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On Alperin's weight conjecture for p-blocks of p-solvable groups

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Abstract. For p-solvable groups, a strong form of Alperin's weight conjecture has been proved by T. Okuyama (unpublished). L. Barker has refined this theorem by taking Green correspondence into account. We prove here a relative version of Barker's theorem.

Introduction.

Let G be a finite group and p a prime. Let k be an algebraically closed field of characteristic p. In the present paper, a block always means a p-block for the prime p. The main result of the present paper is as follows:

THEOREM 1. *Let N be a normal subgroup of G. Let Q be a p-subgroup of G. Let β be a block of $N_G(Q)N$. Assume that G/N is p-solvable. Then the number of isomorphism classes of simple kG-modules with vertex Q whose Green correspondents with respect to $(G, Q, N_G(Q)N)$ lie in β equals the number of isomorphism classes of simple $kN_G(Q)N$-modules with vertex Q lying in β.*

When $N = 1$, Theorem 1 coincides with L. Barker's theorem ([**Ba**, Theorem 1.1]). Thus Theorem 1 is a relative version of Barker's theorem. While Barker's proof of his theorem is based on G-algebra theory and quite involved, our proof of Theorem 1 is module theoretical[1] and straightforward.

Notation and convention.

In this paper all modules are identified with their isomorphic ones. Let $\mathrm{IBr}(G)$ be the set of all simple kG-modules. For a block B of G, let $\mathrm{IBr}(B)$ be the set of all simple kG-modules lying in B. For kG-modules V and W, $V \otimes W$ stands for $V \otimes_k W$. If N is a normal subgroup of G and V is a $k[G/N]$-module, $\mathrm{Inf}_{G/N\to G}(V)$ denotes the inflation of V to G via the natural map $G \to G/N$. For a simple kN-

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[1] There is, however, the only exception, Proposition 7, where we need [**IN**, Theorem 4.3] whose proof depends on character theory.

module X, let $\mathrm{IBr}(G|X)$ be the set of simple kG-modules lying over X and $T_G(X)$ the inertial group of X in G. For a block B of G, let $\mathrm{IBr}(B|X)$ be the set of simple kG-modules in B which lie over X. For a block B of G and a subgroup H of G, $BL(H, B)$ denotes the set of blocks b of H such that $b^G = B$ ([**NT**, p. 320]). For a simple kG-module S, let $B(S)$ be the block of G containing S. For an indecomposable kG-module X, let $\mathrm{vx}(X)$ be a vertex of X.

We introduce the following notation; Let Q be a p-subgroup of G and let H be a subgroup of G containing $N_G(Q)$. Let β be a block of H. For $S \in \mathrm{IBr}(G)$ with a vertex Q, the Green correspondent V with respect to (G, Q, H) is defined ([**NT**, p. 276]). If V lies in β, we write $S \in_Q \beta$.

Let Q be a p-subgroup of G and let β be a block of $N_G(Q)$. Then let

$$l_G(\beta, Q) = \sharp\{S \in \mathrm{IBr}(G); \mathrm{vx}(S) =_G Q, S \in_Q \beta\}.$$

So

$$l_{N_G(Q)}(\beta, Q) = \sharp\{U \in \mathrm{IBr}(\beta); \mathrm{vx}(U) = Q\}.$$

A group G is said to be of Barker type (cf. [**Ba**, Theorem 1.1]) if $l_G(\beta, Q) = l_{N_G(Q)}(\beta, Q)$ for any p-subgroup Q of G and any block β of $N_G(Q)$.

1. Vertices and sources.

In this section we study properties of vertices of indecomposable modules needed in Section 2.

Lemma 2. *Let N be a normal subgroup of G. Let U be a Q-projective kG-module for a subgroup Q of G. Then $\mathrm{Inv}_N(U)$ is a QN/N-projective G/N-module.*

Proof. There is $f \in \mathrm{End}_Q(U)$ such that $\mathrm{id}_U = \mathrm{Tr}_Q^G(f)$. It is easy to see $\mathrm{Tr}_Q^{QN}(f)$ acts on $\mathrm{Inv}_N(U)$. Let φ be the restriction of $\mathrm{Tr}_Q^{QN}(f)$ to $\mathrm{Inv}_N(U)$. Then $\varphi \in \mathrm{End}_{QN/N}(\mathrm{Inv}_N(U))$ and $\mathrm{id}_{\mathrm{Inv}_N(U)} = \mathrm{Tr}_{QN/N}^{G/N}(\varphi)$, so the result follows. $\square$

The following complements Lemma 1.1 of [**Mu**].

Proposition 3. *Let N be a normal subgroup of G. Let W be an indecomposable $k[G/N]$-module. Let V be an indecomposable kG-module such that V_N is indecomposable. Then $V \otimes \mathrm{Inf}_{G/N\to G}(W)$ is indecomposable and $\mathrm{vx}(V \otimes \mathrm{Inf}_{G/N\to G}(W))N/N$ is a vertex of W. In particular, a Sylow p-subgroup of the inverse image in G of $\mathrm{vx}(W)$ is a vertex of $\mathrm{Inf}_{G/N\to G}(W)$.*

Proof. We write $V \otimes W$ instead of $V \otimes \mathrm{Inf}_{G/N \to G}(W)$. By [**HB1**, VII 9.12] (see also [**Mu**, Lemma 1.1]), $V \otimes W$ is indecomposable. It is easy to see that $\mathrm{vx}(V \otimes W)N/N \leq_{G/N} \mathrm{vx}(W)$. Put $Q = \mathrm{vx}(V \otimes W)$. For the dual module V^* of V, $V^* \otimes V \otimes W$ is Q-projective. So $\mathrm{Inv}_N(V^* \otimes V \otimes W)$ is QN/N-projective by Lemma 2. Now $\mathrm{Inv}_N(V^* \otimes V \otimes W) \simeq \mathrm{End}_N(V) \otimes W$. Since V_N is indecomposable, $\mathrm{End}_N(V) = k\,\mathrm{id}_V \oplus J(\mathrm{End}_N(V))$. So $1_G | \mathrm{End}_N(V)$. Therefore $W | \mathrm{Inv}_N(V^* \otimes V \otimes W)$. Thus W is QN/N-projective and $QN/N \geq_{G/N} \mathrm{vx}(W)$.

Put $Q = \mathrm{vx}(\mathrm{Inf}_{G/N \to G}(W))$. By the above we may assume $QN/N = \mathrm{vx}(W)$. So it suffices to show Q is a Sylow p-subgroup of QN. Now $|QN : Q| = |N : Q \cap N|$. Since $(\mathrm{Inf}_{G/N \to G}(W))_N$ is a multiple of 1_N, $Q \geq_G \mathrm{vx}(1_N)$. Since $\mathrm{vx}(1_N)$ is a Sylow p-subgroup of N, we see $|N : Q \cap N|$ is prime to p. The proof is complete. □

If N is a normal subgroup of G, R is a p-subgroup of G and X is an R-invariant simple kN-module, then let $\hat{X}(R)$ be a unique extension of X to RN ([**NT**, Theorem 3.5.11]).

Proposition 4. *Let N be a normal subgroup of G. Let X be a G-invariant simple kN-module. Let S be an indecomposable kG-module such that S_N is a multiple of X. Let P be a vertex of S. Choose an indecomposable $k[PN]$-module U such that $U | S_{PN}$ and $S | U^G$. Then*

(i) *U is determined up to $N_G(PN)$-conjugacy.*
(ii) *There is a unique $k[PN/N]$-module W such that $U = \hat{X}(P) \otimes \mathrm{Inf}_{PN/N \to PN}(W)$. Here, if G/N is p-solvable and S is simple, then $\dim_k W$ is prime to p and P is G-conjugate to a vertex of $\hat{X}(P)$.*

Proof. The existence of U is clear, since S is PN-projective. Then (i) is known and easy to see [**Bu**, Theorem 9]. (In [**Bu**, Definition 3 and Remark, p. 335], PN is called a N-vertex of S and U a N-source of S.) Since U_N is a multiple of X, $\hat{X}(P) \otimes \mathrm{Hom}_N(\hat{X}(P), U) \simeq U$ as $k[PN]$-modules. (The map sending $v \otimes \varphi$ to $\varphi(v)$ is an isomorphism.) Thus it suffices to set $W = \mathrm{Hom}_N(\hat{X}(P), U)$. The uniqueness of W follows from [**HB1**, VII 9.12].

