-\sum_{k=0}^{N} \varphi(\theta)^k - (e^{-\theta^2} - 1)/-i\mu\theta + 2\gamma(\theta)$ is continuously differentiable, $\Psi_1'(\theta)$ is locally integrable. To prove the integrability on $[-1,1]^c$ of $\Psi_1'(\theta)$ we note that the second term $e^{-\theta^2}/-i\mu\theta$ in (3.5) has the derivative that is integrable on $[-1,1]^c$. The derivative of the first term in (3.5) is

$$\frac{d}{d\theta}\frac{\varphi(\theta)^{N+1}}{1-\varphi(\theta)} = \frac{(N+1)\varphi(\theta)^N\varphi'(\theta)}{1-\varphi(\theta)} + \frac{\varphi(\theta)^{N+1}\varphi'(\theta)}{(1-\varphi(\theta))^2}. \qquad (3.8)$$

By $\inf_{|\theta|\geq 1} |1-\varphi(\theta)| > 0$, $|(d/d\theta)(\varphi(\theta)^{N+1}/(1-\varphi(\theta)))| \leq \mathrm{const}\,|\varphi(\theta)|^N$ if $|\theta| \geq 1$. Since $\varphi(\theta) \in L^p \cap L^\infty$ and $N \geq p$, we have $\varphi(\theta)^N \in L^1$ and $(d/d\theta)(\varphi(\theta)^{N+1}/(1-\varphi(\theta)))$ is integrable on $[-1,1]^c$.

Recall that $\mathcal{F}v(\theta) = (\varphi(\theta)-1-\mu(e^{i\theta}-1))/(-i\mu\theta)^2$. By differentiation we have

$$(\mathcal{F}v)'(\theta) = \frac{\varphi'(\theta)-i\mu e^{i\theta}}{(-i\mu\theta)^2} - 2\frac{\varphi(\theta)-1-\mu(e^{i\theta}-1)}{(-i\mu)^2\theta^3}. \qquad (3.9)$$

It is easy to deduce that $(\mathcal{F}v)'(\theta) = O_{\theta\to\pm\infty}(|\theta|^{-2})$ and, in a similar manner, that $(\mathcal{F}w)'(\theta) = O_{\theta\to\pm\infty}(|\theta|^{-2})$. Hence $(\mathcal{F}v)'(\theta)$ and $(\mathcal{F}w)'(\theta)$ are integrable on $[-1,1]^c$.

It remains to prove $\lim_{\theta\to\pm\infty} \Psi_1'(\theta) = 0$. Since $\varphi'(\theta) = \int_{-\infty}^{\infty} e^{i\theta t} itf(t)dt$, we have $\lim_{\theta\to\pm\infty} \varphi'(\theta) = 0$ by the Riemann-Lebesgue theorem. Hence $\lim_{\theta\to\pm\infty} \Psi_1'(\theta) = 0$ follows from (3.5), (3.8), and $|(\mathcal{F}v)'(\theta)| + |(\mathcal{F}w)'(\theta)| = O_{\theta\to\pm\infty}(|\theta|^{-2})$.

(iv) We derivate each term in (3.5), take the difference between θ and $\theta+\varepsilon$, and integrate over $[-1,1]^c$. For $\varepsilon \in (0,1/2)$ and a function $g(\theta)$, let $\Delta_\varepsilon g(\theta) = g(\theta) - g(\theta+\varepsilon)$.

It is easy to deduce from (3.9) and (2.3) that $\sup_{|\theta|\geq 1} \theta^2|\Delta_\varepsilon(\mathcal{F}v)'(\theta)| = o_{\varepsilon\to+0}(\varepsilon^{\alpha-1})$, which implies $\int_{[-1,1]^c} |\Delta_\varepsilon(\mathcal{F}v)'(\theta)|d\theta = o_{\varepsilon\to+0}(\varepsilon^{\alpha-1})$. A similar estimate holds for $(\mathcal{F}w)'$. The estimate concerning $(d/d\theta)(e^{-\theta^2}/-i\mu\theta)$ is immediate: $|\Delta_\varepsilon(d/d\theta)(e^{-\theta^2}/-i\mu\theta)| \leq \mathrm{const}\,|\theta|\exp(-(|\theta|-1/2)^2)\varepsilon$ for any $\varepsilon \in (0,1/2)$ and θ with $|\theta| \geq 1$. Its integral over $[-1,1]^c$ is $O_{\varepsilon\to+0}(\varepsilon)$ and hence $o_{\varepsilon\to+0}(\varepsilon^{\alpha-1})$.

