

Upper triangular parts of conjugacy classes of nilpotent matrices with finite number of B -orbits

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(Received Sep. 10, 2011)

(Revised Jan. 13, 2012)

Abstract. We consider the intersection of the conjugacy class of a nilpotent matrix with the space of upper triangular matrices. We give necessary and sufficient conditions for this intersection to be a union of finitely many orbits for the action by conjugation of the group of invertible upper triangular matrices.

1. Introduction and statement of the main result.

Let \mathbb{K} be an algebraically closed field of characteristic zero. Let $\mathcal{M} = \mathcal{M}_n(\mathbb{K})$ be the space of $n \times n$ matrices. A conjugacy class $\mathcal{O}_x = \{g x g^{-1} : g \in GL_n(\mathbb{K})\}$ is called nilpotent when $x \in \mathcal{M}$ is a nilpotent matrix. The nilpotent conjugacy classes of \mathcal{M} have the following usual combinatorial description. Given $x \in \mathcal{M}$ nilpotent, let $\lambda(x) = (\lambda_1 \geq \dots \geq \lambda_r)$ be the (nonincreasing) sequence of the sizes of its Jordan blocks. Thus, $\lambda(x)$ is a partition of n . The map $\mathcal{O}_x \mapsto \lambda(x)$ is then a bijection between the set of nilpotent conjugacy classes of \mathcal{M} and the set of partitions of n .

Let $B \subset GL_n(\mathbb{K})$ be the subgroup of invertible upper triangular matrices. By restriction, we get an action of B on each conjugacy class. However, this action may have an infinite number of orbits. By the theory of homogeneous spherical varieties (see [1], [20]), we know that \mathcal{O}_x has finitely many B -orbits exactly when it admits a dense B -orbit. Then, invoking [13], we get that a nilpotent conjugacy class $\mathcal{O}_x \subset \mathcal{M}$ consists of a finite number of B -orbits if and only if $x^2 = 0$.

Let $\mathcal{M}^+ \subset \mathcal{M}$ be the subspace of strictly upper triangular matrices. In this article, we study the restriction of the B -action to the intersection $\mathcal{O}_x \cap \mathcal{M}^+$. This action may have infinitely many orbits as well. The intersection $\mathcal{O}_x \cap \mathcal{M}^+$ has a natural structure of quasi-affine algebraic variety. It is not irreducible in general, and its irreducible components are not homogeneous varieties. Thus, the study of the B -action on $\mathcal{O}_x \cap \mathcal{M}^+$ is made more delicate by the fact that, in this situation,

2010 *Mathematics Subject Classification.* Primary 17B08; Secondary 20G99, 05E10.

Key Words and Phrases. nilpotent matrices, nilpotent orbits, spherical varieties, orbital varieties.

one cannot a priori rely on the theory of homogeneous spherical varieties.

The purpose of this article is to show the following classification result. In the statement, the notation 2^k (resp. 1^ℓ) stands for a sequence of k 2's (resp. ℓ 1's).

THEOREM 1. *Let $\mathcal{O}_x \subset \mathcal{M}_n(\mathbb{K})$ be a nilpotent conjugacy class, and let $\mathcal{O}_x \cap \mathcal{M}^+$ be its intersection with the space of strictly upper triangular matrices. The following assertions are equivalent.*

- (a) $\mathcal{O}_x \cap \mathcal{M}^+$ consists of finitely many B -orbits;
- (b) each irreducible component of $\mathcal{O}_x \cap \mathcal{M}^+$ admits a dense B -orbit;
- (c) the partition $\lambda(x)$ is of one of the following types:
 - (i) $\lambda(x) = (\lambda_1, 1^k)$ with $\lambda_1 \geq 1, k \geq 0$;
 - (ii) $\lambda(x) = (2^k, 1^\ell)$ with $k \geq 2, \ell \geq 0$;
 - (iii) $\lambda(x) = (3, 2^k, 1^\ell)$ with $k \geq 1, \ell \geq 0$;
 - (iv) $\lambda(x) = (3, 3)$.