Assume G/N is p-solvable and S is simple. To show that $\dim_k W$ is prime to p, we choose a central extension of G

$$1 \longrightarrow Z \longrightarrow \hat{G} \xrightarrow{f} G \longrightarrow 1$$

with the following properties: $f^{-1}(N) = N_1 \times Z$, $N_1 \lhd \hat{G}$, X extends to $\hat{G}$ under the identification of N_1 with N via f, and Z is a (central) p'-group. Let $\hat{X}$ be an extension of X to $\hat{G}$. Put $\tilde{G} = \hat{G}/N$. There is a unique simple $k\tilde{G}$-module $\tilde{S}$ such that $\mathrm{Inf}_{G \to \hat{G}}(S) = \hat{X} \otimes \mathrm{Inf}_{\tilde{G} \to \hat{G}}(\tilde{S})$. Put $f^{-1}(P) = \hat{P} \times Z$. Then $\hat{P}$ is a vertex of $\mathrm{Inf}_{G\to\hat{G}}(S)$ by Proposition 3. Put $\tilde{P}=\hat{P}N/N$. Then $\tilde{P}$ is a vertex of $\tilde{S}$ by Proposition 3. Let $\tilde{W}$ be a $\tilde{P}$-source of $\tilde{S}$. Let λ be a one dimensional kZ-module (that is, a character of Z) lying under $\hat{X}$. Put $\tilde{Z}=ZN/N$. We regard λ as a character of $\tilde{Z}$ via the natural isomorphism $\tilde{Z}\simeq Z$. Let $L=g^{-1}(\tilde{P}\times\tilde{Z})$, where $g:\hat{G}\to\tilde{G}$ is the natural map. Then $L=\hat{P}NZ=f^{-1}(PN)$.

Since $\tilde{S}|\tilde{W}^{\tilde{G}}=(\tilde{W}^{\tilde{P}\times\tilde{Z}})^{\tilde{G}}$ and $\tilde{S}$ lies over the character λ^{-1} of $\tilde{Z}$, we obtain $\tilde{S}|(\tilde{W}\times\lambda^{-1})^{\tilde{G}}$. Thus

$$\mathrm{Inf}_{G\to\hat{G}}(S)|\hat{X}\otimes\big(\mathrm{Inf}_{\tilde{P}\times\tilde{Z}\to L}(\tilde{W}\times\lambda^{-1})\big)^{\hat{G}}=\big(\hat{X}_L\otimes\mathrm{Inf}_{\tilde{P}\times\tilde{Z}\to L}(\tilde{W}\times\lambda^{-1})\big)^{\hat{G}}.$$

On the other hand, $\tilde{W}|\tilde{S}_{\tilde{P}}$. So $\tilde{W}\times\lambda^{-1}|\tilde{S}_{\tilde{P}\times\tilde{Z}}$, since $\tilde{S}$ lies over λ^{-1}. Thus

$$\hat{X}_L\otimes\mathrm{Inf}_{\tilde{P}\times\tilde{Z}\to L}(\tilde{W}\times\lambda^{-1})|\big(\mathrm{Inf}_{G\to\hat{G}}(S)\big)_L.$$

Hence it follows from (i) that $\hat{X}_L\otimes\mathrm{Inf}_{\tilde{P}\times\tilde{Z}\to L}(\tilde{W}\times\lambda^{-1})=\mathrm{Inf}_{PN\to L}(U^x)$ for some $x\in N_G(PN)$. Considering dimensions we have $\dim_k X\dim_k\tilde{W}=\dim_k U=\dim_k\hat{X}(P)\dim_k W$. So $\dim_k W=\dim_k\tilde{W}$. Since $\tilde{G}$ is p-solvable, by Puig's theorem [**Th**, Theorem 5.30.5], $\tilde{W}$ is an endo-permutation module, so that $\dim_k\tilde{W}$ is prime to p by Lemma 6.4 of Dade [**Da**] ([**Th**, Corollary 5.28.11]). (This fact follows also from Corollary 3 of [**Wa**].) Thus $\dim_k W$ is prime to p.

Clearly $P=_G\mathrm{vx}(U)$. Since $U=\hat{X}(P)\otimes\mathrm{Inf}_{PN/N\to PN}(W)$, $\mathrm{vx}(U)\le_{PN}\mathrm{vx}(\hat{X}(P))$. Since $\dim_k W$ is prime to p, $1_{PN/N}|W^*\otimes W$ ([**Fe**, Lemma III 2.2]). Thus $\hat{X}(P)|U\otimes\mathrm{Inf}_{PN/N\to PN}(W^*)$. So $\mathrm{vx}(\hat{X}(P))\le_{PN}\mathrm{vx}(U)$. Thus $\mathrm{vx}(U)=_{PN}\mathrm{vx}(\hat{X}(P))$. It follows that $P=_G\mathrm{vx}(\hat{X}(P))$. The proof is complete. □

Corollary 5. *Let N be a normal subgroup of G. Let X be a simple kN-module. Let P be a vertex of a simple kG-module lying over X. Assume G/N is p-solvable. Then for some $g\in G$, X is P^g-invariant and P^g is a vertex of $\hat{X}(P^g)$.*

Proof. Let S be a simple kG-module lying over X. Let T be the inertial group of X in G. Let $\tilde{S}$ be the Clifford correspondent of S in T. Then for some $x\in G$, P^x is a vertex of $\tilde{S}$. Then for some $t\in T$, P^{xt} is a vertex of $\hat{X}(P^x)$ by Proposition 4. Then, since $P^{xt}\le P^xN$, we have $P^{xt}N=P^xN$. So it suffices to take $g=xt$. The proof is complete. □

2. A lemma.

In this section we prove a technical lemma. This is a temporary result, which will be refined in Corollary 13.

LEMMA 6. *Let N be a normal subgroup of G such that G/N is p-solvable. Assume that any central extension of G/N is of Barker type* (*cf.* [**Ba**, Theorem 1.1]). *Let Q be a p-subgroup of G. Let β be a block of $N_G(Q)N$. Let X be a G-invariant simple kN-module. Then*

$$\begin{aligned}&\sharp\{S \in \mathrm{IBr}(G|X); \mathrm{vx}(S) =_G Q, S \in_Q \beta\}\\ &\quad = \sharp\{U \in \mathrm{IBr}(\beta|X); \mathrm{vx}(U) =_{N_G(Q)N} Q\}.\end{aligned} \tag{6.1}$$

PROOF. We divide the proof into several parts.

(a) We may assume Q is a vertex of $\hat{X}(Q)$ and β covers $B(X)$.

We assume that for any $g \in G$, Q^g is not a vertex of $\hat{X}(Q^g)$. Then by Corollary 5, the left-hand side (LHS for short) of (6.1) equals 0. Also the right-hand side (RHS for short) of (6.1) equals 0. So we may assume Q^g is a vertex of $\hat{X}(Q^g)$ for some $g \in G$. Both sides remain the same if we replace Q by Q^g and β by β^g. So we may assume Q is a vertex of $\hat{X}(Q)$. If β does not cover $B(X)$, then both sides equal 0. So we may assume β covers $B(X)$.

(b) $N_G(QN) = N_G(Q)N$.

Clearly $N_G(QN) \geq N_G(Q)N$. Since X is G-invariant, $\hat{X}(Q)$ is $N_G(QN)$-invariant. Since Q is a vertex of $\hat{X}(Q)$, Frattini argument shows $N_G(QN) \leq N_G(Q)N$. Thus the equality holds.

(c) For $S \in \mathrm{IBr}(G|X)$, $\mathrm{vx}(S) =_G Q$ if and only if $\mathrm{vx}(S)N =_G QN$.

Indeed one direction is trivial. To show the other direction we may assume $\mathrm{vx}(S)N = QN$. Then by Proposition 4 and (a), $\mathrm{vx}(S) =_G \mathrm{vx}(\hat{X}(Q)) =_G Q$, as required.

(d) For $U \in \mathrm{IBr}(\beta|X)$, $\mathrm{vx}(U) =_{N_G(Q)N} Q$ if and only if $\mathrm{vx}(U)N =_{N_G(Q)N} QN$.