We now have to obtain an estimate concerning the quantity in (3.8). We introduce some notations for the difference between θ and $\theta+\varepsilon$ of the first term (with $N+1$ deleted) in the right hand side of (3.8):

$$\begin{aligned}
&\Delta_\varepsilon \frac{\varphi(\theta)^N \varphi'(\theta)}{1-\varphi(\theta)} \\
&= \frac{\varphi(\theta)^N}{1-\varphi(\theta)} \Delta_\varepsilon \varphi'(\theta) + \frac{\varphi'(\theta+\varepsilon)}{1-\varphi(\theta)} \Delta_\varepsilon \varphi(\theta)^N + \varphi(\theta+\varepsilon)^N \varphi'(\theta+\varepsilon) \Delta_\varepsilon \frac{1}{1-\varphi(\theta)} \\
&=: \Phi_{1,1}(\theta,\varepsilon) + \Phi_{1,2}(\theta,\varepsilon) + \Phi_{1,3}(\theta,\varepsilon).
\end{aligned}$$

Set $A = \sup_{|\theta|\geq 1/2} |1/(1-\varphi(\theta))|$. By (2.3),

$$\int_{[-1,1]^c} |\Phi_{1,1}(\theta,\varepsilon)| d\theta \leq A \left(\int_{[-1,1]^c} |\varphi(\theta)|^N d\theta \right) \sup_{|\theta|\geq 1} |\Delta_\varepsilon \varphi'(\theta)| = o_{\varepsilon\to+0}(\varepsilon^{\alpha-1}).$$

Note that $\Phi_{1,2}(\theta,\varepsilon) = \varphi'(\theta+\varepsilon)\Delta_\varepsilon\varphi(\theta)/(1-\varphi(\theta))\sum_{k=0}^{N-1}\varphi(\theta)^k\varphi(\theta+\varepsilon)^{N-1-k}$ and $|\Delta_\varepsilon\varphi(\theta)| \leq \|\varphi'\|_\infty\varepsilon$. Since $\varphi(\theta) \in L^p \cap L^\infty$ and $p \leq \lceil p \rceil \leq N-1$ we have $\varphi(\theta) \in L^{N-1}$ and $\|\varphi(\theta)^k\varphi(\theta+\varepsilon)^{N-1-k}\|_1 \leq \|\varphi\|_{N-1}^{N-1}$ by the Hölder inequality. Now it is easy to obtain

$$\int_{[-1,1]^c} |\Phi_{1,2}(\theta,\varepsilon)| d\theta \leq A\|\varphi'\|_\infty^2 \varepsilon N \|\varphi\|_{N-1}^{N-1}.$$

It is straightforward to obtain $\int_{[-1,1]^c} |\Phi_{1,3}(\theta,\varepsilon)| d\theta \leq \|\varphi^N\|_1 \|\varphi'\|_\infty^2 A^2 \varepsilon$.

Since $\varepsilon = o_{\varepsilon\to+0}(\varepsilon^{\alpha-1})$, we have established

$$\int_{[-1,1]^c} \left| \Delta_\varepsilon \frac{\varphi(\theta)^N \varphi'(\theta)}{1-\varphi(\theta)} \right| d\theta = o_{\varepsilon\to+0}(\varepsilon^{\alpha-1}).$$

In a similar manner, we can verify the estimate concerning the second term in the right hand side of (3.8):

$$\int_{[-1,1]^c} \left| \Delta_\varepsilon \frac{\varphi(\theta)^{N+1} \varphi'(\theta)}{(1-\varphi(\theta))^2} \right| d\theta = o_{\varepsilon\to+0}(\varepsilon^{\alpha-1}).$$

Now the proof of (iv) is completed.

(v) Let $a, b \in \mathbb{R}$. We denote by $\tilde{o}_{\theta\to 0}(|\theta|^a\varepsilon^b)$ a function $g(\theta,\varepsilon)$ with two variables defined for $\varepsilon \in (0,1/2)$ and θ with $2\varepsilon < |\theta| < 1$ such that

$$\sup_{\varepsilon\in(0,|\theta|/2)} \frac{|g(\theta,\varepsilon)|}{|\theta|^a\varepsilon^b} = o_{\theta\to 0}(1).$$