This work can be viewed as a continuation of [4], where the equivalence (b) \Leftrightarrow (c) of Theorem 1 has been already established. In Section 3, we present a proof of the implication (b) \Rightarrow (c) (by an argument different from the one in [4, Theorem 6.1]). The implication (a) \Rightarrow (b) of the theorem is immediate. Sections 4–5 are then devoted to prove the remaining implication (c) \Rightarrow (a). Specifically one has to check that, in cases (i)–(iv) of Theorem 1 (c), the intersection $\mathcal{O}_x \cap \mathcal{M}^+$ is a union of finitely many B -orbits. This fact is already known for cases (i), (ii), (iv), and the parametrization of the orbits in those cases, as well as some of their properties, is given in Section 4. Our main task is then to show that the orbits are finitely many in case (iii) too. To do this, we give a combinatorial parametrization of the orbits in this case.

Section 6 contains further remarks. In particular, we note through an example that, in general, $\mathcal{O}_x \cap \mathcal{M}^+$ may have irreducible components which contain a dense B -orbit but comprise infinitely many B -orbits.

The irreducible components of $\mathcal{O}_x \cap \mathcal{M}^+$ are also called orbital varieties. Their study is in fact the main motivation of this work, as we explain in the next section.

2. Motivation: orbital varieties.

We deal with a more general situation. Let G be a connected linear reductive algebraic group over \mathbb{K} , and let \mathfrak{g} be its Lie algebra. Let $(g, x) \in G \times \mathfrak{g} \mapsto g \cdot x$ denote the adjoint action. Fix a Borel subgroup $B \subset G$, and let $\mathfrak{n} \subset \mathfrak{b} \subset \mathfrak{g}$ be the corresponding Borel subalgebra and, respectively, its nilradical. The subset $G \cdot \mathfrak{n} \subset \mathfrak{g}$ consists of finitely many adjoint orbits, called *nilpotent orbits*. Given a nilpotent orbit $\mathcal{O}_x = G \cdot x$ (with $x \in G \cdot \mathfrak{n}$), the intersection $\mathcal{O}_x \cap \mathfrak{n}$ has a natural

structure of quasi-affine algebraic variety, it has pure dimension $(1/2) \dim \mathcal{O}_x$ (see [15]), and is in general not irreducible. Its irreducible components are called *orbital varieties*. The orbital varieties arise in representation theory in the study of geometric realizations of simple highest weight modules (see [9]). Up to now, there is however only few information about the geometric structure of the orbital varieties, even in the case of $G = GL_n(\mathbb{K})$.

By restriction of the adjoint action, we get an action of B on $\mathcal{O}_x \cap \mathfrak{n}$, which stabilizes each orbital variety. By determining when this action admits only a finite number of orbits in the case of $GL_n(\mathbb{K})$, our main result (Theorem 1) aims in fact to point out situations which are more favorable for the study of orbital varieties.

REMARK 1. For $x \in G \cdot \mathfrak{n}$, let \mathcal{B}_x be the set of Borel subalgebras $\mathfrak{b}' \subset \mathfrak{g}$ such that $x \in \mathfrak{b}'$. The set \mathcal{B}_x has a natural structure of algebraic projective variety. It is called a *Springer fiber* since it is a fiber of the Springer resolution $G \times_B \mathfrak{n} \rightarrow G \cdot \mathfrak{n}$. The role of Springer fibers in representation theory is underlined in the classical papers [16], [17]. The group $Z_x = \{g \in G : g \cdot x = x\}$ naturally acts on \mathcal{B}_x . In fact, the varieties \mathcal{B}_x and $\mathcal{O}_x \cap \mathfrak{n}$ are closely related. In particular, there is a one to one correspondence between the Z_x -orbits of \mathcal{B}_x and the B -orbits of $\mathcal{O}_x \cap \mathfrak{n}$ (see [15]). Thus, the conditions in Theorem 1 also characterize the cases where \mathcal{B}_x has a finite number of Z_x -orbits, for $G = GL_n(\mathbb{K})$.

