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Resolvent estimates on symmetric spaces of noncompact type

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Abstract. In this article we prove resolvent estimates for the Laplace-Beltrami operator or more general elliptic Fourier multipliers on symmetric spaces of noncompact type. Then the Kato theory implies time-global smoothing estimates for corresponding dispersive equations, especially the Schrödinger evolution equation. For low-frequency estimates, a pseudo-dimension appears as an upper bound of the order of elliptic Fourier multipliers. A key of the proof is to show a weighted L^2-continuity of the modified Radon transform and fractional integral operators.

1. Introduction.

The purpose of this paper is to study resolvent estimates for elliptic Fourier multipliers and smoothing effects for corresponding dispersive equations on symmetric spaces of noncompact type. A typical example of dispersive equations is the Schrödinger evolution equation. It is known that singularities of the solution of the Schrödinger evolution equation propagate along geodesics at infinite speed, since dispersive equations do not have the finite propagation property. Moreover, if each geodesic goes to "infinity", then the solution gains extra smoothness in comparison with the initial data and the forcing term. This phenomenon is called the (local) smoothing effect. In the general case, singularities of a solution of a dispersive equation propagate along the Hamilton flow generated by the principal symbol. For time local smoothing estimates, the main interest is a relationship between the global behavior of classical flows and high-frequency estimates of the solution. However if we consider time global smoothing estimates, careful arguments are required for low-frequency estimates as well as high-frequency estimates. On the Euclidean space it is known that a certain difference appears in time global low-frequency estimates for the solutions of the dispersive equations in connection with the space dimension. The difference is caused by a singularity of the resolvent near the bottom of the continuous spectrum. On symmetric spaces of noncompact type the pseudo-dimension (which will be defined by (1.9) or (2.6)) plays a similar role to the dimension for low-frequency estimates in the Euclidean case.

First let us recall some known results for dispersive equations on Euclidean spaces. For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$, set $x \cdot \xi = x_1\xi_1 + \cdots + x_n\xi_n$, $|x| = \sqrt{x \cdot x}$. We consider the Cauchy problem of the Schrödinger evolution equation:

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$$D_t u + \Delta u = f(t,x) \quad \text{in} \quad \mathbb{R}^{1+n}, \tag{1.1}$$

$$u(0,x) = \phi(x) \qquad \text{in} \quad \mathbb{R}^n, \tag{1.2}$$

where $i = \sqrt{-1}$, $\partial_t = \partial/\partial t$, $D_t = -i\partial_t$, $\partial_j = \partial/\partial x_j$, $\Delta = \sum_{j=1}^n \partial_j^2$. The above Cauchy problem is L^2-well-posed, that is, for any $\phi \in L^2(\mathbb{R}^n)$ and for any $f \in L^1_{\rm loc}(\mathbb{R}; L^2(\mathbb{R}^n))$, (1.1)–(1.2) possess a unique solution $u \in C(\mathbb{R}; L^2(\mathbb{R}^n))$. Moreover, the solution $u(t,x)$ is explicitly given by

$$u(t,x) = e^{-it\Delta}\phi(x) + i\int_0^t e^{-i(t-s)\Delta} f(s,x)ds,$$

$$e^{-it\Delta}\phi(x) = (2\pi)^{-n}\int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi} e^{it|\xi|^2}\phi(y)dyd\xi.$$

In addition, we introduce two operators $|D_x|^s := \mathcal{F}^{-1}|\xi|^s\mathcal{F}$ and $\langle D_x\rangle^s := \mathcal{F}^{-1}\langle\xi\rangle^s\mathcal{F}$ for $s \in \mathbb{R}$, where $\langle\xi\rangle := (1+|\xi|^2)^{1/2}$ and $\mathcal{F}$ denotes the Fourier transform defined by $\mathcal{F}v(\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}v(x)dx$. Then we have the following two types of smoothing estimates for the Schrödinger evolution equation (throughout this paper, different positive constants are denoted by the same letter C).

Type-I: Let $n \geq 2$ and $\delta > 1/2$. Then

$$\left\| \langle x\rangle^{-\delta}|D_x|^{1/2}e^{-it\Delta}\phi \right\|_{L^2(\mathbb{R}^{1+n})} \leq C\|\phi\|_{L^2(\mathbb{R}^n)}, \tag{1.3}$$

$$\left\| \langle x\rangle^{-\delta}|D_x|\int_0^t e^{-i(t-s)\Delta}f(s,\cdot)ds \right\|_{L^2(\mathbb{R}^{1+n})} \leq C\|\langle x\rangle^{\delta}f\|_{L^2(\mathbb{R}^{1+n})}. \tag{1.4}$$

Type-II: Let $n \geq 3$ and $\delta \geq 1$. Then

$$\left\| \langle x\rangle^{-\delta}\langle D_x\rangle^{1/2}e^{-it\Delta}\phi \right\|_{L^2(\mathbb{R}^{1+n})} \leq C\|\phi\|_{L^2(\mathbb{R}^n)}, \tag{1.5}$$

$$\left\| \langle x\rangle^{-\delta}\langle D_x\rangle\int_0^t e^{-i(t-s)\Delta}f(s,\cdot)ds \right\|_{L^2(\mathbb{R}^{1+n})} \leq C\|\langle x\rangle^{\delta}f\|_{L^2(\mathbb{R}^{1+n})}. \tag{1.6}$$

Note that for Type-I estimates the symbols $|\xi|^{1/2}$ and $|\xi|$ vanish at the origin, but for Type-II estimates the symbols $\langle\xi\rangle^{1/2}$ and $\langle\xi\rangle$ never vanish on $\mathbb{R}^n$. Hence the difference between Type-I and Type-II lies in the estimates for low-frequency part of the solutions. TYPE-I estimates (1.3) and (1.4) were studied by many authors (see e.g. [**2**]). TYPE-II estimates (1.5) and (1.6) were first studied by Kato and Yajima [**23**]. We obtain these estimates from uniform resolvent estimates for $-\Delta$ by using the Kato theory. In the lower dimensional case, it is known that careful treatments are needed for the low-frequency part of the solution: On $\mathbb{R}^1$, (1.3) is valid, but we have to replace $|D_x|$ by D_x in (1.4). In addition, TYPE-II estimates (1.5)–(1.6) do not hold for any $\delta > 0$. On $\mathbb{R}^2$, the inequality (1.5) holds if and only if $\delta > 1$ (cf. [**32**]), while (1.6) does not hold for any $\delta > 0$. The invalidity is caused by the singularity of the resolvent $(-\Delta - \zeta)^{-1}$ at $\zeta = 0$, namely, the zero-resonance of $-\Delta$.

The above estimates were generalized to Fourier multipliers of homogeneous real-principal type by Chihara in [**6**], [**7**]. In [**6**], Type-I estimates were generalized to Fourier multipliers defined by homogeneous symbols of real-principal type. Also, in [**7**] Chihara obtained Type-I and Type-II estimates for Fourier multipliers with the real-valued homogeneous elliptic symbol by using uniform resolvent estimates (see Theorem 2.1). For recent developments of smoothing estimates for Fourier multipliers, see e.g. M. Ruzhansky and M. Sugimoto [**26**], [**27**], [**28**]. In [**27**], they deal with the critical case for smoothing estimates and resolvent estimates by using specific smoothing symbols which vanish on the set generated by classical orbits. In [**26**] and [**28**], they established a new method for smoothing estimates in terms of canonical transformations. In those papers, Chihara's results in [**6**] and [**7**] for homogeneous equations were generalized to the much larger class of symbols, e.g. symbols with lower order terms, non-elliptic symbols. In addition, we note that detail results in smoothing estimates are given by [**28**].

Recently, uniform resolvent estimates for the Laplace-Beltrami operator and some kinds of time-global smoothing estimates for the Schrödinger evolution equation have been studied for noncompact complete Reimannian manifolds with several types of end structures, e.g. perturbation of the Euclidean spaces, asymptotically conic manifolds, and asymptotically hyperbolic spaces (cf. [**3**], [**4**], [**5**], [**25**], [**30**], [**31**]). In this paper, we study resolvent estimates and smoothing estimates for symmetric spaces of noncompact type. The class of symmetric spaces of noncompact type contains hyperbolic spaces, which are the most interesting examples of negatively curved Riemannian manifolds. We also deal with higher rank symmetric spaces which are not covered by the above papers. Symmetric spaces of noncompact type are regarded as a generalization of the Euclidean space in the sense that horocycles correspond to planes in the Euclidean space and the Fourier transform is defined through horocycle waves. On these spaces harmonic analysis have been developed and is a powerful tool to investigate invariant differential operators, or more general Fourier multipliers. Therefore it is natural to study resolvent estimates on symmetric spaces of noncompact type. In the estimates of low-frequency part on the Fourier space, the pseudo-dimension necessarily appears. The main results show that the class of symmetric spaces of noncompact type provides a rich amount of examples for the study of low-frequency resolvent estimates.

Let us state our main results. Let $X = G/K$ be a symmetric space of noncompact type. Let Δ_X be the Laplace-Beltrami operator on X with respect to the G-invariant metric. Then the operator $\Delta_X|_{C_0^\infty(X)}$ has an essential selfadjoint extension on $L^2(X)$ which is denoted by the same symbol. It is known that the spectrum of $-\Delta_X$, $\sigma(-\Delta_X)$ consists of absolutely continuous spectrum and that $\sigma(-\Delta_X) = [|\rho|^2, \infty)$ for some positive constant $|\rho|^2$ (ρ is explicitly given in Subsection 2.2). Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and $\mathfrak{a}^*$ its dual space. Let $a(\lambda) \in C(\mathfrak{a}^*) \cap C^\infty(\mathfrak{a}^* \setminus \{0\})$ be a positively homogeneous function of order one which is invariant under the Weyl group action. Suppose that $a(\lambda) > 0$ for $\lambda \neq 0$ and set $p(\lambda) = a(\lambda)^m$ for $m > 1$. In the real rank one case, $p(\lambda)$ is of the form $c|\lambda|^m$ for some positive constant c due to the assumption on $a(\lambda)$. Let $p(D) = \mathcal{F}^{-1}p\mathcal{F}$ be the Fourier multiplier with the symbol p defined by the Fourier transform $\mathcal{F}$ on X (for the details see Subsection 2.2). We consider the Cauchy problem of the form

$$D_t u - p(D)u = f(t,x) \quad \text{in} \quad \mathbb{R}\times X, \tag{1.7}$$
$$u(0,x) = \phi(x) \quad \text{in} \quad X. \tag{1.8}$$

In the case $p(\lambda) = |\lambda|^2$, then $p(D) = -\Delta_X - |\rho|^2$, so (1.7) becomes the (modified) Schrödinger evolution equation. Since $p(\lambda)$ is real-valued, the above Cauchy problem is L^2-well-posed, that is, for any $\phi \in L^2(X)$ and for any $f \in L^1_{\mathrm{loc}}(\mathbb{R}; L^2(X))$, the Cauchy problem (1.7)–(1.8) possess a unique solution $u \in C(\mathbb{R}; L^2(X))$. Now we put

$$|D| = (-\Delta_X - |\rho|^2)^{1/2}, \ \langle D\rangle = (1-\Delta_X - |\rho|^2)^{1/2},$$
$$|x| = d(x,o)^{1/2}, \ \langle x\rangle = (1+d(x,o)^2)^{1/2}, \quad x \in X,$$

where $d(x,o)$ is the Riemannian distance between $x \in X$ and the origin $o = eK$. Let $l \in \mathbb{Z}_{>0}$ be the rank of the symmetric space X, Σ_0^+ be the set of indivisible positive roots of the Lie algebra $\mathfrak{g}$ of G and $|\Sigma_0^+|$ be the its cardinality. Let us define the pseudo-dimension of X by

$$\nu = l + 2|\Sigma_0^+|. \tag{1.9}$$

The above positive integer ν is called the pseudo-dimension of X in [**10**] (ν appears as an upper bound of the exponent in L^p-L^q estimates of the complex power of the resolvent operator, cf. [**10**, Theorem 6.1]).

Then our main results are stated as follows.

Theorem 1.1. (i) *Suppose $m > 1$ and $\delta > 1/2$. Then*

$$\sup_{\zeta\in\mathbb{C}\setminus\mathbb{R}} \left|(|D|^{m-1}(p(D)-\zeta)^{-1}f, g)_{L^2(X)}\right| \le C\|\langle x\rangle^\delta f\|_{L^2(X)}\|\langle x\rangle^\delta g\|_{L^2(X)}. \tag{1.10}$$

(ii) *Suppose $1 < m < \nu$ and $\delta > m/2$. Then*

$$\sup_{\zeta\in\mathbb{C}\setminus\mathbb{R}} \left|(\langle D\rangle^{m-1}(p(D)-\zeta)^{-1}f, g)_{L^2(X)}\right| \le C\|\langle x\rangle^\delta f\|_{L^2(X)}\|\langle x\rangle^\delta g\|_{L^2(X)}. \tag{1.11}$$

Moreover, if $l \ge 2$ and $m < l$, we can take $\delta = m/2$ in (1.11). *Here ν is the pseudo-dimension of X defined by* (2.6).

Theorem 1.1 yields the following time global smoothing estimates.