This is similar to (c).

Take a central extension

$$1 \longrightarrow Z \longrightarrow \hat{G} \xrightarrow{f} G \longrightarrow 1$$

with the following properties: $f^{-1}(N) = N_1 \times Z$, $N_1 \lhd \hat{G}$, X extends to $\hat{G}$ under the identification of N with N_1 via f, and Z is a p'-group. Let $\dot{X}$ be an extension of X to $\hat{G}$. Put $\tilde{G} = \hat{G}/N$ and $\tilde{Z} = ZN/N$. Let λ be a character of Z lying under $\dot{X}$. We regard λ as a character of $\tilde{Z}$ via the natural isomorphism $\tilde{Z} \simeq Z$. Let $f^{-1}(Q) = \hat{Q} \times Z$. Put $\tilde{Q} = \hat{Q}N/N$. For any $L \leq G$ and a kL-module Y, put $\hat{Y} = \mathrm{Inf}_{L \to f^{-1}(L)}(Y)$.

(e) There is a bijection of $\mathrm{IBr}(G|X)$ onto $\mathrm{IBr}(\tilde{G}|\lambda^{-1})$ sending S to $\tilde{S}$ by the rule $\hat{S} = \dot{X} \otimes \mathrm{Inf}_{\tilde{G}\to\hat{G}}(\tilde{S})$. Here $\mathrm{vx}(S) =_G Q$ if and only if $\mathrm{vx}(\tilde{S}) =_{\tilde{G}} \tilde{Q}$.

The first assertion is well-known. The second is proved, since the following conditions are equivalent: (1) $\mathrm{vx}(S) =_G Q$; (2) $\mathrm{vx}(S)N =_G QN$ (by (c)); (3) $\mathrm{vx}(\hat{S})NZ =_{\hat{G}} \hat{Q}NZ$ (by Proposition 3); (4) $\mathrm{vx}(\tilde{S})\tilde{Z} =_{\tilde{G}} \tilde{Q}\tilde{Z}$ (by Proposition 3); (5) $\mathrm{vx}(\tilde{S}) =_{\tilde{G}} \tilde{Q}$.

Let $g : \hat{G} \to \tilde{G}$ be the natural map.

(f) $f^{-1}(N_G(QN)) = N_{\hat{G}}(\hat{Q}N) = g^{-1}(N_{\tilde{G}}(\tilde{Q}))$.

To show the first equality, we note $f^{-1}(N_G(QN)) = N_{\hat{G}}(\hat{Q}NZ)$. The containment $N_{\hat{G}}(\hat{Q}N) \le N_{\hat{G}}(\hat{Q}NZ)$ is clear. Let $\hat{x} \in N_{\hat{G}}(\hat{Q}NZ)$. Then, since $\hat{Q}N$ is a normal subgroup of $\hat{Q}NZ$ of p'-index, we get $\hat{Q}^{\hat{x}} \le \hat{Q}N$. This shows $N_{\hat{G}}(\hat{Q}NZ) \le N_{\hat{G}}(\hat{Q}N)$ and the equality holds. The second equality is clear.

Hereafter, we put $H = N_G(QN)$, $\hat{H} = N_{\hat{G}}(\hat{Q}N)$ and $\tilde{H} = N_{\tilde{G}}(\tilde{Q})$. Let $\hat{\beta}$ be the inflation of β to $\hat{H}$. We see $\hat{\beta}$ covers $B(X)$ by (a). Let $\{\tilde{\beta}_j\}$ be the blocks of $\tilde{H}$ which are $\dot{X}_{\hat{H}}$-dominated by $\hat{\beta}$. (See [**Mu**] for "$\dot{X}_{\hat{H}}$-domination".)

(g) For each $j, \tilde{\beta}_j$ covers λ^{-1}.

For any $k\tilde{H}$-module $\tilde{Y}$ in $\tilde{\beta}_j$, $\dot{X}_{\hat{H}} \otimes \mathrm{Inf}_{\tilde{H}\to\hat{H}}(\tilde{Y})$ lies in $\hat{\beta}$. Since $\hat{\beta}$ covers $1_Z, (\dot{X}_{\hat{H}} \otimes \mathrm{Inf}_{\tilde{H}\to\hat{H}}(\tilde{Y}))_Z$ is a multiple of 1_Z, and the result follows.

(h) There is a bijection of $\mathrm{IBr}(\beta|X)$ onto $\bigcup_j \mathrm{IBr}(\tilde{\beta}_j)$ sending U to $\tilde{U}$ by the rule: $\hat{U} = \dot{X}_{\hat{H}} \otimes \mathrm{Inf}_{\tilde{H}\to\hat{H}}(\tilde{U})$. Here $\mathrm{vx}(U) =_H Q$ if and only if $\mathrm{vx}(\tilde{U}) = \tilde{Q}$.

Given U in $\mathrm{IBr}(\beta|X)$, there is a unique $k\tilde{H}$-module $\tilde{U}$ with $\hat{U} = \dot{X}_{\hat{H}} \otimes \mathrm{Inf}_{\tilde{H}\to\hat{H}}(\tilde{U})$. Then, since $\hat{U}$ lies in $\hat{\beta}$, $\tilde{U}$ lies in $\tilde{\beta}_j$ for some j. Conversely, given $\tilde{U}$ in $\bigcup_j \mathrm{IBr}(\tilde{\beta}_j)$, $\dot{X}_{\hat{H}} \otimes \mathrm{Inf}_{\tilde{H}\to\hat{H}}(\tilde{U})$ is simple, lies in $\hat{\beta}$ and is trivial on Z by (g). Thus $\dot{X}_{\hat{H}} \otimes \mathrm{Inf}_{\tilde{H}\to\hat{H}}(\tilde{U}) = \hat{U}$ for a simple kH-module U. Since $\hat{U}$ lies in $\hat{\beta}$ and Z is a p'-group, U lies in β. Thus $U \in \mathrm{IBr}(H|X)$. The first assertion follows. The second assertion is proved as in (e) (by using (d)).

(i) In the correspondence in (e), $\mathrm{vx}(S) =_G Q$ and $S \in_Q \beta$ if and only if $\mathrm{vx}(\tilde{S}) =_{\tilde{G}} \tilde{Q}$ and $\tilde{S} \in_{\tilde{Q}} \tilde{\beta}_j$ for some j.

We may assume either $\mathrm{vx}(S) =_G Q$ or $\mathrm{vx}(\tilde{S}) =_{\tilde{G}} \tilde{Q}$. Then both hold by (e). Let $\tilde{V}$ be the Green correspondent of $\tilde{S}$ with respect to $(\tilde{G}, \tilde{Q}, \tilde{H})$. Since $\tilde{V}|\tilde{S}_{\tilde{H}}$, $\dot{X}_{\hat{H}} \otimes \mathrm{Inf}_{\tilde{H}\to\hat{H}}(\tilde{V})|\hat{S}_{\hat{H}}$. Therefore $\dot{X}_{\hat{H}} \otimes \mathrm{Inf}_{\tilde{H}\to\hat{H}}(\tilde{V}) = \hat{V}$ for some kH-module V. By [**HB1**, VII 9.12], V is indecomposable. Since $\mathrm{vx}(\tilde{V}) = \tilde{Q}$, we obtain $\mathrm{vx}(V)N =_H QN$ as in the proof of (e). So $\mathrm{vx}(V)N = QN$. Further we have $V|S_H$. On the other hand, we have $\tilde{S}|\tilde{V}^{\tilde{G}}$ and

$$\hat{V}^{\hat{G}} = \left(\dot{X}_{\hat{H}} \otimes \mathrm{Inf}_{\tilde{H}\to\hat{H}}(\tilde{V})\right)^{\hat{G}} = \dot{X} \otimes \mathrm{Inf}_{\tilde{G}\to\hat{G}}(\tilde{V}^{\tilde{G}}).$$

Thus $\hat{S}|\hat{V}^{\hat{G}}$. So $S|V^G$. Therefore $\mathrm{vx}(V) =_G \mathrm{vx}(S) =_G Q$. Put $\mathrm{vx}(V) = Q^g$ for $g \in G$. Then $Q^g N = \mathrm{vx}(V)N = QN$. So $g \in H$. Hence Q is a vertex of V. Since $V|S_H$, V is the Green correspondent of S with respect to (G, Q, H). Now the following conditions are equivalent: (1) $S \in_Q \beta$; (2) V lies in β; (3) $\hat{V}$ lies in $\hat{\beta}$; (4) $\tilde{V}$ lies in $\tilde{\beta}_j$ for some j; (5) $\tilde{S} \in_{\tilde{Q}} \tilde{\beta}_j$ for some j. Thus (i) follows.