The proof of the implication (b) \Rightarrow (c) of Theorem 1 that we present in the next section actually relies on the theory of orbital varieties. Specifically, it focuses on a special family of orbital varieties, called Richardson orbital varieties.

3. Spherical Richardson orbital varieties.

As in Section 2, G denotes a connected reductive linear algebraic group over \mathbb{K} . We consider a parabolic subgroup $P \subset G$ such that $P \supset B$. Let $\mathfrak{u}_P \subset \mathfrak{p} \subset \mathfrak{g}$ be the corresponding parabolic subalgebra and, respectively, its nilradical. There is a unique nilpotent orbit $\mathcal{O} \subset \mathfrak{g}$ such that $\mathcal{O} \cap \mathfrak{u}_P$ is dense in \mathfrak{u}_P . By [18, Section 4], the intersection $\mathcal{O} \cap \mathfrak{u}_P$ is actually an irreducible component of $\mathcal{O} \cap \mathfrak{n}$, hence it is an orbital variety attached to the nilpotent orbit \mathcal{O} . An orbital variety obtained in this way is called a *Richardson orbital variety*.

The Richardson orbital varieties (corresponding to the different choices of parabolic subgroups of G) form a special family of orbital varieties. Compared to general orbital varieties, they present several advantages. First, whereas orbital varieties may be singular in general, we observe that a Richardson orbital variety $\mathcal{O} \cap \mathfrak{u}_P$ is always smooth (because, by definition, $\mathcal{O} \cap \mathfrak{u}_P$ is open in the space \mathfrak{u}_P).

A stronger fact is given by Richardson's Theorem (cf. [14]) which states that

$\mathcal{O} \cap \mathfrak{u}_P$ is homogeneous for the natural action of the parabolic subgroup P . We invoke the following auxiliary result.

LEMMA 1. *Let G be a reductive linear algebraic group over \mathbb{K} , and let $B \subset P$ be respectively a Borel subgroup and a parabolic subgroup of G . Let X be an irreducible algebraic variety, and suppose that X is equipped with a transitive algebraic action of P . Then, the following assertions are equivalent.*

- (a) X consists of finitely many B -orbits;
- (b) X admits a dense B -orbit.

PROOF. We can write $X = P/P_0$, where $P_0 \subset P$ is a closed subgroup. Moreover, let $P = L \ltimes U$ be a Levi decomposition, where $U \subset P$ stands for the unipotent radical. Observe that $M := UP_0$ is a closed subgroup of P . Hence, the quotient $Y := P/M$ is a well defined, L -homogeneous algebraic variety.

Let $\rho : P/P_0 \rightarrow P/M$ be the natural map. Given a B -orbit $\mathcal{Q} = BpP_0 \subset X$, we see that $\rho(\mathcal{Q}) = (L \cap B)pM$ is a $(L \cap B)$ -orbit. If $\rho(BpP_0) = \rho(Bp'P_0)$, then $(L \cap B)pUP_0 = (L \cap B)p'UP_0$, which implies $BpP_0 = Bp'P_0$. Thus, the map $\mathcal{Q} \mapsto \rho(\mathcal{Q})$ is a bijection between the set of B -orbits of X and the set of $(L \cap B)$ -orbits of Y . Since ρ is an equivariant morphism of P -homogeneous varieties, we have in addition

$$\dim X - \dim \mathcal{Q} = \dim Y - \dim \rho(\mathcal{Q})$$

for each B -orbit $\mathcal{Q} \subset X$. Thus, the bijection $\mathcal{Q} \mapsto \rho(\mathcal{Q})$ is codimension preserving.

