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A generalization of twisted modules over vertex algebras

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Abstract. For an arbitrary positive integer T we introduce the notion of a (V,T)-module over a vertex algebra V, which is a generalization of a twisted V-module. Under some conditions on V, we construct an associative algebra $A_m^T(V)$ for $m \in (1/T)\mathbb{N}$ and an $A_m^T(V)$-$A_n^T(V)$-bimodule $A_{n,m}^T(V)$ for $n, m \in (1/T)\mathbb{N}$ and we establish a one-to-one correspondence between the set of isomorphism classes of simple left $A_0^T(V)$-modules and that of simple $(1/T)\mathbb{N}$-graded (V,T)-modules.

1. Introduction.

Twisted modules (or twisted sectors) were introduced in the study of the so-called orbifold models of conformal field theory (cf. [**2**], [**3**]). Let V be a vertex operator algebra and G a finite automorphism group of V. In terms of vertex operator algebras, the study of the orbifold models corresponds to the study of the subalgebra V^G of G-invariants in V. One of the main problems about V^G is to describe the V^G-modules in terms of V and G. Twisted modules have been studied systematically as representations of V related to this problem (cf. [**6**], [**11**], [**13**], [**15**]). For $g \in G$, every g-twisted V-module becomes a V^G-module. Moreover, it is conjectured that under some conditions on V, every simple V^G-module is contained in some simple g-twisted V-module for some $g \in G$ (cf. [**2**]). However, the following easy observation tells us an inconvenience of twisted V-modules from the representation theoretic viewpoint: let g, h be two different elements of G, M a g-twisted V-module and N an h-twisted V-module. Although the direct sum $M \oplus N$ is a V^G-module, this is not a (twisted) V-module in general. This is one of obstructions to develop the representation theory of V^G.

In this paper, for a vertex algebra V and a positive integer T we first introduce the notion of a (V,T)-*module* (cf. Definition 2.1), which is a generalization of a twisted V-module, in order to resolve the inconvenience just mentioned above. Roughly speaking, a (V,T)-module is a "twisted V-module" without automorphisms. We next generalize some results by Zhu[**17**] to (V,T)-modules. In [**17**], if V is a vertex operator algebra, then Zhu constructed an associative algebra $A(V)$ and established a one-to-one correspondence between the set of isomorphism classes of the simple $A(V)$-modules and that of the simple V-modules with some conditions. Some generalizations of $A(V)$ have been obtained in [**4**], [**5**], [**6**], [**7**], [**8**] and they have played an important role in the representation theory of V. We shall show the following results for a vertex algebra V with a grading $V = \bigoplus_{i=\Delta}^{\infty} V_i$

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such that $\Delta \in \mathbb{Z}_{\leq 0}$, $\mathbf{1} \in V_0$ and for all homogeneous element $a \in V$, $a_i V_j \subset V_{\mathrm{wt}\, a-1-i+j}$, where $V_i = 0$ for $i < \Delta$. For every positive integer T and $n, m \in (1/T)\mathbb{N}$, we shall construct an associative algebra $A_m^T(V)$ and an $A_n^T(V)$-$A_m^T(V)$-bimodule $A_{n,m}^T(V)$ in Theorem 4.5. If $T = 1$, then $A_{n,m}^T(V)$ is the same as $A_{n,m}(V)$ in [**5**] and $A_n^T(V)$ is the same as $A_n(V)$ in [**7**]. In particular, $A_0^1(V)$ is the same as $A(V)$ in [**17**]. For an automorphism g of V of finite order, $A_{g,n,m}(V)$ in [**6**], [**8**] is a quotient of $A_{n,m}^{|g|}(V)$. For $m \in (1/T)\mathbb{N}$ and a left $A_m^T(V)$-module U, we shall show in Theorem 5.13 that the $(1/T)\mathbb{N}$-graded vector space $M(U) = \bigoplus_{n \in (1/T)\mathbb{N}} A_{n,m}^T(V) \bigotimes_{A_m^T(V)} U$ has a structure of (V,T)-modules with a universal property. In Corollary 5.14, we establish a one-to-one correspondence between the set of isomorphism classes of simple $A_0^T(V)$-modules and that of simple $(1/T)\mathbb{N}$-graded (V,T)-modules.

The organization of the paper is as follows. In Section 2 we introduce the notion of a (V,T)-module. In Section 3 we introduce a subspace $O_{n,m}^{T,1}(\alpha, \beta; z)$ of $\mathbb{C}[z, z^{-1}]$ for $n, m \in (1/T)\mathbb{N}$ and $\alpha, \beta \in \mathbb{Z}$ and study its properties. In Section 4 we construct an associative algebra $A_m^T(V)$ and an $A_n^T(V)$-$A_m^T(V)$-bimodule $A_m^T(V)$ for $n, m \in (1/T)\mathbb{N}$ by using the results in Section 4. In Section 5 we introduce the notion of a $(1/T)\mathbb{N}$-graded (V,T)-module and study a relation between the left $A_m^T(V)$-modules and the $(1/T)\mathbb{N}$-graded (V,T)-modules. Section 6 consists of two subsections. In Subsection 6.1 we compute the determinant of a matrix used in Section 3. In Subsection 6.2 we improve some results in [**16**]. In Section 7 we list some notations.

2. (V,T)-modules.

We assume that the reader is familiar with the basic knowledge on vertex algebras as presented in [**1**], [**6**], [**14**].

Throughout this paper, $\mathbb{N}$ denotes the set of all non-negative integers, T is a fixed positive integer and $(V, Y, \mathbf{1})$ is a vertex algebra. Recall that V is the underlying vector space, $Y(-,x)$ is the linear map from V to $(\mathrm{End}\, V)[[x, x^{-1}]]$, and $\mathbf{1}$ is the vacuum vector. For $i, j \in \mathbb{Z}$, define

$$\begin{aligned}
\mathbb{Z}_{\leq i} &= \{k \in \mathbb{Z} \mid k \leq i\},\\
\mathbb{Z}_{\geq i} &= \{k \in \mathbb{Z} \mid k \geq i\},\\
\mathbb{C}[x, x^{-1}]_{\leq i} &= \mathrm{Span}_{\mathbb{C}}\{x^k \mid k \leq i\},\\
\mathbb{C}[x, x^{-1}]_{j,i} &= \mathrm{Span}_{\mathbb{C}}\{x^k \mid j \leq k \leq i\}.
\end{aligned}$$

For $f(z) \in \mathbb{C}[z, z^{-1}]$ and $a, b \in V$, $f(z)|_{z^j = a_j b}$ denotes the element of V obtained from $f(z)$ by replacing z^j by $a_j b$ for all $j \in \mathbb{Z}$. For $i, j \in \mathbb{Q}$, define

$$\delta(i \leq j) = \begin{cases} 1 & \text{if } i \leq j, \\ 0 & \text{if } i > j. \end{cases} \tag{2.1}$$

Let M be a vector space over $\mathbb{C}$. Define three linear injective maps

$$\iota_{x,y} : M[[x^{1/T}, y^{1/T}]][x^{-1/T}, y^{-1/T}, (x-y)^{-1}] \to M((x^{1/T}))((y^{1/T})),$$
$$\iota_{y,x} : M[[x^{1/T}, y^{1/T}]][x^{-1/T}, y^{-1/T}, (x-y)^{-1}] \to M((y^{1/T}))((x^{1/T})),$$
$$\iota_{y,x-y} : M[[x^{1/T}, y^{1/T}]][x^{-1/T}, y^{-1/T}, (x-y)^{-1}] \to M((y^{1/T}))((x-y))$$

by

$$\iota_{x,y} f = \sum_{j,k.l} a_{j,k,l} \sum_{i=0}^{\infty} \binom{l}{i} (-1)^i x^{j+l-i} y^{k+i},$$
$$\iota_{y,x} f = \sum_{j,k.l} a_{j,k,l} \sum_{i=0}^{\infty} \binom{l}{i} (-1)^{l-i} y^{k+l-i} x^{j+i},$$
$$\iota_{y,x-y} f = \sum_{j,k.l} a_{j,k,l} \sum_{i=0}^{\infty} \binom{j}{i} y^{k+j-i} (x-y)^{l+i}$$

for $f = \sum_{j,k.l} a_{j,k,l} x^j y^k (x-y)^l \in M[[x^{1/T}, y^{1/T}]][x^{-1/T}, y^{-1/T}, (x-y)^{-1}]$, $a_{j,k,l} \in M$. We can also define the map

$$\iota_{x-y,y} : M[[x^{1/T}, y^{1/T}]][x^{-1/T}, y^{-1/T}, (x-y)^{-1}] \to M(((x-y)^{1/T}))((y^{1/T}))$$

similarly. Since $\iota_{x,y}(x-y)^i = \sum_{j=0}^{\infty} \binom{i}{j} x^{i-j} (-1)^j y^j$ and $\iota_{x-y,y} x^i = \sum_{j=0}^{\infty} \binom{i}{j} (x-y)^{i-j} y^j$, we identify $M(((x-y)^{1/T}))((y^{1/T}))$ with $M((x^{1/T}))((y^{1/T}))$ and $\iota_{x-y,y}$ with $\iota_{x,y}$.

Now we introduce a generalization of a twisted V-module.

Definition 2.1. Let M be a vector space over $\mathbb{C}$ and $Y_M(-, x)$ a linear map from V to $(\operatorname{End}_{\mathbb{C}} M)[[x^{1/T}, x^{-1/T}]]$. We call (M, Y_M) a *(V, T)-module* if

(1) For $a \in V$ and $w \in M$, $Y_M(a, x)w \in M((x^{1/T}))$.
(2) $Y_M(\mathbf{1}, x) = \mathrm{id}_M$.
(3) For $a, b \in V$ and $w \in M$, there is $F(a, b, w|x, y) \in M[[x^{1/T}, y^{1/T}]][x^{-1/T}, y^{-1/T}, (x-y)^{-1}]$ such that

$$\iota_{x,y} F(a, b, w|x, y) = Y_M(a, x) Y_M(b, y) w,$$
$$\iota_{y,x} F(a, b, w|x, y) = Y_M(b, y) Y_M(a, x) w, \quad \text{and}$$
$$\iota_{y,x-y} F(a, b, w|x, y) = Y_M(Y(a, x-y)b, y) w.$$

We note that in Definition 2.1, $F(a, b, w|x, y)$ is uniquely determined by $a, b \in V$ and $w \in M$ since $\iota_{x,y}$ is an injection. For a (V, T)-module M, a subspace N of M is called (V, T)-*submodule* of M if $(N, Y_M|_N)$ is a (V, T)-module, where $Y_M|_N$ is the restriction of Y_M to N. A non-zero (V, T)-module M is called *simple* if there is no submodule of M except 0 and M itself. For a submodule N of a (V, T)-module M, the quotient space M/N is clearly a (V, T)-module. For a set of (V, T)-modules $\{M_i\}_{i \in I}$, the direct sum $\bigoplus_{i \in I} M_i$ is a (V, T)-module.

Remark 2.2. It follows from Lemma 2.4 below that every $(V,1)$-module is a V-module and vice versa and that every g-twisted V-module is a $(V,|g|)$-module for an automorphism g of V of finite order.

Let T' be a positive multiple of T. Then every (V,T)-module is a (V,T')-module. Thus, for positive integers T_1 and T_2 the direct sum of a (V,T_1)-module and a (V,T_2)-module becomes a (V,T_3)-module, where T_3 is a positive common multiple of T_1 and T_2. Thus, (V,T)-modules are closed under direct sums in this sense, while twisted V-modules are not as stated in the introduction.

Example 2.3. We introduce an easy example of simple (V,T)-modules which is not a twisted V-module. Let U be a simple vertex operator algebra. Suppose the symmetric group S_3 of degree 3 is an automorphism group of U. Let $\sigma,\tau\in S_3$ such that $|\sigma|=3$ and $|\tau|=2$ and $M=\bigoplus_{j\in(1/3)\mathbb{N}}M(j)$ a simple σ-twisted U-module [**6**]. It follows from Remark 2.2 that M is a $(U,3)$-module. Restricting Y_M to $U^{\langle\tau\rangle}$, M becomes a $(U^{\langle\tau\rangle},3)$-module. We shall show M is a simple $(U^{\langle\tau\rangle},3)$-module. Let W be a non-zero $(U^{\langle\tau\rangle},3)$-submodule of M. We denote the subspace $\bigoplus_{j\in i/3+\mathbb{N}}M(j)$ of M by $M^i, i=0,1,2$. Since $\tau\sigma\tau=\sigma^{-1}\neq\sigma$, an improvement of [**16**, Theorem 2] (see Subsection 6.2) implies that M^0,M^1 and M^2 are all inequivalent simple U^{S_3}-modules. Thus, W contains at least one of M^0,M^1 and M^2 since $U^{S_3}\subset U^{\langle\tau\rangle}$. We denote the eigenspace $\{u\in U\mid \sigma u=e^{-2\pi\sqrt{-1}r/3}u\}$ of σ by $U^{(\sigma,r)}$, $r=0,1,2$. It follows by [**9**, Proposition 3.3] and [**12**, Theorem 1] that $U^{\langle\tau\rangle}\not\subset U^{\langle\sigma\rangle}$ and hence there exists $a=a^0+a^1+a^2\in U^{\langle\tau\rangle}$, $a^r\in U^{(\sigma,r)}$ such that at least one of a^1,a^2 is not zero. Since

$$Y_M(a,x)=\sum_{i\in\mathbb{Z}}a_i^0x^{-i-1}+\sum_{i\in 1/3+\mathbb{Z}}a_i^1x^{-i-1}+\sum_{i\in 2/3+\mathbb{Z}}a_i^2x^{-i-1}$$

and M is a simple σ-twisted U-module, W contains at least two of M^0,M^1 and M^2. Repeating the same argument, we obtain that M is a simple $(U^{\langle\tau\rangle},3)$-module.

Since at least one of a^1,a^2 above is not zero, M is not a $U^{\langle\tau\rangle}$-module. Suppose M is a g-twisted $U^{\langle\tau\rangle}$-module for some $g\in\operatorname{Aut}U^{\langle\tau\rangle}$ of order 3. Then, the eigenspace $(U^{\langle\tau\rangle})^{(g,r)}=\{v\in U^{\langle\tau\rangle}\mid gv=e^{-2\pi\sqrt{-1}r/3}v\}$ of g is a subspace of $U^{(\sigma,r)}$ for each $r=0,1,2$ since $Y_M(b,x)=\sum_{j\in r/3+\mathbb{Z}}b_jx^{-j-1}$ for $b\in(U^{\langle\tau\rangle})^{(g,r)}$. Therefore, $(U^{\langle\tau\rangle})^{(g,1)}=(U^{\langle\tau\rangle})^{(g,2)}=0$ since there is no representation ρ of S^3 such that $\rho(\sigma)=e^{-2\pi\sqrt{-1}r/3}$ and $\rho(\tau)=1$ for $r=1,2$. This contradicts to that the order of g is equal to 3. We conclude that M is not a twisted $U^{\langle\tau\rangle}$-module.

Let M be a vector space. For $s=0,\dots,T-1$ and $X(x,y)=\sum_{i,j\in(1/T)\mathbb{Z}}X_{ij}x^iy^j\in M[[x^{1/T},x^{-1/T},y^{1/T},y^{-1/T}]]$, $X_{ij}\in M$, we define

$$\begin{aligned}X(x,y)^{s,x}&=\sum_{\substack{i\in s/T+\mathbb{Z}\\ j\in(1/T)\mathbb{Z}}}X_{ij}x^iy^j\quad\text{and}\\ X(x,y)^{s,y}&=\sum_{\substack{i\in(1/T)\mathbb{Z}\\ j\in s/T+\mathbb{Z}}}X_{ij}x^iy^j\end{aligned}\tag{2.2}$$

in $M[[x^{1/T},x^{-1/T},y^{1/T},y^{-1/T}]]$. In the same way, for $s=0,\dots,T-1$ and $X(x,y)=\sum_{i,j\in(1/T)\mathbb{Z}}\sum_{k\in\mathbb{Z}}X_{ijk}x^iy^j(x-y)^k\in M[[x^{1/T},y^{1/T}]][x^{-1/T},y^{-1/T},(x-y)^{-1}]$, we define

$$X(x,y)^{s,x}=\sum_{\substack{i\in s/T+\mathbb{Z}\\ j\in(1/T)\mathbb{Z}}}\sum_{k\in\mathbb{Z}}X_{ijk}x^iy^j(x-y)^k \quad\text{and}$$
$$X(x,y)^{s,y}=\sum_{\substack{i\in(1/T)\mathbb{Z}\\ j\in s/T+\mathbb{Z}}}\sum_{k\in\mathbb{Z}}X_{ijk}x^iy^j(x-y)^k \tag{2.3}$$

in $M[[x^{1/T},y^{1/T}]][x^{-1/T},y^{-1/T},(x-y)^{-1}]$. Clearly

$$\sum_{s=0}^{T-1}X(x,y)^{s,x}=\sum_{s=0}^{T-1}X(x,y)^{s,y}=X(x,y).$$

For $0\le s\le T-1$, $j\in s/T+\mathbb{Z}$, $k\in(1/T)\mathbb{Z}$ and $l\in\mathbb{Z}$, the following fact is well known and straightforward:

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)\iota_{x_1,x_2}\big((x_1^jx_2^kx_0^l)|_{x_0=x_1-x_2}\big)-x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)\iota_{x_2,x_1}\big((x_1^jx_2^kx_0^l)|_{x_0=x_1-x_2}\big)$$
$$=x_1^{-1}\left(\frac{x_2+x_0}{x_1}\right)^{-s/T}\delta\left(\frac{x_2+x_0}{x_1}\right)\iota_{x_2,x_0}\big((x_1^jx_2^kx_0^l)|_{x_1=x_2+x_0}\big). \tag{2.4}$$

The argument in the proof of the following lemma is well known (cf. [**14**, Sections 3.2–3.4]).