Theorem 1.2. (i) *Suppose $m > 1$ and $\delta > 1/2$. Then*

$$\left\|\langle x\rangle^{-\delta}|D|^{(m-1)/2}e^{itp(D)}\phi\right\|_{L^2(\mathbb{R}\times X)} \le C\|\phi\|_{L^2(X)},$$
$$\left\|\langle x\rangle^{-\delta}|D|^{(m-1)}\int_0^t e^{i(t-s)p(D)}f(s,\cdot)ds\right\|_{L^2(\mathbb{R}\times X)} \le C\|\langle x\rangle^\delta f\|_{L^2(\mathbb{R}\times X)}.$$

(ii) *Suppose* $1 < m < \nu$ *and* $\delta > m/2$. *Then*

$$\left\| \langle x \rangle^{-\delta} \langle D \rangle^{(m-1)/2} e^{itp(D)} \phi \right\|_{L^2(\mathbb{R}\times X)} \le C \|\phi\|_{L^2(X)}, \tag{1.12}$$

$$\left\| \langle x \rangle^{-\delta} \langle D \rangle^{(m-1)} \int_0^t e^{i(t-s)p(D)} f(s,\cdot) ds \right\|_{L^2(\mathbb{R}\times X)} \le C \| \langle x \rangle^{\delta} f \|_{L^2(\mathbb{R}\times X)}. \tag{1.13}$$

Moreover, if $l \ge 2$ *and* $m < l$, *we can take* $\delta = m/2$ *in* (1.12)–(1.13).

In the estimates (1.11) and (1.13), the upper bound ν for the order m of the elliptic Fourier multiplier $p(D)$ is sharp in the sense that if $m \ge \nu$ then (1.11) and (1.13) do not hold for any $\delta > 0$. On the other hand, the estimate (1.12) still holds in the critical case $m = \nu$ for $\delta > m/2$ (Theorem 6.2). However in the case $m > \nu$, (1.12) does not hold for any $\delta > 0$. These properties immediately follow from the fact that the vanishing order of the Plancherel measure at the origin is precisely ν (see Section 6).

In the case of dispersive equations on $\mathbb{R}^n$, there are several results on resolvent estimates for low-frequency part. For example, see [**3**], [**4**], [**5**], [**7**]. In these results, the order m satisfies the condition $1 < m < n$. On the other hand, in the case of corresponding dispersive equations on symmetric spaces of noncompact type nothing is known about the resolvent estimates for low-frequency part. Therefore, we stress here that in Theorem 1.1 and 1.2 m has to satisfy the condition $1 < m < \nu$. Namely, the condition on the order of Fourier multiplier is given by the pseudo-dimension ν instead of the dimension of X. The dimension n and pseudo-dimension ν is given by the following (see Subsection 2.2):

$$n = l + \sum_{\alpha\in\Sigma^+} m_\alpha,$$

$$\nu = l + 2|\Sigma_0^+|.$$

In the general case, the pseudo-dimension does not coincide with the dimension (see Remark 1.1). In this sense, the above two theorems show that there is a big difference between the Euclidean case and the case of symmetric spaces of noncompact type.

Remark 1.1. We make some remarks on the properties of the pseudo-dimension ν of X.

(i) Since $\nu \ge 3$ for all symmetric spaces, the above estimates (1.10)–(1.11) are valid for $m = 2$, especially in the case $p(D) = -\Delta_X - |\rho|^2$. Note that the resolvent estimate (1.11) for the Laplace-Beltrami operator are valid on the real hyperbolic plane $H^2(\mathbb{R})$, but not on $\mathbb{R}^2$ for any $\delta > 0$. In fact, in the case $n = 1$ or 2, the kernel of $(-\Delta_{\mathbb{R}^n} - \zeta)^{-1}$ has singularities at $\zeta = 0$ on the other hand that of $(-\Delta_{H^2(\mathbb{R})} - 1/4 - \zeta)^{-1}$ has no singularity at $\zeta = 0$.

(ii) In some special cases, relations between n and ν are as follows:
 (a) If X is of rank one, then $\nu = 3$ irrespective of the dimension.
 (b) If $\mathfrak{g}$ is a normal real form of $\mathfrak{g}_{\mathbb{C}}$, i.e. $m_\alpha = 1$ for all $\alpha \in \Sigma$, then $\nu - n = n - l \ge 1$,

so $\nu > n$.

(c) If X satisfies the even multiplicity condition, then $\nu \leq n$, where the equality holds if and only if G is complex.

The following is the list of irreducible symmetric spaces of noncompact type which satisfy the above conditions respectively (cf. Helgason [**16**, Chapter X, Section 6]):

(a) $SO_0(p,1)/SO(p) \times SO(1)$ $(n = p)$, $SU(p,1)/S(U(p) \times U(1))$ $(n = 2p)$, $Sp(p,1)/Sp(p) \times Sp(1)$ $(n = 4p)$, and $F_{4(-20)}/Spin(9)$ $(n = 16)$. These are real, complex, quaternion hyperbolic space, and the Cayley hyperbolic plane.

(b) $SL(m,\mathbb{R})/SO(m)$, $Sp(m,\mathbb{R})/U(m)$, $SO_0(p,p-1)/SO(p) \times SO(p-1)$, $SO_0(p,p)/SO(p)\times SO(p)$ $(p \geq 2)$, and five exceptional spaces EI, EV, $EVIII$, FI, G. The first four examples have $(n,\nu) = ((m-1)(m+2)/2, m^2-1)$, $(m(m+1), m(2m+1))$, $(p(p-1),(2p-1)(p-1))$, and $(p^2,(2p-1)p)$, respectively.

(c) $SO_0(p,1)/SO(p)\times SO(1)$ (p is odd), $SU^*(2m)/Sp(m)$, $E_{6(-26)}/F_4$, and $K_{\mathbb{C}}/K$, where K is a simple compact Lie group. The first three examples have $(n,\nu) = (p,3)$, $((m-1)(2m+1),(m^2-1))$, and $(26,8)$, respectively.

In the Euclidean case, the Plancherel measure on the Fourier space is also the Euclidean measure. In the case of symmetric spaces, the Plancherel measure is given by $|\boldsymbol{c}(\lambda)|^{-2}d\lambda db$, where $\boldsymbol{c}(\lambda)$ is Harish-Chandra's $\boldsymbol{c}$-function, $d\lambda$ the Euclidean measure on $\mathfrak{a}^*$ and db the normalized invariant measure on a compact Riemannian manifold B. Similarly as in the case of $\mathbb{R}^n$, a sort of Fourier transformation is defined on symmetric spaces of noncompact type. The corresponding Fourier inversion formula is given as follows (we will give the detail in Section 2).

$$f(x) = |W|^{-1} \int_{\mathfrak{a}^*}\int_B e^{(i\lambda+\rho)(A(x,b))}\mathcal{F}f(\lambda,b)|\boldsymbol{c}(\lambda)|^{-2}d\lambda db.$$

The pseudo-dimension relates to the order of the zero of the spectral measure $|\boldsymbol{c}(\lambda)|^{-2}d\lambda$ at the origin. As in [**10**] we can write

$$|\boldsymbol{c}(\lambda)|^{-2}d\lambda = r^{\nu-1}c(r,\omega)drd\sigma(\omega),$$

where $\lambda = r\omega \in \mathfrak{a}^*_+$, $r = |\lambda| \in (0,\infty)$, $\omega = \lambda/|\lambda|$ and $c(r,\omega)$ satisfies $0 < c(r,\omega) \leq C\langle r\rangle^{n-\nu}$, $0 < c(0,\omega)$. Let $k_{\zeta,\sigma}(x)$ be the Schwartz kernel of the Bessel-Green-Riesz operator $(-\Delta_X - |\rho|^2 + \zeta^2)^{-\sigma/2}$ with $\zeta \in \mathbb{C}$, $\operatorname{Im}\zeta > 0$ and $\sigma \in \mathbb{R}$. The kernel $k_{\zeta,\sigma}(x)$ is expressed by the Plancherel measure and the spherical function $\varphi_\lambda(x)$ (for the definition see (2.9)) as follows.

$$k_{\zeta,\sigma}(x) = |W|^{-1}\int_{\mathfrak{a}^*}\varphi_\lambda(x)(|\lambda|^2+\zeta^2)^{-\sigma/2}|\boldsymbol{c}(\lambda)|^{-2}d\lambda.$$

If we take the limit $\zeta \to 0$, we see that the limit of the Riesz kernel $\lim_{\zeta\to 0} k_{\zeta,\sigma}(x)$ exists if and only if $\sigma < \nu$. In other words, let $V^*(r)$ be the volume of the ball with respect to the measure $|\boldsymbol{c}(\lambda)|^{-2}d\lambda$ with radius r centered at the origin. Then we have $V^*(r) \sim r^\nu$

as $r \downarrow 0$ and $V^*(r) \sim r^n$ as $r \to \infty$. Therefore we expect that the pseudo-dimension plays a role of dimension in low-frequency estimates on the Fourier space. We also note that the pseudo-dimension $\nu = l + 2|\Sigma_0^+|$ often appears in certain critical cases of estimates on symmetric spaces of noncompact type (see e.g. [**1**], [**10**], [**21**]).

We turn to some related results for dispersive equations with variable coefficients. There are a lot of results concerning smoothing effects and resolvent estimates for dispersive equations, we only refer a small part of related results. In [**11**], [**12**], [**13**], Doi gave a deep result on the relationship between microlocal smoothing effects and global behavior of geodesic flows (or Hamilton flows). Rodnianski and Tao [**25**] deals with the Schrödinger evolution equation on a three-dimensional Riemannian manifold which is a compact smooth perturbation of the Euclidean space and satisfies the non-trapping condition. They obtained a certain type of time-global smoothing estimates for homogeneous solutions. In [**31**], Vodev proved uniform high-frequency resolvent estimates for the Laplace-Beltrami operator on a large class of noncompact Riemannian manifolds with elliptic ends. The above class of Riemannian manifolds contains long-range perturbation of the Euclidean spaces, asymptotically conic manifolds, and asymptotically hyperbolic spaces. Their estimates with the non-trapping assumption yield high-frequency time global smoothing estimates for solutions of the Schrödinger evolution equation. On the other hand, for low-frequency estimates we refer to [**3**], [**4**], [**5**], [**30**]. In [**3**], [**4**], [**5**], low-frequency resolvent estimates for long range perturbations of the Euclidean Laplacian were obtained. In [**30**], Vasy and Wunsch generalized the results by Bony and Häfner [**3**] to the scattering manifolds.

Finally, this paper is organized as follows. In the former part of Section 2 we give a summary of the result of [**7**]. In the latter part, we review some facts related to harmonic analysis on semi-simple Lie groups and symmetric spaces. In Section 3 we show a weighted L^2-continuity of the modified Radon transform and fractional integral operators. A decomposition formula of Harish-Chandra's $\boldsymbol{c}$-function (Lemma 3.2) and pseudodifferential calculus on the Euclidean space play an essential role in the proof of continuity of the modified Radon transform (Proposition 3.1). Next, we use a weighted L^2-continuity for a convolution operator (Corollary 2.6) and integrability of the Riesz kernel (Corollary 2.8) in order to prove a continuity of fractional integral operators (Proposition 3.3). In Section 4 we show Fourier restriction estimates on symmetric spaces (Lemma 4.1). In the case of rank one we prove directly by using the L^2-continuity of the modified Radon transform. But in the higher rank case those estimates are derived from Fourier restriction estimates on the Euclidean space (Lemma 2 in [**7**] , see Lemma 2.3) by using the modified Radon transform. In Section 5 we prove the first main theorem (Theorem 1.1) according to [**7**] by making use of Lemma 5.1 and Lemma 5.2. In Section 6 we derive a Fourier restriction estimate for low-frequency part in a critical case from a weighted L^2-continuity of a convolution operator (Corollary 2.6). Then this estimate yields time global smoothing estimates (1.12) for homogeneous solutions in the critical case $m = \nu$.

2. Preliminaries.

In this section we give a brief summary of Chihara's result on the resolvent estimates in the Euclidean case. Next we introduce the basic material of harmonic analysis on semisimple Lie groups and symmetric spaces of noncompact type. Moreover, we mention some key estimates for the convolution operator and the Bessel-Green-Riesz kernel.

2.1. Resolvent estimates on the Euclidean space.

First we recall the resolvent estimates on the Euclidean spaces. Let $p(\xi) \in C^1(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \setminus \{0\})$ be a real-valued positively homogeneous symbol of degree $m > 1$ with $\nabla p(\xi) \neq 0$ for $\xi \neq 0$. We consider the Cauchy problem of the form

$$\begin{aligned} D_t u - p(D_x)u &= f(t,x) \quad \text{in} \quad \mathbb{R}^{1+n}, \\ u(0,x) &= \phi(x) \quad\ \ \text{in} \quad \mathbb{R}^n, \end{aligned} \tag{2.1}$$

where $p(D_x) = \mathcal{F}^{-1}p(\xi)\mathcal{F}$. In the case $p(\xi) = |\xi|^2$, i.e. $p(D_x) = -\Delta$, (2.1) is the Schrödinger evolution equation.

Since $p(\xi)$ is real-valued, the above Cauchy problem is L^2-well-posed. It is known that the singularities of the solution of a dispersive equation propagate along the Hamilton flow $\exp(tH_p)(x,\xi) = (x + t\nabla p(\xi), \xi)$ on $T^*\mathbb{R}^n = \mathbb{R}^{2n}$. In the case $p(\xi) = |\xi|^2$, i.e. $p(D_x) = -\Delta$, the Hamilton flow is a geodesic flow on $T^*\mathbb{R}^n$. For the symbol $p(\xi)$, the dispersive condition $\nabla p(\xi) \neq 0$ ($\xi \neq 0$) corresponds to the non-trapping condition for the Hamilton orbit, i.e.

$$\nabla p(\xi) \neq 0 \iff |x + t\nabla p(\xi)| \to \infty \quad \text{as} \quad |t| \to \infty$$

for any $(x,\xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$. Roughly speaking if the dispersive condition is satisfied, then the singularities of the solutions at any point disperse at infinite speed and the solutions gain extra smoothness in comparison with the initial value and the forcing term. In [**19**] Hoshiro proved that for any real-valued polynomial symbol $q(\xi)$ of order $m > 1$, time-local spatially-local smoothing estimates hold for the propagator $e^{itq(D_x)}$ if and only if its principal symbol $q_m(\xi)$ satisfies the dispersive condition.

In [**7**] Chihara proved the following estimates for homogeneous elliptic Fourier multipliers.