If X consists of finitely many B -orbits, then it is immediate that X admits a dense B -orbit. Conversely, assume that X admits a dense B -orbit \mathcal{Q} . Then, $\rho(\mathcal{Q}) \subset Y$ is a $(L \cap B)$ -orbit of maximal dimension, which implies that $\rho(\mathcal{Q})$ is a dense $(L \cap B)$ -orbit of Y . Since Y is L -homogeneous, L is reductive, and $L \cap B \subset L$ is a Borel subgroup, by [1, Theorem 2], or [20, Theorem 1], we have that Y consists of finitely many $(L \cap B)$ -orbits. Therefore, X consists of finitely many B -orbits. □

REMARK 2. In the case of $P = G$, Lemma 1 coincides with [1, Theorem 2]. Theorem 1 in [20] states more generally that an algebraic variety X equipped with a (non necessarily transitive) action of a reductive group G has a finite number of B -orbits whenever it admits a dense B -orbit. In Lemma 1, the assumption of a transitive action of P on X is however necessary (see for example Section 6.3).

The next proposition, which shows that the Richardson orbital varieties are convenient for studying the finiteness of the number of B -orbits, is then a consequence of Lemma 1.

PROPOSITION 1. *The following assertions are equivalent.*

- (a) *The Richardson orbital variety $\mathcal{O} \cap \mathfrak{u}_P$ consists of finitely many B -orbits;*
- (b) *$\mathcal{O} \cap \mathfrak{u}_P$ admits a dense B -orbit;*
- (c) *the nilradical \mathfrak{u}_P admits a dense B -orbit.*

In the case of $G = GL_n(\mathbb{K})$ and $\mathfrak{g} = \mathcal{M}_n(\mathbb{K})$, the Richardson orbital varieties have the following parametrization. A sequence $\eta = (\eta_1, \dots, \eta_r)$ of positive integers such that $\eta_1 + \dots + \eta_r = n$ is called a composition of n . Let $\mathfrak{p}(\eta) \subset \mathfrak{g}$ be the subspace of blockwise upper triangular matrices with blocks of sizes η_1, \dots, η_r along the diagonal. The map $\eta \mapsto \mathfrak{p}(\eta)$ is a one to one correspondence between the compositions of n and the standard parabolic subalgebras of \mathfrak{g} (i.e., the Lie subalgebras which contain the Lie algebra of B). Let $\mathfrak{l}(\eta), \mathfrak{u}(\eta) \subset \mathfrak{p}(\eta)$ be respectively the subspace of blockwise diagonal matrices with blocks of sizes η_1, \dots, η_r along the diagonal and the subspace of blockwise strictly upper triangular matrices with the same frame. Thus, $\mathfrak{l}(\eta)$ and $\mathfrak{u}(\eta)$ are respectively a Levi factor and the nilradical of $\mathfrak{p}(\eta)$, and $\mathfrak{p}(\eta) = \mathfrak{l}(\eta) \oplus \mathfrak{u}(\eta)$ is a Levi decomposition. We denote by $\lambda(\eta) = (\lambda_1 \geq \dots \geq \lambda_r)$ the partition of n obtained by arranging the terms in the sequence η in nonincreasing order, and we denote by $\lambda^*(\eta) = (\lambda_1^* \geq \dots \geq \lambda_{\lambda_1}^*)$ the dual partition, that is,

$$\lambda_j^* = |\{i = 1, \dots, r : \eta_i \geq j\}| \text{ for all } j = 1, \dots, \lambda_1$$

(in other words, if $Y_{\lambda(\eta)}$ denotes the Young diagram with r rows of lengths $\lambda_1, \dots, \lambda_r$, then $\lambda_1^*, \dots, \lambda_{\lambda_1}^*$ are the lengths of the columns of $Y_{\lambda(\eta)}$). It is a standard fact that $\mathcal{O}_x \cap \mathfrak{u}(\eta)$ is dense in $\mathfrak{u}(\eta)$ whenever $x \in \mathfrak{g} = \mathcal{M}_n(\mathbb{K})$ is a nilpotent element of Jordan form $\lambda(x) = \lambda^*(\eta)$ (for a proof, see for instance [4, Section 4.3]). Then, $\mathcal{V}^R(\eta) := \mathcal{O}_x \cap \mathfrak{u}(\eta)$ is a Richardson orbital variety, and every Richardson orbital variety can be obtained in this way, in the case of $G = GL_n(\mathbb{K})$.