LEMMA 2.4. *Let $A(x_1,x_2)\in M((x_1^{1/T}))((x_2^{1/T}))$, $B(x_2,x_1)\in M((x_2^{1/T}))((x_1^{1/T}))$, and $C(x_2,x_0)\in M((x_2^{1/T}))((x_0))$. Then, the three following conditions are equivalent.*

(1) *There is $F\in M[[x_1^{1/T},x_2^{1/T}]][x_1^{-1/T},x_2^{-1/T},(x_1-x_2)^{-1}]$ such that*

$$\iota_{x_1,x_2}F=A(x_1,x_2),\quad \iota_{x_2,x_1}F=B(x_2,x_1),\ \text{and}\ \ \iota_{x_2,x_1-x_2}F=C(x_2,x_1-x_2).$$

(2) *There are $C^{[s]}(x_2,x_0)\in M((x_2^{1/T}))((x_0))$, $s=0,\dots,T-1$ such that $\sum_{s=0}^{T-1}C^{[s]}(x_2,x_0)=C(x_2,x_0)$ and*

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)A(x_1,x_2)^{s,x_1}-x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)B(x_2,x_1)^{s,x_1}$$
$$=x_1^{-1}\left(\frac{x_2+x_0}{x_1}\right)^{-s/T}\delta\left(\frac{x_2+x_0}{x_1}\right)C^{[s]}(x_2,x_0). \tag{2.5}$$

(3) *There are positive integers l,q and $C^{[s]}(x_2,x_0)\in M((x_2^{1/T}))((x_0))$, $s=0,\dots,T-1$ such that $\sum_{s=0}^{T-1}C^{[s]}(x_2,x_0)=C(x_2,x_0)$,*

$$(x_1 - x_2)^l A(x_1, x_2) = (x_1 - x_2)^l B(x_2, x_1) \tag{2.6}$$

in $M[[x_1^{1/T}, x_2^{1/T}, x_1^{-1/T}, x_2^{-1/T}]]$ *and*

$$\begin{aligned} &\iota_{x_0,x_2}(x_0 + x_2)^{-s/T+q}(A(x_1, x_2)^{s,x_1})|_{x_1=x_0+x_2} \\ &\quad = \iota_{x_2,x_0}(x_2 + x_0)^{-s/T+q} C^{[s]}(x_2, x_0) \end{aligned} \tag{2.7}$$

in $M[[x_0, x_2^{1/T}, x_0^{-1}, x_2^{-1/T}]]$.

In this case, F *and* $C^{[s]}(x_2, x_0)$, $s = 0, \dots, T-1$ *are uniquely determined by* $A(x_1, x_2)$, $B(x_2, x_1)$ *and* $C(x_2, x_0)$.

PROOF. We show (1) implies (2). Define $C^{[s]}(x_2, x_0) \in M((x_2^{1/T}))((x_0))$ by $C^{[s]}(x_2, x_1 - x_2) = \iota_{x_2,x_1-x_2} F^{s,x_1} \in M((x_2^{1/T}))((x_1 - x_2))$ for $s = 0, \dots, T-1$. Clearly, $\sum_{s=0}^{T-1} C^{[s]}(x_2, x_0) = C(x_2, x_0)$. Since $\iota_{x_1,x_2} F^{s,x_1} = A(x_1, x_2)^{s,x_1}$ and $\iota_{x_2,x_1} F^{s,x_1} = B(x_2, x_1)^{s,x_1}$ for $s = 0, \dots, T-1$, (2.5) follows from (2.4).

We show (2) implies (3). Let l be a positive integer such that $x_0^l C^{[s]}(x_2, x_0) \in M((x_2^{1/T}))[[x_0]]$ for all $s = 0, \dots, T-1$. Multiplying (2.5) by x_0^l and then taking Res_{x_0}, we have $(x_1 - x_2)^l A(x_1, x_2)^{s,x_1} = (x_1 - x_2)^l B(x_2, x_1)^{s,x_1}$ and hence (2.6). Let q be a positive integer such that $x_1^{-s/T+q} B(x_2, x_1)^{s,x_1} \in M((x_2^{1/T}))[[x_1]]$ for all $s = 0, \dots, T-1$. Multiplying (2.5) by $x_1^{-s/T+q}$ and then taking Res_{x_1}, we have (2.7).

We show (3) implies (1). Since the left-hand side of (2.6) is an element of $M((x_1^{1/T}))((x_2^{1/T}))$ and the right-hand side of (2.6) is an element of $M((x_2^{1/T}))((x_1^{1/T}))$, $G = (x_1 - x_2)^l A(x_1, x_2) (= (x_1 - x_2)^l B(x_2, x_1))$ is an element of $M[[x_1^{1/T}, x_2^{1/T}]][x_1^{-1/T}, x_2^{-1/T}]$. Define

$$F = (x_1 - x_2)^{-l} G \in M[[x_1^{1/T}, x_2^{1/T}]][x_1^{-1/T}, x_2^{-1/T}, (x_1 - x_2)^{-1}].$$

It is clear that $\iota_{x_1,x_2} F = A(x_1, x_2)$ and $\iota_{x_2,x_1} F = B(x_2, x_1)$. Applying the same argument to (2.7), we obtain $H_s \in M[[x_1^{1/T}, x_2^{1/T}]][x_1^{-1/T}, x_2^{-1/T}, (x_1 - x_2)^{-1}]$, $s = 0, \dots, T-1$ such that

$$\begin{aligned} &\iota_{x_1-x_2,x_2} H_s = A(x_1, x_2)^{s,x_1} = A((x_1 - x_2) + x_2, x_2)^{s,x_1} \text{ and} \\ &\iota_{x_2,x_1-x_2} H_s = C^{[s]}(x_2, x_1 - x_2). \end{aligned}$$

Since $M(((x_1 - x_2)^{1/T}))((x_2^{1/T})) = M((x_1^{1/T}))((x_2^{1/T}))$ and ι_{x_1,x_2} is injective, we have $F^{s,x_1} = H_s$ for all $s = 0, \dots, T-1$ and therefore $\iota_{x_2,x_1-x_2} F = C(x_2, x_1 - x_2)$.

We show F and $C^{[s]}(x_2, x_0)$, $s = 0, \dots, T-1$ are uniquely determined. Since ι_{x_1,x_2} is injective and $\iota_{x_1,x_2} F = A(x_1, x_2)$, F is uniquely determined. In the above argument that (3) implies (1), we have constructed F such that $\iota_{x_1,x_2} F = A(x_1, x_2)$ and $\iota_{x_2,x_1-x_2} F^{s,x_1} = C^{[s]}(x_2, x_1 - x_2)$. Thus, $C^{[s]}(x_2, x_0)$, $s = 0, \dots, T-1$ in (3) are uniquely determined. A similar argument shows that $C^{[s]}(x_2, x_0)$, $s = 0, \dots, T-1$ in (2) are uniquely determined and that $C^{[s]}(x_2, x_0)$ in (2) is the same as that in (3) for each s. $\square$

REMARK 2.5. The following facts for (2.5) are well known and straightforward.

(1) A direct computation shows that (2.5) is equivalent to

$$\begin{aligned} & x_0^{-1}\delta\Big(\frac{x_1-x_2}{x_0}\Big)A(x_1,x_2) - x_0^{-1}\delta\Big(\frac{-x_2+x_1}{x_0}\Big)B(x_2,x_1) \\ &\quad = \frac{1}{T}\sum_{j=0}^{T-1} x_1^{-1}\delta\Big(e^{2\pi\sqrt{-1}j/T}\frac{(x_2+x_0)^{1/T}}{x_1^{1/T}}\Big)\sum_{s=0}^{T-1} e^{2\pi\sqrt{-1}js/T}C^{[s]}(x_2,x_0). \end{aligned} \tag{2.8}$$

(2) If we write $A(x_1,x_2)=\sum_{p,q}A_{(p,q)}x_1^{-p-1}x_2^{-q-1}$, $B(x_2,x_1)=\sum_{p,q}B_{(p,q)}x_2^{-p-1}x_1^{-q-1}$ and $C^{[s]}(x_2,x_0)=\sum_{p,q}C^{[s]}_{(p,q)}x_2^{-p-1}x_0^{-q-1}$, where $A_{(p,q)},B_{(p,q)},C_{(p,q)}\in M$, then we have

$$\sum_{i=0}^{\infty}\binom{l}{i}(-1)^i\big(A_{(l+j-i,k+i)}+(-1)^{l+1}B_{(l+k-i,j+i)}\big)=\sum_{i=0}^{\infty}\binom{j}{i}C^{[-s]}_{(j+k-i,l+i)} \tag{2.9}$$

for $0\le s\le T-1$, $j\in s/T+\mathbb{Z}, k\in(1/T)\mathbb{Z}$ and $l\in\mathbb{Z}$ by comparing the coefficients of both sides of (2.5). Thus, a direct computation shows that (2.5) is also equivalent to the condition that

$$\begin{aligned} & \mathrm{Res}_{x_1} A(x_1,x_2)\iota_{x_1,x_2}\big(x_1^j x_2^k(x_1-x_2)^l\big) - \mathrm{Res}_{x_1} B(x_2,x_1)\iota_{x_2,x_1}\big(x_1^j x_2^k(x_1-x_2)^l\big) \\ &\quad = \mathrm{Res}_{x_1-x_2} C^{[-s]}(x_2,x_1-x_2)\iota_{x_2,x_1-x_2}\big(x_1^j x_2^k(x_1-x_2)^l\big) \end{aligned} \tag{2.10}$$

in $M[[x_2^{1/T},x_2^{-1/T}]]$ for all $0\le s\le T-1$, $j\in s/T+\mathbb{Z}$, $k\in(1/T)\mathbb{Z}$ and $l\in\mathbb{Z}$. Here, Res_x is defined by

$$\mathrm{Res}_x f(x)=f_{-1}$$

for $f(x)=\sum_{i\in(1/T)\mathbb{Z}}f_i x^i\in M[[x^{1/T},x^{-1/T}]]$.

REMARK 2.6. For $q\in\mathbb{Z}$ we denote by $M(\!(x_2^{1/T})\!)(\!(x_0)\!)_{\ge q}$ the set of all elements in $M(\!(x_2^{1/T})\!)(\!(x_0)\!)$ of the form $\sum_{\substack{i\in(1/T)\mathbb{Z}\\ j\in\mathbb{Z}_{\ge q}}}X_{ij}x_2^i x_0^j$. Suppose $C(x_2,x_0)$ in Lemma 2.4 is an element of $M(\!(x_2^{1/T})\!)(\!(x_0)\!)_{\ge q}$. Since $\iota_{x_2,x_1-x_2}x_1^j x_2^k(x_1-x_2)^l=\sum_{i=0}^{\infty}\binom{j}{i}x_2^{k+j-i}(x_1-x_2)^{l+i}$, we see that F in Lemma 2.4 (1) has the form $F=(x_1-x_2)^qG$, where $G\in M[[x_1^{1/T},x_2^{1/T}]][x_1^{-1/T},x_2^{-1/T}]$. Thus, $C^{[s]}(x_2,x_1-x_2)=\iota_{x_2,x_1-x_2}F^{s,x_1}\in M(\!(x_2^{1/T})\!)(\!(x_1-x_2)\!)_{\ge q}$ and hence $C^{[s]}(x_2,x_0)\in M(\!(x_2^{1/T})\!)(\!(x_0)\!)_{\ge q}$ for all $s=0,\dots,T-1$.

Let M be a (V,T)-module. For $a\in V$ and $s=0,\dots,T-1$, we define $Y_M^s(a,x)$ by

$$Y_M^s(a,x)=\sum_{i\in s/T+\mathbb{Z}}a_i x^{-i-1}. \tag{2.11}$$

Let $a,b\in V$ and $w\in M$. We apply Lemma 2.4 to $A(x_1,x_2)=Y_M(a,x_1)Y_M(b,x_2)w$, $B(x_2,x_1)=Y_M(b,x_2)Y_M(a,x_1)w$ and $C(x_2,x_0)=Y_M(Y(a,x_0)b,x_2)w$. In this case F in Lemma 2.4 (1) is equal to $F(a,b,w|x_1,x_2)$ in Definition 2.1 (3). We denote by $Y_M^{(-s)}(a,b|x_2,x_0)(w)$ the element $C^{[s]}(x_2,x_0)$ of $M(\!(x_2^{1/T})\!)(\!(x_0)\!)$, $s=0,\dots,T-1$ in this case. That is,

$$Y_M^{(s)}(a,b|x_2,x_1-x_2)(w)=\iota_{x_2,x_1-x_2}(F(a,b,w|x_1,x_2)^{-s,x_1}), \tag{2.12}$$

where $F(a,b,w|x_1,x_2)^{-s,x_1}$ is defined by (2.3). The conditions in Lemma 2.4 (2) become

$$\sum_{s=0}^{T-1}Y_M^{(s)}(a,b|x_2,x_0)(w)=Y_M(Y(a,x_0)b,x_2)w \tag{2.13}$$

and

$$\begin{aligned}&x_0^{-1}\delta\Big(\frac{x_1-x_2}{x_0}\Big)Y_M^s(a,x_1)Y_M(b,x_2)w-x_0^{-1}\delta\Big(\frac{-x_2+x_1}{x_0}\Big)Y_M(b,x_2)Y_M^s(a,x_1)w\\&\quad=x_1^{-1}\Big(\frac{x_2+x_0}{x_1}\Big)^{s/T}\delta\Big(\frac{x_2+x_0}{x_1}\Big)Y_M^{(s)}(a,b|x_2,x_0)(w).\end{aligned} \tag{2.14}$$

The uniqueness of $F(a,b,w|x_1,x_2)$ for each $a,b\in V$ and $w\in M$ implies that for fixed $a,b\in V$ the map $Y_M^{(s)}(a,b|x_2,x_0):M\to M(\!(x_2^{1/T})\!)(\!(x_0)\!)$ is linear and that the map $V\times V\ni(a,b)\mapsto Y_M^{(s)}(a,b|x_2,x_0)\in\mathrm{Hom}_{\mathbb{C}}(M,M(\!(x_2^{1/T})\!)(\!(x_0)\!))$ is bilinear. We write

$$Y_M^{(s)}(a,b|x_2,x_0)=\sum_{i\in(1/T)\mathbb{Z}}\sum_{j\in\mathbb{Z}}Y_M^{(s)}(a,b;i,j)x_2^{-i-1}x_0^{-j-1},$$

where $Y_M^{(s)}(a,b;i,j)\in\mathrm{End}_{\mathbb{C}}M$.

Remark 2.7. Let g be an automorphism of V of finite order, t a positive multiple of $|g|$ and (M,Y_M) a g-twisted V-module. As stated in Remark 2.2, (M,Y_M) is a (V,t)-module by Lemma 2.4. We explain what is $Y_M^{(s)}(a,b|x_2,x_0)$ for $a,b\in V$ and $s=0,\dots,t-1$ in this case. We denote by $V^{(g,r)}$, $r=0,\dots,t-1$ the eigenspace $\{v\in V\mid gv=e^{-2\pi\sqrt{-1}r/t}v\}$ of g. For $a\in V$, we denote by $a^{(g,r)}$ the r-th component of a in the decomposition $V=\bigoplus_{r=0}^{t-1}V^{(g,r)}$, that is, $a=\sum_{r=0}^{t-1}a^{(g,r)}$, $a^{(g,r)}\in V^{(g,r)}$.