Theorem 2.1 ([**7**, Theorem 1]). *Suppose $n \geq 2$. Let $p(\xi) \in C^1(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \setminus \{0\})$ be a real-valued, elliptic, and positively homogeneous symbol of degree $m > 1$.*

Type-I: *Suppose $m > 1$ and $\delta > 1/2$. Then*

$$\big\| \langle x\rangle^{-\delta} |D_x|^{(m-1)/2} e^{itp(D_x)}\phi \big\|_{L^2(\mathbb{R}^{1+n})} \leq C\|\phi\|_{L^2(\mathbb{R}^n)},$$

$$\left\| \langle x\rangle^{-\delta} |D_x|^{(m-1)} \int_0^t e^{i(t-s)p(D_x)} f(s,\cdot)ds \right\|_{L^2(\mathbb{R}^{1+n})} \leq C\|\langle x\rangle^{\delta} f\|_{L^2(\mathbb{R}^{1+n})}.$$

Type-II: *Suppose $1 < m < n$. Then*

$$\big\| \langle x\rangle^{-m/2} \langle D_x\rangle^{(m-1)/2} e^{itp(D_x)}\phi \big\|_{L^2(\mathbb{R}^{1+n})} \leq C\|\phi\|_{L^2(\mathbb{R}^n)},$$

$$\left\| \langle x\rangle^{-m/2}\langle D_x\rangle^{(m-1)}\int_0^t e^{i(t-s)p(D_x)}f(s,\cdot)ds\right\|_{L^2(\mathbb{R}^{1+n})}\le C\|\langle x\rangle^{m/2}f\|_{L^2(\mathbb{R}^{1+n})}.$$

THEOREM 2.2 ([**7**, Theorem 2]). *Suppose $n\ge 2$. Let $p(\xi)\in C^1(\mathbb{R}^n)\cap C^\infty(\mathbb{R}^n\setminus\{0\})$ be a real-valued, elliptic, and positively homogeneous symbol of degree $m>1$.*

Type-I: *Suppose $m>1$ and $\delta>1/2$. Then*

$$\sup_{\zeta\in\mathbb{C}\setminus\mathbb{R}}\left|(|D_x|^{m-1}(p(D_x)-\zeta)^{-1}f,g)_{L^2(\mathbb{R}^n)}\right|\le C\|\langle x\rangle^\delta f\|_{L^2(\mathbb{R}^n)}\|\langle x\rangle^\delta g\|_{L^2(\mathbb{R}^n)}.$$

Type-II: *Suppose $1<m<n$. Then*

$$\sup_{\zeta\in\mathbb{C}\setminus\mathbb{R}}\left|(\langle D_x\rangle^{m-1}(p(D_x)-\zeta)^{-1}f,g)_{L^2(\mathbb{R}^n)}\right|\le C\|\langle x\rangle^{m/2}f\|_{L^2(\mathbb{R}^n)}\|\langle x\rangle^{m/2}g\|_{L^2(\mathbb{R}^n)}. \tag{2.2}$$

The key of the proof of Theorem 2.2 is the following Fourier restriction estimates and L^2-continuity of some singular integrals. We use the following two lemmas in the proof of Lemma 4.1 and Lemma 5.1.

LEMMA 2.3 ([**7**, Lemma 1]). *Let $n\ge 2$. Let $a(\xi)\in C(\mathbb{R}^n)\cap C^\infty(\mathbb{R}^n\setminus\{0\})$ be an elliptic positively homogeneous symbol of degree 1 with $a(\xi)>0$ for $\xi\ne 0$. Put*

$$\Sigma(\tau)=\{\xi\in\mathbb{R}^n;a(\xi)=\tau\},\quad \tau\ge 0.$$

Then we have the following three estimates:

(i) (*Uniform trace estimates*) *Suppose $\theta>0$. Then*

$$\|\hat f\|_{L^2(\Sigma(\tau))}\le C\|\langle x\rangle^{1/2+\theta}f\|_{L^2(\mathbb{R}^n)}. \tag{2.3}$$

(ii) (*Hölder continuity*) *Suppose $0<\theta\le 1/2$ for $n=2$, and $0<\theta<1$ for $n\ge 3$. Then*

$$\left\|\tau^{(n-1)/2}\hat f(\tau\cdot)-\sigma^{(n-1)/2}\hat f(\sigma\cdot)\right\|_{L^2(\Sigma(1))}\le C|\tau-\sigma|^\theta\|\langle x\rangle^{1/2+\theta}f\|_{L^2(\mathbb{R}^n)}. \tag{2.4}$$

(iii) (*Low frequency trace estimates*) *Suppose $0<\theta<(n-1)/2$. Then*

$$\|\hat f\|_{L^2(\Sigma(\tau))}\le C\tau^\theta\|\langle x\rangle^{1/2+\theta}f\|_{L^2(\mathbb{R}^n)}. \tag{2.5}$$

LEMMA 2.4 ([**7**, Lemma 2]). *Let δ satisfy $0<\delta\le 1$ for $n\ge 3$ and $0<\delta<1$ for $n=2$. Suppose that $q(\xi)\in C^\infty(\mathbb{R}^n\setminus\{0\})$ is homogeneous of degree zero. Then*

$$\|(|x|^\delta q(D_x)-q(D_x)|x|^\delta)f\|_{L^2(\mathbb{R}^n)}\le C\||x|^\delta f\|_{L^2(\mathbb{R}^n)},$$
$$\|\langle x\rangle^\delta q(D_x)f\|_{L^2(\mathbb{R}^n)}\le C\|\langle x\rangle^\delta f\|_{L^2(\mathbb{R}^n)}.$$

2.2. Harmonic analysis on symmetric spaces.

Notation. Let G be a noncompact, connected, semisimple Lie group with finite center, $K \subset G$ a maximal compact subgroup, and $X = G/K$ the associated symmetric space. Let θ be the Cartan involution associated with the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ at the Lie algebra level. Then the Killing form induces the G-invariant metric on $X = G/K$. Let Δ_X be the Laplace-Beltrami operator with respect to the G-invariant metric on X and dx be the corresponding measure. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and $\mathfrak{a}^*$ its dual space. For $\alpha \in \mathfrak{a}^*$ we put $\mathfrak{g}_\alpha = \{Y \in \mathfrak{g}; [H, Y] = \alpha(H)Y \text{ for all } H \in \mathfrak{a}\}$. We set $\Sigma = \{\alpha \in \mathfrak{a}^* \setminus \{0\}; \mathfrak{g}_\alpha \neq \{0\}\}$, the set of restricted roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$. Let $\mathfrak{m}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Then we have the following root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{a}$.

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus \{\oplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha\}.$$

We select a connected component in $\mathfrak{a}$ such that $\alpha \neq 0$ for all $\alpha \in \Sigma$. Let $\mathfrak{a}^+$ denote the connected component, called a positive Weyl chamber. Also, define positive roots and positive indivisible roots by $\Sigma^+ = \{\alpha \in \Sigma; \alpha > 0 \text{ on } \mathfrak{a}^+\}$ and $\Sigma_0^+ = \{\alpha \in \Sigma^+; \alpha/2 \notin \Sigma^+\}$ respectively. Put the nilpotent subalgebra $\mathfrak{n} = \oplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$. Let $\rho = 1/2 \sum_{\alpha \in \Sigma^+} m_\alpha \alpha$ be the half sum of positive roots, counted with multiplicities $m_\alpha = \dim \mathfrak{g}_\alpha$. Let $l = \dim \mathfrak{a}$ be the rank of X. Let n be the dimension of X and ν the pseudo-dimension of X:

$$\begin{aligned} n &= l + \sum_{\alpha \in \Sigma^+} m_\alpha, \\ \nu &= l + 2|\Sigma_0^+|. \end{aligned} \tag{2.6}$$

Set $A = \exp \mathfrak{a}$, $N = \exp \mathfrak{n}$ and M denotes the centralizer of A in K. Let M' be the normalizer of A in K, and W the factor group M'/M, called the Weyl group of X. The group W acts as a group of linear transformations on $\mathfrak{a}^*$ by $(s\lambda)(H) = \lambda(s^{-1} \cdot H)$ for $H \in \mathfrak{a}$, $\lambda \in \mathfrak{a}^*$ and $s \in W$, where $g \cdot X = \mathrm{Ad}(g)X$ for $g \in G$, $X \in \mathfrak{g}$. Let $|W|$ denote the order of W.

Then we have the Iwasawa decomposition of G as follows;

$$G = KAN.$$

Each $g \in G$ is uniquely written as $g = k(g) \exp(H(g)) n(g)$, where $k(g) \in K$, $H(g) \in \mathfrak{a}$ and $n(g) \in N$. Those mappings $g \to k(g)$, $g \to H(g)$, and $g \to n(g)$ are smooth. We have the following integral formula on X.

$$\int_X f(x)dx = \iint_{A \times N} f(an \cdot o)da dn, \tag{2.7}$$

where da, dn are the Haar measure on A, N, respectively. Put $B = K/M = G/MAN$, B is the "boundary" of X. Let db be the normalized left K-invariant measure on B. Set $A(x, b) = -H(g^{-1}k)$ for $(x, b) = (gK, kM) \in X \times B$.

We define the modified Radon transform $\mathcal{R}$ and the Helgason Fourier transform $\mathcal{F}$ for $f \in C_0^\infty(X)$ respectively.

$$\mathcal{R}f(H, kM) = e^{\rho(H)} \int_N f(ke^H n \cdot o) dn, \qquad (H, kM) \in \mathfrak{a} \times B,$$

$$\mathcal{F}f(\lambda, b) = \int_X e^{(-i\lambda+\rho)(A(x,b))} f(x) dx, \qquad (\lambda, b) \in \mathfrak{a}^* \times B.$$

On the other hand, let dH, $d\lambda$ be normalized Euclidean measure with the constant factor $(2\pi)^{-l/2}$ on $\mathfrak{a}$, $\mathfrak{a}^*$, respectively. Then the Fourier transform on $\mathfrak{a}$ and its inverse are given respectively by

$$\mathcal{F}_{\mathfrak{a}}\phi(\lambda) = \int_{\mathfrak{a}} e^{-i\lambda(H)} \phi(H) dH, \quad \mathcal{F}_{\mathfrak{a}}^{-1}\psi(H) = \int_{\mathfrak{a}^*} e^{i\lambda(H)} \psi(\lambda) d\lambda.$$

Then by using (2.7) we have

$$\mathcal{F}f(\lambda, b) = \mathcal{F}_{\mathfrak{a}}[\mathcal{R}f(\cdot, b)](\lambda).$$

Namely, the Helgason Fourier transform is expressed as the composition of the modified Radon transform and the Euclidean Fourier transform on $\mathfrak{a}$.

Let $\boldsymbol{c}(\lambda)$ denote Harish-Chandra's $\boldsymbol{c}$-function defined by the integral

$$\boldsymbol{c}(\lambda) = \int_{\bar{N}} e^{-(i\lambda+\rho)(H(\bar{n}))} d\bar{n},$$

where $\bar{N} = \theta N$, and $d\bar{n}$ is the Haar measure on $\bar{N}$ normalized by $\int_{\bar{N}} e^{-2\rho(H(\bar{n}))} d\bar{n} = 1$. An explicit expression for Harish-Chandra's $\boldsymbol{c}$-function was obtained by Gindikin and Karpelevič as follows (cf. Helgason [**14**, Chapter IV, Section 6, Theorem 6.14]).

$$\boldsymbol{c}(\lambda) = c_0 \prod_{\alpha \in \Sigma_o^+} \frac{2^{-\langle i\lambda, \alpha_0 \rangle} \Gamma(\langle i\lambda, \alpha_0 \rangle)}{\Gamma((1/2)((1/2)m_\alpha + 1 + \langle i\lambda, \alpha_0 \rangle))\Gamma((1/2)((1/2)m_\alpha + m_{2\alpha} + \langle i\lambda, \alpha_0 \rangle))},$$

where $\alpha_0 = \alpha / \langle \alpha, \alpha \rangle$, the constant c_0 is given so that $\boldsymbol{c}(-i\rho) = 1$, and $\Gamma(z)$ is the gamma function. Also, let $L^2_W(\mathfrak{a}^* \times B, |W|^{-1}|\boldsymbol{c}(\lambda)|^{-2} d\lambda db)$ denote the closed subspace of $L^2(\mathfrak{a}^* \times B, |W|^{-1}|\boldsymbol{c}(\lambda)|^{-2} d\lambda db)$ whose elements satisfy the following condition for the Weyl group:

$$\int_B e^{(is\lambda+\rho)(A(x,b))} \psi(s\lambda, b) db = \int_B e^{(i\lambda+\rho)(A(x,b))} \psi(\lambda, b) db$$

for all $s \in W$, and a.e. $x \in X$, $b \in B$. Let $C_W(\mathfrak{a}^*)$ denote the set of all continuous functions on $\mathfrak{a}^*$ which are invariant for the Weyl group action. We define function spaces $C_W^\infty(\mathfrak{a}^*)$ and $C_W^\infty(\mathfrak{a}^* \setminus \{0\})$ in the same manner. For the Helgason Fourier transform the following are well-known (see for example Helgason [**15**, Chapter 3, Section 1]):

(i) (The inversion formula for $\mathcal{F}$)

$$f(x) = |W|^{-1} \int_{\mathfrak{a}^* \times B} e^{(i\lambda+\rho)(A(x,b))} \mathcal{F}f(\lambda, b) |\boldsymbol{c}(\lambda)|^{-2} d\lambda db, \quad f \in C_0^\infty(X).$$

(ii) (The Plancherel theorem) We have the following unitary isomorphism

$$\mathcal{F} : L^2(X) \longrightarrow L^2_W(\mathfrak{a}^* \times B, |W|^{-1} |\boldsymbol{c}(\lambda)|^{-2} d\lambda db).$$