(j) If $\tilde{S} \in \mathrm{IBr}(\tilde{G})$, $\mathrm{vx}(\tilde{S}) =_{\tilde{G}} \tilde{Q}$ and $\tilde{S} \in_{\tilde{Q}} \tilde{\beta}_j$ for some j, then $\tilde{S} \in \mathrm{IBr}(\tilde{G}|\lambda^{-1})$. This follows from (g).

Now by (e), (i) and (j), the LHS of (6.1) equals

$$\sum_j \sharp\{\tilde{S} \in \mathrm{IBr}(\tilde{G}); \mathrm{vx}(\tilde{S}) = \tilde{Q}, \tilde{S} \in_{\tilde{Q}} \tilde{\beta}_j\} = \sum_j l_{\tilde{G}}(\tilde{\beta}_j, \tilde{Q}).$$

On the other hand, by (h), the RHS of (6.1) equals $\sum_j l_{N_{\tilde{G}}(\tilde{Q})}(\tilde{\beta}_j, \tilde{Q})$. Since $\tilde{G}$ is of Barker type by assumption, $l_{\tilde{G}}(\tilde{\beta}_j, \tilde{Q}) = l_{N_{\tilde{G}}(\tilde{Q})}(\tilde{\beta}_j, \tilde{Q})$ for each j. Thus the equality (6.1) holds. The proof is complete. $\square$

3. Barker's theorem.

In this section we prove Barker's theorem [**Ba**, Theorem 1.1] by using a result of Isaacs and Navarro [**IN**]. For a while we follow the notation of Isaacs-Navarro (although we use simple modules instead of irreducible Brauer characters). For a normal subgroup K of G and a simple kK-module X, let $n(G, X)$ be the number of isomorphism classes of simple kG-modules lying over X. For a p-subgroup Q of G, let $n(G, X, Q)$ be the number of isomorphism classes of simple kG-modules lying over X with vertex Q.

The following proposition is a special case of Proposition 6.4 of [**IN**]. Our proof is a variant of the proof of Proposition 6.5 of [**IN**].

Proposition 7 (Isaacs-Navarro). *Let Q be a p-subgroup of a p-solvable group G. Let K be a normal p'-subgroup of G. Assume that $G = N_G(Q)K$. Let X be a G-invariant simple kK-module. Let $Y \in \mathrm{IBr}(C_K(Q))$ be the Glauberman correspondent of X with respect to the action of Q on K* ([**Is**, Theorem 13.1]). *Assume that any central extension of any subgroup of G/K is of Barker type* (cf. [**Ba**, Theorem 1.1]). *Then $n(G, X, Q) = n(N_G(Q), Y, Q)$.*

Proof. We argue by induction on $|G : Q|$. Put

$$\mathcal{P} = \{P; P \text{ is a } p\text{-subgroup such that } Q \leq P \leq N_G(Q)\}.$$

Let $\mathcal{P}_0$ be a set of representatives of $N_G(Q)$-conjugacy classes of $\mathcal{P}$. Let S be a simple kG-module lying over X. We claim that S has a unique vertex in $\mathcal{P}_0$. Indeed, let B be the block of G containing S. Let b be a unique block of QK covering the block of K containing X. Since X is QK-invariant, Q is a defect group of b. Since B covers b, there is a defect group D of B such that $D \cap QK = Q$ by Knörr's theorem [**NT**, Theorem 5.5.16 (ii)]. If we choose $\mathrm{vx}(S)$ so that $\mathrm{vx}(S) \leq D$, then $\mathrm{vx}(S) \cap QK \leq D \cap QK = Q$. On the other hand, since S lies over $\hat{X}$, $\mathrm{vx}(\hat{X}) \leq_G \mathrm{vx}(S) \cap QK$, where $\hat{X}$ is the extension of X to QK. Since $\hat{X}$ has p'-degree, $\mathrm{vx}(\hat{X}) =_{QK} Q$. Thus $\mathrm{vx}(S) \cap QK = Q$, and $\mathrm{vx}(S) \in \mathcal{P}$.

Next we show: P, $P^g \in \mathcal{P}$, $g \in G$ implies $g \in N_G(Q)$. Indeed, since $P \geq Q$, $QK \geq P \cap QK \geq Q$. So $P \cap QK = Q$. Likewise, $P^g \cap QK = Q$. Hence $Q^g = P^g \cap QK = Q$, so that $g \in N_G(Q)$. Thus the claim is proved.

The same thing holds for any $U \in \mathrm{IBr}(N_G(Q))$ lying over Y.

Since $n(G, X) = n(N_G(Q), Y)$ by Theorem 4.3 of [**IN**], it follows that

$$\sum_{P \in \mathcal{P}_0} n(G, X, P) = \sum_{P \in \mathcal{P}_0} n(N_G(Q), Y, P).$$

If Q is a Sylow p-subgroup of $N_G(Q)$, then $\mathcal{P}_0 = \{Q\}$. So $n(G, X, Q) = n(N_G(Q), Y, Q)$. Assume that Q is not a Sylow p-subgroup of $N_G(Q)$. We show

$$(*) \quad \text{For any } P \in \mathcal{P}_0,\ P \neq Q,\ n(G, X, P) = n(N_G(Q), Y, P).$$

From $(*)$ it will follow that $n(G, X, Q) = n(N_G(Q), Y, Q)$.

Let $P \in \mathcal{P}_0$, $P \neq Q$. Let $Z \in \mathrm{IBr}(C_K(P))$ be the Glauberman correspondent of Y with respect to the action of P on $C_K(Q)$. Put $L = C_K(Q)$. Note that Y is $N_G(Q)$-invariant and that $N_G(P)L \leq N_G(Q)$, since $P \cap QK = Q$ as above. To prove $(*)$, it suffices to show the following equalities:

(1) $n(N_G(P)K, X, P) = n(N_G(P), Z, P)$.
(2) $n(N_G(P)L, Y, P) = n(N_G(P), Z, P)$.
(3) $n(G, X, P) = n(N_G(P)K, X, P)$.
(4) $n(N_G(Q), Y, P) = n(N_G(P)L, Y, P)$.

(1) Since $N_G(P)K/K \leq G/K$ and $|N_G(P)K : P| < |G : Q|$, the equality holds by induction. (Note that Z is the Glauberman correspondent of X with respect to the action of P on K.)

(2) Since $N_G(P)L/L \simeq N_G(P)/N_G(P) \cap L = N_G(P)/C_K(P) \simeq N_G(P)K/K \leq G/K$ and $|N_G(P)L : P| < |G : Q|$, the equality holds by induction.

(3) By our assumption, we can use Lemma 6 to obtain that

$$\begin{aligned}&\sharp\{S \in \mathrm{IBr}(G|X); \mathrm{vx}(S) =_G P, S \in_P \beta\}\\&\quad = \sharp\{U \in \mathrm{IBr}(\beta|X); \mathrm{vx}(U) =_{N_G(P)K} P\}\end{aligned}$$

for all blocks β of $N_G(P)K$. Summing this equality for all β, we obtain (3).

(4) Since $N_G(Q)/L = N_G(Q)/N_G(Q) \cap K \simeq N_G(Q)K/K = G/K$, the proof is similar to that of (3).

The proof is complete. □

In the following, by abuse of notation, the block idempotent of kG corresponding to a block of G will be denoted by the same letter when necessary. For the notation and terminology, we refer the reader to [**AB**], [**Th**]. In particular, for each p-subgroup Q of G, let $\mathrm{Br}_Q : (kG)^Q \to kC_G(Q)$ be the Brauer homomorphism, where

$$(kG)^Q = \{a \in kG; ax = xa \text{ for all } x \in Q\}.$$

Until Proposition 10, we use the following notation. Let N be a normal subgroup of G and let e be a block of N. Let B be a block of G covering e. Let T be the inertial group of e in G. Let b be the Fong-Reynolds correspondent of B in T over e ([**NT**, Theorem 5.5.10]).