Let $0\le s\le t-1$, $a,b\in V$ and $w\in M$. Since

$$\begin{aligned}&(Y_M(a,x_1)Y_M(b,x_2)w)^{-s,x_1}=Y_M(a^{(g,s)},x_1)Y_M(b,x_2)w\quad\text{and}\\&(Y_M(b,x_2)Y_M(a,x_1)w)^{-s,x_1}=Y_M(b,x_2)Y_M(a^{(g,s)},x_1)w,\end{aligned}$$

it follows by (2.5) that

$$Y_M^{(s)}(a,b|x_2,x_0)(w)=Y_M(Y(a^{(g,s)},x_0)b,x_2)w. \tag{2.15}$$

Let $a,b\in V$, $w\in M$, $j,k\in(1/T)\mathbb{Z}$, $l\in\mathbb{Z}$ and s the integer uniquely determined by the conditions $0\le s\le T-1$ and $s/T\equiv j \pmod{\mathbb{Z}}$. It follows by (2.9) or by comparing the coefficients of both sides of (2.14) that

$$\begin{aligned}&\sum_{i=0}^{\infty}\binom{j}{i}Y_M^{(s)}(a,b;j+k-i,l+i)(w)\\&\quad=\sum_{i=0}^{\infty}\binom{l}{i}(-1)^i(a_{l+j-i}b_{k+i}+(-1)^{l+1}b_{l+k-i}a_{j+i})w.\end{aligned}\tag{2.16}$$

It follows by (2.10) that

$$\begin{aligned}&\operatorname{Res}_{x_1-x_2}\iota_{x_2,x_1-x_2}\big(x_1^jx_2^k(x_1-x_2)^l\big)Y_M^{(s)}(a,b|x_2,x_1-x_2)(w)\\&\quad=\operatorname{Res}_{x_1}\iota_{x_1,x_2}\big(x_1^jx_2^k(x_1-x_2)^l\big)Y_M(a,x_1)Y_M(b,x_2)w\\&\qquad-\operatorname{Res}_{x_1}\iota_{x_2,x_1}\big(x_1^jx_2^k(x_1-x_2)^l\big)Y_M(b,x_2)Y_M(a,x_1)w\\&\quad=\operatorname{Res}_{x_1}\iota_{x_1,x_2}\big(x_1^jx_2^k(x_1-x_2)^l\big)Y_M^s(a,x_1)Y_M(b,x_2)w\\&\qquad-\operatorname{Res}_{x_1}\iota_{x_2,x_1}\big(x_1^jx_2^k(x_1-x_2)^l\big)Y_M(b,x_2)Y_M^s(a,x_1)w.\end{aligned}\tag{2.17}$$

Lemma 2.8. *We use the notation above. Let L be an integer such that $a_ib=0$ for all $i\in\mathbb{Z}_{\ge L+1}$. Then*

$$\begin{aligned}&Y_M^{(s)}(a,b;j+k,l)(w)\\&=\sum_{m=0}^{L-l}\binom{-j}{m}\sum_{i=0}^{\infty}\binom{l+m}{i}(-1)^i(a_{l+m+j-i}b_{k-m+i}+(-1)^{l+m+1}b_{l+k-i}a_{j+i})w.\end{aligned}\tag{2.18}$$

Proof. It follows from Remark 2.6 that $Y_M^{(s)}(a,b|x_2,x_0)(w)\in M((x_2^{1/T}))\cdot((x_0))_{\ge -L-1}$. Thus, if $l>L$, then the both-sides of (2.18) are equal to 0. Suppose $l\le L$. Define

$$R(m)=\sum_{i=0}^{\infty}\binom{m}{i}(-1)^i(a_{m+j-i}b_{k-m+l+i}+(-1)^{m+1}b_{l+k-i}a_{j+i})w$$

for $m\in\mathbb{Z}_{\le L}$. Since

$$\begin{pmatrix}1&0&\cdots&\cdots&0\\\binom{j}{1}&1&\ddots&&\vdots\\\binom{j}{2}&\ddots&\ddots&\ddots&\vdots\\\vdots&\ddots&\ddots&\ddots&0\\\binom{j}{L-l}&\cdots&\binom{j}{2}&\binom{j}{1}&1\end{pmatrix}\begin{pmatrix}Y_M^{(s)}(a,b;j+k+l-L,L)(w)\\Y_M^{(s)}(a,b;j+k+l-L+1,L-1)(w)\\\vdots\\Y_M^{(s)}(a,b;j+k,l)(w)\end{pmatrix}=\begin{pmatrix}R(L)\\R(L-1)\\\vdots\\R(l)\end{pmatrix}$$

by (2.16), we have

$$\begin{pmatrix} Y_M^{(s)}(a,b;j+k+l-L,L)(w) \\ Y_M^{(s)}(a,b;j+k+l-L+1,L-1)(w) \\ \vdots \\ Y_M^{(s)}(a,b;j+k,l)(w) \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ \binom{-j}{1} & 1 & \ddots & & \vdots \\ \binom{-j}{2} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \binom{-j}{L-l} & \cdots & \binom{-j}{2} & \binom{-j}{1} & 1 \end{pmatrix} \begin{pmatrix} R(L) \\ R(L-1) \\ \vdots \\ R(l) \end{pmatrix}.$$

This implies (2.18). $\square$

Let $a,b \in V$, $w \in M$, $j,k \in (1/T)\mathbb{Z}$, $l \in \mathbb{Z}$ and s the integer uniquely determined by the conditions $0 \leq s \leq T-1$ and $s/T \equiv j \pmod{\mathbb{Z}}$. It follows by Lemma 2.4 that $F(a,\mathbf{1},w|x_1,x_2) = Y_M(a,x_1)w$ since

$$\begin{aligned} Y_M(a,x_1)Y_M(\mathbf{1},x_2)w &= Y_M(a,x_1)w \\ &\in M[[x_1^{1/T},x_2^{1/T}]][x_1^{-1/T},x_2^{-1/T},(x_1-x_2)^{-1}]. \end{aligned}$$

Comparing the coefficients of

$$\begin{aligned} &\iota_{x_2,x_1-x_2} x_1^j Y_M^{(s)}(a,\mathbf{1}|x_2,x_1-x_2)(w) \\ &= \sum_{k\in(1/T)\mathbb{Z}} \sum_{l\in\mathbb{Z}} \sum_{i=0}^{\infty} \binom{j}{i} Y_M^{(s)}(a,\mathbf{1};j+k-i,l+i)(w) x_2^{-k-1}(x_1-x_2)^{-l-1} \end{aligned}$$

and

$$\begin{aligned} &\iota_{x_2,x_1-x_2} x_1^j Y_M^s(a,x_1)w \\ &= \sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} \binom{k}{-l-1} (-1)^{l+1} a_{j+k+l+1} w x_2^{-k-1}(x_1-x_2)^{-l-1}, \end{aligned}$$

we have

$$\begin{aligned} &\sum_{i=0}^{-l-1} \binom{j}{i} Y_M^{(s)}(a,\mathbf{1};j+k-i,l+i)(w) \\ &= \begin{cases} \binom{k}{-l-1}(-1)^{l+1} a_{j+k+l+1} w & \text{if } k \in \mathbb{Z}, \\ 0 & \text{if } k \notin \mathbb{Z}. \end{cases} \end{aligned} \tag{2.19}$$

Here, we used $Y_M^{(s)}(a,\mathbf{1}|x_2,x_1-x_2)(w) \in M(\!(x_2^{1/T})\!)[[x_1-x_2]]$ by Remark 2.6. We can also obtain (2.19) by taking $b=\mathbf{1}$ in (2.16). Taking $l=-1$ in (2.19), we have

$$Y_M^{(s)}(a,\mathbf{1};i,-1)(w)=\begin{cases}a_iw & \text{if } i\in s/T+\mathbb{Z},\\ 0 & \text{if } i\notin s/T+\mathbb{Z}.\end{cases} \tag{2.20}$$

By a similar argument, we have $F(\mathbf{1},a,w|x_1,x_2)=Y_M(a,x_2)w$, $Y_M^{(s)}(\mathbf{1},a|x_2,x_1-x_2)(w)=\delta_{s,0}Y_M(a,x_2)w$ and hence

$$Y_M^{(s)}(\mathbf{1},a;k,l)(w)=\delta_{s,0}\delta_{l,-1}a_kw. \tag{2.21}$$

Lemma 2.9. *Let M be a (V,T)-module. Then, $Y_M(a_{-2}\mathbf{1},x)=(d/dx)Y_M(a,x)$.*

Proof. Let $a\in V$, $w\in M$, $k\in(1/T)\mathbb{Z}$ and $s\in\mathbb{Z}$ with $0\le s\le T-1$. Taking $j=s/T$ and $l=-2$ in (2.19), we have

$$\begin{aligned}&Y_M^{(s)}\Big(a,\mathbf{1};\frac{s}{T}+k,-2\Big)w+\frac{s}{T}Y_M^{(s)}\Big(a,\mathbf{1};\frac{s}{T}+k-1,-1\Big)w\\ &\quad=\begin{cases}-ka_{s/T+k-1}w & \text{if } k\in\mathbb{Z},\\ 0 & \text{if } k\notin\mathbb{Z}.\end{cases}\end{aligned} \tag{2.22}$$

Let $r\in\mathbb{Z}$ with $0\le r\le T-1$ and $n\in r/T+\mathbb{Z}$. By (2.20) and (2.22), we have

$$\begin{aligned}(a_{-2}\mathbf{1})_nw&=\sum_{s=0}^{T-1}Y_M^{(s)}(a,\mathbf{1};n,-2)(w)\\ &=\sum_{s=0}^{T-1}Y_M^{(s)}\Big(a,\mathbf{1};\frac{s}{T}+\Big(-\frac{s}{T}+n\Big),-2\Big)(w)\\ &=\sum_{s=0}^{T-1}\frac{-s}{T}Y_M^{(s)}\Big(a,\mathbf{1};\frac{s}{T}+\Big(-\frac{s}{T}+n-1\Big),-1\Big)(w)-\Big(-\frac{r}{T}+n\Big)a_{n-1}w\\ &=\frac{-r}{T}a_{n-1}w-\Big(-\frac{r}{T}+n\Big)a_{n-1}w\\ &=-na_{n-1}w.\end{aligned}$$

□

3. Subspaces of $\mathbb{C}[z,z^{-1}]$.

Throughout this section we fix a non-positive integer Δ. This is the lowest weight of a graded vertex algebra $V=\bigoplus_{i=\Delta}^{\infty}V_i$ which will be discussed in Section 4. In this section we introduce a subspace $O_{n,m}^{T,1}(\alpha,\beta;z)$ of $\mathbb{C}[z,z^{-1}]$ (see (3.11) below) for $n,m\in(1/T)\mathbb{N}$ and $\alpha,\beta\in\mathbb{Z}$ and we study its properties. The subspace $O_{n,m}^{T,1}(\alpha,\beta;z)$ will be used to define the subspace $O_{n,m}^{T,1}(V)$ of V in Section 4.

For $N,q\in\mathbb{Z}$ and $Q\in\mathbb{Q}$, $O(N,Q,q;z)$ denotes the subspace of $\mathbb{C}[z,z^{-1}]$ spanned by

$$\mathrm{Res}_x\left((1+x)^Q x^{q+j}\sum_{i\in\mathbb{Z}_{\le N}} z^i x^{-i-1}\right)=\sum_{i=0}^{N-q-j}\binom{Q}{i}z^{i+q+j},\quad j=0,-1,\ldots \tag{3.1}$$

and $z^i, i\in\mathbb{Z}_{\ge N+1}$. We note that if $N\le q$, then $O(N,Q,q;z)=\mathbb{C}[z,z^{-1}]$. We also note that

$$O(N,Q,q;z)\subset O(N,Q,q+1;z)\subset\cdots. \tag{3.2}$$

A similar computation as in the proof of Lemma 2.8 shows the following lemma (or see [**16**, Proof of Lemma 2]).

Lemma 3.1. *Fix $N,q\in\mathbb{Z}$, $Q\in\mathbb{Q}$ and $i\in\mathbb{Z}_{\le N}$. Then*

$$z^i\equiv\sum_{k=1}^{N-q}\sum_{j=1}^{k}\binom{-Q}{-i+q+j}\binom{Q}{k-j}z^{q+k}\pmod{O(N,Q,q;z)}. \tag{3.3}$$

The proof of the following lemma is similar to that of [**16**, Lemma 3].

Lemma 3.2. *Let $N\in\mathbb{Z}, q_0,\ldots,q_{T-1}\in\mathbb{Z}$ and $Q_0,\ldots,Q_{T-1}\in\mathbb{Q}$ such that $Q_i\not\equiv Q_j$ (mod $\mathbb{Z}$) for all $i\ne j$. The diagonal map $\mathbb{C}[z,z^{-1}]\ni f\mapsto(f,\ldots,f)\in\mathbb{C}[z,z^{-1}]^{\oplus T}$ induces an isomorphism*

$$\mathbb{C}[z,z^{-1}]/\bigcap_{s=0}^{T-1}O(N,Q_s,q_s;z)\to\bigoplus_{s=0}^{T-1}\mathbb{C}[z,z^{-1}]/O(N,Q_s,q_s;z)$$

as vector spaces.

Proof. It is sufficient to show that the induced map is surjective. Note that $\mathbb{C}[z,z^{-1}]_{\ge N+1}$ is a subspace of $O(N,Q_s,q_s;z)$ for each s. Fix an integer q such that $q\le\min\{q_0,\ldots,q_{T-1}\}$. We may assume $q\le N$ from the comment right after (3.1). Since $O(N,Q_s,q;z)$ is a subspace of $O(N,Q_s,q_s;z)$ for each $s=0,\ldots,T-1$, it is sufficient to show that the diagonal map

$$\mathbb{C}[z,z^{-1}]_{N+1-T(N-q),N}\ni f\mapsto(f+O(N,Q_s,q;z))_{s=0}^{T-1}\in\bigoplus_{s=0}^{T-1}\mathbb{C}[z,z^{-1}]/O(N,Q_s,q;z)$$

is surjective. For a Laurent polynomial $\Lambda(z)=\sum_{i=N+1-T(N-q)}^{N}\lambda_i z^i\in\mathbb{C}[z,z^{-1}]_{N+1-T(N-q),N}$, it follows by (3.3) that

$$\Lambda(z)\equiv\sum_{i=N+1-T(N-q)}^{N}\lambda_i\sum_{k=1}^{N-q}\sum_{j=1}^{k}\binom{-Q_s}{-i+q+j}\binom{Q_s}{k-j}z^{q+k}\pmod{O(N,Q_s,q;z)} \tag{3.4}$$

for $s = 0, \dots, T-1$. We denote $\sum_{j=1}^{k} \binom{-Q_s}{-i+q+j}\binom{Q_s}{k-j}$ by $\alpha^{s,k}_{-i+q}$ for $0 \le s \le T-1$, $1 \le k \le N-q$ and $i \in \mathbb{Z}$. Define T $T(N-q) \times (N-q)$-matrices Γ_s, $s = 0, \dots, T-1$ by

$$\Gamma_s = \begin{pmatrix} \alpha^{s,1}_{(T-1)(N-q)-1} & \alpha^{s,2}_{(T-1)(N-q)-1} & \cdots & \alpha^{s,N-q}_{(T-1)(N-q)-1} \\ \alpha^{s,1}_{(T-1)(N-q)-2} & \alpha^{s,2}_{(T-1)(N-q)-2} & \cdots & \alpha^{s,N-q}_{(T-1)(N-q)-2} \\ \vdots & \vdots & & \vdots \\ \alpha^{s,1}_{-N+q} & \alpha^{s,2}_{-N+q} & \cdots & \alpha^{s,N-q}_{-N+q} \end{pmatrix}.$$

Since

$$(\lambda_{N+1-T(N-q)}, \lambda_{N+2-T(N-q)}, \dots, \lambda_N)\Gamma_s \begin{pmatrix} z^{q+1} \\ \vdots \\ z^N \end{pmatrix}$$

is equal to the right-hand side of (3.4) for $s = 0, \dots, T-1$, it is sufficient to show that the square matrix

$$\Gamma = (\Gamma_0 \ \Gamma_1 \ \cdots \ \Gamma_{T-1}) \tag{3.5}$$

of order $T(N-q)$ is non-singular. It is proved in Subsection 6.1 that Γ is non-singular. $\square$

For $N \in \mathbb{Z}$ and $\gamma \in \mathbb{Q}$, define a linear automorphism $\varphi_{N,\gamma}$ of $\mathbb{C}[z, z^{-1}]$ by

$$\varphi_{N,\gamma}(z^i) = \begin{cases} (-1)^{i+1} \operatorname{Res}_x \left((1+x)^{\gamma-i} x^i \displaystyle\sum_{j \in \mathbb{Z}_{\le N}} z^j x^{-j-1} \right) & \text{for } i \le N, \\ z^i & \text{for } i \ge N+1. \end{cases} \tag{3.6}$$

Lemma 3.3.

$$\begin{aligned} &\varphi_{N,\gamma}\left(\operatorname{Res}_x \left((1+x)^k x^i \sum_{j \in \mathbb{Z}_{\le N}} z^j x^{-j-1} \right) \right) \\ &\quad = (-1)^{i+1} \operatorname{Res}_x \left((1+x)^{\gamma-k-i} x^i \sum_{j \in \mathbb{Z}_{\le N}} z^j x^{-j-1} \right) \end{aligned} \tag{3.7}$$

for $k \in \mathbb{Q}$ and $i \in \mathbb{Z}_{\le N}$. In particular, $\varphi^2_{N,\gamma} = \mathrm{id}_{\mathbb{C}[z,z^{-1}]}$ and

$$\varphi_{N,\gamma}(O(N, Q, q; z)) = O(N, \gamma - Q - q, q; z)$$

for $Q \in \mathbb{Q}$ and $q \in \mathbb{Z}$.