We remark that zeros of $\boldsymbol{c}(\lambda)^{-1}$ on $\mathfrak{a}^*$ is precisely the Weyl walls:

$$\bigcup_{\alpha \in \Sigma_0^+} \{\lambda \in \mathfrak{a}^*; \langle \alpha, \lambda \rangle = 0\}.$$

Here we define three kinds of Fourier multipliers on $\mathfrak{a}$, $\mathfrak{a}^*$, and X. The Fourier multiplier $p(D_H)$ on $\mathfrak{a}$ (resp. $q(D_\lambda)$ on $\mathfrak{a}^*$) with the symbol $p(\lambda)$ (resp. $q(H)$) is defined by

$$p(D_H) = \mathcal{F}_{\mathfrak{a}}^{-1} p(\lambda) \mathcal{F}_{\mathfrak{a}} \quad (\text{resp. } q(D_\lambda) = \mathcal{F}_{\mathfrak{a}} q(H) \mathcal{F}_{\mathfrak{a}}^{-1}).$$

In a similar manner, for a W-invariant function $p(\lambda)$ the Fourier multiplier $p(D)$ with the symbol $p(\lambda)$ is defined by

$$\begin{aligned} p(D)f(x) &= \mathcal{F}^{-1}[p(\lambda)\mathcal{F}f](x) \\ &= |W|^{-1} \int_{\mathfrak{a}^* \times B} e^{(i\lambda+\rho)(A(x,b))} p(\lambda) \mathcal{F}f(\lambda, b) |\boldsymbol{c}(\lambda)|^{-2} d\lambda db. \end{aligned}$$

Then for any Fourier multiplier $p(D)$ with the symbol $p(\lambda)$ we have

$$\mathcal{F}[p(D)f](\lambda, b) = \mathcal{F}_{\mathfrak{a}}[p(D_H)\mathcal{R}f(\cdot, b)](\lambda).$$

For the modified Radon transform we have the following:

(i) (The inversion formula for $\mathcal{R}$) Put $J = \boldsymbol{c}^{-1}(D_H)$. Then we have

$$f = |W|^{-1} \mathcal{R}^* \bar{J} J \mathcal{R} f, \quad f \in C_0^\infty(X). \tag{2.8}$$

(ii) We have an isometric operator

$$J\mathcal{R} : L^2(X) \longrightarrow L^2(\mathfrak{a} \times B, |W|^{-1} dH db).$$

2.3. Continuity of convolutions.

In this subsection, we state some results on a convolution operator, in order to prove a weighted L^2-continuity of fractional integral operators.

For two functions f_1 and f_2 the convolution $f_1 * f_2$ is defied by

$$f_1 * f_2(g) = \int_G f_1(h) f_2(h^{-1}g) dh,$$

if the above integral converges. In [**17**] Herz proved the following weighted L^2-estimate for the convolution.

THEOREM 2.5 ([**17**, Theorem 1]). *Let $\varphi_\lambda(g)$ denote the elementary spherical function defined by*

$$\varphi_\lambda(g) = \int_K e^{(i\lambda - \rho)(H(gk))} dk, \quad \lambda \in \mathfrak{a}^*_{\mathbb{C}}. \tag{2.9}$$

Let w be a positive, continuous, and bi-K-invariant function on G satisfying

$$\left(\int_G \varphi_0^2(g) w^{-1}(g) dg \right)^{1/2} =: N_1 < \infty.$$

Then we have

$$\|f_1 * f_2\|_{L^2(G,dg)} \le N_1 \|f_1\|_{L^2(G,wdg)} \|f_2\|_{L^2(G,dg)}.$$

In particular, for some $N_2 > 0$ we can take w such that

$$w(g) \le C\{\log(e + V_r)\}^{N_2}, \quad d(g \cdot o, o) \le r$$

for some $N_2 > 0$. Here V_r denotes the volume of balls with radius r (with respect to the pseudo-distance on G) with the Haar measure dg.

REMARK 2.1. Theorem 2.5 is related to the Kunze-Stein phenomenon for the convolution operator. For the Kunze-Stein phenomenon and Herz majorizing principle refer [**8**], [**9**], [**17**], [**24**]. After his result more general and sharp estimates for the convolution operator have been studied, see [**20**], [**21**].

For a function $f(x)$ on X, we define its lift function on G by $\tilde{f}(g) = f(gK)$. For two functions f_1, f_2 on X set the bilinear operator $\times$ as follows (cf. Helgason [**15**, Chapter II, Section 3]).

$$(f_1 \times f_2)(x) = \tilde{f}_1 * \tilde{f}_2(g), \quad x = gK.$$

We put $L^{2,\delta}(X) = \langle x \rangle^{-\delta} L^2(X)$ for $\delta \in \mathbb{R}$.

For the elementary spherical function we have

$$\begin{aligned} \varphi_0(\exp H) &\asymp \Big\{ \prod_{\alpha \in \Sigma_0^+} (1 + \langle \alpha, H \rangle) \Big\} e^{-\rho(H)} \\ &\le C \langle H \rangle^{|\Sigma_0^+|} e^{-\rho(H)} \end{aligned}$$

for any $H \in \overline{\mathfrak{a}^+}$ (cf. [**1**, Proposition 2.2.12]). Here for positive functions f, g, $f \asymp g$ means that $C^{-1}g \leq f \leq Cg$ uniformly for some positive constant C. Then we can take $w(g) = \langle g \cdot o \rangle^{2\delta}$ for any $\delta > \nu/2$ in Theorem 2.5 and obtain the following corollary.

Corollary 2.6. *For any $\delta > \nu/2$ we have*

$$\|f_1 \times f_2\|_{L^2(X)} \leq C\|f_1\|_{L^{2,\delta}(X)}\|f_2\|_{L^2(X)}.$$

2.4. The Bessel-Green-Riesz kernel and the fractional integrals.

For $\zeta \in [0, \infty)$, $\sigma \in \mathbb{R}$ let $k_{\zeta,\sigma}(x)$ be the Schwartz kernel of the operator $(-\Delta_X - |\rho|^2 + \zeta^2)^{-\sigma/2}$ if it exists. We call $k_{\zeta,\sigma}(x)$ the Bessel-Green-Riesz kernel. In the limiting case $\zeta = 0$, we call $|D|^{-\sigma} = (-\Delta_X - |\rho|^2)^{-\sigma/2}$ and $k_{0,\sigma}(x)$ the fractional integral operators and the Riesz kernel respectively. In [**1**] J. -Ph. Anker and Li. Ji obtained the optimal upper and lower bounds for the heat kernel, Poisson kernel, and Bessel-Green-Riesz kernels. We use the estimate for the Riesz kernel to prove a certain weighted L^2-continuity of the fractional integral operator.

The following is due to Anker and Ji [**1**].

Theorem 2.7 ([**1**, Theorem 4.2.2]). *For the Bessel-Green-Riesz kernel $k_{\zeta,\sigma}$ we have the following.*

(i) *The behavior of $k_{\zeta,\sigma}$ away from the origin*:

$$k_{\zeta,\sigma}(x) \asymp \begin{cases} |x|^{(\sigma-l-1)/2-|\Sigma_0^+|}\varphi_0(x)e^{-\zeta|x|} & \text{if} \quad \zeta > 0,\ \sigma > 0, \\ |x|^{\sigma-l-2|\Sigma_0^+|}\varphi_0(x) & \text{if} \quad \zeta = 0,\ 0 < \sigma < \nu. \end{cases}$$

(ii) *The behavior of $k_{\zeta,\sigma}$ near the origin*:

$$k_{\zeta,\sigma}(x) \asymp \begin{cases} |x|^{\sigma-n} & \text{if} \quad 0 < \sigma < n \\ \log(1/|x|) & \text{if} \quad \sigma = n \\ 1 & \text{if} \quad \sigma > n \end{cases}$$

with the restriction $0 < \sigma < \nu$ if $\zeta = 0$.

As a direct consequence of Theorem 2.7, we have the following corollary for the Riesz kernel.

Corollary 2.8. (i) *Let χ_0 be the characteristic function of the set $\{x \in X; |x| \leq 1\}$. Then for any $0 < \sigma < \nu$ we have*

$$\chi_0 k_{0,\sigma} \in L^1(X).$$

(ii) *Put $\chi_1 = 1 - \chi_0$. Then for any $0 < \sigma < \nu$, $\delta < \nu/2 - \sigma$ we have*

$$\langle x \rangle^{\delta} \chi_1 k_{0,\sigma} \in L^2(X).$$

3. Key tools.

In this section we prove a global continuity of the modified Radon transform and the fractional integral operator on symmetric spaces of noncompact type.

Put $L^{2,\delta}(\mathfrak{a}\times B)=\langle H\rangle^{-\delta}L^2(\mathfrak{a}\times B,|W|^{-1}dHdb)$ for $\delta\in\mathbb{R}$. Then we have a global continuity of the modified Radon transform as follows.

Proposition 3.1. *For any $\delta>0$, we have the following continuous map.*

$$J\mathcal{R}:L^{2,\delta}(X)\to L^{2,\delta}(\mathfrak{a}\times B).$$

Here and in the proof of Proposition 3.1, we use the pseudodifferential calculus. For the details see [**18**, Chapter XVIII, Section 18.4–18.6].

Generally, the inverse of Harish-Chandra's $\boldsymbol{c}$-function $\boldsymbol{c}^{-1}(\lambda)$ vanishes on each Weyl wall. This fact causes many difficulties in the global analysis on $\mathfrak{a}^*$ and $\mathfrak{a}$. To avoid these difficulties and to prove Proposition 3.1 we need the following factorization lemma.

Lemma 3.2. *Put*

$$\begin{aligned}\boldsymbol{\pi}(\lambda)&=\Pi_{\alpha\in\Sigma_0^+}\langle\lambda,\alpha\rangle,\\ \boldsymbol{b}(\lambda)&=\boldsymbol{\pi}(i\lambda)\boldsymbol{c}(\lambda),\\ \boldsymbol{b}_0(\lambda)&=\Pi_{\alpha\in\Sigma_0^+}(1+\langle\lambda,\alpha_0\rangle^2)^{-(m_\alpha+m_{2\alpha}-2)/4},\end{aligned}$$

where $\alpha_0=\alpha/\langle\alpha,\alpha\rangle$. Then we have $\boldsymbol{c}^{-1}(\lambda)=\boldsymbol{\pi}(i\lambda)\boldsymbol{b}^{-1}(\lambda)$ and $\boldsymbol{b}^{\pm1}(\lambda)\in S(\boldsymbol{b}_0^{\pm1}(\lambda),\langle H\rangle^{-2}|dH|^2+|d\lambda|^2)$ (here $S(m,g)$ denotes a symbol class for a slowly varying metric g and a g-continuous positive function m, see [**18**, Chapter XVIII, Definitions 18.4.1 and 18.4.2]). *In particular, $\boldsymbol{b}(\lambda)$ has no singularity on each Weyl wall.*

Proof. It is sufficient to prove the following properties:

$$\begin{gathered}|\boldsymbol{b}(\lambda)|\asymp\boldsymbol{b}_0(\lambda),\\ |p_0(D_\lambda)\boldsymbol{b}(\lambda)|\le C|\boldsymbol{b}_0(\lambda)|\end{gathered}$$

for any polynomial function $p_0(H)$ on $\mathfrak{a}$. These estimates are essentially obtained in the proof of Lemma 3.5 and Lemma 3.6 of [**15**, Chapter II, Section 3]. □

We prove Proposition 3.1 by using Lemma 3.2 and pseudodifferential calculus on $\mathfrak{a}$.

Proof of Proposition 3.1. First we take a dyadic decomposition $\{\phi_{0,j}\}_{j=0}^\infty\subset C_0^\infty(\mathbb{R})$ such that $0\le\phi_{0,j}\le1$, $\phi_{0,0}=1$ if $|t|\le1/2$, $=0$ if $|t|\ge1$, $\phi_{0,1}=1$ if $1\le|t|\le2$, $=0$ if $|t|\le1/2$ or $4\le|t|$, and $\phi_{0,j}(t)=\phi_{0,1}(t/2^{j-1})$ $(j\ge1)$, $\sum_{j=0}^\infty\phi_{0,j}=1$. Also put $\phi_{1,j}(x)=\phi_{0,j}(|x|)$, $\phi_{2,j}(H)=\phi_{0,j}(|H|)$ for any $j\ge1$. Then we have the following equivalent norms on $L^{2,\delta}(X)$ and $L^{2,\delta}(\mathfrak{a}\times B)$.

$$\|\langle x\rangle^\delta f\|_{L^2(X)} \asymp \left\|\{2^{j\delta}\|\phi_{1,j}f\|_{L^2(X)}\}_{j=0}^\infty\right\|_{l^2},$$
$$\|\langle H\rangle^\delta \varphi\|_{L^2(\mathfrak{a}\times B)} \asymp \left\|\{2^{j\delta}\|\phi_{2,j}\varphi\|_{L^2(\mathfrak{a}\times B)}\}_{j=0}^\infty\right\|_{l^2}.$$

Here we have put $l^2 = \{\{x_j\}_{j=0}^\infty \subset \mathbb{C}; \|\{x_j\}_{j=0}^\infty\|_{l^2} < \infty\}$ and $\|\{x_j\}_{j=0}^\infty\|_{l^2} = (\sum_{j=0}^\infty |x_j|^2)^{1/2}$. In this proof we use the latter norm on each space. Take cut-off functions $\{\chi_{0,j}\}_{j=0}^\infty \subset C_0^\infty(\mathfrak{a})$ such that $\chi_{0,0} = 1$ if $|H| \leq 1$, $= 0$ if $|H| \geq 2$, and $\chi_{0,j}(H) = \chi_{0,0}(H/2^j)$ for $j \geq 1$. For any $f \in C_0^\infty(X)$ we have $f = \sum_{k=0}^\infty \phi_{1,k} f$ and

$$\|\phi_{2,j} J\mathcal{R} f\|_{L^2(\mathfrak{a}\times B)} \leq \sum_{k=0}^\infty \|\phi_{2,j} J\mathcal{R}(\phi_{1,k} f)\|_{L^2(\mathfrak{a}\times B)}. \tag{3.1}$$