The function $g(\theta,\varepsilon)$ may vary each time $\tilde{o}_{\theta\to 0}(|\theta|^a\varepsilon^b)$ appears. We will take advantage of the fact that for any $g(\theta,\varepsilon) = \tilde{o}_{\theta\to 0}(|\theta|^a\varepsilon^b)$, $\int_{\{2\varepsilon<|\theta|<1\}} |g(\theta,\varepsilon)|\, d\theta = o_{\varepsilon\to+0}(\varepsilon^{1+a+b})$ if $a < -1$; $\int_{\{2\varepsilon<|\theta|<1\}} |g(\theta,\varepsilon)|\, d\theta = o_{\varepsilon\to+0}(\varepsilon^b \log(1/\varepsilon))$ if $a = -1$; $\int_{\{2\varepsilon<|\theta|<1\}} |g(\theta,\varepsilon)|\, d\theta = O_{\varepsilon\to+0}(\varepsilon^b)$ if $a > -1$.

We define $\tilde{o}_{\theta\to 0}(|\theta+\varepsilon|^a\varepsilon^b)$ similarly. Since $1/2 < |\theta+\varepsilon|/|\theta| < 3/2$ for any $\varepsilon \in (0,1/2)$ and any θ with $2\varepsilon < |\theta| < 1$, we have similar estimates for $\int_{\{2\varepsilon<|\theta|<1\}} |g(\theta,\varepsilon)|\, d\theta$ if

$g(\theta,\varepsilon)=\tilde{o}_{\theta\to 0}(|\theta+\varepsilon|^a\varepsilon^b)$.

We denote by $\tilde{o}_{\varepsilon\to+0}(|\theta|^a\varepsilon^b)$ and $\tilde{O}(|\theta|^a\varepsilon^b)$ a function $g(\theta,\varepsilon)$ with two variables defined for $\varepsilon\in(0,1/2)$ and θ with $2\varepsilon<|\theta|<1$ such that

$$\sup_{|\theta|\in(2\varepsilon,1)}\frac{|g(\theta,\varepsilon)|}{|\theta|^a\varepsilon^b}=o_{\varepsilon\to+0}(1)\ \text{ and }\ \sup_{\varepsilon\in(0,1/2)}\sup_{|\theta|\in(2\varepsilon,1)}\frac{|g(\theta,\varepsilon)|}{|\theta|^a\varepsilon^b}<\infty,$$

respectively. The function $g(\theta,\varepsilon)$ may vary from place to place.

We rely on (3.6) to obtain a bound of $|\Delta_\varepsilon\Psi_1'(\theta)|$ that is valid for $2\varepsilon<|\theta|<1$.

Since $\gamma(\theta)$ is smooth we have $|\Delta_\varepsilon\gamma'(\theta)|=\tilde{O}(\varepsilon)$. A similar bound holds for $(e^{-\theta^2}-1)/-i\mu\theta$. As for $\sum_{k=0}^N\varphi(\theta)^k$, we have

$$\Delta_\varepsilon\biggl(\sum_{k=0}^N k\varphi(\theta)^{k-1}\varphi'(\theta)\biggr)=\tilde{o}_{\varepsilon\to+0}(\varepsilon^{\alpha-1})$$

by (2.3). Since we easily obtain $\int_{\{2\varepsilon<|\theta|<1\}}|\tilde{O}(\varepsilon)|d\theta=O_{\varepsilon\to+0}(\varepsilon)\le o_{\varepsilon\to+0}(\varepsilon^{\alpha-1})$ and $\int_{\{2\varepsilon<|\theta|<1\}}|\tilde{o}_{\varepsilon\to+0}(\varepsilon^{\alpha-1})|d\theta=o_{\varepsilon\to+0}(\varepsilon^{\alpha-1})$, these terms are harmless in the proof of (v).

We introduce some notations for the difference between θ and $\theta+\varepsilon$ of the first term (with a coefficient deleted) in the right hand side of (3.7):

$$\Delta_\varepsilon\frac{(\varphi(\theta)-1-i\mu\theta)^2(\varphi'(\theta)-i\mu)}{(1-\varphi(\theta))\theta^3}=\Phi_{2,1}(\theta,\varepsilon)+\Phi_{2,2}(\theta,\varepsilon)+\Phi_{2,3}(\theta,\varepsilon)+\Phi_{2,4}(\theta,\varepsilon)\quad(3.10)$$

with

$$\Phi_{2,1}(\theta,\varepsilon)=\frac{(\varphi(\theta)-1-i\mu\theta)^2(\varphi'(\theta)-i\mu)}{(1-\varphi(\theta))}\Delta_\varepsilon\biggl(\frac{1}{\theta^3}\biggr),$$