The drawback of the Richardson orbital varieties is that they are highly susceptible to admit no dense B -orbit. The next proposition gives a characterization of the Richardson orbital varieties which do not admit a dense B -orbit in the case of $G = GL_n(\mathbb{K})$. It uses the following notation: when $\eta = (\eta_1, \dots, \eta_r)$ and $\rho = (\rho_1, \dots, \rho_s)$ are two sequences of positive integers, we write $\eta \geq \rho$ if there are $1 \leq i_1 < \dots < i_s \leq r$ such that $\eta_{i_j} \geq \rho_j$ for each $j \in \{1, \dots, s\}$.

PROPOSITION 2. *Let η be a composition of n , and let $\mathcal{V}^R(\eta)$ be the corresponding Richardson orbital variety. The following conditions are equivalent.*

- (a) *$\mathcal{V}^R(\eta)$ contains infinitely many B -orbits;*
- (b) *$\mathcal{V}^R(\eta)$ has no dense B -orbit;*
- (c) *$\eta \geq (1, 2, 2, 1)$ or $\eta \geq (2, 3, 2)$.*

PROOF. The equivalence between (a) and (b) follows from Proposition 1. The equivalence between (b) and (c) is proved in [4, Theorem 6.3]. The proof can be summed up as follows. If $\eta \not\geq (1, 2, 2, 1)$ and $\eta \not\geq (2, 3, 2)$, then one can construct the representative of a dense B -orbit in the Richardson orbital variety $\mathcal{V}^R(\eta)$, following the recipe described in [4, Propositions 7.1, 7.4]. Thus, (b) implies (c). If $\eta \geq (1, 2, 2, 1)$ or $\eta \geq (2, 3, 2)$, then the arguments in [4, Propositions 7.5, 7.6] allow to construct infinitely many pairwise distinct B -orbits in $\mathcal{V}^R(\eta)$. Whence, (c) implies (a). \square

COROLLARY 1. *Assume that $x \in \mathcal{M}_n(\mathbb{K})$ is nilpotent with a Jordan form $\lambda(x) = (\lambda_1 \geq \dots \geq \lambda_r)$ satisfying*

$$(\lambda_1 \geq 4 \text{ and } \lambda_2 \geq 2) \quad \text{or} \quad (\lambda_1, \lambda_2 \geq 3 \text{ and } \lambda_3 \geq 1).$$

Then, there is a Richardson orbital variety $\mathcal{V}^R(\eta) \subset \mathcal{O}_x$ which has no dense B -orbit.

PROOF. Under the assumption of the corollary, we can produce a composition $\eta = (\eta_1, \dots, \eta_{\lambda_1})$ of n such that $\lambda^*(\eta) = \lambda(x)$ and with $\eta \geq (1, 2, 2, 1)$ or $\eta \geq (2, 3, 2)$. Then, by Proposition 2, the corresponding Richardson orbital variety $\mathcal{V}^R(\eta) \subset \mathcal{O}_x$ has no dense B -orbit. \square

Corollary 1 implies that, apart from cases (i)–(iv) of Theorem 1 (c), the intersection $\mathcal{O}_x \cap \mathcal{M}^+$ contains a Richardson orbital variety (in particular, an irreducible component) which has no dense B -orbit. This proves the implication (b) \Rightarrow (c) in Theorem 1.