Since $|H| \leq |k\exp(H)n\cdot o|$, we see that

$$\operatorname{supp}\mathcal{R}(\phi_{1,k}f)(\cdot, b) \subset \{H \in \mathfrak{a}; |H| \leq 2^{k+1}\}, \quad b \in B.$$

By using the local property of the partial differential operator $\boldsymbol{\pi}(iD_H)$, $\operatorname{supp}\boldsymbol{\pi}(iD_H)\varphi \subset \operatorname{supp}\varphi$ for any $\varphi \in C_0^\infty(\mathfrak{a})$, we have

$$\begin{aligned} J\mathcal{R}(\phi_{1,k}f) &= \boldsymbol{b}^{-1}(D_H)\boldsymbol{\pi}(iD_H)\mathcal{R}(\phi_{1,k}f) \\ &= \boldsymbol{b}^{-1}(D_H)\chi_{0,k}\boldsymbol{\pi}(iD_H)\mathcal{R}(\phi_{1,k}f) \\ &= \boldsymbol{b}^{-1}(D_H)\chi_{0,k}\boldsymbol{b}(D_H)J\mathcal{R}(\phi_{1,k}f). \end{aligned} \tag{3.2}$$

Combining (3.1), (3.2) and the equality

$$\|J\mathcal{R}(\phi_{1,k}f)\|_{L^2(\mathfrak{a}\times B, |W|^{-1}dHdb)} = \|\phi_{1,k}f\|_{L^2(X)},$$

we obtain

$$2^{\delta j}\|\phi_{2,j}J\mathcal{R}f\|_{L^2(\mathfrak{a}\times B,|W|^{-1}dHdb)} \leq \sum_{k=0}^\infty a_{jk}(2^{\delta k}\|\phi_{1,k}f\|_{L^2(X)}),$$

where we put

$$a_{jk} = 2^{\delta(j-k)}\|\phi_{2,j}\boldsymbol{b}^{-1}(D_H)\chi_{0,k}\boldsymbol{b}(D_H)\|_{\mathcal{L}(L^2(\mathfrak{a}))}.$$

Here $\mathcal{L}(L^2(\mathfrak{a}))$ denotes the Banach space of all bounded linear operators on $L^2(\mathfrak{a})$ and $\|\cdot\|_{\mathcal{L}(L^2(\mathfrak{a}))}$ its operator norm. From the Schur lemma for sequences, it suffices to show that $\sum_{j=0}^\infty a_{jk} \leq C$ and $\sum_{k=0}^\infty a_{jk} \leq C$ uniformly in j, k for some positive constant C. Since the symbols $\{\phi_{2,j}\}$, $\{\chi_{0,k}\}$ are uniformly bounded in $S(1, \langle H\rangle^{-2}|dH|^2 + |d\lambda|^2)$, the family of pseudodifferential operators

$$\{\phi_{2,j}\boldsymbol{b}^{-1}(D_H)\chi_{0,k}\boldsymbol{b}(D_H)\}_{j,k}$$

is uniformly bounded in $\mathcal{L}(L^2(\mathfrak{a}))$ by the L^2-boundedness theorem for pseudodifferential operators (cf. [**18**, Theorem 18.6.3]). Hence we have

$$\sum_{j=0}^{k} a_{jk} \le C \sum_{j=0}^{k} 2^{\delta(j-k)} \le \frac{2^\delta C}{2^\delta - 1}.$$

On the other hand, we have the uniformly bounded symbols

$$\begin{aligned} 2^{(\delta+1)j}\phi_{2,j} &\in S(\langle H\rangle^{\delta+1}, \langle H\rangle^{-2}|dH|^2 + |d\lambda|^2), \\ 2^{-(\delta+1)k}\chi_{0,k} &\in S(\langle H\rangle^{-(\delta+1)}, \langle H\rangle^{-2}|dH|^2 + |d\lambda|^2). \end{aligned}$$

So the pseudodifferential operators $2^{(\delta+1)(j-k)}\phi_{2,j}\boldsymbol{b}^{-1}(D_H)\chi_{0,k}\boldsymbol{b}(D_H)$ are uniformly bounded in $\mathcal{L}(L^2(\mathfrak{a}))$. Therefore

$$\begin{aligned} \sum_{j=k+1}^{\infty} a_{jk} &= \sum_{j=k+1}^{\infty} 2^{(k-j)} 2^{(\delta+1)(j-k)} \|\phi_{2,j}\boldsymbol{b}^{-1}(D_H)\chi_{0,k}\boldsymbol{b}(D_H)\|_{\mathcal{L}(L^2(\mathfrak{a}))} \\ &\le C \sum_{j=k+1}^{\infty} 2^{(k-j)} \\ &= C. \end{aligned}$$

The uniform boundedness for the sum in j is proved in the same way. □

Next, we prove a weighted L^2-continuity of the fractional integral operators on X defined in Subsection 2.4. Here the pseudo-dimension ν introduced in Section 2 (2.6) appears in upper bounds of the exponent of the fractional integral operator. The following proposition plays an important role in the estimates for low-frequency part.

Proposition 3.3. *For any $(\sigma, \delta, \delta')$ with $0 < \sigma < \nu/2 + \min\{0, \delta, \delta'\}$, $\delta + \delta' > \sigma$, we have*

$$\big\|\langle x\rangle^{-\delta}|D|^{-\sigma}\langle x\rangle^{-\delta'} f\big\|_{L^2(X)} \le C\|f\|_{L^2(X)}.$$

Proof. First, we have

$$\langle x\rangle^{-\delta}|D|^{-\sigma}\langle x\rangle^{-\delta'} f(x) = \int_X \langle x\rangle^{-\delta} k_{0,\sigma}(x,y)\langle y\rangle^{-\delta'} f(y)dy,$$

where $k_{0,\sigma}(g\cdot o, h\cdot o) = k_{0,\sigma}(h^{-1}g\cdot o) = k_{0,\sigma}(g^{-1}h\cdot o)$. For a positive constant C_0 put

$$\begin{aligned} X_0 &= \{y \in X; d(x,y) \le 1\}, \\ X_1 &= \{y \in X; \langle d(x,y)\rangle^{\delta} \le C_0\langle x\rangle^{\delta}, \langle d(x,y)\rangle^{\delta'} \le C_0\langle y\rangle^{\delta'}\} \setminus X_0, \end{aligned}$$

$$X_2 = \{y \in X; \langle d(x,y)\rangle^{\delta} > C_0\langle x\rangle^{\delta}\} \setminus X_0,$$
$$X_3 = \{y \in X; \langle d(x,y)\rangle^{\delta'} > C_0\langle y\rangle^{\delta'}\} \setminus X_0.$$

Then $\{X_j\}_{j=0}^3$ is a covering of X. Now we set

$$I_j(x) = \int_{X_j} \langle x\rangle^{-\delta} k_{0,\sigma}(x,y)\langle y\rangle^{-\delta'}|f(y)|dy$$

for $j = 0, 1, 2, 3$. Clearly we get

$$|\langle x\rangle^{-\delta}|D|^{-\sigma}\langle x\rangle^{-\delta'} f(x)| \leq \sum_{j=0}^{3} I_j(x).$$

On X_0 we have $\langle x\rangle^{-\delta}\langle y\rangle^{-\delta'} \leq C$. Hence

$$I_0(x) \leq C(\chi_0 k_{0,\sigma}) \times |f|(x).$$

By the assumption that $0 < \sigma < \nu$ and the fact that $f_1 \times f_2 \in L^2(X)$ for $f_1 \in L^1(X)$ and for $f_2 \in L^2(X)$, we have

$$\|I_0\|_{L^2(X)} \leq C\|\chi_0 k_{0,\sigma}\|_{L^1(X)}\|f\|_{L^2(X)}. \tag{3.3}$$

It suffices to show that in the cases (a) $\delta, \delta' \geq 0$ and $0 < \delta + \delta' \leq \nu/2$, (b) $0 \leq \delta \leq \nu/2$ and $\delta' \leq 0$.

Put $\delta_1 = \{\nu + (\delta + \delta' - \sigma)\}/2$, $\delta_2 = \{\nu - \sigma - (\delta + \delta')\}/2$ so that $\delta + \delta' = \delta_1 - \delta_2$. We have $\delta_1 > \nu/2$, $\delta_2 \in (0, \nu/2 - \delta)$ in case (a), and $\delta_1 > \nu/2$, $\delta_2 \in (-\delta', \nu/2 - \delta)$ in case (b). On X_1 we have $\langle x\rangle^{-\delta}\langle y\rangle^{-\delta'} \leq C_0^2\langle d(x,y)\rangle^{-\delta_1+\delta_2}$. Hence

$$I_1(x) \leq C_0^2\big(\langle x\rangle^{-\delta_1+\delta_2}\chi_1 k_{0,\sigma}\big) \times |f|(x).$$

Since $\delta_1 > \nu/2$ and $\delta_2 < \nu/2 - \delta$, from Corollary 2.6 and Corollary 2.8 we get

$$\|I_1\|_{L^2(X)} \leq C\|\langle x\rangle^{\delta_2}\chi_1 k_{0,\sigma}\|_{L^2(X)}\|f\|_{L^2(X)}. \tag{3.4}$$

On X_2 for a sufficiently large C_0 we have $C^{-1}\langle x\rangle \leq \langle y\rangle \leq C\langle d(x,y)\rangle$ and thus $\langle x\rangle^{-\delta}\langle y\rangle^{-\delta'} \leq \langle x\rangle^{-\delta_1}\langle d(x,y)\rangle^{\delta_2}$. Hence

$$I_2(x) \leq C\langle x\rangle^{-\delta_1}\big(\langle x\rangle^{\delta_2}\chi_1 k_{0,\sigma}\big) \times |f|(x).$$

By Corollary 2.6 and Corollary 2.8 we obtain

$$\|I_2\|_{L^2(X)} \leq C\|\langle x\rangle^{\delta_2}\chi_1 k_{0,\sigma}\|_{L^2(X)}\|f\|_{L^2(X)}. \tag{3.5}$$

On X_3 we have

$$\langle x\rangle^{-\delta}\langle y\rangle^{-\delta'} \leq C \begin{cases} \langle d(x,y)\rangle^{\delta_2}\langle y\rangle^{-\delta_1} & \text{in case (a),} \\ \langle d(x,y)\rangle^{-\delta_1+\delta_2} & \text{in case (b).} \end{cases}$$

Therefore we get

$$\|I_3\|_{L^2(X)} \leq C\|\langle x\rangle^{\delta_2}\chi_1 k_{0,\sigma}\|_{L^2(X)}\|f\|_{L^2(X)}. \tag{3.6}$$

By combining (3.3)–(3.6), we complete the proof. □

Finally we note that the fractional integral operators on X are different from those on $\mathbb{R}^n$ in the following sense. In [**29**] Stein and Weiss gave the following estimate.

THEOREM 3.4 ([**29**, Theorem B^*]). *Suppose* $0 < \alpha < n$, $\beta < n/2$, $\gamma < n/2$ *and* $\alpha = \beta + \gamma$. *Then*

$$\left\||x|^{-\beta}|D_x|^{-\alpha}|x|^{-\gamma}f\right\|_{L^2(\mathbb{R}^n)} \leq C\|f\|_{L^2(\mathbb{R}^n)}.$$

COROLLARY 3.5. *Suppose* $0 < \alpha < n/2 + \min\{0,\beta,\gamma\}$ *and* $\beta + \gamma \geq \alpha$. *Then*

$$\left\|\langle x\rangle^{-\beta}|D_x|^{-\alpha}\langle x\rangle^{-\gamma}f\right\|_{L^2(\mathbb{R}^n)} \leq C\|f\|_{L^2(\mathbb{R}^n)}.$$

If we compare the upper bounds of the exponents in Proposition 3.3 and Corollary 3.5, we see that the pseudo-dimension ν corresponds to the dimension n. As a consequence of Proposition 3.3, if $\nu \geq n$, then we have a better estimate about the L^2-continuity of the fractional integral operators on X (e.g. if $X = H^2(\mathbb{R})$, then $n = 2$ but $\nu = 3$).

4. Restriction estimates.

In this section we prove the Fourier restriction estimates by using Proposition 3.1 and Proposition 2.3. For a positively homogeneous function of order one $a(\lambda)$ with $a \in C_W(\mathfrak{a}^*) \cap C_W^\infty(\mathfrak{a}^* \setminus \{0\})$ and $a(\lambda) > 0$ for $\lambda \neq 0$, let $\Sigma(\tau)$ be the level set defined by

$$\Sigma(\tau) = \{(\lambda, b) \in \mathfrak{a}^* \times B; a(\lambda) = \tau\} \tag{4.1}$$

with the measure $2^{-1}|\boldsymbol{c}(\lambda)|^{-2}d\delta_{|\lambda|=\tau}db$ for $l = 1$ (here $d\delta_{|\lambda|=\tau}$ denotes the Dirac measure on the set $\{|\lambda| = \tau\}$), and the surface measure induced by $|W|^{-1}|\boldsymbol{c}(\lambda)|^{-2}d\lambda db$ for $l \geq 2$. By using the co-area formula for the family of level sets $\{\Sigma(\tau)\}_{\tau\in(0,\infty)}$ in $\mathfrak{a}^* \times B$, the L^2-norm of $\mathcal{F}f$ on the Fourier space is expressed as

$$|W|^{-1}\iint_{\mathfrak{a}^*\times B}|\mathcal{F}f(\lambda,b)|^2|\boldsymbol{c}(\lambda)|^{-2}d\lambda db = \int_0^\infty \left\||\nabla a|^{-1/2}\mathcal{F}f\right\|^2_{L^2(\Sigma(\tau))}d\tau.$$

We prove the following Fourier restriction estimates for the Fourier transform on X.