Proof. We simply write $\varphi = \varphi_{N,\gamma}$. Let $i \in \mathbb{Z}_{\leq N}$. Since

$$\varphi(z^i) = (-1)^{i+1} \sum_{j=0}^{N-i} \binom{\gamma - i}{j} z^{i+j},$$

we have

$$\begin{aligned}
&\varphi\Bigg(\operatorname{Res}_x (1+x)^k x^i \sum_{j \in \mathbb{Z}_{\leq N}} z^j x^{-j-1}\Bigg) = \sum_{j=0}^{N-i} \binom{k}{j} \varphi(z^{i+j}) \\
&= \sum_{j=0}^{N-i} \binom{k}{j} (-1)^{i+j+1} \sum_{m=0}^{N-i-j} \binom{\gamma - i - j}{m} z^{i+j+m} \\
&= \sum_{j=0}^{N-i} \binom{k}{j} (-1)^{i+j+1+m} \sum_{m=0}^{N-i-j} \binom{-\gamma + i + j + m - 1}{m} z^{i+j+m} \\
&= \sum_{l=0}^{N-i} (-1)^{i+1+l} z^{i+l} \sum_{\substack{0 \leq j, m \leq N-i \\ j+m=l}} \binom{k}{j} \binom{-\gamma + i + l - 1}{m} \\
&= \sum_{l=0}^{N-i} (-1)^{i+1+l} z^{i+l} \binom{k - \gamma + i + l - 1}{l} \\
&= (-1)^{i+1} \sum_{l=0}^{N-i} \binom{-k + \gamma - i}{l} z^{i+l} \\
&= (-1)^{i+1} \operatorname{Res}_x \Bigg((1+x)^{\gamma - k - i} x^i \sum_{j \in \mathbb{Z}_{\leq N}} z^j x^{-j-1} \Bigg).
\end{aligned}$$

By this, $\varphi^2(z^j) = z^j$ for $j \in \mathbb{Z}$. Since

$$\begin{aligned}
&\varphi(\operatorname{Res}_x \Bigg((1+x)^Q x^{q+d} \sum_{j \in \mathbb{Z}_{\leq N}} z^j x^{-j-1} \Bigg) \\
&\quad = (-1)^{q+d+1} \operatorname{Res}_x \Bigg((1+x)^{\gamma - Q - q - d} x^{q+d} \sum_{j \in \mathbb{Z}_{\leq N}} z^j x^{-j-1} \Bigg) \\
&\quad = (-1)^{q+d+1} \sum_{m=0}^{-d} \binom{-d}{m} \operatorname{Res}_x \Bigg((1+x)^{\gamma - Q - q} x^{q+d+m} \sum_{j \in \mathbb{Z}_{\leq N}} z^j x^{-j-1} \Bigg)
\end{aligned}$$

for $d \in \mathbb{Z}_{\leq 0}$, we have $\varphi_{N,\gamma}(O(N, Q, q; z)) = O(N, \gamma - Q - q, q; z)$. $\square$

Throughout the rest of this section, $m = l_1 + i_1/T$, $p = l_2 + i_2/T$, $n = l_3 + i_3/T \in$

$(1/T)\mathbb{N}$ with $l_1, l_2, l_3 \in \mathbb{N}$ and $0 \leq i_1, i_2, i_3 \leq T-1$. We always denote m, p, n as above until further notice. For $i, j \in (1/T)\mathbb{Z}$, $r(i,j)$ denotes the integer uniquely determined by the conditions

$$0 \leq r(i,j) \leq T-1 \text{ and } i - j \equiv \frac{r(i,j)}{T} \pmod{\mathbb{Z}}. \tag{3.8}$$

For $0 \leq s \leq T-1$, $s^\vee$ denotes the integer uniquely determined by the conditions

$$0 \leq s^\vee \leq T-1 \text{ and } i_1 - i_3 \equiv s + s^\vee \pmod{T}. \tag{3.9}$$

For $s = 0, \dots, T-1$ and $\alpha, \beta \in \mathbb{Z}$, define

$$\begin{aligned} O^{(T;s),1}_{n,m}(\alpha, \beta; z) = O\Big(&\alpha + \beta - 1 - \Delta, \alpha - 1 + l_1 + \delta(s \leq i_1) + \frac{s}{T}, \\ &- l_1 - l_3 - \delta(s \leq i_1) - \delta(T \leq s + i_3) - 1; z\Big) \end{aligned} \tag{3.10}$$

and

$$O^{T,1}_{n,m}(\alpha, \beta; z) = \bigcap_{s=0}^{T-1} O^{(T;s),1}_{n,m}(\alpha, \beta; z), \tag{3.11}$$

where Δ is the fixed non-positive integer as stated at the beginning of this section and $\delta(i \leq j)$ is defined in (2.1). For $\alpha, \beta \in \mathbb{Z}$, $j \in \mathbb{Z}_{\leq 0}$ and $s = 0, \dots, T-1$, $\Psi^{(T;s)}_{n,m}(\alpha, \beta, j; z)$ denotes the Laurent polynomial in $O^{(T;s),1}_{n,m}(\alpha, \beta; z)$ defined by (3.1), that is,

$$\begin{aligned} &\Psi^{(T;s)}_{n,m}(\alpha, \beta, j; z) \\ &= \operatorname{Res}_x \Bigg((1+x)^{\alpha - 1 + l_1 + \delta(s \leq i_1) + s/T} x^{-l_1 - l_3 - \delta(s \leq i_1) - \delta(T \leq s + i_3) - 1 + j} \sum_{\substack{i \in \mathbb{Z} \\ i \leq \alpha + \beta - 1 - \Delta}} z^i x^{-i-1}\Bigg) \\ &= \sum_{i=0}^{\alpha + \beta - \Delta + l_1 + l_3 + \delta(s \leq i_1) + \delta(T \leq s + i_3) - j} \binom{\alpha - 1 + l_1 + \delta(s \leq i_1) + \frac{s}{T}}{i} \\ &\quad \times z^{i - l_1 - l_3 - \delta(s \leq i_1) - \delta(T \leq s + i_3) - 1 + j}. \end{aligned} \tag{3.12}$$

The disjoint union $\{\Psi^{(T;s)}_{n,m}(\alpha, \beta, j; z) \mid j = 0, -1, \dots\} \cup \{z^i \mid i \geq \alpha + \beta - \Delta\}$ spans $O^{(T;s),1}_{n,m}(\alpha, \beta; z)$.

LEMMA 3.4. *Let $m' = l'_1 + i'_1/T$, $n' = l'_3 + i'_3/T \in (1/T)\mathbb{N}$ with $l'_1, l'_3 \in \mathbb{N}$ and $0 \leq i'_1$, $i'_3 \leq T-1$. If $m' \leq m$ and $n' \leq n$, then $O^{(T;s),1}_{n,m}(\alpha, \beta; z) \subset O^{(T;s),1}_{n',m'}(\alpha, \beta; z)$ for $\alpha, \beta \in \mathbb{Z}$ and $s = 0, \dots, T-1$. In particular, $O^{T,1}_{n,m}(\alpha, \beta; z) \subset O^{T,1}_{n',m'}(\alpha, \beta; z)$.*

Proof. Let $\rho_1 = l_1 + \delta(s \le i_1) - (l_1' + \delta(s \le i_1'))$ and $\rho_3 = l_3 + \delta(T \le s + i_3) - (l_3' + \delta(T \le s + i_3'))$ for $s = 0, \dots, T-1$. It follows by $m' \le m$ and $n' \le n$ that ρ_1 and ρ_3 are non-negative integers. Since

$$\Psi_{n,m}^{(T;s)}(\alpha, \beta, j; z) = \sum_{i=0}^{\rho_1} \binom{\rho_1}{i} \Psi_{n',m'}^{(T;s)}(\alpha, \beta, j - \rho_1 - \rho_3 + i; z),$$

the proof is complete. $\square$

A direct computation shows

$$\frac{-s - s^\vee + i_1 - i_3}{T} + \delta(T \le s^\vee + i_3) = \delta(s \le i_1) - 1 \tag{3.13}$$

for $s = 0, \dots, T-1$ and hence

$$\delta(s^\vee \le i_1) + \delta(T \le s^\vee + i_3) = \delta(s \le i_1) + \delta(T \le s + i_3). \tag{3.14}$$

For a non-positive integer j, it follows by (3.7), (3.13) and (3.14) that

$$\begin{aligned}
&\varphi_{\alpha+\beta-1-\Delta, \alpha+\beta+m-n-2}(\Psi_{n,m}^{(T;s^\vee)}(\beta, \alpha, j; z)) \\
&\quad = (-1)^{-l_1 - l_3 - \delta(s \le i_1) - \delta(T \le s + i_3) + j} \\
&\qquad \times \operatorname{Res}_x \Bigg((1+x)^{\alpha - 1 + l_1 + \delta(s \le i_1) + s/T - j} x^{-l_1 - l_3 - \delta(s \le i_1) - \delta(T \le s + i_3) - 1 + j} \\
&\qquad\qquad \times \sum_{\substack{i \in \mathbb{Z} \\ i \le \alpha + \beta - 1 - \Delta}} z^i x^{-i-1} \Bigg) \\
&\quad = (-1)^{-l_1 - l_3 - \delta(s \le i_1) - \delta(T \le s + i_3) + j} \sum_{k=0}^{-j} \binom{-j}{k} \Psi_{n,m}^{(T;s)}(\alpha, \beta, j + k; z)
\end{aligned}$$

and hence

$$\varphi_{\alpha+\beta-1-\Delta, \alpha+\beta+m-n-2}(O_{n,m}^{(T;s^\vee),1}(\beta, \alpha; z)) = O_{n,m}^{(T;s),1}(\alpha, \beta; z). \tag{3.15}$$

Thus, $\varphi_{\alpha+\beta-1-\Delta, \alpha+\beta+m-n-2}$ induces an isomorphism

$$\mathbb{C}[z, z^{-1}]/O_{n,m}^{(T;s^\vee),1}(\beta, \alpha; z) \to \mathbb{C}[z, z^{-1}]/O_{n,m}^{(T;s),1}(\alpha, \beta; z) \tag{3.16}$$

and hence

$$\mathbb{C}[z, z^{-1}]/\bigcap_{s=0}^{T-1} O_{n,m}^{(T;s),1}(\beta, \alpha; z) \cong \mathbb{C}[z, z^{-1}]/\bigcap_{s=0}^{T-1} O_{n,m}^{(T;s),1}(\alpha, \beta; z) \tag{3.17}$$

by Lemma 3.1.

For $r = 0, \dots, T-1$ and $i \in \mathbb{Z}_{\leq \alpha+\beta-1-\Delta}$, it follows by the argument in the proof of Lemma 3.2 that there exists a unique Laurent polynomial in $\mathbb{C}[z, z^{-1}]_{\alpha+\beta-\Delta-T(\alpha+\beta-\Delta+l_1+l_3+2), \alpha+\beta-1-\Delta}$, which we denote by $E^{(T;r)}_{n,m}(\alpha, \beta, i; z)$, such that

$$\begin{aligned} &E^{(T;r)}_{n,m}(\alpha, \beta, i; z) \equiv \delta_{r,s} z^i \\ &\quad \left(\operatorname{mod} O\left(\alpha+\beta-1-\Delta, \alpha-1+l_1+\delta(s \leq i_1) + \frac{s}{T}, -l_1-l_3-3; z \right) \right), \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad s = 0, \dots, T-1. \end{aligned} \tag{3.18}$$

We also define

$$E^{(T;r)}_{n,m}(\alpha, \beta, i; z) = 0 \text{ for } i \in \mathbb{Z}_{\geq \alpha+\beta-\Delta} \tag{3.19}$$

for convenience. Since

$$-l_1 - l_3 - 3 \leq -l_1 - l_3 - \delta(s \leq i_1) - \delta(T \leq s + i_3) - 1,$$

it follows by (3.2), (3.18) and (3.19) that

$$E^{(T;r)}_{n,m}(\alpha, \beta, i; z) \equiv \delta_{r,s} z^i \quad (\operatorname{mod} O^{(T;s),1}_{n,m}(\alpha, \beta; z)) \tag{3.20}$$

for $r, s = 0, \dots, T-1$ and $i \in \mathbb{Z}$. It follows from Lemma 3.2 that

$$\sum_{s=0}^{T-1} E^{(T;s)}_{n,m}(\alpha, \beta, i; z) \equiv z^i \quad (\operatorname{mod} O^{T,1}_{n,m}(\alpha, \beta; z)) \tag{3.21}$$

for $i \in \mathbb{Z}$. Define a Laurent polynomial $\Phi^T_{n,p,m}(\alpha, \beta; z)$ by

$$\begin{aligned} &\Phi^T_{n,p,m}(\alpha, \beta; z) \\ &\quad = \sum_{i=0}^{l_2} \binom{-l_1 - l_3 + l_2 - \delta(r(p,n) \leq i_1) - \delta(T \leq r(p,n) + i_3)}{i} \\ &\qquad \times \operatorname{Res}_x \Bigg((1+x)^{\alpha-1+l_1+\delta(r(p,n) \leq i_1) + r(p,n)/T} \\ &\qquad\qquad \times x^{-l_1-l_3+l_2-\delta(r(p,n)\leq i_1)-\delta(T \leq r(p,n)+i_3)-i} \sum_{j \in \mathbb{Z}} E^{(T;r(p,n))}_{n,m}(\alpha, \beta, j; z) x^{-j-1} \Bigg) \\ &\quad \in \mathbb{C}[z, z^{-1}]_{\alpha+\beta-\Delta-T(\alpha+\beta-\Delta+l_1+l_3+2), \alpha+\beta-1-\Delta}, \end{aligned} \tag{3.22}$$

where $r(p, n)$ is defined in (3.8). This is used to define the product $*^T_{n,p,m}$ on a vertex algebra in Section 4.

We denote $\mathrm{Span}_{\mathbb{C}}\{z^i \in \mathbb{C}[z,z^{-1}] \mid i \neq -1\}$ by $\mathbb{C}[z,z^{-1}]_{\neq -1}$. The following two results will be used to compute $\mathbf{1} *^T_{n,p,m} a$ for $a \in V$ in Section 4.

Lemma 3.5. *Let $\alpha, r \in \mathbb{Z}$ with $0 \leq r \leq T-1$. Then*

$$\sum_{j\in\mathbb{Z}} E^{(T;r)}_{n,m}(0,\alpha,j;z)x^{-j-1} \equiv \delta_{r,0}z^{-1} \pmod{(\mathbb{C}[z,z^{-1}]_{\neq -1})((x))}.$$

Proof. Since

$$\Psi^{(T;0)}_{n,m}(0,\alpha,j;z) = \sum_{i=0}^{\alpha+\Delta+l_1+l_3+1-j} \binom{l_1}{i} z^{i-l_1-l_3-2+j} \in \mathbb{C}[z,z^{-1}]_{\leq -2}$$

for all $j \in \mathbb{Z}_{\leq 0}$, $O^{(T;0),1}_{n,m}(0,\alpha;z)$ is a subspace of $\mathbb{C}[z,z^{-1}]_{\neq -1}$. By (3.20), we have the desired result. □

Lemma 3.6. *For $\alpha \in \mathbb{Z}$, we have*

$$\Phi^T_{n,p,m}(0,\alpha;z) \equiv \delta_{n,p}z^{-1} \pmod{\mathbb{C}[z,z^{-1}]_{\neq -1}}.$$

Proof. If $n \not\equiv p \pmod{\mathbb{Z}}$, then it follows by Lemma 3.5 that

$$\Phi^T_{n,p,m}(0,\alpha;z) \equiv 0 \pmod{\mathbb{C}[z,z^{-1}]_{\neq -1}}.$$

Suppose $n \equiv p \pmod{\mathbb{Z}}$. By Lemma 3.5 again, the same computation as in the proof of [**4**, Lemma 4.7] shows

$$\Phi^T_{n,p,m}(0,\alpha;z) \equiv \delta_{n,p}z^{-1} \pmod{\mathbb{C}[z,z^{-1}]_{\neq -1}}. \qquad \square$$

The following result will be used in order to obtain Lemma 4.3, which induces the commutator formula in Lemma 5.9.

Lemma 3.7. *For $\alpha, \beta \in \mathbb{Z}$, we have*

$$\begin{aligned}\Phi^T_{n,p,m}(\alpha,\beta;z) &- \varphi_{\alpha+\beta-1-\Delta,\alpha+\beta+m-n-2}(\Phi^T_{n,m+n-p,m}(\beta,\alpha;z))\\ &- \mathrm{Res}_x(1+x)^{\alpha-1+p-n}\sum_{j\in\mathbb{Z}} E^{(T;r(p,n))}_{n,m}(\alpha,\beta,j;z)x^{-j-1} \in O^{T,1}_{n,m}(\alpha,\beta;z).\end{aligned} \tag{3.23}$$

Proof. The proof is similar to that of [**5**, Lemma 3.4]. We simply write $r = r(p,n)$ and $\varphi = \varphi_{\alpha+\beta-1-\Delta,\alpha+\beta+m-n-2}$. It follows by

$$(m+n-p) - n \equiv \frac{i_1 - i_2}{T} \equiv \frac{r^\vee}{T} \pmod{\mathbb{Z}}$$

that $\Phi^T_{n,m+n-p,p}(\beta,\alpha;z) \in \bigcap_{s\neq r^\vee} O^{(T;s),1}_{n,m}(\beta,\alpha;z)$, where $r^\vee$ is defined in (3.9). Since $\varphi(\Phi^T_{n,m+n-p,p}(\beta,\alpha;z)) \in \bigcap_{s\neq r} O^{(T;s),1}_{n,m}(\alpha,\beta;z)$ by (3.15), we have

$$\begin{aligned}&\Phi^T_{n,p,m}(\alpha,\beta;z) - \varphi(\Phi^T_{n,m+n-p,m}(\beta,\alpha;z))\\ &\quad - \mathrm{Res}_x(1+x)^{\alpha-1+p-n}\sum_{j\in\mathbb{Z}} E^{(T;r)}_{n,m}(\alpha,\beta,j;z)x^{-j-1} \in \bigcap_{s\neq r} O^{(T;s),1}_{n,m}(\alpha,\beta;z).\end{aligned}$$

Thus, it is sufficient to show (3.23) modulo $O^{(T;r),1}_{n,m}(\alpha,\beta;z)$ by Lemma 3.2. Define

$$\varepsilon = \begin{cases} 1 & \text{if } T \leq i_1+i_3-i_2,\\ 0 & \text{if } 0 \leq i_1+i_3-i_2 < T,\\ -1 & \text{if } i_1+i_3-i_2 < 0.\end{cases} \tag{3.24}$$

It follows by the formula of ε in the proof of [**5**, Lemma 3.4] and (3.14) that

$$\begin{aligned}&\Phi^T_{n,m+n-p,m}(\beta,\alpha;z)\\ &= \sum_{i=0}^{l_1+l_3-l_2+\varepsilon}\binom{-l_1-l_3+(l_1+l_3-l_2+\varepsilon)-\delta(r^\vee\leq i_1)-\delta(T\leq r^\vee+i_3)}{i}\\ &\quad\times \mathrm{Res}_x(1+x)^{\beta-1+l_1+\delta(r^\vee\leq i_1)+r^\vee/T}\\ &\quad\times x^{-l_1-l_3+(l_1+l_3-l_2+\varepsilon)-\delta(r^\vee\leq i_1)-\delta(T\leq r^\vee+i_3)-i}\sum_{j\in\mathbb{Z}} E^{(T;r^\vee)}_{n,m}(\beta,\alpha,j;z)x^{-j-1}\\ &= \sum_{i=0}^{l_1+l_3-l_2+\varepsilon}\binom{-l_2-1}{i}\mathrm{Res}_x(1+x)^{\beta-1+l_1+\delta(r^\vee\leq i_1)+r^\vee/T}x^{-l_2-1-i}\\ &\quad\times\sum_{j\in\mathbb{Z}} E^{(T;r^\vee)}_{n,m}(\beta,\alpha,j;z)x^{-j-1}.\end{aligned}$$