4. Description of the cases with finitely many B -orbits.

This section, together with Section 5, is concerned with the proof of the implication (c) \Rightarrow (a) of Theorem 1. As in Theorem 1, fix a nilpotent matrix $x \in \mathcal{M}_n(\mathbb{K})$ and consider $\mathcal{O}_x \cap \mathcal{M}^+$, the intersection of its conjugacy class with the space of strictly upper triangular matrices. The proof requires to check that in cases (i)–(iv) of Theorem 1 (c), $\mathcal{O}_x \cap \mathcal{M}^+$ consists of finitely many B -orbits. Note that this fact is immediate in case (ii), i.e., for $\lambda(x) = (2^k, 1^\ell)$: here we have $x^2 = 0$ hence the conjugacy class \mathcal{O}_x itself comprises finitely many B -orbits (cf. [13]). In the other cases (i), (iii), (iv), we will get that $\mathcal{O}_x \cap \mathcal{M}^+$ contains only finitely many B -orbits though the conjugacy class \mathcal{O}_x has infinitely many ones.

Before focusing on cases (i)–(iv) of Theorem 1 (c), we emphasize a particular family of B -orbits: the B -orbits which contain a matrix with a maximal number of coefficients equal to zero. This family of orbits will play a role in our arguments.

4.1. A particular family of B -orbits.

A matrix $y \in \mathcal{M}^+$ such that

- (a) each coefficient of y is either 0 or 1, and
- (b) each row and each column of y contains at most one coefficient equal to 1

will be called a Jordan matrix. This means that the canonical basis (e_1, \dots, e_n) of \mathbb{K}^n is a Jordan basis for y (in the sense that $y(e_i) \in \{0, e_1, \dots, e_{i-1}\}$ for each i , and $y(e_i) = y(e_j) \neq 0$ implies $i = j$). Correspondingly, we will say that the B -orbit of y is a Jordan B -orbit.

Let $x \in \mathcal{M}_n(\mathbb{K})$ be a nilpotent matrix of Jordan form $\lambda(x) = (\lambda_1 \geq \dots \geq \lambda_r)$. The Jordan B -orbits which lie in the intersection $\mathcal{O}_x \cap \mathcal{M}^+$ can be parametrized as follows. Let Π_x denote the set of partitions of $\{1, \dots, n\}$ into r subsets of cardinals $\lambda_1, \dots, \lambda_r$. Hence, an element $\pi \in \Pi_x$ can be written as a sequence $\pi = (I_1, \dots, I_r)$ where $I_j \subset \{1, \dots, n\}$ is a subset with λ_j elements, and $\{1, \dots, n\} = I_1 \sqcup \dots \sqcup I_r$. Let $\alpha_\pi : \{1, \dots, n\} \rightarrow \{0, 1, \dots, n\}$ be the map defined by setting $\alpha_\pi(i) = 0$ if $i = \min I_j$ for some j , and by letting $\alpha_\pi(i)$ be the predecessor of i in the set I_j if $i \in I_j \setminus \{\min I_j\}$. Let $y_\pi \in \mathcal{M}^+$ be the matrix defined by

$$(y_\pi)_{j,i} = \begin{cases} 1 & \text{if } j = \alpha_\pi(i), \\ 0 & \text{otherwise.} \end{cases}$$

Finally, let $\mathcal{Q}_\pi = B \cdot y_\pi$.

PROPOSITION 3. *The map $\pi \mapsto y_\pi$ (resp. $\pi \mapsto \mathcal{Q}_\pi$) establishes a bijection between Π_x and the set of Jordan matrices of $\mathcal{O}_x \cap \mathcal{M}^+$ (resp. the set of Jordan B -orbits which lie in $\mathcal{O}_x \cap \mathcal{M}^+$).*

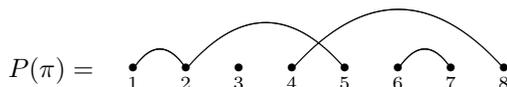
PROOF. Only the injectivity of the map $\pi \mapsto \mathcal{Q}_\pi$ is not immediate. Given a matrix $y \in \mathcal{M}_n(\mathbb{K})$ and $1 \leq j \leq i \leq n$, we let $y_{[j,i]} = (y_{s,t})_{(s,t) \in \{j, \dots, i\}^2} \in \mathcal{M}_{i-j+1}(\mathbb{K})$ be the submatrix contained between the j -th and i -th rows and columns. We can see that, whenever $y \in \mathcal{Q}_\pi$, we have

$$\alpha_\pi(i) = \begin{cases} \max\{j = 1, \dots, i-1 : \text{rk } y_{[j,i]} > \text{rk } y_{[j,i-1]}\} & \text{if } \text{rk } y_{[1,i]} > \text{rk } y_{[1,i-1]}, \\ 0 & \text{otherwise.} \end{cases}$$