Lemma 4.1. (i) (*Uniform trace estimates*) *Suppose* $\theta > 0$. *Then*

$$\|\mathcal{F}f\|_{L^2(\Sigma(\tau))} \leq C\|\langle x\rangle^{1/2+\theta} f\|_{L^2(X)}, \quad \tau > 0.$$

(ii) (*Hölder continuity*) *Suppose* $0 < \theta < 1/2$ *for* $l = 2$, *and* $0 < \theta < 1$ *for* $l = 1$, *or* $l \geq 3$. *Then*

$$\left\|s(\tau,\cdot)\mathcal{F}f(\tau\cdot,\cdot) - s(\sigma,\cdot)\mathcal{F}f(\sigma\cdot,\cdot)\right\|_{L^2(\Sigma(1))} \leq C|\tau-\sigma|^{\theta}\|\langle x\rangle^{1/2+\theta} f\|_{L^2(X)} \tag{4.2}$$

for $\tau, \sigma > 0$. *Here we put* $s(\tau,\lambda) = \tau^{(l-1)/2}\boldsymbol{c}^{-1}(\tau\lambda)/\boldsymbol{c}^{-1}(\lambda)$ *for* $\tau > 0$, $\lambda \in \mathfrak{a}^*$.

(iii) (*Low frequency trace estimates*) *Suppose* $0 < \theta < 1$ *for* $l = 1$, *and* $0 < \theta < (l-1)/2$ *for* $l \geq 2$. *Then*

$$\|\mathcal{F}f\|_{L^2(\Sigma(\tau))} \leq C\tau^{\theta}\|\langle x\rangle^{1/2+\theta} f\|_{L^2(X)}, \quad \tau > 0.$$

Remark 4.1. Proposition 3.3 does not cover the critical case $\sigma = \delta+\delta'$. Therefore, in order to deal with the critical case $\delta = m/2$ in Theorem 1.1 (ii), we take the approach of translating the result for the Euclidean spaces into symmetric spaces, but this gives the constraint for the order of $p(\lambda)$. If the inequality in Proposition 3.3 holds also for $\sigma = \delta + \delta'$, then it enables us to prove Theorem 1.1 (ii) in the critical case by applying the lemma above as in [**7**].

Proof. (i) In the case $l = 1$, we have

$$c^{-1}(\lambda)\mathcal{F}f(\lambda,b) = \int_{\mathfrak{a}} e^{-i\lambda(H)} J\mathcal{R}f(H,b)dH.$$

For any $\theta > 0$, by using Hölder inequality we have

$$\begin{aligned}|c^{-1}(\lambda)\mathcal{F}f(\lambda,b)| &\leq \int_{\mathfrak{a}} \langle H\rangle^{-(1/2+\theta)}\langle H\rangle^{1/2+\theta}|J\mathcal{R}f(H,b)|dH \\ &\leq \left(\int_{\mathfrak{a}} \langle H\rangle^{-(1+2\theta)}dH\right)^{1/2} \|\langle H\rangle^{1/2+\theta}J\mathcal{R}f(\cdot,b)\|_{L^2(\mathfrak{a})}.\end{aligned}$$

Hence by using Proposition 3.1 we obtain

$$\begin{aligned}\left(\int_B |\mathcal{F}f(\lambda,b)|^2|c^{-1}(\lambda)|^{-2}db\right)^{1/2} &\leq C\|\langle H\rangle^{1/2+\theta}J\mathcal{R}f\|_{L^2(\mathfrak{a}\times B)} \\ &\leq C\|\langle x\rangle^{1/2+\theta}f\|_{L^2(X)}.\end{aligned}$$

In the case $l \geq 2$, from the uniform trace estimate (2.3) and applying Proposition 3.1 we have

$$\begin{aligned}
\|\mathcal{F}f\|_{L^2(\Sigma(\tau))} &= \left(\int_B \int_{a(\lambda)=\tau} |\mathcal{F}_{\mathfrak{a}}[J\mathcal{R}f(\cdot,b)](\lambda)|^2 d\sigma(\lambda)db\right) \\
&\leq \left(\int_B C^2\|\langle H\rangle^{1/2+\theta} J\mathcal{R}f(\cdot,b)\|^2_{L^2(\mathfrak{a})}db\right) \\
&= C\|\langle H\rangle^{1/2+\theta} J\mathcal{R}f\|_{L^2(\mathfrak{a}\times B)} \\
&\leq C\|\langle x\rangle^{1/2+\theta} f\|_{L^2(X)}. \qquad (4.3)
\end{aligned}$$

(ii) In the case $l=1$, we use the following Hölder estimate on $\mathbb{R}^n$ for $0<\gamma<1$:

$$\sup_{\xi\neq\eta} \frac{|\mathcal{F}f(\xi)-\mathcal{F}f(\eta)|}{|\xi-\eta|^\gamma} \leq C_{\gamma,n}\||x|^{n/2+\gamma} f\|_{L^2(\mathbb{R}^n)}.$$

So for any $0<\theta<1$, we have

$$\begin{aligned}
\left|c^{-1}(\lambda_1)\mathcal{F}f(\lambda_1,b) - c^{-1}(\lambda_2)\mathcal{F}f(\lambda_2,b)\right| &= \left|\mathcal{F}_{\mathfrak{a}}[J\mathcal{R}f(\cdot,b)](\lambda_1) - \mathcal{F}_{\mathfrak{a}}[J\mathcal{R}f(\cdot,b)](\lambda_2)\right| \\
&\leq C|\lambda_1-\lambda_2|^\theta \|\langle H\rangle^{1/2+\theta} J\mathcal{R}f(\cdot,b)\|_{L^2(\mathfrak{a})}.
\end{aligned}$$

Therefore by using Proposition 3.1 we obtain

$$\begin{aligned}
&\left(\int_B \left|c^{-1}(\lambda_1)\mathcal{F}f(\lambda_1,b) - c^{-1}(\lambda_2)\mathcal{F}f(\lambda_2,b)\right|^2 db\right)^{1/2} \\
&\leq C|\lambda_1-\lambda_2|^\theta \|\langle H\rangle^{1/2+\theta} J\mathcal{R}f\|_{L^2(\mathfrak{a}\times B)} \\
&\leq C|\lambda_1-\lambda_2|^\theta \|\langle x\rangle^{1/2+\theta} f\|_{L^2(X)}. \qquad (4.4)
\end{aligned}$$

For $l\geq 2$, we can prove the inequality (4.2) by using (2.4) in the same way as (4.3) in (i).

(iii) For $l=1$, by taking $\lambda_2=0$ in (4.4) we have

$$\left(\int_B \left|c^{-1}(\lambda)\mathcal{F}f(\lambda,b)\right|^2 db\right)^{1/2} \leq |\lambda|^\theta \|\langle x\rangle^{1/2+\theta} f\|_{L^2(X)}.$$

Since $a(\lambda)=c|\lambda|$, for any $0<\theta<1$ we have

$$\|\mathcal{F}f\|_{L^2(\Sigma(\tau))} \leq C\tau^\theta \|\langle x\rangle^{1/2+\theta} f\|_{L^2(X)}.$$

For $l\geq 2$, we can prove the assertion by using (2.5) and applying Proposition 3.1 in the same way as (4.3) in (i). □

5. Resolvent estimates.

In this section we prove the resolvent estimates (1.10)–(1.11) in Theorem 1.1. First we introduce a certain function space. Let $\mathcal{B}^\infty_W(\mathfrak{a}^*)$ denote the Fréchet space defined by

$$\mathcal{B}_W^\infty(\mathfrak{a}^*) \ni \psi \Leftrightarrow \psi \in C_W^\infty(\mathfrak{a}^*) \text{ and } \sup_{\lambda\in\mathfrak{a}^*} |p_0(D_\lambda)\psi(\lambda)| < \infty$$
$$\text{for any polynomial function } p_0(H). \tag{5.1}$$

We prove the following lemma which is a modification of Theorem 1.1 (i).

Lemma 5.1. *Suppose $m > 1$ and $\delta > 1/2$. Then for any $\psi \in \mathcal{B}_W^\infty(\mathfrak{a}^*)$ we have*

$$\sup_{\zeta\in\mathbb{C}\setminus\mathbb{R}} \left|(|D|^{m-1}(p(D)-\zeta)^{-1}\psi(D)f, \psi(D)g)_{L^2(X)}\right| \le C\|\langle x\rangle^\delta f\|_{L^2(X)}\|\langle x\rangle^\delta g\|_{L^2(X)}, \tag{5.2}$$

where $\psi(D)$ is the Fourier multiplier with the symbol $\psi(\lambda)$.

Proof. The following argument is almost the same as in the proof of Lemma 4 in [**7**]. It suffices to show the estimate (5.2) in the case $f = g$ and $1/2 < \delta < 1$. We put $\zeta = \mu \pm i\eta$, $\eta > 0$, $\mu \in \mathbb{R}$. Using the Plancherel formula and the co-area formula, we have

$$\begin{aligned}
&\big(|D|^{(m-1)}(p(D)-\zeta)^{-1}\psi(D)f, \psi(D)f\big)_{L^2(X)} \\
&\quad = |W|^{-1}\int_{\mathfrak{a}^*\times B} \frac{|\lambda|^m}{p(\lambda)-\zeta}|\psi\mathcal{F}f(\lambda,b)|^2|\boldsymbol{c}(\lambda)|^{-2}d\lambda db \\
&\quad = \int_0^\infty \frac{(\tau-\mu)\pm i\eta}{(\tau-\mu)^2+\eta^2}\|q\psi\mathcal{F}f\|^2_{L^2(\Sigma(\tau^{1/m}))}d\tau \quad (\Sigma(\tau) \text{ is given by (4.1)}),
\end{aligned} \tag{5.3}$$

where we put $q(\lambda) = |\lambda|^{(m-1)/2}|\nabla p(\lambda)|^{-1/2}$. Note that $q(\lambda)$ is bounded. Applying Lemma 4.1 (i) to the imaginary part of (5.3), we have

$$\begin{aligned}
&\left|\operatorname{Im}(|D|^{m-1}(p(D)-\zeta)^{-1}\psi(D)f, \psi(D)f)_{L^2(X)}\right| \\
&\quad = \int_0^\infty \frac{\eta}{(\tau-\mu)^2+\eta^2}\|q\psi\mathcal{F}f\|^2_{L^2(\Sigma(\tau^{1/m}))}d\tau \\
&\quad \le C\int_0^\infty \frac{\eta}{(\tau-\mu)^2+\eta^2}\|\mathcal{F}f\|^2_{L^2(\Sigma(\tau^{1/m}))}d\tau \\
&\quad \le C\|\langle x\rangle^\delta f\|^2_{L^2(X)}\int_0^\infty \frac{\eta}{(\tau-\mu)^2+\eta^2}d\tau \quad \text{(by Lemma 4.1 (i))} \\
&\quad \le C\|\langle x\rangle^\delta f\|^2_{L^2(X)}.
\end{aligned}$$

For the real part of (5.3), according to $\tau \in \mathbb{R}$, we consider the following three cases:

(Case I): In the case $\mu \le 0$. By Proposition 3.3 we have

$$\begin{aligned}
&\left|\operatorname{Re}(|D|^{m-1}(p(D)-\zeta)^{-1}\psi(D)f, \psi(D)f)_{L^2(X)}\right| \\
&\quad \le \int_0^\infty \frac{(\tau-\lambda)}{(\tau-\mu)^2+\eta^2}\|q\psi\mathcal{F}f\|^2_{L^2(\Sigma(\tau^{1/m}))}d\tau
\end{aligned}$$

$$
\begin{aligned}
&\le C\int_0^\infty \frac{1}{\tau}\|\mathcal{F}f\|_{L^2(\Sigma(\tau^{1/m}))}d\tau\\
&= C\int_{\mathfrak{a}^*\times B}\frac{|\nabla p(\lambda)|}{|p(\lambda)|}|\mathcal{F}f(\lambda,b)|^2|\boldsymbol{c}(\lambda)|^{-2}d\lambda db\\
&\le C\big\||D|^{-1/2}f\big\|^2_{L^2(X)}\\
&\le C\|\langle x\rangle^\delta f\|^2_{L^2(X)}.
\end{aligned}
$$

(Case II): In the case $0<\mu\le 2^m$. We split the real part into three parts.

$$
\begin{aligned}
&\mathrm{Re}\left(|D|^{m-1}(p(D)-\zeta)^{-1}\psi(D)f,\psi(D)f\right)_{L^2(X)}\\
&=\int_0^\infty \frac{(\tau-\mu)}{(\tau-\mu)^2+\eta^2}\|q\psi\mathcal{F}f\|^2_{L^2(\Sigma(\tau^{1/m}))}d\tau\\
&=\int_0^{\mu/2}+\int_{\mu/2}^{3\mu/2}+\int_{3\mu/2}^\infty\\
&=: I_1+I_2+I_3.
\end{aligned}
$$

For $\tau\in(0,\infty)\setminus(\mu/2,3\mu/2)$ we have

$$
\frac{|\tau-\mu|}{(\tau-\mu)^2+\eta^2}\le\frac{1}{|\tau-\mu|}\le\frac{3}{\tau}.
$$

As in the Case I we get

$$
|I_1|,|I_3|\le C\big\||D|^{-1/2}f\big\|^2_{L^2(X)}\le C\big\|\langle x\rangle^\delta f\big\|^2_{L^2(X)}.
$$

For $\tau,\mu>0$ we put $F(\tau,\mu)=\|q\psi\mathcal{F}f\|^2_{L^2(\Sigma(\tau^{1/m}))}-\|q\psi\mathcal{F}f\|^2_{L^2(\Sigma(\mu^{1/m}))}$. Since

$$
\int_{\mu/2}^{3\mu/2}\frac{(\tau-\mu)}{(\tau-\mu)^2+\eta^2}d\tau=0,
$$

we have

$$
I_2=\int_{\mu/2}^{3\mu/2}\frac{(\tau-\mu)}{(\tau-\mu)^2+\eta^2}F(\tau,\mu)d\tau.
$$