Thus, it follows by (3.7) that

$$\begin{aligned}&\varphi(\Phi^T_{n,m+n-p,m}(\beta,\alpha;z))\\ &\quad= \sum_{i=0}^{l_1+l_3-l_2+\varepsilon}\binom{-l_2-1}{i}(-1)^{-l_2-i}\,\mathrm{Res}_x(1+x)^{\alpha-1+p-n+i}x^{-l_2-1-i}\\ &\quad\times\sum_{j\in\mathbb{Z}} E^{(T;r)}_{n,m}(\alpha,\beta,j;z)x^{-j-1}\end{aligned} \tag{3.25}$$

and therefore

$$
\begin{aligned}
&\varphi(\Phi^T_{n,m+n-p,m}(\beta,\alpha;z)) \\
&\quad\equiv \sum_{i=0}^{l_1+l_3-l_2+\varepsilon} \binom{-l_2-1}{i} (-1)^{-l_2-i} \operatorname{Res}_x (1+x)^{\alpha-1+p-n+i} x^{-l_2-1-i} \\
&\qquad\times \sum_{\substack{j\in\mathbb{Z}\\ j\le \alpha+\beta-1-\Delta}} z^j x^{-j-1} \qquad (\mathrm{mod}\ O^{(T;r),1}_{n,m}(\alpha,\beta;z)).
\end{aligned}
$$

The same argument as in the proof of [**5**, Lemma 3.4] shows

$$
\begin{aligned}
&\Phi^T_{n,p,m}(\alpha,\beta;z) - \varphi(\Phi^T_{n,m+n-p,m}(\beta,\alpha;z)) \\
&\quad\equiv \sum_{i=0}^{l_2} \binom{-l_1-l_3+l_2-\varepsilon-1}{i} \\
&\qquad\times \operatorname{Res}_x \Bigg((1+x)^{\alpha-1+l_1+\delta(r\le i_1)+r/T} x^{-l_1-l_3+l_2-\varepsilon-1-i} \sum_{\substack{j\in\mathbb{Z}\\ j\le \alpha+\beta-1-\Delta}} z^j x^{-j-1} \Bigg) \\
&\qquad - \sum_{i=0}^{l_1+l_3-l_2+\varepsilon} \binom{-l_2-1}{i} (-1)^{-l_2-i} \operatorname{Res}_x (1+x)^{\alpha-1+p-n+i} x^{-l_2-1-i} \sum_{\substack{j\in\mathbb{Z}\\ j\le \alpha+\beta-1-\Delta}} z^j x^{-j-1} \\
&\qquad\qquad (\mathrm{mod}\ O^{(T;r),1}_{n,m}(\alpha,\beta;z)) \\
&\quad= \operatorname{Res}_x (1+x)^{\alpha-1+p-n} \sum_{\substack{j\in\mathbb{Z}\\ j\le \alpha+\beta-1-\Delta}} z^j x^{-j-1}.
\end{aligned}
$$

The proof is complete. □

Let $l \in (1/T)\mathbb{N}$ with $l \le n, m$. Then, it follows by Lemma 3.4 that

$$
E^{(T;r)}_{n,m}(\alpha,\beta,i;z) \equiv E^{(T;r)}_{n-l,m-l}(\alpha,\beta,i;z) \quad (\mathrm{mod}\ O^{T,1}_{n-l,m-l}(\alpha,\beta;z))
$$

for $\alpha, \beta, i \in \mathbb{Z}$. The same computation as in the proof of [**5**, Proposition 4.3] shows the following lemma.

Lemma 3.8. *Let $l \in (1/T)\mathbb{N}$ with $l \le n, m$. Then*

$$
\Phi^T_{n,p,m}(\alpha,\beta;z) \equiv \Phi^T_{n-l,p-l,m-l}(\alpha,\beta;z) \quad (\mathrm{mod}\ O^{T,1}_{n-l,m-l}(\alpha,\beta;z))
$$

for $\alpha, \beta \in \mathbb{Z}$.

Let T' be a positive multiple of T and $\alpha, \beta \in \mathbb{Z}$. Set $d = T'/T$. We note that $m = l_1 + di_1/T'$, $p = l_2 + di_2/T'$ and $n = l_3 + di_3/T'$. Thus it follows by (3.10) that

$$O_{n,m}^{(T';dr),1}(\alpha,\beta;z)=O_{n,m}^{(T;r),1}(\alpha,\beta;z) \tag{3.26}$$

for $r=0,\dots,T-1$. By this and (3.20), we have

$$E_{n,m}^{(T';dr)}(\alpha,\beta,i;z)\equiv\delta_{r,s}z^i \pmod{O_{n,m}^{(T;s),1}(\alpha,\beta,i;z)}$$

for $i\in\mathbb{Z}$ and $r,s=0,\dots,T-1$. Therefore, Lemma 3.2 implies

$$E_{n,m}^{(T';dr)}(\alpha,\beta,i;z)\equiv E_{n,m}^{(T;r)}(\alpha,\beta,i;z) \pmod{O_{n,m}^{T,1}(\alpha,\beta;z)}$$

for $i\in\mathbb{Z}$ and $r=0,\dots,T-1$. By (3.22), we have the following result.

Lemma 3.9. *Let T' be a positive multiple of T and $\alpha,\beta\in\mathbb{Z}$. Then*

$$\Phi_{n,p,m}^{T'}(\alpha,\beta;z)\equiv\Phi_{n,p,m}^{T}(\alpha,\beta;z) \pmod{O_{n,m}^{T,1}(\alpha,\beta;z)}.$$

4. Associative algebras $A_m^T(V)$ and bimodules $A_{n,m}^T(V)$.

Throughout the rest of this paper, we always assume the following properties for a vertex algebra V: V has a grading $V=\bigoplus_{i=\Delta}^{\infty}V_i$ such that $\Delta\in\mathbb{Z}_{\le 0}$, $\mathbf{1}\in V_0$ and for any homogeneous element $a\in V$, $a_iV_j\subset V_{\mathrm{wt}\,a-1-i+j}$, where $V_i=0$ for $i<\Delta$. Every vertex operator algebra satisfies these properties. Throughout this section, we fix $m=l_1+i_1/T$, $p=l_2+i_2/T$, $n=l_3+i_3/T\in(1/T)\mathbb{N}$ with $l_1,l_2,l_3\in\mathbb{N}$ and $0\le i_1,i_2,i_3\le T-1$.

In this section, we first define a product $*_{n,p,m}^T$ on V and a quotient space $A_{n,m}^T(V)$ of V. In the following, we shall use a similar argument as in [**4**, Section 3]. For $a\in V_i$, we denote i by $\mathrm{wt}\,a$. Define

$$\hat{E}_{n,m}^{(T;s)}(a,b,i)=E_{n,m}^{(T;s)}(\mathrm{wt}\,a,\mathrm{wt}\,b,i;z)|_{z^j=a_jb}\in V \tag{4.1}$$

for homogeneous elements a,b of V and $i\in\mathbb{Z}$, where $E_{n,m}^{(T;s)}(\mathrm{wt}\,a,\mathrm{wt}\,b,i;z)$ is defined in (3.18), and extend $\hat{E}_{n,m}^{(T;s)}(a,b,i)$ for arbitrary $a,b\in V$ by linearity.

Let $O_{n,m}^{T,0}(V)$ be the subspace of V spanned by

$$\{a_{-2}\mathbf{1}+(\mathrm{wt}\,a+m-n)a\in V\mid \text{homogeneous } a\in V\} \tag{4.2}$$

and $O_{n,m}^{T,1}(V)$ the subspace of V spanned by

$$\left\{P(z)|_{z^j=a_jb}\in V\;\middle|\;\begin{array}{l}\text{homogeneous } a,b\in V \text{ and}\\ P(z)\in O_{n,m}^{T,1}(\mathrm{wt}\,a,\mathrm{wt}\,b;z)\end{array}\right\}. \tag{4.3}$$

A similar argument as in the proof of [**17**, Lemma 2.1.3] shows the following lemma as stated in the proof of [**4**, Lemma 2.3].

Lemma 4.1. *For homogeneous $a,b\in V$, we have*

$$\operatorname{Res}_x(1+x)^i x^j Y(b,x)a$$
$$\equiv (-1)^{j+1}\operatorname{Res}_x(1+x)^{\mathrm{wt}\,a+\mathrm{wt}\,b+m-n-2-i-j}x^j Y(a,x)b \pmod{O^{T,0}_{n,m}(V)}$$

for $i\in\mathbb{Q}, j\in\mathbb{Z}$ and homogeneous $a,b\in V$.

By (3.21), we have

$$\sum_{s=0}^{T-1}\hat{E}^{(T;s)}_{n,m}(a,b,i)\equiv a_i b \pmod{O^{T,1}_{n,m}(V)} \tag{4.4}$$

for $i\in\mathbb{Z}$. Define

$$a *^T_{n,p,m} b = \Phi^T_{n,p,m}(\mathrm{wt}\,a,\mathrm{wt}\,b;z)|_{z^j=a_j b}\in V \tag{4.5}$$

for homogeneous $a,b\in V$, where $\Phi^T_{n,p,m}$ is defined in (3.22), and extend $a *^T_{n,p,m} b$ for arbitrary $a,b\in V$ by linearity. By $Y(\mathbf{1},x)=\mathrm{id}_V$ and Lemma 3.6, we have

$$\mathbf{1} *^T_{n,p,m} a = \delta_{n,p}a \tag{4.6}$$

for $a\in V$.

Definition 4.2. Let $O^{T,2}_{n,m}(V)$ be the subspace of V spanned by

$$u *^T_{n,p_3,m}\big((a *^T_{p_3,p_2,p_1} b) *^T_{p_3,p_1,m} c - a *^T_{p_3,p_2,m}(b *^T_{p_2,p_1,m} c)\big)$$

for all $a,b,c,u\in V$ and all $p_1,p_2,p_3\in(1/T)\mathbb{N}$. Define

$$O^{T,3}_{n,m}(V)=\sum_{p_1,p_2\in(1/T)\mathbb{N}} V *^T_{n,p_2,p_1}(O^{T,0}_{p_2,p_1}(V)+O^{T,1}_{p_2,p_1}(V)) *^T_{n,p_1,m} V$$

and

$$O^T_{n,m}(V)=O^{T,0}_{n,m}(V)+O^{T,1}_{n,m}(V)+O^{T,2}_{n,m}(V)+O^{T,3}_{n,m}(V).$$

By (4.6), we have

$$(a *^T_{n,p_2,p_1} b) *^T_{n,p_1,m} c - a *^T_{n,p_2,m}(b *^T_{p_2,p_1,m} c)\in O^{T,2}_{n,m}(V)$$

for $a,b,c\in V$ and $p_1,p_2\in(1/T)\mathbb{N}$.

Lemma 4.3. *For $a,b\in V$, we have*

$$a *^T_{n,p,m} b - b *^T_{n,m+n-p,m} a - \operatorname{Res}_x(1+x)^{\mathrm{wt}\,a-1+p-n}\sum_{j\in\mathbb{Z}}\hat{E}^{(T;r(p,n))}_{n,m}(a,b,j)x^{-j-1}$$
$$\in O^{T,0}_{n,m}(V)+O^{T,1}_{n,m}(V),$$

where $r(p,n)$ *is defined in* (3.8).

Proof. We may assume a and b to be homogeneous elements of V. We simply write $r = r(p,n)$. Let ε be the integer defined in (3.24). By Lemma 4.1 and (3.25), we have

$$\begin{aligned}
&b *^T_{n,m+p-n,m} a \\
&\quad \equiv \sum_{i=0}^{l_1+l_3-l_2+\varepsilon} \binom{-l_2-1}{i} (-1)^{-l_2-i} \operatorname{Res}_x (1+x)^{\mathrm{wt}\, a-1+p-n+i} x^{-l_2-1-i} \\
&\qquad \times \sum_{j\in\mathbb{Z}} \hat{E}^{(T;r)}_{n,m}(a,b,j) x^{-j-1} \qquad (\mathrm{mod}\ O^{T,0}_{n,m}(V) + O^{T,1}_{n,m}(V)) \\
&\quad = \sum_{i=0}^{l_1+l_3-l_2+\varepsilon} \binom{-l_2-1}{i} (-1)^{-l_2-i} \operatorname{Res}_x (1+x)^{\mathrm{wt}\, a-1+p-n+i} x^{-l_2-1-i} \\
&\qquad \times \sum_{j\in\mathbb{Z}} E^{(T;r)}_{n,m}(\mathrm{wt}\, a, \mathrm{wt}\, b, j; z)|_{z^k = a_k b}\ x^{-j-1} \\
&\quad = \varphi_{\mathrm{wt}\, a+\mathrm{wt}\, b-1-\Delta, \mathrm{wt}\, a+\mathrm{wt}\, b+m-n-2}(\Phi^T_{n,m+n-p,m}(\mathrm{wt}\, b, \mathrm{wt}\, a; z))|_{z^k=a_k b},
\end{aligned}$$

where $\varphi_{\mathrm{wt}\, a+\mathrm{wt}\, b-1-\Delta, \mathrm{wt}\, a+\mathrm{wt}\, b+m-n-2}$ is defined by (3.6). Thus, the assertion follows from Lemma 3.7. □

By (4.6) and Lemmas 3.5 and 4.3, we have

$$a *^T_{n,m,m} \mathbf{1} \equiv a \quad (\mathrm{mod}\ O^{T,0}_{n,m}(V) + O^{T,1}_{n,m}(V)) \tag{4.7}$$

for $a \in V$.

The same argument as in the proof of [**5**, Lemma 3.8] shows the following lemma.

Lemma 4.4. *For* $m,p,n \in (1/T)\mathbb{Z}$, *we have* $V *^T_{n,p,m} O^T_{p,m}(V) \subset O^T_{n,m}(V)$ *and* $O^T_{n,p}(V) *^T_{n,p,m} V \subset O^T_{n,m}(V)$.

We define

$$A^T_{n,m}(V) = V/O^T_{n,m}(V). \tag{4.8}$$

If $m = n$, we simply write $A^T_m(V) = A^T_{m,m}(V)$. By Definition 4.2, (4.6), (4.7) and Lemma 4.4, we have the following result.

Theorem 4.5. *Let* $m,n \in (1/T)\mathbb{N}$. *Then,* $(A^T_m(V), *^T_{m,m,m})$ *is an associative* $\mathbb{C}$-*algebra and* $A^T_{n,m}(V)$ *is an* $A^T_n(V)$-$A^T_m(V)$-*bimodule, where the left action of* $A^T_n(V)$ *is given by* $*^T_{n,n,m}$ *and the right action of* $A^T_m(V)$ *is given by* $*^T_{n,m,m}$.

Lemmas 3.4 and 3.8 imply the following result.

Proposition 4.6. *Let $l, m, n \in (1/T)\mathbb{N}$ with $l \le n, m$. Then $O^{T,1}_{n,m}(V)$ is a subspace of $O^{T,1}_{n-l,m-l}(V)$. Moreover, the identity map on V induces a surjective algebra homomorphism $A^T_m(V) \to A^T_{m-l}(V)$ and a surjective $A^T_n(V)$-$A^T_m(V)$-bimodule homomorphism $A^T_{n,m}(V) \to A^T_{n-l,m-l}(V)$.*

Lemma 3.9 and (3.26) imply the following result.

Proposition 4.7. *Let $m, n \in (1/T)\mathbb{N}$ and T' a positive multiple of T. Then $O^{T,1}_{n,m}(V)$ is a subspace of $O^{T',1}_{n,m}(V)$. Moreover, the identity map on V induces a surjective algebra homomorphism $A^{T'}_m(V) \to A^T_m(V)$ and a surjective $A^{T'}_n(V)$-$A^{T'}_m(V)$-bimodule homomorphism $A^{T'}_{n,m}(V) \to A^T_{n,m}(V)$.*

Remark 4.8. Suppose V is a vertex operator algebra. Let g be an automorphism of V of finite order t. In [**5**], a product $*^n_{g,m,p}$ on V and a quotient space $A_{g,n,m}(V) = V/O_{g,n,m}(V)$ of V are constructed for each $n, p, m \in (1/t)\mathbb{N}$. If $g = \mathrm{id}_V$, then $*^n_{g,m,p} = *^n_{m,p}$ and $A_{g,n,m}(V) = A_{n,m}(V)$, where $*^n_{m,p}$ is a product on V and $A_{n,m}(V)$ is a quotient space of V constructed in [**4**].

We shall discuss a relation between $A_{g,n,m}(V)$ and $A^T_{n,m}(V)$. Suppose $T = 1$. Then $*^1_{n,p,m} = *^n_{m,p}$ by the definition. Moreover, $O^{1,0}_{n,m}(V) + O^{1,1}_{n,m}(V) = O'_{n,m}(V)$ by (3.10) and (3.11), where $O'_{n,m}(V)$ is the subspace of V defined on p. 801 in [**4**]. Thus, $O^1_{n,m}(V) = O_{n,m}(V)$ and $A^1_{n,m}(V) = A_{n,m}(V)$.

We shall use the notation in Remark 2.7 and [**5**]. For homogeneous $a, b \in V$ and $P(z) \in O^{t,1}_{n,m}(\mathrm{wt}\, a, \mathrm{wt}\, b; z)$, the definition of $O^{t,1}_{n,m}(\mathrm{wt}\, a, \mathrm{wt}\, b; z)$ implies

$$P(z)|_{z^j = a_j b} = \sum_{r=0}^{t-1} P(z)\Big|_{z^j = a^{(g,r)}_j b} \in O'_{g,n,m}(V),$$

where $O'_{g,n,m}(V)$ is the subspace of V defined on p. 4240 in [**5**]. Thus, $O^{t,1}_{n,m}(V)$ is a subspace of $O'_{g,n,m}(V)$. We simply write $r = r(p,n)$, which is defined in (3.8). For $s = 0, \dots, t-1$, we have

$$\hat{E}^{(t;r)}_{n,m}(a^{(g,s)}, b, i) - \delta_{r,s} a^{(g,s)}_i b = \big(E^{(t;r)}_{n,m}(\mathrm{wt}\, a^{(g,s)}, \mathrm{wt}\, b, i; z) - \delta_{r,s} z^i\big)\big|_{z^j = a^{(g,s)}_j b} \in O'_{g,n,m}(V)$$

since $E^{(t;r)}_{n,m}(\mathrm{wt}\, a^{(g,s)}, \mathrm{wt}\, b, i; z) - \delta_{r,s} z^i \in O^{(t;s),1}_{n,m}(\mathrm{wt}\, a^{(g,s)}, \mathrm{wt}\, b; z)$ by (3.20). Therefore, by (3.22) and (4.5) we have

$$\begin{aligned} a *^t_{n,p,m} b &= \sum_{s \ne r} a^{(g,s)} *^t_{n,p,m} b + a^{(g,r)} *^t_{n,p,m} \\ &\equiv a^{(g,r)} *^n_{g,m,p} b \pmod{O'_{g,n,m}(V)}. \end{aligned}$$

We conclude that $O^{t,1}_{n,m}(V) \subset O_{g,n,m}(V)$ and $A_{g,n,m}(V)$ is a quotient space of $A^t_{n,m}(V)$.