Part of the following proposition are similar to part of Theorem 1 of Puig [**Pu**].

PROPOSITION 8. *The following holds.*

(1) *For any b-subpair (Q, b_Q), $b_Q^{C_G(Q)}$ is defined, and $(Q, b_Q^{C_G(Q)})$ is a B-subpair.*
(2) *Two b-subpairs (Q, b_Q) and (R, b_R) are T-conjugate if and only if $(Q, b_Q^{C_G(Q)})$ and $(R, b_R^{C_G(R)})$ are G-conjugate.*
(3) *Any B-subpair is G-conjugate to $(Q, b_Q^{C_G(Q)})$ for some b-subpair (Q, b_Q).*
(4)[2] *For any b-subpair (Q, b_Q), $N_G(Q, b_Q^{C_G(Q)}) = N_T(Q, b_Q)C_G(Q)$. In particular, $N_G(Q, b_Q^{C_G(Q)})/C_G(Q) \simeq N_T(Q, b_Q)/C_T(Q)$.*
(5) *Let Q be a p-subgroup of T. For any $b' \in BL(N_T(Q), b)$, $b'^{N_G(Q)}$ is defined and $b'^{N_G(Q)} \in BL(N_G(Q), B)$.*
(6) *Let Q be a p-subgroup of G. For any $B' \in BL(N_G(Q), B)$, there exist $R \leq T$ and $b' \in BL(N_T(R), b)$ such that $R = Q^g$ and that $b'^{N_G(R)} = B'^g$ for some $g \in G$.*

PROOF. For each b-subpair (Q, b_Q), let e_Q be a block of $C_N(Q)$ which is

[2]This is not necessary in the present paper. It is included here for future use.

covered by b_Q. It holds that $\mathrm{Br}_Q(e)e_Q = e_Q$. Indeed, since b covers e, $eb = b$. Since $\mathrm{Br}_Q(b)b_Q = b_Q$, we get $\mathrm{Br}_Q(e)b_Q = b_Q$. Thus there is a block e'_Q of $C_N(Q)$ which is covered by b_Q and $\mathrm{Br}_Q(e)e'_Q = e'_Q$. Then, since e'_Q is $C_T(Q)$-conjugate to e_Q, we get $\mathrm{Br}_Q(e)e_Q = e_Q$.

(1) and (4). Since $\mathrm{Br}_Q(e)e_Q = e_Q$, the inertial group of e_Q in $N_G(Q)$ is contained in T. In particular, the inertial group of e_Q in $C_G(Q)$ is contained in $C_T(Q)$. Therefore, by the Fong-Reynolds theorem, $b_Q^{C_G(Q)}$ is defined. Put $B_Q = b_Q^{C_G(Q)}$. We have $Bb = b$ by [**NT**, 5.5.11]. So $B_Q \mathrm{Br}_Q(B)\mathrm{Br}_Q(b)b_Q = B_Q \mathrm{Br}_Q(b)b_Q = B_Q b_Q$. Here $B_Q b_Q \neq 0$ by [**NT**, 5.3.9]. Hence $B_Q \mathrm{Br}_Q(B) \neq 0$, and (Q, B_Q) is a B-subpair. Thus (1) is proved.

To prove (4), let $t \in N_G(Q, B_Q)$. Let e_Q be as above. Then B_Q covers e_Q and e_Q^t, so $e_Q^t = e_Q^x$ for some $x \in C_G(Q)$. Then $tx^{-1} \in T$ (as above) and if T_1 is the inertial group of e_Q in $N_T(Q)$, $tx^{-1} \in T_1$. Thus $N_G(Q, B_Q) \leq (N_G(Q, B_Q) \cap T_1)C_G(Q)$. By the Fong-Reynolds theorem, we get $N_G(Q, B_Q) \cap T_1 \leq N_T(Q, b_Q)$. Hence $N_G(Q, B_Q) \leq N_T(Q, b_Q)C_G(Q)$. Since the reverse containment is clear, the equality holds.

(2) "only if" part is clear. Assume that $(Q, b_Q^{C_G(Q)})^x = (R, b_R^{C_G(R)})$ for $x \in G$. If e_Q and e_R are as above, then $b_R^{C_G(R)}$ covers both e_Q^x and e_R. So we have $e_Q^x = e_R^c$ for some $c \in C_G(R)$. Put $y = xc^{-1}$. Then $(Q, b_Q^{C_G(Q)})^y = (R, b_R^{C_G(R)})$ and $e_Q^y = e_R$. Then, since $\mathrm{Br}_Q(e)e_Q = e_Q$, we have $\mathrm{Br}_R(e^y)e_Q^y = e_Q^y$; that is, $\mathrm{Br}_R(e^y)e_R = e_R$. Since $\mathrm{Br}_R(e)e_R = e_R$, we get $e = e^y$ and $y \in T$. Then by the Fong-Reynolds theorem, $(Q, b_Q)^y = (R, b_R)$. Thus (2) holds.

(3) Let (Q, B_Q) be a B-subpair. Write $\sum_{x \in T\backslash G} e^x = \sum_i e_i$, where e_i are blocks of QN. Since $\sum_i e_i B = B$, we have

$$
\begin{aligned}
B_Q = \mathrm{Br}_Q(B)B_Q &= \mathrm{Br}_Q\left(\sum_i e_i\right)\mathrm{Br}_Q(B)B_Q \\
&= \mathrm{Br}_Q\left(\sum_i e_i\right)B_Q = \sum_i \mathrm{Br}_Q(e_i)B_Q.
\end{aligned}
$$

Thus, for some i, $\mathrm{Br}_Q(e_i)B_Q \neq 0$ and then a defect group of e_i contains Q. Then the block of N covered by this e_i is QN-invariant. Then we have $e_i = e^x$ for some $x \in G$. Then $Q^{x^{-1}} \leq T$. Put $(Q, B_Q)^{x^{-1}} = (R, B_R)$. Then $\mathrm{Br}_R(e)B_R \neq 0$. Thus there is a block e'_R of $C_N(R)$ which is covered by B_R and $\mathrm{Br}_R(e)e'_R = e'_R$. As in the proof of (1), we see that the inertial group of e'_R in $C_G(R)$ is contained in $C_T(R)$. Let b_R be the Fong-Reynolds correspondent of B_R over e'_R in $C_T(R)$. Then $B_R = b_R^{C_G(R)}$.

It remains to show that (R, b_R) is a b-subpair. Let (R, b_R) be a b_1-subpair for a block b_1 of T. Then

$$0 \neq b_R e'_R = b_R \operatorname{Br}_R(b_1) \operatorname{Br}_R(e) e'_R = e_R \operatorname{Br}_R(b_1 e) e'_R.$$

So $b_1 e \neq 0$, and b_1 covers e. Let B_1 be the Fong-Reynolds correspondent of b_1 over e in G. Applying (1) with B_1 in place of B, we see (R, B_R) is a B_1-subpair. So $B_1 = B$. Thus $b_1 = b$ by the Fong-Reynolds theorem, and (R, b_R) is a b-subpair. Thus (3) holds.

(5) Let b_Q be a block of $C_T(Q)$ covered by b'. Then (Q, b_Q) is a b-subpair. Let e_Q be as above. Then the inertial group of e_Q in $N_G(Q)$ is contained in $N_T(Q)$ by the proof of (1). Since b' covers e_Q, $b'^{N_G(Q)}$ is defined by the Fong-Reynolds theorem. Then $(b'^{N_G(Q)})^G = (b'^T)^G = b^G = B$.

(6) Let B_Q be a block of $C_G(Q)$ covered by B'. Then (Q, B_Q) is a B-subpair. So, by (3), there is a b-subpair (R, b_R) such that $(Q, B_Q)^g = (R, b_R^{C_G(R)})$ for some $g \in G$. Then (6) holds with $b' = b_R^{N_T(R)}$. Indeed, since B'^g covers $b_R^{C_G(R)}$, $B'^g = (b_R^{C_G(R)})^{N_G(R)} = b_R^{N_G(R)} = b'^{N_G(R)}$.