Here for simplicity we set $g=\mathcal{F}^{-1}q\psi\mathcal{F}f$. Then by Lemma 4.1 (i), we have

$$
\begin{aligned}
|F(\tau,\mu)|&=\big|\|\mathcal{F}g\|_{L^2(\Sigma(\tau^{1/m}))}-\|\mathcal{F}g\|_{L^2(\Sigma(\mu^{1/m}))}\big|\\
&\quad\times\left(\|\mathcal{F}g\|_{L^2(\Sigma(\tau^{1/m}))}+\|\mathcal{F}g\|_{L^2(\Sigma(\mu^{1/m}))}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\le \left\| s(\tau^{1/m},\cdot)\mathcal{F}g(\tau^{1/m}\cdot,\cdot) - s(\mu^{1/m},\cdot)\mathcal{F}g(\mu^{1/m}\cdot,\cdot)\right\|_{L^2(\Sigma(1))} \\
&\quad \times C\big(\|\mathcal{F}f\|_{L^2(\Sigma(\tau^{1/m}))} + \|\mathcal{F}f\|_{L^2(\Sigma(\mu^{1/m}))}\big) \\
&\le \left\| s(\tau^{1/m},\cdot)\mathcal{F}g(\tau^{1/m}\cdot,\cdot) - s(\mu^{1/m},\cdot)\mathcal{F}g(\mu^{1/m}\cdot,\cdot)\right\|_{L^2(\Sigma(1))} \\
&\quad \times C\|\langle x\rangle^{\delta} f\|_{L^2(X)}. \qquad (5.4)
\end{aligned}
$$

By applying Lemma 2.3 (ii) to $J\mathcal{R}g(\cdot,b)$, we have

$$
\begin{aligned}
&\left\| s(\tau^{1/m},\cdot)\mathcal{F}g(\tau^{1/m}\cdot,\cdot) - s(\mu^{1/m},\cdot)\mathcal{F}g(\mu^{1/m}\cdot,\cdot)\right\|_{L^2(\Sigma(1))} \\
&= \bigg(\int_B \int_{a(\lambda)=1} \big|\tau^{(l-1)/2m}\mathcal{F}_{\mathfrak{a}}[J\mathcal{R}g(\cdot,b)](\tau^{1/m}\lambda) \\
&\qquad - \mu^{(l-1)/2m}\mathcal{F}_{\mathfrak{a}}[J\mathcal{R}g(\cdot,b)](\mu^{1/m}\lambda)\big|^2 d\sigma(\lambda)db\bigg)^{1/2} \\
&\le C\big|\tau^{1/m}-\mu^{1/m}\big|^{\theta}\bigg(\int_B \|\langle H\rangle^{\delta} J\mathcal{R}g(\cdot,b)\|^2_{L^2(\mathfrak{a})}db\bigg)^{1/2}, \qquad (5.5)
\end{aligned}
$$

where we put $\theta = \delta - 1/2$. By using Lemma 2.4 (note that if $l = 1$ then q is constant) and the L^2-continuity of the pseudodifferential operator $\langle H\rangle^{\delta}\psi(D_H)\langle H\rangle^{-\delta}$ in the integrand on the right hand side of the inequality (5.5), we get

$$
\begin{aligned}
\left\|\langle H\rangle^{\delta} J\mathcal{R}g(\cdot,b)\right\|^2_{L^2(\mathfrak{a})} &= \left\|\langle H\rangle^{\delta} q(D_H)\psi(D_H)J\mathcal{R}f(\cdot,b)\right\|^2_{L^2(\mathfrak{a})} \\
&\le C\left\|\langle H\rangle^{\delta}\psi(D_H)J\mathcal{R}f(\cdot,b)\right\|^2_{L^2(\mathfrak{a})} \quad \text{(by Lemma 2.4)} \\
&\le C\left\|\langle H\rangle^{\delta} J\mathcal{R}f(\cdot,b)\right\|^2_{L^2(\mathfrak{a})}.
\end{aligned}
$$

Then by using the weighted L^2-continuity of the modified Radon transform, we have

$$
\begin{aligned}
\bigg(\int_B \left\|\langle H\rangle^{\delta} J\mathcal{R}g(\cdot,b)\right\|^2_{L^2(\mathfrak{a})}db\bigg)^{1/2} &\le C\|\langle H\rangle^{\delta} J\mathcal{R}f\|_{L^2(\mathfrak{a}\times B)} \\
&\le C\|\langle x\rangle^{\delta} f\|_{L^2(X)}. \qquad (5.6)
\end{aligned}
$$

By (5.5) and (5.6), we have

$$
\begin{aligned}
&\left\| s(\tau^{1/m},\cdot)\mathcal{F}g(\tau^{1/m}\cdot,\cdot) - s(\mu^{1/m},\cdot)\mathcal{F}g(\mu^{1/m}\cdot,\cdot)\right\|_{L^2(\Sigma(1))} \\
&\le C|\tau^{1/m}-\mu^{1/m}|^{\theta}\|\langle x\rangle^{\delta} f\|_{L^2(X)}. \qquad (5.7)
\end{aligned}
$$

By (5.4) and (5.7), we obtain

$$|F(\tau, \mu)| \leq C|\tau^{1/m} - \mu^{1/m}|^{\theta} \|\langle x\rangle^{\delta} f\|^2_{L^2(X)}. \tag{5.8}$$

Here the mean value theorem shows that for $\mu/2 \leq \tau \leq 3\mu/2$

$$\left|\tau^{1/m} - \mu^{1/m}\right| \leq C\mu^{-(m-1)/m}|\tau - \mu|. \tag{5.9}$$

Substituting (5.9) into (5.8), we have

$$|F(\tau, \mu)| \leq C\mu^{-\theta(m-1)/m}|\tau - \mu|^{\theta} \|\langle x\rangle^{\delta} f\|^2_{L^2(X)} \tag{5.10}$$

for $\mu/2 \leq \tau \leq 3\mu/2$. Hence we obtain

$$\begin{aligned}
|I_2| &\leq C\mu^{-\theta(m-1)/m} \|\langle x\rangle^{\delta} f\|^2_{L^2(X)} \int_{\mu/2}^{3\mu/2} |\tau - \mu|^{\theta - 1} d\tau \\
&= 2C\mu^{-\theta(m-1)/m} \|\langle x\rangle^{\delta} f\|^2_{L^2(X)} \int_0^{\mu/2} \sigma^{\theta-1} d\sigma \\
&= \frac{2C}{\theta} \mu^{\theta/m} \|\langle x\rangle^{\delta} f\|^2_{L^2(X)} \\
&\leq \frac{2^{1+\theta} C}{\theta} \|\langle x\rangle^{\delta} f\|^2_{L^2(X)}.
\end{aligned}$$

(Case III): In the case $\mu > 2^m$. We split the real part of (5.3) into four parts as follows.

$$\begin{aligned}
&\operatorname{Re}\left(|D|^{m-1}(p(D) - \zeta)^{-1}\psi(D)f, \psi(D)f\right)_{L^2(X)} \\
&= \int_0^{\infty} \frac{(\tau - \mu)}{(\tau - \mu)^2 + \eta^2} \|q\psi\mathcal{F}f\|^2_{L^2(\Sigma(\tau^{1/m}))} d\tau \\
&= \int_0^{\mu - \mu^{(m-1)/m}} + \int_{\mu - \mu^{(m-1)/m}}^{\mu + \mu^{(m-1)/m}} + \int_{\mu + \mu^{(m-1)/m}}^{2\mu} + \int_{2\mu}^{\infty} \\
&=: I_4 + I_5 + I_6 + I_7.
\end{aligned}$$

It is easy to estimate I_4, I_6 and I_7. Since $\mu - \tau \geq \tau^{(m-1)/m}$ for $0 \leq \tau \leq \mu - \mu^{(m-1)/m}$, and $\tau - \mu \leq (\tau/2)^{(m-1)/m}$ for $\mu + \mu^{(m-1)/m} \leq \tau \leq 2\mu$,

$$\frac{|\tau - \mu|}{(\tau - \mu)^2 + \eta^2} \leq \frac{1}{|\tau - \mu|} \leq \left(\frac{2}{\tau}\right)^{(m-1)/m}$$

for $0 \leq \tau \leq \mu - \mu^{(m-1)/m}$ and $\mu + \mu^{(m-1)/m} \leq \tau \leq 2\mu$. Hence, we deduce

$$
\begin{aligned}
|I_4|, |I_6| &\leq C \int_0^\infty \tau^{-(m-1)/m} \|\mathcal{F}f\|^2_{L^2(\Sigma(\tau^{1/m}))} d\tau \\
&= C \int_{\mathfrak{a}^* \times B} |p(\lambda)|^{-(m-1)/m} |\nabla p(\lambda)| |\mathcal{F}f(\lambda, b)|^2 |\boldsymbol{c}(\lambda)|^{-2} d\lambda db \\
&\leq C \|f\|^2_{L^2(X)}. \qquad (5.11)
\end{aligned}
$$

Since $\tau - \mu \geq \tau/2$ for $\tau \geq 2\mu$, we get

$$
|I_7| \leq C \big\| |D|^{-1/2} f \big\|^2_{L^2(X)} \leq C \big\| \langle x \rangle^\delta f \big\|^2_{L^2(X)}. \qquad (5.12)
$$

As in the Case II, since

$$
\int_{\mu - \mu^{(m-1)/m}}^{\mu + \mu^{(m-1)/m}} \frac{(\tau - \mu)}{(\tau - \mu)^2 + \eta^2} d\tau = 0,
$$

we have

$$
I_5 = \int_{\mu - \mu^{(m-1)/m}}^{\mu + \mu^{(m-1)/m}} \frac{(\tau - \mu)}{(\tau - \mu)^2 + \eta^2} F(\tau, \mu) d\tau.
$$

Here we remark that for $\mu > 2^m$

$$
\left(\mu - \mu^{(m-1)/m}\right) - \frac{\mu}{2} = \mu^{(m-1)/m} \left(\frac{\mu^{1/m}}{2} - 1 \right) > 0.
$$

Then (5.9) is also valid for $\mu > 2^m$ and $|\tau - \mu| \leq \mu^{(m-1)/m}$, so (5.10) holds for $\mu > 2^m$ and $|\tau - \mu| \leq \mu^{(m-1)/m}$. Hence we obtain

$$
\begin{aligned}
|I_5| &\leq C \mu^{-\theta(m-1)/m} \big\| \langle x \rangle^\delta f \big\|_{L^2(X)} \int_0^{\mu^{(m-1)/m}} \sigma^{\theta - 1} d\sigma \\
&= \frac{C}{\theta} \big\| \langle x \rangle^\delta f \big\|^2_{L^2(X)}. \qquad (5.13)
\end{aligned}
$$

Combining the inequalities (5.11), (5.12) and (5.13), we obtain the assertion. □

Next, in order to control the low-frequency part we need the following lemma.

Lemma 5.2. (i) *Suppose $m > 1$ and $\delta > 1/2$. Then for any $\phi \in \mathcal{B}^\infty_W(\mathfrak{a}^*)$ with $\phi(\lambda) = 0$ near 0 in $\mathfrak{a}^*$, we have*

$$
\sup_{\zeta \in \mathbb{C} \setminus \mathbb{R}} \left| (\phi(D) \langle D \rangle^{m-1} (p(D) - \zeta)^{-1} f, g)_{L^2(X)} \right| \leq C \| \langle x \rangle^\delta f \|_{L^2(X)} \| \langle x \rangle^\delta g \|_{L^2(X)}, \qquad (5.14)
$$

where $\phi(D)$ is the Fourier multiplier with the symbol $\phi(\lambda)$ on X.

(ii) *Suppose* $1 < m < \nu$ *and* $\delta > m/2$. *Then for any* $\psi \in \mathcal{B}^\infty_W(\mathfrak{a}^*)$ *we have*

$$\sup_{\zeta \in \mathbb{C}\setminus\mathbb{R}} \left|((p(D)-\zeta)^{-1}\psi(D)f, \psi(D)g)_{L^2(X)}\right| \le C\|\langle x\rangle^\delta f\|_{L^2(X)}\|\langle x\rangle^\delta g\|_{L^2(X)}, \quad (5.15)$$

where $\psi(D)$ *is the Fourier multiplier with the symbol* $\psi(\lambda)$ *on* X.

Here $\mathcal{B}^\infty_W(\mathfrak{a}^*)$ *denotes the Fréchet space defined by* (5.1).