$$\Phi_{2,2}(\theta,\varepsilon)=\frac{(\varphi(\theta)-1-i\mu\theta)^2(\varphi'(\theta)-i\mu)}{(\theta+\varepsilon)^3}\Delta_\varepsilon\biggl(\frac{1}{1-\varphi(\theta)}\biggr),$$

$$\Phi_{2,3}(\theta,\varepsilon)=\frac{(\varphi'(\theta)-i\mu)}{(1-\varphi(\theta+\varepsilon))(\theta+\varepsilon)^3}\Delta_\varepsilon\bigl((\varphi(\theta)-1-i\mu\theta)^2\bigr),$$

$$\Phi_{2,4}(\theta,\varepsilon)=\frac{(\varphi(\theta+\varepsilon)-1-i\mu(\theta+\varepsilon))^2}{(1-\varphi(\theta+\varepsilon))(\theta+\varepsilon)^3}\Delta_\varepsilon(\varphi'(\theta)-i\mu).$$

By Lemma 2.1 and $\Delta_\varepsilon(1/\theta^3)=\tilde{O}(|\theta|^2\varepsilon)/\theta^3(\theta+\varepsilon)^3=\tilde{O}(|\theta|^{-4}\varepsilon)$, we have

$$|\Phi_{2,1}(\theta,\varepsilon)|=o_{\theta\to0}(|\theta|^{2\alpha})o_{\theta\to0}(|\theta|^{\alpha-1})O(|\theta|^{-1})\tilde{O}(|\theta|^{-4}\varepsilon)=\tilde{o}_{\theta\to0}(|\theta|^{3\alpha-6}\varepsilon).$$

If $\alpha\in(3/2,5/3)$, we have $3\alpha-6<-1$, $3\alpha-4>\alpha-1$, and

$$\int_{\{2\varepsilon<|\theta|<1\}}\tilde{o}_{\theta\to0}(|\theta|^{3\alpha-6}\varepsilon)d\theta=o_{\varepsilon\to+0}(\varepsilon^{3\alpha-4})\le o_{\varepsilon\to+0}(\varepsilon^{\alpha-1}).$$

Otherwise, we have $\alpha \in [5/3, 2)$, $3\alpha - 6 \in [-1, 0)$, and

$$\int_{\{2\varepsilon<|\theta|<1\}} \tilde{o}_{\theta\to 0}(|\theta|^{3\alpha-6}\varepsilon)d\theta = o_{\varepsilon\to+0}(\varepsilon\log(1/\varepsilon)) \le o_{\varepsilon\to+0}(\varepsilon^{\alpha-1}).$$

Hence $\int_{\{2\varepsilon<|\theta|<1\}} |\Phi_{2,1}(\theta,\varepsilon)|d\theta = o_{\varepsilon\to+0}(\varepsilon^{\alpha-1})$.

Since $\Delta_\varepsilon(1/(1-\varphi(\theta))) = \tilde{O}(|\theta|^{-2}\varepsilon)$ the argument concerning $\Phi_{2,2}(\theta,\varepsilon)$ is similar.

Note that

$$\begin{aligned}&\Delta_\varepsilon(\varphi(\theta) - 1 - i\mu\theta)^2 \\ &\quad = ((\varphi(\theta) - 1 - i\mu\theta) + (\varphi(\theta+\varepsilon) - 1 - i\mu(\theta+\varepsilon)))(\varphi(\theta) - \varphi(\theta+\varepsilon) + i\mu\varepsilon)\end{aligned}$$

and

$$\begin{aligned}\varphi(\theta) - \varphi(\theta+\varepsilon) + i\mu\varepsilon &= \int_\theta^{\theta+\varepsilon} (\varphi'(0) - \varphi'(x))dx \\ &= \int_\theta^{\theta+\varepsilon} o_{x\to 0}(|x|^{\alpha-1})dx = \tilde{O}(|\theta|^{\alpha-1}\varepsilon)\end{aligned}$$

since $1/2 < |\theta+\varepsilon|/|\theta| < 3/2$. Combining these estimates we have

$$\begin{aligned}|\Phi_{2,3}(\theta,\varepsilon)| &= \tilde{o}_{\theta\to 0}(|\theta|^{\alpha-1})\tilde{O}(|\theta+\varepsilon|^{-4})\tilde{O}(|\theta|^\alpha + |\theta+\varepsilon|^\alpha)\tilde{O}(|\theta|^{\alpha-1}\varepsilon) \\ &= \tilde{o}_{\theta\to 0}(|\theta|^{3\alpha-6}\varepsilon)\end{aligned}$$

and its integral over $\{2\varepsilon < |\theta| < 1\}$ is $o_{\varepsilon\to+0}(\varepsilon^{\alpha-1})$.