The proof is complete. $\square$

Proposition 9. *Let Q be a subgroup of a defect group of B. Let $\{Q^{x_i}\}$ be a set of representatives of T-conjugacy classes of $\{Q^g; Q^g \leq T, g \in G\}$. Put $BL(N_T(Q^{x_i}), b) = \{b_{ij}\}$. Put $\beta_{ij} = \{{b_{ij}}^{N_G(Q^{x_i})}\}^{x_i^{-1}}$. Then $BL(N_G(Q), B) = \{\beta_{ij}\}$, where no duplication occurs.*

Proof. By (5) of Proposition 8, $b_{ij}^{N_G(Q^{x_i})}$ is defined. Then clearly $\beta_{ij} \in BL(N_G(Q), B)$. Conversely, let $\beta \in BL(N_G(Q), B)$. By (6) of Proposition 8 there exist $R \leq T$ and $b' \in BL(N_T(R), b)$ such that $R = Q^g$ and that $b'^{N_G(R)} = \beta^g$ for some $g \in G$. Then $Q^{gt} = Q^{x_i}$ for some $t \in T$ and some i. Then $b'^t \in BL(N_T(Q^{x_i}), b)$. So $b'^t = b_{ij}$ for some j. Then $b_{ij}^{N_G(Q^{x_i})}$ is defined by (5) of Proposition 8. Since $\beta^g = b'^{N_G(Q^g)}$, we have $\beta^{gt} = b_{ij}^{N_G(Q^{x_i})}$. Then $(b_{ij}^{N_G(Q^{x_i})})^{x_i^{-1}} = \beta^{gtx_i^{-1}} = \beta$, since $gtx_i^{-1} \in N_G(Q)$. Thus $\beta = \beta_{ij}$. Hence $BL(N_G(Q), B) = \{\beta_{ij}\}$.

Assume $\beta_{ij} = \beta_{lm}$. Let $b_{Q^{x_i}}$ (resp. $b_{Q^{x_l}}$) be a block of $C_T(Q^{x_i})$ (resp. $C_T(Q^{x_l})$) covered by b_{ij} (resp. b_{lm}). Then as before, b_{ij} is the Fong-Reynolds correspondent of $b_{ij}^{N_G(Q^{x_i})}$ over $e_{Q^{x_i}}$. A similar thing holds for $b_{Q^{x_l}}$. So $b_{lm}^{N_G(Q^{x_l})}$ covers $e_{Q^{x_l}}$. By assumption, $(b_{ij}^{N_G(Q^{x_i})})^{x_i^{-1}x_l} = b_{lm}^{N_G(Q^{x_l})}$. So $b_{lm}^{N_G(Q^{x_l})}$ covers $(e_{Q^{x_i}})^{x_i^{-1}x_l}$. Thus $(e_{Q^{x_i}})^{x_i^{-1}x_l} = (e_{Q^{x_l}})^n$ for some $n \in N_G(Q^{x_l})$. Put $y = x_i^{-1}x_l n^{-1}$. Then $(e_{Q^{x_i}})^y = e_{Q^{x_l}}$ and $(Q^{x_i})^y = Q^{x_l}$. Then as in the proof of (2) of Proposition 8, we have $y \in T$. Then $i = l$. By the Fong-Reynolds theorem, $b_{ij} = b_{im}$. So $j = m$. The proof is complete. $\square$

Proposition 10. *Let $\beta \in BL(N_G(Q), B)$. Then $\beta = \beta_{ij}$ for a unique (i, j) by Proposition 9.*

(i) *For $S \in \mathrm{IBr}(B)$, let $\tilde{S}$ be the Fong-Reynolds correspondent of S in b. Then the following are equivalent.*

(a) *Q is a vertex of S and $S \in_Q \beta$.*

(b) *Q^{x_i} is a vertex of $\tilde{S}$ and $\tilde{S} \in_{Q^{x_i}} b_{ij}$.*

(ii) $\sharp\{X \in \mathrm{IBr}(\beta); \mathrm{vx}(X) = Q\} = \sharp\{Y \in \mathrm{IBr}(b_{ij}); \mathrm{vx}(Y) = Q^{x_i}\}$.

Proof. (i) We may assume Q^{x_l} is a vertex of $\tilde{S}$ for some l. Let V be the Green correspondent of $\tilde{S}$ with respect to $(T, Q^{x_l}, N_T(Q^{x_l}))$. Then V lies in b_{lm} for some m by Nagao-Green theorem [**NT**, Theorem 5.3.12]. Now $V^{N_G(Q^{x_l})}$ is indecomposable, lies in $b_{lm}^{N_G(Q^{x_l})}$ and Q^{x_l} is a vertex of $V^{N_G(Q^{x_l})}$ by the Fong-Reynolds theorem. By Mackey decomposition, $V^{N_G(Q^{x_l})}$ is a direct summand of $S_{N_G(Q^{x_l})}$. Thus $V^{N_G(Q^{x_l})}$ is the Green correspondent of S with respect to $(G, Q^{x_l}, N_G(Q^{x_l}))$. Therefore $(V^{N_G(Q^{x_l})})^{x_l^{-1}}$ is the Green correspondent of S with respect to $(G, Q, N_G(Q))$ and it lies in β_{lm}. Thus (a) holds if and only if $(l, m) = (i, j)$ if and only if (b) holds.

(ii) We have $\beta^{x_i} = b_{ij}^{N_G(Q^{x_i})}$. So conjugation by x_i defines a bijection of $\{X \in \mathrm{IBr}(\beta); \mathrm{vx}(X) = Q\}$ and $\{Z \in \mathrm{IBr}(b_{ij}^{N_G(Q^{x_i})}); \mathrm{vx}(Z) = Q^{x_i}\}$. Further, $\{Z \in \mathrm{IBr}(b_{ij}^{N_G(Q^{x_i})}); \mathrm{vx}(Z) = Q^{x_i}\}$ corresponds bijectively to $\{Y \in \mathrm{IBr}(b_{ij}); \mathrm{vx}(Y) = Q^{x_i}\}$ by Fong-Reynolds theorem. Thus (ii) holds. The proof is complete. □

Lemma 11. *Let G be a p-solvable group and let Q be a p-subgroup of G. Let β be a block of $N_G(Q)$. Put $B = \beta^G$ and $K = \mathrm{O}_{p'}(G)$. Let X be a G-invariant simple kK-module. Let $Y \in \mathrm{IBr}(C_K(Q))$ be the Glauberman correspondent of X with respect to the action of Q on K. Assume B covers $B(X)$. Then*

(i) *β is a unique block of $N_G(Q)$ covering $B(Y)$.*

(ii) *$BL(N_G(Q), B) = \{\beta\}$.*

Proof. (i) It is well known that $\mathrm{Br}_Q(B(X)) = B(Y)$. By Fong's theorem $B = B(X)$. So we have $0 \neq \mathrm{Br}_Q(B)\beta = \mathrm{Br}_Q(B(X))\beta = B(Y)\beta$. So β covers $B(Y)$. We see Y is $N_G(Q)$-invariant. Since G is p-solvable, $C_K(Q) = \mathrm{O}_{p'}(N_G(Q))$ ([**HB2**, X 1.6]). Therefore, by Fong's theorem, (i) follows.

(ii) Let $\gamma \in BL(N_G(Q), B)$. For the same reason as above, γ covers $B(Y)$. So $\gamma = \beta$ by (i). □

Theorem 12 (Barker [**Ba**, Theorem 1.1]). *Any p-solvable group is of Barker type.*

Proof. Let G be a p-solvable group. Let Q be a p-subgroup of G and let β be a block of $N_G(Q)$. We argue by induction firstly on $|G/Z(G)|$ and secondly on $|G|$.

Put $B = \beta^G$ and $K = \mathrm{O}_{p'}(G)$. Let e be a block of K covered by B. Let X be a unique simple kK-module in e.

Step 1: We may assume $G = T_G(X)$.

Put $T = T_G(X)$. Assume $G \neq T$. By applying Proposition 9 with $N = K$, we have $\beta = \beta_{ij}$ for a unique (i,j). By Proposition 10 (i) and Nagao-Green theorem [**NT**], $l_G(\beta, Q) = l_T(b_{ij}, Q^{x_i})$. By Proposition 10 (ii), $l_{N_G(Q)}(\beta, Q) = l_{N_T(Q^{x_i})}(b_{ij}, Q^{x_i})$. Since $|G : Z(G)| > |T : Z(T)|$, $l_T(b_{ij}, Q^{x_i}) = l_{N_T(Q^{x_i})}(b_{ij}, Q^{x_i})$ by induction. Therefore, $l_G(\beta, Q) = l_{N_G(Q)}(\beta, Q)$, as required.