Proof. (i) First, a slight modification of the proof of Theorem 2 in [**7**] gives the following high-frequency resolvent estimate for any $l \ge 1$:

$$\sup_{\zeta \in \mathbb{C}\setminus\mathbb{R}} \left|(\phi(D_H)\langle D_H\rangle^{m-1}(p(D_H)-\zeta)^{-1}f, g)_{L^2(\mathfrak{a})}\right| \le C\|\langle H\rangle^\delta f\|_{L^2(\mathfrak{a})}\|\langle H\rangle^\delta g\|_{L^2(\mathfrak{a})}.$$

By the Plancherel theorem and the continuity of the modified Radon transform we obtain

$$\begin{aligned}
&\sup_{\zeta \in \mathbb{C}\setminus\mathbb{R}} \left|(\phi(D)\langle D\rangle^{m-1}(p(D)-\zeta)^{-1}f, g)_{L^2(X)}\right| \\
&\quad = \sup_{\zeta \in \mathbb{C}\setminus\mathbb{R}} \left|\int_B (\phi(D_H)\langle D_H\rangle^{m-1}(p(D_H)-\zeta)^{-1}J\mathcal{R}f(\cdot,b), J\mathcal{R}g(\cdot,b))_{L^2(\mathfrak{a})}db\right| \\
&\quad \le C\left\|\langle H\rangle^\delta J\mathcal{R}f\right\|_{L^2(\mathfrak{a}\times B)}\left\|\langle H\rangle^\delta J\mathcal{R}g\right\|_{L^2(\mathfrak{a}\times B)} \\
&\quad \le C\left\|\langle x\rangle^\delta f\right\|_{L^2(X)}\left\|\langle x\rangle^\delta g\right\|_{L^2(X)}.
\end{aligned}$$

(ii) For $1 < m < \nu$, $\delta > m/2$, we can take $\delta' \in \mathbb{R}_+$ such that $1/2 < \delta' < \min\{(\nu - m)/2, \delta - m/2\} + 1/2$. Then $\langle x\rangle^{\delta'}|D|^{-(m-1)/2}\langle x\rangle^{-\delta}$ is a bounded linear operator on $L^2(X)$ due to Proposition 3.3. By (i) for $\delta' > 1/2$, we obtain

$$\begin{aligned}
&\sup_{\zeta \in \mathbb{C}\setminus\mathbb{R}} \left|((p(D)-\zeta)^{-1}\psi(D)f, \psi(D)g)_{L^2(X)}\right| \\
&\quad \le C\left\|\langle x\rangle^{\delta'}|D|^{-(m-1)/2}f\right\|_{L^2(X)}\left\|\langle x\rangle^{\delta'}|D|^{-(m-1)/2}g\right\|_{L^2(X)} \\
&\quad \le C\left\|\langle x\rangle^\delta f\right\|_{L^2(X)}\left\|\langle x\rangle^\delta g\right\|_{L^2(X)}.
\end{aligned}$$

□

Now we prove Theorem 1.1.

Proof of Theorem 1.1. (i) Take $\psi = 1$ on $\mathfrak{a}^*$ in Lemma 5.1 (i).

(ii) In the case $1 < m < \nu$ and $\delta > m/2$. Choose $\psi_0(\lambda) \in \mathcal{B}^\infty_W(\mathfrak{a}^*;\mathbb{R})$ such that $0 \le \psi_0(\lambda) \le 1$, $\psi_0(\lambda) = 1$ if $|\lambda| \le 1/2$, and $\psi_0(\lambda) = \langle\lambda\rangle^{-(m-1)/2}$ if $|\lambda| \ge 1$. Set $\psi(\lambda) = \langle\lambda\rangle^{(m-1)/2}\psi_0(\lambda)$, $\phi(\lambda) = 1 - \psi_0^2(\lambda)$. Then we have

$$\langle D\rangle^{(m-1)} = \phi(D)\langle D\rangle^{(m-1)} + \psi_0(D)^*\langle D\rangle^{(m-1)}\psi_0(D).$$

Applying the estimate (5.14) to $\phi(\lambda)$ and (5.15) to $\psi(\lambda)$ respectively, we have the following two estimates.

$$\sup_{\zeta\in\mathbb{C}\setminus\mathbb{R}}\left|((1-\psi_0(D)^2)\langle D\rangle^{m-1}(p(D)-\zeta)^{-1}f,g)_{L^2(X)}\right| \le C\|\langle x\rangle^\delta f\|_{L^2(X)}\|\langle x\rangle^\delta g\|_{L^2(X)},$$

$$\sup_{\zeta\in\mathbb{C}\setminus\mathbb{R}}\left|(\psi_0(D)^2\langle D\rangle^{m-1}(p(D)-\zeta)^{-1}f,g)_{L^2(X)}\right| \le C\|\langle x\rangle^\delta f\|_{L^2(X)}\|\langle x\rangle^\delta g\|_{L^2(X)}.$$

Combining above two inequalities, we obtain the estimate (1.11). In the case $l \ge 2$ and $m < l$. Use the following expression for the resolvent operator.

$$\langle D\rangle^{m-1}(p(D)-\zeta)^{-1} = |W|^{-1}\mathcal{R}^*\bar{J}\langle D_H\rangle^{m-1}(p(D_H)-\zeta)^{-1}J\mathcal{R}, \quad \zeta\in\mathbb{C}\setminus\mathbb{R}.$$

By the estimate (2.2) in the Euclidean case and the continuity of the modified Radon transform we obtain the assertion. □

Finally, we show how Theorem 1.1 yields Theorem 1.2. The following is well-known (see [**22**]).

Proposition 5.3. *Let H be a selfadjoint operator in the Hilbert space $\mathcal{H}$. Suppose that A is a densely defined, closed operator from $\mathcal{H}$ to another Hilbert space $\mathcal{H}_1$.*

(i) *The following conditions are equivalent*:

(i.1)

$$\sup_{\zeta\in\mathbb{C}\setminus\mathbb{R}}\left|\operatorname{Im}((H-\zeta)^{-1}A^*v,A^*v)_{\mathcal{H}}\right| \le C_1\|v\|^2_{\mathcal{H}_1}, \quad v\in\mathcal{D}(A^*).$$

(i.2)

$$\int_{-\infty}^{\infty}\left\|Ae^{-itH}u\right\|^2_{\mathcal{H}_1}dt \le 2C_1\|u\|^2_{\mathcal{H}}, \quad u\in\mathcal{H}. \tag{5.16}$$

(ii) *Assume that there exists $C_2 > 0$ such that*

$$\sup_{\zeta\in\mathbb{C}\setminus\mathbb{R}}\left\|A(H-\zeta)^{-1}A^*v\right\|_{\mathcal{H}_1} \le C_2\|v\|_{\mathcal{H}_1}$$

for $v\in\mathcal{D}(A^*)$. Then we have

$$\left\|\int_0^t Ae^{-i(t-s)H}A^*f(s)ds\right\|_{L^2(\mathbb{R}_t;\mathcal{H}_1)} \le C_2\|f\|_{L^2(\mathbb{R}_t;\mathcal{H}_1)} \tag{5.17}$$

for each $f(t)\in L^2(\mathbb{R}_t;\mathcal{D}(A^*))$.

Then time global smoothing estimates in Theorem 1.2 are direct consequence of the resolvent estimates in Theorem 1.1 and the proposition above. By applying the inequalities (5.16) and (5.17) to the our case $\mathcal{H} = \mathcal{H}_1 = L^2(X)$, $H = p(D)$, $A = \langle x\rangle^{-\delta}|D|^{(m-1)/2}$ in Theorem 1.1 (i), $= \langle x\rangle^{-\delta}\langle D\rangle^{(m-1)/2}$ in Theorem 1.1 (ii) respectively, we obtain the assertion in Theorem 1.2.

6. Some remarks.

First we remark that the pseudo-dimension ν does not appear for low-frequency estimates in Lemma 4.1 (iii). But in the case $l \geq 2$ for $(l-1)/2 \leq \theta < (\nu-1)/2$, $0 < \theta' < 1/2$, and $\sigma = \theta - \theta'$ we see easily that

$$\|\mathcal{F}f\|_{L^2(\Sigma(\tau))} \leq C\tau^{\theta} \big\|\langle x\rangle^{(1/2+\theta')}|D|^{-\sigma}\langle x\rangle^{-(1/2+\theta)}\big\|_{\mathcal{L}(L^2(X))} \big\|\langle x\rangle^{1/2+\theta} f\big\|_{L^2(X)}.$$

Hence if $\|\langle x\rangle^{(1/2+\theta')}|D|^{-\sigma}\langle x\rangle^{-(1/2+\theta)}\|_{\mathcal{L}(L^2(X))} < \infty$ (this corresponds to the critical case $\sigma = \delta + \delta'$ in Proposition 3.3), then Lemma 4.1 (iii) holds for $0 < \theta < (\nu-1)/2$.

Here we prove restriction estimates for low-frequency part in the critical case $\theta = (\nu-1)/2$. As in introduced in Section 1, let $a(\lambda) \in C_W(\mathfrak{a}^*) \cap C_W^\infty(\mathfrak{a}^* \setminus \{0\})$ be a positively homogeneous function of order one and $\Sigma(\tau)$ be the level set of $a(\lambda)$ defined by (4.1).

Lemma 6.1. *Let $\delta > \nu/2$. Then we have*

$$\|\mathcal{F}f\|_{L^2(\Sigma(\tau))} \leq C\tau^{(\nu-1)/2}\|\langle x\rangle^{\delta} f\|_{L^2(X)}. \tag{6.1}$$

The above lemma and uniform trace estimates immediately yield the following time global smoothing estimates for homogeneous solutions.

Theorem 6.2. *Let $p(\lambda) = a(\lambda)^\nu$ and $\delta > \nu/2$. Then*

$$\big\|\langle x\rangle^{-\delta}\langle D\rangle^{(\nu-1)/2} e^{itp(D)}\phi\big\|_{L^2(\mathbb{R}\times X)} \leq C\|\phi\|_{L^2(X)}.$$

Proof of Lemma 6.1. It is sufficient to prove the estimate (6.1) for $0 < \tau < 1$ and $f \in C_0^\infty(X)$. Let $\chi(\tau) \in C_0^\infty(0,\infty)$ and set $\psi(\lambda) = \chi(a(\lambda)) \in C_{0,W}^\infty(\mathfrak{a}^* \setminus \{0\})$ and $f_0 = \mathcal{F}^{-1}\psi$. Since f_0 is K-invariant, we have $\mathcal{F}(f \times f_0)(\lambda, b) = \mathcal{F}f(\lambda, b)\psi(\lambda)$. By using Corollary 2.6 and the Plancherel formula, we get

$$|W|^{-1}\int_{\mathfrak{a}^*\times B} |\mathcal{F}f(\lambda,b)|^2|\psi(\lambda)|^2|\boldsymbol{c}(\lambda)|^{-2}d\lambda db \leq C\|f\|^2_{L^{2,\delta}(X)}\|f_0\|^2_{L^2(X)}.$$

Applying the co-area formula to the left hand side of the inequality above, we obtain

$$\begin{aligned}
&|W|^{-1}\int_{\mathfrak{a}^*\times B} |\mathcal{F}f(\lambda,b)|^2|\psi(\lambda)|^2|\boldsymbol{c}(\lambda)|^{-2}d\lambda db \\
&= \int_0^\infty |\chi(\tau)|^2 \int_{\Sigma(\tau)} |\mathcal{F}f(\lambda,b)|^2|\boldsymbol{c}(\lambda)|^{-2}|\nabla a(\lambda)|^{-1}d\sigma(\lambda)dbd\tau \\
&\geq C^{-1}\int_0^\infty |\chi(\tau)|^2\|\mathcal{F}f\|^2_{L^2(\Sigma(\tau))}d\tau.
\end{aligned}$$

In a similar way, we have

$$\begin{aligned}\|f_0\|^2_{L^2(X)} &= |W|^{-1}\int_{\mathfrak{a}^*}|\psi(\lambda)|^2|\boldsymbol{c}(\lambda)|^{-2}d\lambda \\ &\le C\int_0^\infty |\chi(\tau)|^2\|1\|^2_{L^2(\Sigma(\tau))}d\tau \\ &\le C\int_0^\infty |\chi(\tau)|^2\tau^{(\nu-1)}\langle\tau\rangle^{(n-\nu)}d\tau.\end{aligned}$$

Therefore, we obtain

$$\int_0^\infty |\chi(\tau)|^2\|\mathcal{F}f\|^2_{L^2(\Sigma(\tau))}d\tau \le C\int_0^\infty |\chi(\tau)|^2\tau^{(\nu-1)}\langle\tau\rangle^{(n-\nu)}d\tau\|f\|^2_{L^{2,\delta}(X)}.$$

This inequality implies that

$$\|\mathcal{F}f\|_{L^2(\Sigma(\tau))} \le C\tau^{(\nu-1)/2}\langle\tau\rangle^{(n-\nu)/2}\|f\|_{L^{2,\delta}(X)}, \quad \tau > 0.$$

We complete the proof. □

Next, we briefly show the optimality of the pseudo-dimension ν for the resolvent estimate and smoothing estimates corresponding to low-frequency part. For any radially symmetric function $\psi(\lambda) \in C_0^\infty(\mathfrak{a}^*)$ with $\psi(0) > 0$, we put $f = \mathcal{F}^{-1}\langle\lambda\rangle^{-(m-1)/2}\psi$. Then we have $f \in L^{2,\delta}(X)$ for any $\delta > 0$. If (1.11) or (1.13) holds, the following must be finite.

$$\sup_{\varepsilon>0}\left|(\langle D\rangle^{(m-1)}(p(D)+\varepsilon)^{-1}f, f)\right| = \int_0^\infty \left\||\nabla p|^{-1/2}\psi\right\|^2_{L^2(\Sigma(\mu^{1/m}))}\mu^{-1}d\mu.$$

By using $\|1\|_{L^2(\Sigma(\mu))} \asymp \mu^{\nu-1}$ as $\mu \downarrow 0$, we have

$$\left\||\nabla p|^{-1/2}\psi\right\|^2_{L^2(\Sigma(\mu^{1/m}))} \asymp \mu^{(\nu-m)/m} \quad \text{as} \quad \mu\downarrow 0. \tag{6.2}$$

Therefore if (1.11) or (1.13) holds, then m must satisfy $m < \nu$. From Proposition 5.3 (i), if the estimate (1.12) is valid then the following must be finite for any $\mu > 0$.

$$\mathrm{Im}(\langle D\rangle^{(m-1)}(p(D)-(\mu+i0))^{-1}f, f) = \pi\left\||\nabla p|^{-1/2}\psi\right\|^2_{L^2(\Sigma(\mu^{1/m}))}.$$

Then from (6.2), m must satisfy $m \le \nu$.

Finally, we note that for positively homogeneous (not necessarily elliptic) symbols of real-principal-type time global smoothing estimates, as in Theorem 1.2, also hold. The proof is done by combining the Euclidean estimate (Theorem 1.1 in [**6**]), Proposition 3.1, and Proposition 3.3 as in the proof of Theorem 1.1 (ii).

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