For an automorphism group G of V of finite order, the same argument as above shows $A_{G,n}(V)$ in [**16**] is a quotient space of $A^{|G|}_n(V)$.

5. $(1/T)\mathbb{N}$-graded (V,T)-modules and $A^T_{n,m}(V)$.

Throughout this section, we always assume the properties mentioned at the beginning of Section 4 for a vertex algebra V as stated there. In this section, for $m \in (1/T)\mathbb{N}$ we describe a relation between the $A^T_m(V)$-modules and the $(1/T)\mathbb{N}$-graded (V,T)-modules defined below.

Definition 5.1. *A $(1/T)\mathbb{N}$-graded (V,T)-module M is a (V,T)-module with a $(1/T)\mathbb{N}$-grading $M = \bigoplus_{n\in(1/T)\mathbb{N}} M(n)$ such that*

$$a_i M(n) \subset M(n + \mathrm{wt}\, a - i - 1)$$

for homogeneous $a \in V$ and $i, n \in (1/T)\mathbb{N}$, where $M(n) = 0$ for $n < 0$.

For a $(1/T)\mathbb{N}$-graded (V,T)-module M, a (V,T)-submodule N of M is called *$(1/T)\mathbb{N}$-graded (V,T)-submodule* of M if N is a $(1/T)\mathbb{N}$-graded (V,T)-module such that every homogeneous subspace of N is contained in some homogeneous subspace of M. A non-zero $(1/T)\mathbb{N}$-graded (V,T)-module M is called *simple* if there is no $(1/T)\mathbb{N}$-graded submodule of M except 0 and M itself.

In the following, we shall use a similar argument as in [**4**, Section 4]. Throughout this section, $m = l_1 + i_1/T$, $n = l_3 + i_3/T \in (1/T)\mathbb{N}$ with $l_1, l_2 \in \mathbb{N}$ and $0 \le i_1,\ i_3 \le T-1$. Until Proposition 5.7, $M = \bigoplus_{i\in(1/T)\mathbb{N}} M(i)$ is a $(1/T)\mathbb{N}$-graded (V,T)-module. Without loss of generality, we can shift the grading of a $(1/T)\mathbb{N}$-graded (V,T)-module M so that $M(0) \neq 0$ if $M \neq 0$.

Define a linear map $o_{n,m} : V \to \mathrm{Hom}_{\mathbb{C}}(M(m), M(n))$ by

$$o_{n,m}(a) = a_{\mathrm{wt}\, a + m - n - 1} \tag{5.1}$$

for homogeneous $a \in V$ and extend $o_{n,m}(a)$ for an arbitrary $a \in V$ by linearity. If $m = n$, we simply write $o = o_{m,m}$. Define a linear map $Z^{(s)}_{M,n,m}(a,b;-) : \mathbb{C}[z,z^{-1}] \to \mathrm{Hom}_{\mathbb{C}}(M(m), M)$ by

$$Z^{(s)}_{M,n,m}(a,b;z^i) = Y^{(s)}_M(a,b;\mathrm{wt}\, a + \mathrm{wt}\, b + m - n - 2 - i, i) \tag{5.2}$$

for $s = 0,\dots,T-1$ and homogeneous $a, b \in V$ and extend $Z^{(s)}_{M,n,m}(a,b;-)$ for arbitrary elements $a, b \in V$ by linearity. Lemma 2.8 implies that the image of $Z^{(s)}_{M,n,m}(a,b;f(z)) : M(m) \to M$ is contained in $M(n)$ for $f(z) \in \mathbb{C}[z,z^{-1}]$. That is, $Z^{(s)}_{M,n,m}(a,b;-) : \mathbb{C}[z,z^{-1}] \to \mathrm{Hom}_{\mathbb{C}}(M(m), M(n))$.

Lemma 5.2. *For $s = 0,\dots,T-1$ and homogeneous $a, b \in V$, $Z^{(s)}_{M,n,m}(a,b;-) = 0$ on $O^{(T;s),1}_{n,m}(\mathrm{wt}\, a, \mathrm{wt}\, b; z)$.*

Proof. It is sufficient to show that $Z^{(s)}_{M,n,m}(a,b;\Psi^{(T;s)}_{n,m}(\mathrm{wt}\, a, \mathrm{wt}\, b, d; z)) = 0$ for all $d \in \mathbb{Z}_{\le 0}$. Let $w \in M(m)$. Since $Y_M(Y(a,x_0)b, x_2)w \in M((x_2^{1/T}))((x_0))_{\ge -\mathrm{wt}\, a - \mathrm{wt}\, b + \Delta}$, it follows by Remark 2.6 that

$$Y_M^{(s)}(a,b|x_2,x_0)(w) \in M((x_2^{1/T}))((x_0))_{\geq -\operatorname{wt} a - \operatorname{wt} b + \Delta}. \tag{5.3}$$

Let

$$\begin{aligned} j &= \operatorname{wt} a - 1 + l_1 + \delta(s \leq i_1) + \frac{s}{T}, \\ k &= \operatorname{wt} b - 1 + l_1 + \delta(s^\vee \leq i_1) + \frac{s^\vee}{T} - d \quad \text{and} \\ l &= -l_1 - l_3 - \delta(s \leq i_1) - \delta(T \leq s + i_3) - 1 + d, \end{aligned}$$

where $s^\vee$ is defined in (3.9). Since $a_{j+i} = b_{k+i} = 0$ on $M(m)$ for all $i \in \mathbb{N}$, it follows by (2.16), (3.13) and (5.3) that

$$\begin{aligned} &Z_{M,n,m}^{(s)}(a,b;\Psi_{n,m}^{(T;s)}(\operatorname{wt} a, \operatorname{wt} b, d; z))(w) \\ &= \sum_{i=0}^{\operatorname{wt} a + \operatorname{wt} b - 1 - \Delta - l} \binom{j}{i} Y_M^{(s)}(a,b;j+k-i,l+i)(w) \\ &= \sum_{i=0}^{\infty} \binom{j}{i} Y_M^{(s)}(a,b;j+k-i,l+i)(w) \\ &= 0. \end{aligned}$$

□

Lemma 5.3. *For $u \in O_{n,m}^{T,0}(V) + O_{n,m}^{T,1}(V)$, $o_{n,m}(u) = 0$ on $M(m)$.*

Proof. Let a, b be homogeneous elements of V. It follows by Lemma 2.9 that $o_{n,m}(a_{-2}\mathbf{1} + (\operatorname{wt} a + m - n)a) = 0$ on $M(m)$. Let $P(z) = \sum_{i\in\mathbb{Z}} \lambda_i z^i \in O_{n,m}^{T,1}(\operatorname{wt} a, \operatorname{wt} b; z)$. It follows by Lemma 5.2 that on $M(m)$

$$\begin{aligned} o_{n,m}\left(\sum_{i\in\mathbb{Z}} \lambda_i a_i b\right) &= \sum_{i\in\mathbb{Z}} \lambda_i o_{n,m}(a_i b) \\ &= \sum_{s=0}^{T-1} \sum_{i\in\mathbb{Z}} \lambda_i Y_M^{(s)}(a,b;\operatorname{wt} a + \operatorname{wt} b + m - n - 2 - i, i) \\ &= \sum_{s=0}^{T-1} \sum_{i\in\mathbb{Z}} \lambda_i Z_{M,n,m}^{(s)}(a,b;z^i) \\ &= \sum_{s=0}^{T-1} Z_{M,n,m}^{(s)}(a,b;P(z)) \\ &= 0. \end{aligned}$$

□

LEMMA 5.4. *For $a, b \in V$ and $w \in M(m)$*

$$o_{n,m}(a *^T_{n,p,m} b)w = o_{n,p}(a)o_{p,m}(b)w.$$

PROOF. We may assume a and b to be homogeneous elements of V. We simply write $r = r(p,n)$, which is defined in (3.8). By (3.20) and Lemma 5.2, we have

$$\begin{aligned} &Z^{(r)}_{M,n,m}(a, b; E^{(T;r)}_{n,m}(\mathrm{wt}\, a, \mathrm{wt}\, b, j; z))(w) \\ &\quad = Z^{(r)}_{M,n,m}(a, b; z^j)(w) \\ &\quad = Y^{(r)}_M(a, b; \mathrm{wt}\, a + \mathrm{wt}\, b + m - n - 2 - j, j) \end{aligned} \tag{5.4}$$

for $j \in \mathbb{Z}$. We write $\Phi^T_{n,p,m}(\mathrm{wt}\, a, \mathrm{wt}\, b; z) = \sum_{i \in \mathbb{Z}} \lambda_i z^i, \lambda_i \in \mathbb{C}$. By (5.4), we have

$$\begin{aligned} o_{n,m}(a *^T_{n,p,m} b)w &= \sum_{i \in \mathbb{Z}} \lambda_i o_{n,m}(a_i b)w \\ &= \sum_{s=0}^{T-1} \sum_{i \in \mathbb{Z}} \lambda_i Y^{(s)}_M(a, b; \mathrm{wt}\, a + \mathrm{wt}\, b + m - n - 2 - i, i)(w) \\ &= \sum_{s=0}^{T-1} Z^{(s)}_{M,n,m}(a, b; \Phi^T_{n,p,m}(\mathrm{wt}\, a, \mathrm{wt}\, b; z))(w) \\ &= Z^{(r)}_{M,n,m}(a, b; \Phi^T_{n,p,m}(\mathrm{wt}\, a, \mathrm{wt}\, b; z))(w) \\ &= \sum_{i=0}^{l_2} \binom{-l_1 - l_3 + l_2 - \delta(r \le i_1) - \delta(T \le r + i_3)}{i} \\ &\quad \times \mathrm{Res}_x \Bigg((1+x)^{\mathrm{wt}\, a - 1 + l_1 + \delta(r \le i_1) + r/T} x^{-l_1 - l_3 + l_2 - \delta(r \le i_1) - \delta(T \le r + i_3) - i} \\ &\qquad\qquad \times \sum_{j \in \mathbb{Z}} Z^{(r)}_{M,n,m}(a, b; E^{(T;r)}_{n,m}(\mathrm{wt}\, a, \mathrm{wt}\, b, j; z))(w) x^{-j-1} \Bigg) \\ &= \sum_{i=0}^{l_2} \binom{-l_1 - l_3 + l_2 - \delta(r \le i_1) - \delta(T \le r + i_3)}{i} \\ &\quad \times \mathrm{Res}_x \Bigg((1+x)^{\mathrm{wt}\, a - 1 + l_1 + \delta(r \le i_1) + r/T} x^{-l_1 - l_3 + l_2 - \delta(r \le i_1) - \delta(T \le r + i_3) - i} \\ &\qquad\qquad \times \sum_{j \in \mathbb{Z}} Y^{(r)}_M(a, b; \mathrm{wt}\, a + \mathrm{wt}\, b + m - n - 2 - j, j)(w) x^{-j-1} \Bigg). \end{aligned} \tag{5.5}$$

Let $\mu = -l_1 - l_3 + l_2 - \delta(r \le i_1) - \delta(T \le r + i_3)$ and $i \in \mathbb{Z}$. Then

$$
\begin{aligned}
&\mathrm{Res}_x \biggl((1+x)^{\mathrm{wt}\, a-1+l_1+\delta(r\le i_1)+r/T} x^{\mu-i} \\
&\qquad \times \sum_{j\in\mathbb{Z}} Y_M^{(r)}(a,b;\mathrm{wt}\, a+\mathrm{wt}\, b+m-n-2-j,j)(w)x^{-j-1}\biggr) \\
&\quad = \sum_{k=0}^{\infty} \binom{\mathrm{wt}\, a-1+l_1+\delta(r\le i_1)+r/T}{k} \\
&\quad\quad \times Y_M^{(r)}(a,b;\mathrm{wt}\, a+\mathrm{wt}\, b+m-n-2-\mu+i-k,\mu-i+k)(w) \\
&\quad = \sum_{k=0}^{\infty} \binom{\mathrm{wt}\, a-1+l_1+\delta(r\le i_1)+r/T}{k} \\
&\quad\quad \times \mathrm{Res}_{x_2}\mathrm{Res}_{x_1-x_2}\bigl(x_2^{\mathrm{wt}\, a+\mathrm{wt}\, b+m-n-2-\mu+i-k}(x_1-x_2)^{\mu-i+k} Y_M^{(r)}(a,b|x_2,x_1-x_2)(w)\bigr) \\
&\quad = \mathrm{Res}_{x_2}\mathrm{Res}_{x_1-x_2}\bigl(x_1^{\mathrm{wt}\, a-1+l_1+\delta(r\le i_1)+r/T} x_2^{\mathrm{wt}\, b-1+l_1+\delta(r^\vee\le i_1)+r^\vee/T-l_2+i-1} \\
&\qquad\qquad\qquad \times (x_1-x_2)^{-l_1-l_3+l_2-\delta(r\le i_1)-\delta(T\le r+i_3)-i} Y_M^{(r)}(a,b|x_2,x_1-x_2)(w)\bigr),
\end{aligned}
$$

where we used (3.13) in the last step and $r^\vee$ is defined in (3.9). Thus, (5.5) becomes

$$
\begin{aligned}
&o_{n,m}(a *_{n,p,m}^T b)w \\
&\quad = \sum_{i=0}^{l_2} \binom{-l_1-l_3+l_2-\delta(r\le i_1)-\delta(T\le r+i_3)}{i} \\
&\quad\quad \times \mathrm{Res}_{x_2}\mathrm{Res}_{x_1-x_2}\bigl(x_1^{\mathrm{wt}\, a-1+l_1+\delta(r\le i_1)+r/T} x_2^{\mathrm{wt}\, b-1+l_1+\delta(r^\vee\le i_1)+r^\vee/T-l_2+i-1} \\
&\qquad\qquad \times (x_1-x_2)^{-l_1-l_3+l_2-\delta(r\le i_1)-\delta(T\le r+i_3)-i} \\
&\qquad\qquad \times Y_M^{(r)}(a,b|x_2,x_1-x_2)(w)\bigr). \qquad (5.6)
\end{aligned}
$$

The rest of the proof is the same as that of [**5**, Lemma 5.1] by (2.17). □

The following result is a direct consequence of Lemma 5.4.

COROLLARY 5.5. *If M is generated by one homogeneous element w as a (V,T)-module, then* $M=\{a_i w \mid a\in V, i\in(1/T)\mathbb{Z}\}$.

We define an $A_n^T(V)$-$A_m^T(V)$-bimodule structure on $\mathrm{Hom}_{\mathbb{C}}(M(m),M(n))$ by

$$(afb)(w)=a(f(bw))$$

for $f\in\mathrm{Hom}_{\mathbb{C}}(M(m),M(n))$, $a\in A_n^T(V), b\in A_m^T(V)$ and $w\in M(m)$. For a (V,T)-module W and $m\in(1/T)\mathbb{N}$, define

$$\Omega_m(W)=\{w\in W \mid a_{\mathrm{wt}\, a-1+k}w=0 \text{ for all homogeneous } a\in V \text{ and } k>m\}.$$

Clearly, $\bigoplus_{i=0}^{m} M(i) \subset \Omega_m(M)$.

Lemmas 5.3 and 5.4 imply the following results.

Lemma 5.6. *For $u \in O^T_{n,m}(V)$, $o_{n,m}(u) = 0$ on $M(m)$. The linear map $o_{n,m} : V \to \mathrm{Hom}_{\mathbb{C}}(M(m), M(n))$ induces an $A^T_n(V)$-$A^T_m(V)$-bimodule homomorphism from $A^T_{n,m}(V)$ to $\mathrm{Hom}_{\mathbb{C}}(M(m), M(n))$.*

Proposition 5.7. *Let W be a (V, T)-module. Then $o : V \to \mathrm{End}_{\mathbb{C}}(\Omega_m(W))$ induces a representation of $A^T_m(V)$. In particular, $M(m)$ is a left $A^T_m(V)$-module.*

For a left $A^T_m(V)$-module U, set

$$M(U) = \bigoplus_{n \in (1/T)\mathbb{N}} A^T_{n,m}(V) \bigotimes_{A^T_m(V)} U$$

and $M(U)(n) = A^T_{n,m}(V) \bigotimes_{A^T_m(V)} U$ for every $n \in (1/T)\mathbb{N}$. For homogeneous $a \in V$ and $i \in (1/T)\mathbb{Z}$, define an operator a_i from $M(U)(n)$ to $M(U)(n + \mathrm{wt}\, a - i - 1)$ by

$$a_i(b \otimes u) = \begin{cases} (a *^T_{n+\mathrm{wt}\, a-i-1,n,m} b) \otimes u & \text{if } n + \mathrm{wt}\, a - i - 1 \geq 0, \\ 0 & \text{if } n + \mathrm{wt}\, a - i - 1 < 0 \end{cases} \tag{5.7}$$

for $b \otimes u \in M(U)(n)$ with $b \in V$ and $u \in U$. This operation is well-defined (cf. [**4**, p. 815]). We extend a_i for an arbitrary $a \in V$ by linearity and set

$$Y_{M(U)}(a, x) = \sum_{i \in (1/T)\mathbb{Z}} a_i x^{-i-1} : M(U) \to M(U)(\!(x^{1/T})\!).$$

We shall show $(M(U), Y_{M(U)})$ is a $(1/T)\mathbb{N}$-graded (V, T)-module. For homogeneous $a, b \in V$, $s \in \mathbb{Z}$ with $0 \leq s \leq T - 1$, $i \in (1/T)\mathbb{Z}$ and $j \in \mathbb{Z}$, define a linear map $Y^{(s)}_{M(U)}(a, b; i, j) : M(U)(n) \to M(U)(\mathrm{wt}\, a + \mathrm{wt}\, b - i - j - 2 + n)$ by

$$\begin{aligned} &Y^{(s)}_{M(U)}(a, b; i, j)(c \otimes u) \\ &\quad = \big(\hat{E}^{(T;s)}_{\mathrm{wt}\, a+\mathrm{wt}\, b-i-j-2+n,n}(a, b, j) *^T_{\mathrm{wt}\, a+\mathrm{wt}\, b-i-j-2+n,n,m} c\big) \otimes u \end{aligned} \tag{5.8}$$

for $c \otimes u \in M(U)(n)$ with $c \in V$ and $u \in U$. This operation is also well-defined. We extend $Y^{(s)}_{M(U)}(a, b; i, j)$ for arbitrary $a, b \in V$ by linearity and set

$$Y^{(s)}_{M(U)}(a, b|x_2, x_0) = \sum_{i \in (1/T)\mathbb{Z}} \sum_{j \in \mathbb{Z}} Y^{(s)}_{M(U)}(a, b; i, j) x_2^{-i-1} x_0^{-j-1}.$$

It follows by (3.19) and (4.1) that $Y^{(s)}_{M(U)}(a, b|x_2, x_0)$ is a linear map from $M(U)$ to $M(U)(\!(x_2^{1/T})\!)(\!(x_0)\!)$.