It follows from $\Delta_\varepsilon\varphi'(\theta) = \tilde{o}_{\varepsilon\to+0}(\varepsilon^{\alpha-1})$ that $|\Phi_{2,4}(\theta,\varepsilon)| = \tilde{o}_{\varepsilon\to+0}(|\theta|^{2\alpha-4}\varepsilon^{\alpha-1})$. Since $2\alpha - 4 > -1$, $\int_{\{2\varepsilon<|\theta|<1\}} |\Phi_{2,4}(\theta,\varepsilon)|d\theta = o_{\varepsilon\to+0}(\varepsilon^{\alpha-1})$ and hence the integral of (3.10) over $\{2\varepsilon < |\theta| < 1\}$ is $o_{\varepsilon\to+0}(\varepsilon^{\alpha-1})$.

Although we omit the proof, we can obtain similar estimates corresponding to other two terms in the right hand side of (3.7). Combining these estimates we have

$$\int_{\{2\varepsilon<|\theta|<1\}} \left|\Delta_\varepsilon \frac{d}{d\theta}\frac{(\varphi(\theta) - 1 - i\mu\theta)^3}{(1-\varphi(\theta))(-i\mu\theta)^3}\right| d\theta = o_{\varepsilon\to+0}(\varepsilon^{\alpha-1})$$

and the proof of (v) is completed. □

Proof of Theorem 1.1. Recall $N = \lceil p \rceil + 1$. By (3.3) we have, if $|t| \ge 1$,

$$u(t) - \frac{1}{\mu}1_{[0,\infty)}(t) - \sum_{n=1}^N f_n(t) - r(t) - r_1(t) = u_1(t) + \frac{1}{2\mu}(\mathrm{Erf}(t/2) - \mathrm{sgn}(t))$$

and $\mathrm{Erf}(t/2) - \mathrm{sgn}(t) = o_{t\to\pm\infty}(|t|^{-\alpha})$. Hence (1.1) is reduced to the estimate $u_1(t) = o_{t\to\pm\infty}(|t|^{-\alpha})$.

Using (3.4), $\Psi_1(-\theta) = \overline{\Psi_1(\theta)}$, and Lemma 3.1 (ii) we have

$$u_1(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Re\big(e^{-it\theta}\Psi_1(\theta)\big)d\theta = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-it\theta}\Psi_1(\theta)d\theta. \tag{3.11}$$

Integrating by parts we have, by Lemma 3.1 (i)–(iii),

$$2\pi itu_1(t) = \int_{-\infty}^{\infty} e^{-it\theta}\Psi_1'(\theta)d\theta.$$

We set $\varepsilon = \pi/|t|$. Since $\int_{-\infty}^{\infty} e^{-it\theta}\Psi_1'(\theta)d\theta = \int_{-\infty}^{\infty} e^{-it(\theta+\varepsilon)}\Psi_1'(\theta+\varepsilon)d\theta = -\int_{-\infty}^{\infty} e^{-it\theta}\Psi_1'(\theta+\varepsilon)d\theta$, the sum of the leftmost and the rightmost sides yields

$$4\pi itu_1(t) = \int_{-\infty}^{\infty} e^{-it\theta}(\Psi_1'(\theta) - \Psi_1'(\theta+\varepsilon))d\theta. \tag{3.12}$$

By Lemma 3.1 (iv) and (v),

$$\begin{aligned}
|4\pi tu_1(t)| &\le \int_{-\infty}^{\infty} |\Psi_1'(\theta) - \Psi_1'(\theta+\varepsilon)|d\theta \\
&= \int_{\{|\theta|\le 2\varepsilon\}} |\Psi_1'(\theta) - \Psi_1'(\theta+\varepsilon)|d\theta + o_{\varepsilon\to+0}(\varepsilon^{\alpha-1}) \\
&\le 2\int_{\{|\theta|\le 3\varepsilon\}} |\Psi_1'(\theta)|d\theta + o_{\varepsilon\to+0}(\varepsilon^{\alpha-1}).
\end{aligned}$$