Step 2: $BL(N_G(Q), B) = \{\beta\}$.

This follows from Step 1 and Lemma 11.

Step 3: We may assume $Q \geq \mathrm{O}_p(G)$.

We assume $Q \not\geq \mathrm{O}_p(G)$ and show that $l_G(\beta, Q) = l_{N_G(Q)}(\beta, Q) = 0$. Assume $l_G(\beta, Q) \neq 0$. If S is a simple kG-module with vertex Q, then $Q \geq \mathrm{O}_p(G)$, a contradiction. Hence $l_G(\beta, Q) = 0$. On the other hand, assume $l_{N_G(Q)}(\beta, Q) \neq 0$. If U is a simple $kN_G(Q)$-module with vertex Q, then $\mathrm{O}_p(N_G(Q)) = Q$. So $Q = \mathrm{O}_p(N_G(Q)) \geq N_G(Q) \cap Q\,\mathrm{O}_p(G) \geq Q$, so that $N_G(Q) \cap Q\,\mathrm{O}_p(G) = Q$. Hence $Q\,\mathrm{O}_p(G) = Q$ and $Q \geq \mathrm{O}_p(G)$, a contradiction. Hence $l_{N_G(Q)}(\beta, Q) = 0$.

Step 4: We may assume $\mathrm{O}_p(G) = 1$.

Assume $\mathrm{O}_p(G) \neq 1$. Put $\bar{G} = G/\mathrm{O}_p(G)$. Then $|\bar{G} : Z(\bar{G})| \leq |G : Z(G)\,\mathrm{O}_p(G)| \leq |G : Z(G)|$ and $|\bar{G}| < |G|$. So $\bar{G}$ is of Barker type by induction. Let $\{\bar{\beta}_j\}$ be the set of blocks of $\overline{N_G(Q)} = N_{\bar{G}}(\bar{Q})$ dominated by β. For a simple kG-module S, let $\bar{S}$ be a simple $k\bar{G}$-module corresponding to S. Then by Proposition 3 S has vertex Q if and only if $\bar{S}$ has vertex $\bar{Q}$. (A similar thing holds for a simple $kN_G(Q)$-module.) Further $S \in_Q \beta$ if and only if $\bar{S} \in_{\bar{Q}} \bar{\beta}_j$ for some j. Therefore $l_G(\beta, Q) = \sum_j l_{\bar{G}}(\bar{\beta}_j, \bar{Q}) = \sum_j l_{N_{\bar{G}}(\bar{Q})}(\bar{\beta}_j, \bar{Q}) = l_{N_G(Q)}(\beta, Q)$, as required.

Step 5: We may assume $G = N_G(Q)K$ and $Z(G) < K$.

Since $\mathrm{O}_p(G) = 1$, $Z(G) \leq K$. If $Z(G) = K$, then $\mathrm{O}_{p'p}(G) = Z(G)$. So $G = Z(G)$ and we are done. So we may assume $Z(G) < K$. Then, by induction any central extension of G/K is of Barker type. Let $\beta_1 = \beta^{N_G(Q)K}$. By Lemma 6,

$$\begin{aligned}&\sharp\{S \in \mathrm{IBr}(G|X); \mathrm{vx}(S) =_G Q, S \in_Q \beta_1\}\\ &\quad = \sharp\{U \in \mathrm{IBr}(\beta_1|X); \mathrm{vx}(U) =_{N_G(Q)K} Q\}. \qquad (12.1)\end{aligned}$$

For $S \in \mathrm{IBr}(G)$ with a vertex Q, $S \in_Q \beta_1$ if and only if $S \in B$ if and only if $S \in_Q \beta$ by Nagao-Green theorem [**NT**] and Step 2. And if these conditions hold, then S lies over X. Thus the LHS of (12.1) equals $l_G(\beta, Q)$. On the other hand, since β_1

covers e, the RHS of (12.1) equals $l_{N_G(Q)K}(\beta, Q)$ by Nagao-Green theorem [**NT**] and Step 2. If $N_G(Q)K < G$, then by induction $l_{N_G(Q)K}(\beta, Q) = l_{N_G(Q)}(\beta, Q)$. Thus $l_G(\beta, Q) = l_{N_G(Q)}(\beta, Q)$, as required.

Step 6: Conclusion.

By Step 1 and Fong's theorem, B is a unique block of G covering e. So $l_G(\beta, Q) = n(G, X, Q)$ by Step 2 and Nagao-Green theorem [**NT**]. Let Y be as in Lemma 11. By Lemma 11 β is a unique block of $N_G(Q)$ covering $B(Y)$. So $l_{N_G(Q)}(\beta, Q) = n(N_G(Q), Y, Q)$. Let $H/K \leq G/K$ and $\hat{H}$ be a central extension of H/K. Then $|\hat{H} : Z(\hat{H})| \leq |H : K| \leq |G : K| < |G : Z(G)|$. Thus by induction $\hat{H}$ is of Barker type. Therefore the assumption of Proposition 7 holds by Step 5. Hence $n(G, X, Q) = n(N_G(Q), Y, Q)$. So $l_G(\beta, Q) = l_{N_G(Q)}(\beta, Q)$. The proof is complete. $\square$

Now we can refine Lemma 6. (Similarly we could refine Proposition 7.)

Corollary 13. *Assume that G/N is p-solvable. Let Q be a p-subgroup of G. Let β be a block of $N_G(Q)N$. Let X be a G-invariant simple kN-module. Then*

$$\begin{aligned}&\sharp\{S \in \mathrm{IBr}(G|X); \mathrm{vx}(S) =_G Q, S \in_Q \beta\}\\ &\quad = \sharp\{U \in \mathrm{IBr}(\beta|X); \mathrm{vx}(U) =_{N_G(Q)N} Q\}.\end{aligned}$$

Proof. Use Lemma 6 and Theorem 12. $\square$

Corollary 14. *Assume that G/N is p-solvable. Let Q be a p-subgroup of G. Let X be a G-invariant simple kN-module. Then*

$$\begin{aligned}&\sharp\{S \in \mathrm{IBr}(G|X); \mathrm{vx}(S) =_G Q\}\\ &\quad = \sharp\{U \in \mathrm{IBr}(N_G(Q)N|X); \mathrm{vx}(U) =_{N_G(Q)N} Q\}.\end{aligned}$$

Proof. Sum the equality of Corollary 13 over all blocks β of $N_G(Q)N$. $\square$

Remark 15. When G is p-solvable and N is a p'-group, Corollary 14 is a special case of Theorem 6.3 of [**IN**].

4. Proof of Theorem 1.

The following extends Corollary 13.

Proposition 16. *Use the notation in Theorem 1. Let X be a simple kN-module and let T be the inertial group of X in G. Let $\{Q^{x_i}\}$ be a set of represen-*

tatives of T-conjugacy classes of $\{Q^g; Q^g \le T, g \in G\}$. *Then*

$$\begin{aligned}&\sharp\{S \in \mathrm{IBr}(G|X); \mathrm{vx}(S) =_G Q, S \in_Q \beta\}\\&\quad = \sum_i \sharp\big\{U \in \mathrm{IBr}(\beta|X^{x_i^{-1}}); \mathrm{vx}(U) =_{N_G(Q)N} Q\big\}.\end{aligned} \tag{16.1}$$

Proof. For $S \in \mathrm{IBr}(G|X)$, let $\tilde{S}$ be the Clifford correspondent of S in T. By Clifford's theorem the LHS of (16.1) equals

$$\sum_i \sharp\big\{\tilde{S} \in \mathrm{IBr}(T|X); \mathrm{vx}(\tilde{S}) =_T Q^{x_i}, S \in_Q \beta\big\}. \tag{16.2}$$