From now on, we simply write $M = M(U)$. By (4.4) we have

$$\begin{aligned}
&\sum_{s=0}^{T-1} Y_M^{(s)}(a,b;i,j)(c\otimes u)\\
&\quad = \sum_{s=0}^{T-1} (\hat{E}^{(T;s)}_{\mathrm{wt}\,a+\mathrm{wt}\,b-i-j-2+n,n}(a,b,j) *^T_{\mathrm{wt}\,a+\mathrm{wt}\,b-i-j-2+n,n,m} c)\otimes u\\
&\quad = ((a_j b) *^T_{\mathrm{wt}\,a+\mathrm{wt}\,b-i-j-2+n,n,m} c)\otimes u\\
&\quad = (a_j b)_i (c\otimes u)
\end{aligned}$$

for homogeneous $a, b \in V$ and $c\otimes u \in M(n)$ with $c \in V$ and $u \in U$. Thus

$$\sum_{s=0}^{T-1} Y_M^{(s)}(a,b|x_2,x_0)(w) = Y_M(Y(a,x_0)b,x_2)w \tag{5.9}$$

for $w \in M$.

Lemma 5.8. (1) $a_i M(n) = 0$ *for homogeneous* $a \in V$ *and* $i > \mathrm{wt}\,a - 1 + n$.
(2) $Y_M(\mathbf{1},x) = \mathrm{id}_M$.

Proof. Clearly, (1) holds. Let $a\otimes u \in M(n)$ with $a \in V$ and $u \in U$. By Lemma 3.6, we have

$$\begin{aligned}
\mathbf{1}_i(a\otimes u) &= (\mathbf{1} *^T_{-i-1+n,n,m} a)\otimes u\\
&= \delta_{i,-1}(\mathbf{1}_{-1}a)\otimes u = \delta_{i,-1} a\otimes u
\end{aligned}$$

for $i \in (1/T)\mathbb{Z}$. □

Lemma 5.9. *Let* a, b *be homogeneous elements of* V, $i, j \in (1/T)\mathbb{Z}$ *and* r *the integer uniquely determined by the conditions* $0 \le r \le T-1$ *and* $r/T \equiv i \pmod{\mathbb{Z}}$. *Then*

$$[a_i, b_j]w = \sum_{k=0}^{\infty} \binom{i}{k} Y_M^{(r)}(a,b;i+j-k,k)(w)$$

for $w \in M$. *In particular,*

$$(x_1 - x_2)^l [Y_M(a,x_1), Y_M(b,x_2)] = 0$$

for $l \in \mathbb{Z}_{\ge \max\{\mathrm{wt}\,a+\mathrm{wt}\,b-\Delta,0\}}$.

Proof. Let $c\otimes u \in M(n)$ with $c \in V$ and $u \in U$. By Lemma 4.3, we have

$$\begin{aligned}
&a_i b_j(c\otimes u) - b_j a_i(c\otimes u)\\
&\quad = (a *^T_{\mathrm{wt}\,a+\mathrm{wt}\,b-i-j-2+n,\mathrm{wt}\,b-1-j+n,m} (b *^T_{\mathrm{wt}\,b-1-j+n,n,m} c))\otimes u\\
&\qquad - (b *^T_{\mathrm{wt}\,a+\mathrm{wt}\,b-i-j-2+n,\mathrm{wt}\,a-1-i+n,m} (a *^T_{\mathrm{wt}\,a-1-i+n,n,m} c))\otimes u
\end{aligned}$$

$$
\begin{aligned}
&= ((a *^{T}_{\mathrm{wt}\,a+\mathrm{wt}\,b-i-j-2+n,\mathrm{wt}\,b-1-j+n,n} b) *^{T}_{\mathrm{wt}\,a+\mathrm{wt}\,b-i-j-2+n,n,m} c) \otimes u \\
&\quad - ((b *^{T}_{\mathrm{wt}\,a+\mathrm{wt}\,b-i-j-2+n,\mathrm{wt}\,a-1-i+n,n} a) *^{T}_{\mathrm{wt}\,a+\mathrm{wt}\,b-i-j-2+n,n,m} c) \otimes u \\
&= \left(\mathrm{Res}_x (1+x)^i \left(\sum_{p\in\mathbb{Z}} \hat{E}^{(T;r)}_{\mathrm{wt}\,a+\mathrm{wt}\,b-i-j-2+n,n}(a,b,p)x^{-p-1} \right) *^{T}_{\mathrm{wt}\,a+\mathrm{wt}\,b-i-j-2+n,n,m} c \right) \otimes u \\
&= \left(\sum_{k=0}^{\infty} \binom{i}{k} \hat{E}^{(T;r)}_{\mathrm{wt}\,a+\mathrm{wt}\,b-i-j-2+n,n}(a,b,k) *^{T}_{\mathrm{wt}\,a+\mathrm{wt}\,b-i-j-2+n,n,m} c \right) \otimes u \\
&= \sum_{k=0}^{\infty} \binom{i}{k} Y^{(r)}_M(a,b;i+j-k,k)(c\otimes u).
\end{aligned}
$$

The last formula follows from this and Remark 2.6 (cf. [**14**, Remark 3.1.13]). □

We recall that $Y^r_M(a,x)$ denotes $\sum_{i\in r/T+\mathbb{Z}} a_i x^{-i-1}$ for $a\in V$ (cf. (2.11)).

LEMMA 5.10. *Let $a,b\in V$ with a being homogeneous, $l,r\in\mathbb{N}$ with $0\le r\le T-1$ and $n=l_3+i_3/T\in(1/T)\mathbb{N}$ with $l_3,i_3\in\mathbb{N}$ and $0\le i_3\le T-1$. Then*

$$
\begin{aligned}
&\mathrm{Res}_{x_0}\, x_0^{l}(x_2+x_0)^{\mathrm{wt}\,a-1+l_3+\delta(r\le i_3)+r/T} Y^{(r)}_M(a,b|x_2,x_0)(w) \\
&\quad = \mathrm{Res}_{x_0}\, x_0^{l}(x_0+x_2)^{\mathrm{wt}\,a-1+l_3+\delta(r\le i_3)+r/T} Y^{r}_M(a,x_0+x_2)Y_M(b,x_2)w
\end{aligned}
$$

for $w\in M(n)$.

PROOF. Using Lemma 5.9, we obtain the formula by the same computation as in the proof of [**5**, Lemma 5.9]. □

LEMMA 5.11. *Let $a,b\in V$ with a being homogeneous, $r\in\mathbb{N}$ with $0\le r\le T-1$ and $n=l_3+i_3/T\in(1/T)\mathbb{N}$ with $l_3\in\mathbb{N}$ and $0\le i_3\le T-1$. Then*

$$
\begin{aligned}
&\mathrm{Res}_{x_0}\, x_0^{-l}(x_2+x_0)^{\mathrm{wt}\,a-1+l_3+\delta(r\le i_3)+r/T} Y^{(r)}_M(a,b|x_2,x_0)(w) \\
&\quad = \mathrm{Res}_{x_0}\, x_0^{-l}(x_0+x_2)^{\mathrm{wt}\,a-1+l_3+\delta(r\le i_3)+r/T} Y^{r}_M(a,x_0+x_2)Y_M(b,x_2)w
\end{aligned}
$$

for $w\in M(n)$.

PROOF. Let $c\otimes u\in M(n)$ with $c\in V$ and $u\in U$. We may assume b to be a homogeneous element of V. We shall show

$$
\begin{aligned}
&\mathrm{Res}_{x_0}\, x_0^{-l}(x_2+x_0)^{\mathrm{wt}\,a+q} x_2^{\mathrm{wt}\,b-q} Y^{(r)}_M(a,b|x_2,x_0)(c\otimes u) \\
&\quad = \mathrm{Res}_{x_0}\, x_0^{-l}(x_0+x_2)^{\mathrm{wt}\,a+q} x_2^{\mathrm{wt}\,b-q} Y^{r}_M(a,x_0+x_2)Y_M(b,x_2)c\otimes u,
\end{aligned}
$$

where $q=-1+l_3+\delta(r\le i_3)+r/T$. We have

$$\begin{aligned}
&\operatorname{Res}_{x_0} x_0^{-l}(x_2+x_0)^{\mathrm{wt}\,a+q}x_2^{\mathrm{wt}\,b-q}Y_M^{(r)}(a,b|x_2,x_0)(c\otimes u)\\
&=\sum_{j=0}^{\infty}\sum_{k\in(1/T)\mathbb{Z}}\binom{\mathrm{wt}\,a-1+l_3+\delta(r\le i_3)+r/T}{j}x_2^{-k-1+\mathrm{wt}\,a+\mathrm{wt}\,b-j}\\
&\quad\times Y_M^{(r)}(a,b;k,j-l)(c\otimes u)\\
&=\sum_{j=0}^{\infty}\sum_{k\in(1/T)\mathbb{N}}\binom{\mathrm{wt}\,a-1+l_3+\delta(r\le i_3)+r/T}{j}x_2^{-l+k-n+1}\\
&\quad\times Y_M^{(r)}(a,b;\mathrm{wt}\,a+\mathrm{wt}\,b-j+l-k+n-2,j-l)(c\otimes u)\\
&=\sum_{j=0}^{\infty}\sum_{k\in(1/T)\mathbb{N}}\binom{\mathrm{wt}\,a-1+l_3+\delta(r\le i_3)+r/T}{j}x_2^{-l+k-n+1}\\
&\quad\times(\hat{E}_{k,n}^{(T;r)}(a,b,j-l)*_{k,n,m}^{T}c)\otimes u\\
&=\sum_{k\in(1/T)\mathbb{N}}x_2^{-l+k-n+1}\Bigg(\operatorname{Res}_x x^{-l}(1+x)^{\mathrm{wt}\,a-1+l_3+\delta(r\le i_3)+r/T}\\
&\qquad\qquad\times\Bigg(\sum_{j\in\mathbb{Z}}\hat{E}_{k,n}^{(T;r)}(a,b,j)x^{-j-1}\Bigg)*_{k,n,m}^{T}c\Bigg)\otimes u. \qquad (5.10)
\end{aligned}$$

On the other hand, applying the same computation as in the proof of [**5**, Lemma 5.10] to

$$\begin{aligned}
&\operatorname{Res}_{x_0} x_0^{-l}(x_0+x_2)^{\mathrm{wt}\,a+q}x_2^{\mathrm{wt}\,b-q}Y_M^{r}(a,x_0+x_2)Y_M(b,x_2)(c\otimes u)\\
&\quad=\sum_{s=0}^{T-1}\operatorname{Res}_{x_0} x_0^{-l}(x_0+x_2)^{\mathrm{wt}\,a+q}x_2^{\mathrm{wt}\,b-q}Y_M^{r}(a,x_0+x_2)Y_M^{s}(b,x_2)(c\otimes u),
\end{aligned}$$

we have

$$\begin{aligned}
&\operatorname{Res}_{x_0} x_0^{-l}(x_0+x_2)^{\mathrm{wt}\,a+q}x_2^{\mathrm{wt}\,b-q}Y_M^{r}(a,x_0+x_2)Y_M(b,x_2)(c\otimes u)\\
&=\sum_{s=0}^{T-1}\sum_{\substack{k\in(i_3-r-s)/T+\mathbb{Z}\\0\le k}}x_2^{-l+k-n+1}\\
&\quad\times\sum_{\substack{j\in(i_3-s)/T+\mathbb{Z}\\0\le j\le k+l_3+\delta(r\le i_3)+r/T-l}}\binom{-l}{-j+l_3+\delta(r\le i_3)+r/T-l+k}\\
&\quad\times(-1)^{-j+l_3+\delta(r\le i_3)+r/T-l+k}(a*_{k,j,m}^{T}(b*_{j,n,m}^{T}c))\otimes u
\end{aligned}$$

$$\equiv \sum_{s=0}^{T-1} \sum_{\substack{k\in(i_3-r-s)/T+\mathbb{Z}\\ 0\le k}} x_2^{-l+k-n+1}$$
$$\times \sum_{\substack{j\in(i_3-s)/T+\mathbb{Z}\\ 0\le j\le k+l_3+\delta(r\le i_3)+r/T-l}} \binom{-l}{-j+l_3+\delta(r\le i_3)+r/T-l+k}$$
$$\times (-1)^{-j+l_3+\delta(r\le i_3)+r/T-l+k}((a *_{k,j,n}^T b) *_{k,n,m}^T c)\otimes u$$
$$\pmod{O_{n,m}^T(V)(\!(x_2)\!)}. \tag{5.11}$$

Moreover, for each $k = l_4 + (i_3 - r - s)/T \in (i_3 - r - s)/T + \mathbb{Z}$ with $k \ge 0$ and $l_4 \in \mathbb{Z}$, we have

$$\sum_{\substack{j\in(i_3-s)/T+\mathbb{Z}\\ 0\le j\le k+l_3+\delta(r\le i_3)+r/T-l}} \binom{-l}{-j+l_3+\delta(r\le i_3)+r/T-l+k}$$
$$\times (-1)^{-j+l_3+\delta(r\le i_3)+r/T-l+k} a *_{k,j,n}^T b$$
$$= \sum_{p=0}^{l_4+l_3+\delta(r\le i_3)+\delta(s\le i_3)-l-1} \binom{-l}{p}(-1)^p \sum_{i=0}^{l_4+l_3+\delta(r\le i_3)+\delta(s\le i_3)-l-1-p} \binom{-p-l}{i}$$
$$\times \operatorname{Res}_x (1+x)^{\mathrm{wt}\, a-1+l_3+\delta(r\le i_3)+r/T} x^{-p-l-i} \sum_{j\in\mathbb{Z}} \hat{E}_{k,n}^{(T;r)}(a,b,j)x^{-j-1}$$
$$= \operatorname{Res}_x x^{-l}(1+x)^{\mathrm{wt}\, a-1+l_3+\delta(r\le i_3)+r/T} \sum_{j\in\mathbb{Z}} \hat{E}_{k,n}^{(T;r)}(a,b,j)x^{-j-1} \tag{5.12}$$

by [**4**, Proposition 5.3]. By (5.10)–(5.12) the proof is complete. □

By Lemmas 5.10 and 5.11, we have the following result.

LEMMA 5.12. *Let $a, b \in V$ with a being homogeneous, $r \in \mathbb{N}$ with $0 \le r \le T-1$ and $n = l_3 + i_3/T \in (1/T)\mathbb{N}$ with $l_3, i_3 \in \mathbb{N}$ and $0 \le i_3 \le T-1$. Then*

$$(x_2+x_0)^{\mathrm{wt}\, a-1+l_3+\delta(r\le i_3)+r/T} Y_M^{(r)}(a,b|x_2,x_0)$$
$$= (x_0+x_2)^{\mathrm{wt}\, a-1+l_3+\delta(r\le i_3)+r/T} Y_M^r(a,x_0+x_2)Y_M(b,x_2)$$

on $M(n)$.

By (5.9) and Lemmas 2.4, 5.8, 5.9 and 5.12, the same argument as in the proof of [**4**, Theorem 4.13] shows the following theorem.

THEOREM 5.13. *Let U be a left $A_m^T(V)$-module. Then $M(U) = \bigoplus_{n\in(1/T)\mathbb{N}} \cdot A_{n,m}^T(V) \bigotimes_{A_m^T(V)} U$ is a $(1/T)\mathbb{N}$-graded (V,T)-module with $M(U)(n) = A_{n,m}^T(V)$*

$\cdot \bigotimes_{A_m^T(V)} U$ *and the following universal property*: *for a* (V,T)*-module* W *and an* $A_m^T(V)$*-homomorphism* $\sigma : U \to \Omega_m(W)$, *there is a unique homomorphism* $\bar{\sigma} : M(U) \to W$ *of* (V,T)*-modules that extends* σ. *Moreover, if* U *cannot factor through* $A_{m-1/T}^T(V)$, *then* $M(U)(0) \neq 0$.

The following result immediately follows from Theorem 5.13 (cf. [**7**, Theorem 4.9]).