This implies that, if $\mathcal{Q}_\pi = \mathcal{Q}_{\pi'}$, then α_π and $\alpha_{\pi'}$ must coincide. \square

It is convenient to represent an element $\pi \in \Pi_x$ by a graph. Let $P(\pi)$ be the graph with n vertices labeled with the integers $1, \dots, n$, and with an edge between the vertices j and $i > j$ if and only if $\alpha_\pi(i) = j$.

EXAMPLE 1. Let $x \in \mathcal{M}_8(\mathbb{K})$ be a nilpotent matrix such that $\lambda(x) = (3, 2, 2, 1)$. Then, $\pi = (\{1, 2, 5\}, \{4, 8\}, \{6, 7\}, \{3\}) \in \Pi_x$ is represented by the graph



The Jordan B -orbits form a finite family of B -orbits, which makes impossible that they cover all the possible B -orbits in general. The orbital varieties which admit a dense Jordan B -orbit enjoy some good properties (cf. [2]).

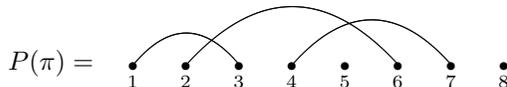
Our goal in the following subsections is to propose parametrizations of the B -orbits of $\mathcal{O}_x \cap \mathcal{M}^+$ in cases (i)–(iv) of Theorem 1 (c). To do this, we rely on the combinatorics introduced above, as in each case we will describe the B -orbits as slight deformations of Jordan B -orbits. The Jordan B -orbits are especially prominent in case (ii). The parametrization of the orbits in this case is known after the work of A. Melnikov, and we recall it in the next subsection.

4.2. B -orbits in case (ii).

Assume in this subsection that $\lambda(x) = (2^k, 1^\ell)$ with k, ℓ nonnegative integers such that $2k + \ell = n$. Thus, $x^2 = 0$, hence the conjugacy class \mathcal{O}_x is spherical (see [13]) so that we can say a priori that $\mathcal{O}_x \cap \mathcal{M}^+$ consists of a finite number of B -orbits. These orbits can be described as follows.

Here, an element $\pi \in \Pi_x$ is a partition of $\{1, \dots, n\}$ into k pairs and ℓ singletons, and it can be written $\pi = (\{i_1, j_1\}, \dots, \{i_k, j_k\}, \{m_1\}, \dots, \{m_\ell\})$ with $i_p < j_p$. This implies that the graph $P(\pi)$ has k edges connecting the pairs $\{i_p, j_p\}$ ($p = 1, \dots, k$) and ℓ fixed points (i.e., not incident with an arc) m_1, \dots, m_ℓ .

- Two edges $(i_p, j_p), (i_q, j_q)$ are said to have a crossing if $i_p < i_q < j_p < j_q$. A pair formed by an edge (i_p, j_p) and a fixed point m_q is said to form a bridge if $i_p < m_q < j_p$. Let $N_c(\pi)$ (resp. $N_b(\pi)$) denote the number of crossings (resp. of bridges) associated to the partition π . For example, π such that



has two crossings (the pairs $((1, 3), (2, 6))$ and $((2, 6), (4, 7))$) and two bridges (the pairs $((2, 6), 5)$ and $((4, 7), 5)$).

- Given $1 \leq s < t \leq n$, we let $R_{s,t}(\pi)$ be the number of pairs $\{i_p, j_p\}$ such that $s \leq i_p < j_p \leq t$.

The following proposition is due to A. Melnikov [10], [11].