For each i, let $\{\gamma_{ij}\}$ be the set of blocks γ of $N_T(Q^{x_i})N$ such that γ covers the block of N containing X and $\gamma^{N_G(Q^{x_i})N} = \beta^{x_i}$. We claim that if Q^{x_i} is a vertex of $\tilde{S}$, then $S \in_Q \beta$ if and only if $\tilde{S} \in_{Q^{x_i}} \gamma_{ij}$ for some j. Here $S \in_Q \beta$ if and only if $S \in_{Q^{x_i}} \beta^{x_i}$ by conjugation. So it suffices to show that if Q^{x_i} is a vertex of $\tilde{S}$, then $S \in_{Q^{x_i}} \beta^{x_i}$ if and only if $\tilde{S} \in_{Q^{x_i}} \gamma_{ij}$ for some j. Let $\tilde{V}$ be the Green correspondent of $\tilde{S}$ with respect to $(T, Q^{x_i}, N_T(Q^{x_i})N)$. Then $\tilde{V}|S_{N_T(Q^{x_i})N}$, so that there is an indecomposable $kN_G(Q^{x_i})N$-module V such that $V|S_{N_G(Q^{x_i})N}$ and $\tilde{V}|V_{N_T(Q^{x_i})N}$. Then we can choose vertices so that $\mathrm{vx}(S) \ge \mathrm{vx}(V) \ge \mathrm{vx}(\tilde{V}) = Q^{x_i}$. Since $\mathrm{vx}(S) =_G Q$, we obtain $\mathrm{vx}(V) = Q^{x_i}$. Thus V is the Green correspondent of S with respect to $(G, Q^{x_i}, N_G(Q^{x_i})N)$. Let γ be the block containing $\tilde{V}$. Let Y be a simple submodule of $\tilde{V}$. Then Y is a simple submodule of $V_{N_T(Q^{x_i})N}$. Thus

$$0 \ne \mathrm{Hom}_{N_T(Q^{x_i})N}\big(Y, V_{N_T(Q^{x_i})N}\big) \simeq \mathrm{Hom}_{N_G(Q^{x_i})N}\big(Y^{N_G(Q^{x_i})N}, V\big).$$

Since $\tilde{V}_N$ is a multiple of X, so is Y_N. Therefore $Y^{N_G(Q^{x_i})N}$ is a simple module in $\gamma^{N_G(Q^{x_i})N}$ by Lemma 3.1 of [**Mu**]. Then the followig conditions are equivalent: (1) $S \in_{Q^{x_i}} \beta^{x_i}$; (2) V lies in β^{x_i}; (3) $Y^{N_G(Q^{x_i})N}$ lies in β^{x_i}; (4) $\gamma^{N_G(Q^{x_i})N} = \beta^{x_i}$; (5) $\tilde{V}$ lies in γ_{ij} for some j; (6) $\tilde{S} \in_{Q^{x_i}} \gamma_{ij}$ for some j. The claim is proved.

Thus (16.2) equals $\sum_{i,j} \sharp\{\tilde{S} \in \mathrm{IBr}(T|X); \mathrm{vx}(\tilde{S}) =_T Q^{x_i}, \tilde{S} \in_{Q^{x_i}} \gamma_{ij}\}$.

On the other hand, by conjugation the RHS of (16.1) equals

$$\sum_i \sharp\big\{U \in \mathrm{IBr}(\beta^{x_i}|X); \mathrm{vx}(U) =_{N_G(Q^{x_i})N} Q^{x_i}\big\}.$$

Thus the equality follows if we show the following for each i:

$$\sum_j \sharp\{\tilde{S} \in \mathrm{IBr}(T|X); \mathrm{vx}(\tilde{S}) =_T Q^{x_i}, \tilde{S} \in_{Q^{x_i}} \gamma_{ij}\}$$
$$= \sharp\{U \in \mathrm{IBr}(\beta^{x_i}|X); \mathrm{vx}(U) =_{N_G(Q^{x_i})N} Q^{x_i}\}.$$

By Corollary 13 we obtain for each j

$$\sharp\{\tilde{S} \in \mathrm{IBr}(T|X); \mathrm{vx}(\tilde{S}) =_T Q^{x_i}, \tilde{S} \in_{Q^{x_i}} \gamma_{ij}\}$$
$$= \sharp\{\tilde{U} \in \mathrm{IBr}(\gamma_{ij}|X); \mathrm{vx}(\tilde{U}) =_{N_T(Q^{x_i})N} Q^{x_i}\}.$$

Therefore the equality above follows from Clifford's theorem and [**Mu**, Lemma 3.1]. The proof is complete. □

Corollary 17. *Use the notation in Theorem* 1. *Let X be a simple kN-module and let T be the inertial group of X in G. Let $\{Q^{x_i}\}$ be a set of representatives of T-conjugacy classes of $\{Q^g; Q^g \leq T, g \in G\}$. Then*

$$\sharp\{S \in \mathrm{IBr}(G|X); \mathrm{vx}(S) =_G Q\}$$
$$= \sum_i \sharp\{U \in \mathrm{IBr}(N_G(Q)N|X^{x_i^{-1}}); \mathrm{vx}(U) =_{N_G(Q)N} Q\}.$$

Proof. Sum the equality of Proposition 16 over all blocks β of $N_G(Q)N$. □

Remark 18. A result similar to Corollary 17 is proved in Theorem of Laradji [**La**] when G itself is p-solvable.

Proof of Theorem 1. Let $\{X_j\}$ be a complete set of representatives of the G-conjugacy classes of $\mathrm{IBr}(N)$. We have

$$(*)\quad \sharp\{S \in \mathrm{IBr}(G); \mathrm{vx}(S) =_G Q, S \in_Q \beta\}$$
$$= \sum_j \sharp\{S \in \mathrm{IBr}(G|X_j); \mathrm{vx}(S) =_G Q, S \in_Q \beta\}.$$

For each j let $\{Q^{x_{ji}}\}$ be a complete set of representatives of $T_G(X_j)$-conjugacy classes of $\{Q^g; Q^g \leq T_G(X_j), g \in G\}$. Then we obtain by Proposition 16 that the RHS of $(*)$ equals

$$\sum_{j,i} \sharp\{U \in \mathrm{IBr}\left(\beta|X_j^{x_{ji}^{-1}}\right); \mathrm{vx}(U) =_{N_G(Q)N} Q\}.$$

Now we claim that if $Y \in \mathrm{IBr}(N)$ is an irreducible constituent of U_N for some $U \in \mathrm{IBr}(N_G(Q)N)$ with a vertex Q, then Y is $N_G(Q)N$-conjugate to $X_j^{x_{ji}^{-1}}$ for some j, i. To see this we first show that Y is QN-invariant. Let $\tilde{U}$ be the Clifford correspondent of U in $T_{N_G(Q)N}(Y)$. Then $\tilde{U}$ has a vertex Q^x for some $x \in N_G(Q)N$. So $T_{N_G(Q)N}(Y) \geq Q^x$ and $T_G(Y) \geq Q$, as required. We can write $Y = X_j^g$ for some j and some $g \in G$. Then $Q \leq T_G(X_j)^g$. So $Q^{g^{-1}} \leq T_G(X_j)$. Hence $Q^{g^{-1}} = Q^{x_{ji}t}$ for some i and some $t \in T_G(X_j)$. This yields $x_{ji}tg =: y \in N_G(Q)$. So $Y = X_j^g = (X_j^{x_{ji}^{-1}})^y$. The claim is proved.

Next we claim that if $(j, i) \neq (j', i')$, then $X_j^{x_{ji}^{-1}}$ and $X_{j'}^{x_{j'i'}^{-1}}$ are not $N_G(Q)N$-conjugate. Indeed, assume $X_j^{x_{ji}^{-1}} = X_{j'}^{x_{j'i'}^{-1}y}$ for $y \in N_G(Q)N$. Then X_j and $X_{j'}$ are G-conjugate, so $j = j'$. Thus $X_j^{x_{ji}^{-1}} = X_j^{x_{ji'}^{-1}y}$. So $x_{ji'}^{-1}yx_{ji} =: t \in T_G(X_j)$. Put $y^{-1} = mn$ with $m \in N_G(Q)$ and $n \in N$. Then $Q^{x_{ji}} = Q^{y^{-1}x_{ji'}t} = Q^{nx_{ji'}t} = Q^{x_{ji'}n^{x_{ji'}}t}$. Since $n^{x_{ji'}}t \in T_G(X_j)$, we obtain $i = i'$. The claim is proved.

Therefore the required equality follows by Clifford's theorem. The proof is complete. $\square$

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