Corollary 5.14. *For every* $m \in (1/T)\mathbb{N}$, *there is a bijection between the set of isomorphism classes of simple left* $A_m^T(V)$*-modules which cannot factor through* $A_{m-1/T}^T(V)$ *and that of simple* $(1/T)\mathbb{N}$*-graded* (V,T)*-modules.*

6. Appendix.

6.1. The determinant of a matrix.

In this subsection we shall show that the matrix Γ in (3.5) is non-singular. Let b, t be positive integers and $x_0, \dots, x_{t-1}$ indeterminates. We denote by E_n the $n \times n$ identity matrix. Define $\alpha_i^k(x_s) = \sum_{j=1}^k \binom{x_s}{i+j}\binom{-x_s}{k-j} \in \mathbb{C}[x_s]$ for $0 \leq s \leq t-1$, $1 \leq k \leq b$ and $i \in \mathbb{Z}$. Note that

$$\deg \alpha_i^k(x_s) = i + k \tag{6.1}$$

for $i \in \mathbb{N}$. Define t $bt \times b$-matrices $A_s, s = 0, \dots, t-1$ by

$$A_s = \begin{pmatrix} \alpha^1_{(t-1)b-1}(x_s) & \alpha^2_{(t-1)b-1}(x_s) & \cdots & \alpha^b_{(t-1)b-1}(x_s) \\ \alpha^1_{(t-1)b-2}(x_s) & \alpha^2_{(t-1)b-2}(x_s) & \cdots & \alpha^b_{(t-1)b-2}(x_s) \\ \vdots & \vdots & & \vdots \\ \alpha^1_{-b}(x_s) & \alpha^2_{-b}(x_s) & \cdots & \alpha^b_{-b}(x_s) \end{pmatrix} \tag{6.2}$$

and set $A = (A_0 \cdots A_{t-1})$. The following result implies Γ is non-singular.

Proposition 6.1.

$$\det A = \prod_{0 \leq i < j \leq t-1} \prod_{k=-b+1}^{b-1} \left(\frac{x_i - x_j + k}{b(j-i) + k} \right)^{b-|k|}.$$

Proof. Since

$$\begin{aligned} &(\alpha_i^1(x_s), \dots, \alpha_i^b(x_s)) \\ &= \left(\binom{x_s}{i+1}, \binom{x_s}{i+2}, \dots, \binom{x_s}{i+b} \right) \begin{pmatrix} 1 & \binom{-x_s}{1} & \binom{-x_s}{2} & \cdots & \binom{-x_s}{b} \\ 0 & 1 & \binom{-x_s}{1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \binom{-x_s}{2} \\ \vdots & & \ddots & \ddots & \binom{-x_s}{1} \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}, \end{aligned}$$

the determinant of A is equal to that of $B = (B_0 \cdots B_{t-1})$, where

$$B_s = \begin{pmatrix} \binom{x_s}{(t-1)b} & \binom{x_s}{(t-1)b+1} & \cdots & \binom{x_s}{tb-1} \\ \binom{x_s}{(t-1)b-1} & \binom{x_s}{(t-1)b} & \cdots & \binom{x_s}{tb-2} \\ \vdots & \vdots & & \vdots \\ \binom{x_s}{-b+1} & \binom{x_s}{-b+2} & \cdots & \binom{x_s}{0} \end{pmatrix}, \ s = 0, \dots, t-1.$$

The same argument as in the proof of [**16**, Proposition 9] shows that $(x_i - x_j + k)^{b-|k|}$ is a factor of $\det B$ for each $0 \le i < j \le t-1$ and $-b+1 \le k \le b-1$. Thus, there is $c \in \mathbb{C}[x_0, \dots, x_{t-1}]$ such that

$$\det B = c \prod_{0 \le i < j \le t-1} \prod_{k=-b+1}^{b-1} (x_i - x_j + k)^{b-|k|}.$$

Since $\alpha_i^k(x_s) = \delta_{i+k,0}$ for $i < 0$, we have $A_s = \binom{A'_s}{E_b}$, $s = 0, \dots, T-1$, where

$$A'_s = \begin{pmatrix} \alpha^1_{(t-1)b-1}(x_s) & \alpha^2_{(t-1)b-1}(x_s) & \cdots & \alpha^b_{(t-1)b-1}(x_s) \\ \alpha^1_{(t-1)b-2}(x_s) & \alpha^2_{(t-1)b-2}(x_s) & \cdots & \alpha^b_{(t-1)b-2}(x_s) \\ \vdots & \vdots & & \vdots \\ \alpha^1_0(x_s) & \alpha^2_0(x_s) & \cdots & \alpha^b_0(x_s) \end{pmatrix}.$$

It follows by

$$\begin{pmatrix} E_{(t-1)b} & -A'_0 \\ O & E_b \end{pmatrix} A = \begin{pmatrix} O & A'_1 - A'_0 & \cdots & A'_{t-1} - A'_0 \\ E_b & E_b & \cdots & E_b \end{pmatrix} \tag{6.3}$$

that $\det A = (-1)^{(t-1)b^2} \det(A'_1 - A'_0 \ \cdots \ A'_{t-1} - A'_0)$. Thus, the degree of $\det A \in \mathbb{C}[x_0, \dots, x_{t-1}]$ is at most $\binom{t}{2} b^2$ by (6.1). Since the degree of $\prod_{0 \le i < j \le t-1} \prod_{k=-b+1}^{b-1} (x_i - x_j + k)^{b-|k|}$ is equal to $\binom{t}{2} b^2$, we have $c \in \mathbb{C}$.

Substituting $((t-1)b, (t-2)b, \dots, 0)$ for $(x_0, x_1, \dots, x_{t-1})$, we obtain

$$1 = c \prod_{0 \le i < j \le t-1} \prod_{k=-b+1}^{b-1} (b(j-i) + k)^{b-|k|}.$$

The proof is complete. □

6.2. Some improvements of results on $A_{G,n}(V)$.

The purpose of this subsection is to improve Theorems 1 and 2 in [**16**]. Let $V = \bigoplus_{j=\Delta}^{\infty} V_j$ be a vertex operator algebra and G an automorphism group of V of finite order t. For $g \in G$ and $n \in (1/t)\mathbb{N}$, $O_{g,n}(V)$ is the subspace of V defined in [**8**].

In [**16**], under the condition that $\Delta = 0$, we constructed an associative algebra

$A_{G,n}(V)$ for each $n \in (1/t)\mathbb{Z}$ in Theorem 1 and got a duality theorem of Schur–Weyl type in Theorem 2 by using $A_{G,n}(V)$. The condition that $\Delta = 0$ was used in order to show the non-singularity of a matrix in [**16**, Lemma 3].

We shall show [**16**, Theorems 1 and 2] without assuming $\Delta = 0$. To do this, it is sufficient to show the following lemma, which is an improvement of [**16**, Lemma 3], by using $\hat{E}^{(t;s)}_{n,m}(a,b,i)$ defined in (4.1). We note that the existence of $\hat{E}^{(t;s)}_{n,m}(a,b,i)$ follows from Lemma 3.2 and Proposition 6.1. We use the notation in Remark 2.7.

Lemma 6.2. *For $a, b \in V = \bigoplus_{j=\Delta}^{\infty} V_j$, $0 \le r \le t-1$, $p \in \mathbb{Z}$, $n \in (1/t)\mathbb{N}$ and $g \in G$, we have*

$$\hat{E}^{(t;r)}_{n,n}(a,b,p) \equiv a^{(g,r)}_p b \pmod{O_{g,n}(V)}.$$

Proof. We may assume a, b to be homogeneous. We write $n = l + i/t$ with $l, i \in \mathbb{N}$ and $0 \le i \le t-1$. We use the notation in Section 3. It follows from (3.1) that the image of the subspace $O(\operatorname{wt} a + \operatorname{wt} b - 1 - \Delta, \operatorname{wt} a - 1 + l + \delta(s \le i) + s/t, -2l-3; z)$ of $\mathbb{C}[z, z^{-1}]$ under the map $\mathbb{C}[z,z^{-1}] \ni f \mapsto f|_{z^j = a^{(g,s)}_j b} \in V$ is contained in $O_{g,n}(V)$ for $s = 0, \dots, t-1$. By (3.18), we have

$$\begin{aligned}\hat{E}^{(t;r)}_{n,n}(a,b,p) &= E^{(t;r)}_{n,n}(\operatorname{wt} a, \operatorname{wt} b, p; z)|_{z^j = a_j b}\\ &= \sum_{s \ne r} E^{(t;r)}_{n,n}(\operatorname{wt} a, \operatorname{wt} b, p; z)|_{z^j = a^{(g,s)}_j b} + E^{(t;r)}_{n,n}(\operatorname{wt} a, \operatorname{wt} b, p; z)|_{z^j = a^{(g,r)}_j b}\\ &\equiv a^{(g,r)}_p b \pmod{O_{g,n}(V)}. \qquad \square\end{aligned}$$

7. List of Notations.

$\delta(i \le j)$ $= \begin{cases} 1 & \text{if } i \le j, \\ 0 & \text{if } i > j. \end{cases}$

$Y^s_M(a,x)$ $= \sum_{i \in s/T + \mathbb{Z}} a_i x^{-i-1}$ where $Y_M(a,x) = \sum_{i \in (1/T)\mathbb{Z}} a_i x^{-i-1}$.

$O(N,Q,q;z)$ the subspace of $\mathbb{C}[z,z^{-1}]$ spanned by $\operatorname{Res}_x \big((1+x)^Q x^{q+j} \sum_{i \in \mathbb{Z}_{\le N}} \cdot z^i x^{-i-1}\big)$, $j = 0, -1, \dots$ and $z^i, i \in \mathbb{Z}_{\ge N+1}$ where $N, q \in \mathbb{Z}$ and $Q \in \mathbb{Q}$.

$\varphi_{N,\gamma}$ $\varphi_{N,\gamma}(z^i) = \begin{cases} (-1)^{i+1} \operatorname{Res}_x \big((1+x)^{\gamma - i} x^i \sum_{j \in \mathbb{Z}_{\le N}} z^j x^{-j-1}\big) & \text{for } i \le N, \\ z^i & \text{for } i \ge N+1, \end{cases}$
for $z^i \in \mathbb{C}[z,z^{-1}]$.

$r(i,j)$ the integer uniquely determined by the conditions that $0 \le r(i,j) \le T-1$ and $i - j \equiv r(i,j)/T \pmod{\mathbb{Z}}$ where $i, j \in (1/T)\mathbb{Z}$ and $T \in \mathbb{Z}_{>0}$.

$s^\vee$ the integer uniquely determined by the conditions that $0 \le s^\vee \le T-1$ and $i_1 - i_3 \equiv s + s^\vee \pmod{T}$ where $T \in \mathbb{Z}_{>0}$ and $i_1, i_3, s \in \mathbb{Z}$ with $0 \le i_1, i_3, s \le T$.

$$
\begin{array}{ll}
O_{n,m}^{(T;s),1}(\alpha,\beta;z) & = O\big(\alpha+\beta-1-\Delta, \alpha-1+l_1+\delta(s\le i_1)+s/T, \\
& \qquad -l_1-l_3-\delta(s\le i_1)-\delta(T\le s+i_3)-1; z\big). \\
O_{n,m}^{T,1}(\alpha,\beta;z) & = \bigcap_{s=0}^{T-1} O_{n,m}^{(T;s),1}(\alpha,\beta;z). \\
\Psi_{n,m}^{(T;s)}(\alpha,\beta,j;z) & = \mathrm{Res}_x \big((1+x)^{\alpha-1+l_1+\delta(s\le i_1)+s/T} x^{-l_1-l_3-\delta(s\le i_1)-\delta(T\le s+i_3)-1+j} \\
& \qquad \times \sum_{\substack{i\in\mathbb{Z} \\ i\le\alpha+\beta-1-\Delta}} z^i x^{-i-1}\big) \qquad \text{(cf. (3.12))}. \\
E_{n,m}^{(T;r)}(\alpha,\beta,i;z) & \text{the Laurent polynomial in } \mathbb{C}[z,z^{-1}]_{\alpha+\beta-\Delta-T(\alpha+\beta-\Delta+l_1+l_3+2),\alpha+\beta-1-\Delta} \\
& \text{uniquely determined by the condition (3.18)}. \\
\Phi_{n,p,m}^{T}(\alpha,\beta;z) & = \sum_{i=0}^{l_2} \binom{-l_1-l_3+l_2-\delta(r(p,n)\le i_1)-\delta(T\le r(p,n)+i_3)}{i} \\
& \quad \times \mathrm{Res}_x \big((1+x)^{\alpha-1+l_1+\delta(r(p,n)\le i_1)+r(p,n)/T} \\
& \qquad \times x^{-l_1-l_3+l_2-\delta(r(p,n)\le i_1)-\delta(T\le r(p,n)+i_3)-i} \\
& \qquad \times \sum_{j\in\mathbb{Z}} E_{n,m}^{(T;r(p,n))}(\alpha,\beta,j;z) x^{-j-1}\big) \qquad \text{(cf. (3.22))}. \\
\hat{E}_{n,m}^{(T;s)}(a,b,i) & = E_{n,m}^{(T;s)}(\mathrm{wt}\,a,\mathrm{wt}\,b,i;z)|_{z^j=a_jb} \in V. \\
a *_{n,p,m}^{T} b & = \Phi_{n,p,m}^{T}(\mathrm{wt}\,a,\mathrm{wt}\,b;z)|_{z^j=a_jb} \in V. \\
O_{n,m}^{T,0}(V) & \text{the subspace of } V \text{ spanned by } \{a_{-2}\mathbf{1}+(\mathrm{wt}\,a+m-n)a \in V \mid \\
& \text{homogeneous } a\in V\}. \\
O_{n,m}^{T,1}(V) & \text{the subspace of } V \text{ spanned by} \\
& \left\{ P(z)|_{z^j=a_jb} \in V \;\middle|\; \begin{array}{l} \text{homogeneous } a,b\in V \text{ and} \\ P(z)\in O_{n,m}^{T,1}(\mathrm{wt}\,a,\mathrm{wt}\,b;z) \end{array} \right\}. \\
O_{n,m}^{T,2}(V) & \text{the subspace of } V \text{ spanned by } u *_{n,p_3,m}^{T} ((a *_{p_3,p_2,p_1}^{T} b) *_{p_3,p_1,m}^{T} c - \\
& a *_{p_3,p_2,m}^{T} (b *_{p_2,p_1,m}^{T} c)) \text{ for all } a,b,c,u\in V \text{ and all } p_1,p_2,p_3\in(1/T)\mathbb{N}. \\
O_{n,m}^{T,3}(V) & = \sum_{p_1,p_2\in(1/T)\mathbb{N}} (V *_{n,p_2,p_1}^{T} (O_{p_2,p_1}^{T,0}(V)+O_{p_2,p_1}^{T,1}(V)) *_{n,p_1,m}^{T} V). \\
O_{n,m}^{T}(V) & = O_{n,m}^{T,0}(V)+O_{n,m}^{T,1}(V)+O_{n,m}^{T,2}(V)+O_{n,m}^{T,3}(V). \\
Z_{M,n,m}^{(s)}(a,b;z^i) & = Y_M^{(s)}(a,b;\mathrm{wt}\,a+\mathrm{wt}\,b+m-n-2-i,i).
\end{array}
$$

References

[1] R. Borcherds, Vertex algebras, Kac-Moody algebras, and the Monster, Proc. Nat. Acad. Sci. U.S.A., **83** (1986), 3068–3071.

[2] R. Dijkgraaf, C. Vafa, E. Verlinde and H. Verlinde, The operator algebra of orbifold models, Comm. Math. Phys., **123** (1989), 485–526.

[3] L. Dixon, J. A. Harvey, C. Vafa and E. Witten, String on orbifolds, Nucl. Phys. B, **261** (1985), 620–678; String on orbifolds II, Nucl. Phys. B, **274** (1986), 285–314.

[4] C. Dong and C. Jiang, Bimodules associated to vertex operator algebras, Math. Z., **259** (2008), 799–826.

[5] C. Dong and C. Jiang, Bimodules and g-rationality of vertex operator algebras, Trans. Amer. Math. Soc., **360** (2008), 4235–4262.

[6] C. Dong, H. S. Li and G. Mason, Twisted representations of vertex operator algebras, Math. Ann., **310** (1998), 571–600.

[7] C. Dong, H. S. Li and G. Mason, Vertex operator algebras and associative algebras, J. Algebra, **206** (1998), 67–96.

[8] C. Dong, H. S. Li and G. Mason, Twisted representations of vertex operator algebras and associative algebras, Internat. Math. Res. Notices, (1998), 389–397.

[9] C. Dong and G. Mason, On quantum Galois theory, Duke Math. J., **86** (1997), 305–321.

[10] E. Frenkel and D. Ben-Zvi, Vertex algebras and algebraic curves, 2nd edn. Mathematical Surveys and Monographs, **88**, Amer. Math. Soc., Providence, 2004.

[11] I. B. Frenkel, J. Lepowsky and A. Meurman, Vertex Operator Algebras and the Monster, Pure and Applied Math., **134**, Academic Press, 1988.

[12] A. Hanaki, M. Miyamoto and D. Tambara, Quantum Galois theory for finite groups, Duke Math. J., **97** (1999), 541–544.

[13] J. Lepowsky, Calculus of twisted vertex operators, Proc. Nat. Acad. Sci. U.S.A., **82** (1985), 8295–8299.

[14] J. Lepowsky and H. S. Li, Introduction to Vertex Operator Algebras and their Representations, Progress in Mathematics, **227**, Birkhauser Boston, Inc., Boston, MA, 2004.

[15] H. S. Li, Local systems of twisted vertex operators, vertex operator superalgebras and twisted modules, Contemp. Math., **193** (1996), 203–236.

[16] M. Miyamoto and K. Tanabe, Uniform product of $A_{g,n}(V)$ for an orbifold model V and G-twisted Zhu algebra, J. Algebra, **274** (2004), 80–96.

[17] Y. Zhu, Modular invariance of characters of vertex operator algebras, J. Amer. Math. Soc., **9** (1996), 237–302.

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