Since $-\sum_{k=0}^{N}\varphi(\theta)^k - (e^{-\theta^2}-1)/-i\mu\theta + 2\gamma(\theta)$ is continuously differentiable, we have $|\Psi_1'(\theta)| = o_{\theta\to 0}(|\theta|^{3\alpha-5}) + O_{\theta\to 0}(1)$ by Lemma 3.1 (iii). If $\alpha \in (3/2, 2)$ we have $\alpha - 1 < 3\alpha - 4$ and $\int_{\{|\theta|\le 3\varepsilon\}} |\Psi_1'(\theta)|d\theta = o_{\varepsilon\to+0}(\varepsilon^{3\alpha-4}) + O_{\varepsilon\to+0}(\varepsilon) \le o_{\varepsilon\to+0}(\varepsilon^{\alpha-1})$. Finally $|tu_1(t)| = o_{\varepsilon\to+0}(\varepsilon^{\alpha-1}) = o_{t\to\pm\infty}(|t|^{1-\alpha})$ and the proof is completed. □

Sketch of proof of Theorem 1.2. Recall $N = \lceil p\rceil + \lfloor\alpha\rfloor$. Set

$$u_2(t) = u(t) - \frac{1+\mathrm{Erf}(t/2)}{2\mu} - \sum_{n=1}^{N} f_n(t) - v(t) - w(t).$$

If $|t| \ge 1$,

$$u(t) - \frac{1}{\mu}1_{[0,\infty)}(t) - \sum_{n=1}^{N} f_n(t) - r(t) - r_1(t) = u_2(t) + \frac{1}{2\mu}(\mathrm{Erf}(t/2) - \mathrm{sgn}(t)).$$

By Remark 2.1 and $\mathrm{Erf}(t/2) - \mathrm{sgn}(t) = o_{t\to\pm\infty}(|t|^{-\alpha})$, we can reduce (1.2) to the estimate $u_2(t) = o_{t\to\pm\infty}(|t|^{-\alpha})$.

We define a function $\Psi_2(\theta)$ by

$$\Psi_2(\theta) = \frac{\varphi(\theta)^{N+1}}{1-\varphi(\theta)} - \frac{e^{-\theta^2}}{-i\mu\theta} - \mathcal{F}v(\theta) - \mathcal{F}w(\theta) \tag{3.13}$$

for $\theta \neq 0$ and $\Psi_2(0) = -N - 1 + 2\gamma(0)$.

By the same argument in the proof of Lemma 3.1 (i) we have, for $\theta \neq 0$,

$$\begin{aligned}\Psi_2(\theta) &= \frac{(\varphi(\theta)-1-i\mu\theta)^3}{(1-\varphi(\theta))(-i\mu\theta)^3} - \sum_{k=0}^{N} \varphi(\theta)^k - \frac{e^{-\theta^2}-1}{-i\mu\theta} + 2\gamma(\theta) \\ &= (-i\mu)^2 \frac{\theta^2 \mathcal{F}r(\theta)^3}{\mathcal{F}q(\theta)} - \sum_{k=0}^{N} \varphi(\theta)^k - \frac{e^{-\theta^2}-1}{-i\mu\theta} + 2\gamma(\theta)\end{aligned} \tag{3.14}$$

since it holds $(1-\varphi(\theta))/\theta = (-i\mu)\mathcal{F}q(\theta)$ and $(\varphi(\theta)-1-i\mu\theta)/\theta^2 = (-i\mu)^2\mathcal{F}r(\theta)$ for $\theta \neq 0$ as is seen in Lemma 2.4 (i) and (ii).

We can prove $\Psi_2(\theta)$ is integrable and

$$u_2(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-it\theta}\Psi_2(\theta)d\theta \tag{3.15}$$

in a manner similar to the proofs of Lemma 3.1 (ii) and (3.11).

We now introduce the notation $m := \lfloor \alpha \rfloor \geq 2$ and $\beta := \alpha - m \in [0,1)$. By [**10**, pp. 333–334], it holds

$$\begin{aligned}&\varphi(\theta) \text{ is of class } C^m(\mathbb{R}),\\ &\mathcal{F}q(\theta) \text{ is of class } C^{m-1}(\mathbb{R}) \cap C^m(\mathbb{R}\setminus\{0\}),\\ &(\mathcal{F}q)^{(m)}(\theta) = o_{\theta\to 0}(|\theta|^{-1}),\\ &\mathcal{F}r(\theta) \text{ is of class } C^{m-2}(\mathbb{R}) \cap C^m(\mathbb{R}\setminus\{0\}),\\ &(\mathcal{F}r)^{(m-1)}(\theta) = o_{\theta\to 0}(|\theta|^{-1}),\\ &(\mathcal{F}r)^{(m)}(\theta) = o_{\theta\to 0}(|\theta|^{-2}).\end{aligned}$$

Now it is straightforward to verify that $\Psi_2(\theta)$ is of class $C^m(\mathbb{R})$ using (3.14) and the Leibniz rule. As a consequence $\Psi_2^{(k)}(\theta)$ is integrable on $[-1,1]$ for $k \in \{1,\ldots,m\}$. We next derive the integrability on $[-1,1]^c$ from (3.13). By (2.1) and $\inf_{|\theta|\geq 1}|1-\varphi(\theta)| > 0$ we have

$$\frac{d^k}{d\theta^k}\frac{\varphi(\theta)^{N+1}}{1-\varphi(\theta)} \leq \text{const}\,|\varphi(\theta)|^{N+1-k}$$

for any $\theta \in [-1,1]^c$ and any $k \in \{1,\ldots,m\}$. By its definition, $N \geq p+m$ and hence $|\varphi(\theta)|^{N+1-k} \in L^1(\mathbb{R})$. By Lemma 2.4 (iii) and Lemma 2.5 (iv) we easily deduce that $|(\mathcal{F}v)^{(k)}(\theta)| + |(\mathcal{F}w)^{(k)}(\theta)| = O_{\theta\to\pm\infty}(|\theta|^{-2})$ for any $k \in \{1,\ldots,m\}$. Now it is an easy conclusion that $\Psi_2^{(k)}(\theta)$ is integrable on $[-1,1]^c$ and $\lim_{\theta\to\pm\infty}\Psi_2^{(k)}(\theta)=0$ for any $k\in\{1,\dots,m\}$. Hence by the integration by parts of (3.15),

$$2\pi(it)^m u_2(t)=\int_{-\infty}^{\infty}e^{-it\theta}\Psi_2^{(m)}(\theta)d\theta.$$

We now complete the proof for the case $\beta=0$: By the Riemann-Lebesgue theorem we have $t^m u_2(t)=o_{t\to\pm\infty}(1)$ and this is equivalent with $u_2(t)=o_{t\to\pm\infty}(|t|^{-\alpha})$.

We continue the proof for the case $0<\beta<1$. Recall the notation $\Delta_\varepsilon g(\theta)=g(\theta)-g(\theta+\varepsilon)$ for $\varepsilon\in(0,1/2)$ and a function $g(\theta)$. The estimate

$$\sup_{\theta\in\mathbb{R}}\left|\Delta_\varepsilon\varphi^{(m)}(\theta)\right|=o_{\varepsilon\to+0}(\varepsilon^\beta)$$

can be verified by using $\varphi^{(m)}(\theta)=\int_{-\infty}^{\infty}e^{i\theta t}i^m t^m f(t)dt$ and the method of proof of (2.3). We set $\varepsilon=\pi/|t|$ for t with $|t|>2\pi$. By the same argument for (3.12) we have

$$4\pi(it)^m u_2(t)=\int_{-\infty}^{\infty}e^{-it\theta}\Delta_\varepsilon\Psi_2^{(m)}(\theta)d\theta. \tag{3.16}$$

Since $\alpha=m+\beta$, (1.2) is reduced to the estimate $\int_{-\infty}^{\infty}|\Delta_\varepsilon\Psi_2^{(m)}(\theta)|d\theta=o_{\varepsilon\to+0}(\varepsilon^\beta)$. The estimate

$$\int_{\{|\theta|\le 2\varepsilon\}}|\Delta_\varepsilon\Psi_2^{(m)}(\theta)|d\theta\le\int_{\{|\theta|\le 3\varepsilon\}}2|\Psi_2^{(m)}(\theta)|d\theta=O_{\varepsilon\to+0}(\varepsilon)\le o_{\varepsilon\to+0}(\varepsilon^\beta)$$

follows from $\Psi_2(\theta)\in C^m(\mathbb{R})$.

We can obtain the estimate $\int_{[-1,1]^c}|\Delta_\varepsilon\Psi_2^{(m)}(\theta)|d\theta=o_{\varepsilon\to+0}(\varepsilon^\beta)$ by the same argument as the proof of Lemma 3.1 (iv) using (3.13) and the following estimates that can be obtained in an elementary way:

$$\sup_{\theta\in[-1,1]^c}\frac{1}{|\varphi(\theta)|^{\lceil p\rceil}+|\varphi(\theta+\varepsilon)|^{\lceil p\rceil}}\left|\Delta_\varepsilon\frac{d^m}{d\theta^m}\frac{\varphi(\theta)^{N+1}}{1-\varphi(\theta)}\right|=o_{\varepsilon\to+0}(\varepsilon^\beta),$$

$$\sup_{\theta\in[-1,1]^c}|\theta|^2\left(|\Delta_\varepsilon(\mathcal{F}v)^{(m)}(\theta)|+|\Delta_\varepsilon(\mathcal{F}w)^{(m)}|\right)=o_{\varepsilon\to+0}(\varepsilon^\beta).$$

We proceed to the estimate of integral over $\{2\varepsilon<|\theta|<1\}$. By inspecting more carefully than [**10**, pp. 333–334] we have

$$|(\mathcal{F}q)^{(m)}(\theta)|=o_{\theta\to0}(|\theta|^{-1+\beta}),$$

$$|(\mathcal{F}r)^{(m-1)}(\theta)|=o_{\theta\to0}(|\theta|^{-1+\beta}),$$

$$|(\mathcal{F}r)^{(m)}(\theta)|=o_{\theta\to0}(|\theta|^{-2+\beta}).$$

It is elementary and tedious to deduce the estimates

$$
\begin{aligned}
|\Delta_\varepsilon(\mathcal{F}q)^{(m-1)}(\theta)| &= \tilde{o}_{\varepsilon\to+0}(|\theta|^{-1+\beta}\varepsilon) \le \tilde{o}_{\varepsilon\to+0}(\varepsilon^\beta),\\
|\Delta_\varepsilon(\mathcal{F}q)^{(m)}(\theta)| &= \tilde{o}_{\theta\to0}(\varepsilon^\beta|\theta|^{-1}),\\
|\Delta_\varepsilon(\mathcal{F}r)^{(m-2)}(\theta)| &= \tilde{o}_{\varepsilon\to+0}(|\theta|^{-1+\beta}\varepsilon) \le \tilde{o}_{\varepsilon\to+0}(\varepsilon^\beta),\\
|\Delta_\varepsilon(\mathcal{F}r)^{(m-1)}(\theta)| &= \tilde{o}_{\varepsilon\to+0}(|\theta|^{-2+\beta}\varepsilon),\\
|\Delta_\varepsilon(\mathcal{F}r)^{(m)}(\theta)| &= \tilde{o}_{\varepsilon\to+0}(\varepsilon^\beta|\theta|^{-2}),
\end{aligned}
$$

that are valid when $\varepsilon \in (0,1/2)$ and $2\varepsilon < |\theta| < 1$. Among many terms that appear in the mth derivative of (3.14) we investigate here the term $\theta^2(\mathcal{F}r)^{(m)}(\theta)\chi(\theta)$ where $\chi(\theta)$ is a continuous function that satisfies $\Delta_\varepsilon\chi(\theta) = \tilde{o}_{\varepsilon\to+0}(\varepsilon^\beta)$. The difference of this term is estimated as follows using $1/2 < |\theta+\varepsilon|/|\theta| < 3/2$:

$$
\begin{aligned}
&\Delta_\varepsilon(\theta^2(\mathcal{F}r)^{(m)}(\theta)\chi(\theta))\\
&\quad= \Delta_\varepsilon(\theta^2)(\mathcal{F}r)^{(m)}(\theta)\chi(\theta) + (\theta+\varepsilon)^2\Delta_\varepsilon((\mathcal{F}r)^{(m)}(\theta))\chi(\theta)\\
&\qquad+ (\theta+\varepsilon)^2(\mathcal{F}r)^{(m)}(\theta+\varepsilon)\Delta_\varepsilon(\chi(\theta))\\
&\quad= \tilde{o}_{\varepsilon\to+0}(|\theta|^{-1+\beta}\varepsilon) + \tilde{o}_{\varepsilon\to+0}(\varepsilon^\beta) + \tilde{o}_{\varepsilon\to+0}(|\theta|^\beta\varepsilon^\beta)\\
&\quad= \tilde{o}_{\varepsilon\to+0}(\varepsilon^\beta).
\end{aligned}
$$

By similar investigation we can deduce that $\Delta_\varepsilon\Psi_2^{(m)}(\theta) = \tilde{o}_{\varepsilon\to+0}(\varepsilon^\beta)$ and hence

$$
\int_{\{2\varepsilon\le|\theta|\le1\}} |\Delta_\varepsilon\Psi_2^{(m)}(\theta)|d\theta = \tilde{o}_{\varepsilon\to+0}(\varepsilon^\beta). \qquad \square
$$

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