PROPOSITION 4. Assume $\lambda(x) = (2^k, 1^\ell)$ with $2k + \ell = n$.

- (a) Each B -orbit in $\mathcal{O}_x \cap \mathcal{M}^+$ is a Jordan B -orbit. Hence, the map $\pi \mapsto \mathcal{Q}_\pi$ establishes a bijection between Π_x and the set of B -orbits in $\mathcal{O}_x \cap \mathcal{M}^+$.
- (b) $\dim \mathcal{Q}_\pi = k(n-k) - N_c(\pi) - N_b(\pi)$ for all $\pi \in \Pi_x$. In particular, the irreducible components of $\mathcal{O}_x \cap \mathcal{M}^+$ are the closures of \mathcal{Q}_π for π such that $N_c(\pi) = N_b(\pi) = 0$.
- (c) Given $\pi, \rho \in \Pi_x$, we have $\mathcal{Q}_\pi \subset \overline{\mathcal{Q}_\rho}$ if and only if $R_{s,t}(\pi) \leq R_{s,t}(\rho)$ for all s, t .
- (d) Let $\pi_0 = (\{i, n-k+i\}_{i=1}^k, \{j\}_{j=k+1}^{n-k})$. Then, \mathcal{Q}_{π_0} is the unique closed B -orbit of $\mathcal{O}_x \cap \mathcal{M}^+$, and we have $\dim \mathcal{Q}_{\pi_0} = (1/2)k(k+1)$.

Part (a) of the proposition will be used in Section 5.

REMARK 3. Proposition 4 (a) almost characterizes case (ii). The parametrizations of the orbits in the next subsections will testify that, for x general, $\mathcal{O}_x \cap \mathcal{M}^+$ admits non-Jordan B -orbits. In fact, one can check that all the B -orbits of $\mathcal{O}_x \cap \mathcal{M}^+$ are Jordan B -orbits if and only if $x^2 = 0$ (i.e., $\lambda(x) = (2^k, 1^\ell)$ as above) or x is regular (i.e., $\lambda(x) = (n)$). In the latter case, $\mathcal{O}_x \cap \mathcal{M}^+$ consists of a single (Jordan) B -orbit.

4.3. B-orbits in case (i).

Now, we assume that $\lambda(x) = (\ell, 1^k)$ with $\ell \geq 3, k \geq 0, \ell + k = n$. The fact that $\mathcal{O}_x \cap \mathcal{M}^+$ has a finite number of B -orbits in this case is known (see [19], or [4, Section 6.4]). Here, we recall the description of the orbits from [4] and we complete it by providing more information on the topological properties of the orbits (dimension and inclusion relations between orbit closures).

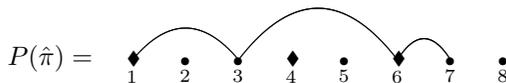
An element $\pi \in \Pi_x$ is a partition of $\{1, \dots, n\}$ of the form

$$\pi = (\{a_1, \dots, a_\ell\}, \{m_1\}, \dots, \{m_k\})$$

where $a_1 < \dots < a_\ell$. Let $\hat{\Pi}_x$ be the set of pairs $\hat{\pi} = (\pi, (b_p)_{p=1}^{\ell-1})$ where $\pi = (\{a_1 < \dots < a_\ell\}, \{m_p\}_{p=1}^k) \in \Pi_x$ is a partition and $(b_p)_{p=1}^{\ell-1}$ is a sequence of integers such that $b_1 = a_1$ and $a_p \leq b_p < a_{p+1}$ for each $p = 2, \dots, \ell - 1$. We then write

$$a_p(\hat{\pi}) = a_p \quad \forall p \in \{1, \dots, \ell\}, \quad b_p(\hat{\pi}) = b_p \quad \forall p \in \{1, \dots, \ell - 1\}.$$

We can represent $\hat{\pi}$ by the graph denoted $P(\hat{\pi})$ obtained from $P(\pi)$ by marking the vertices of labels $b_1, \dots, b_{\ell-1}$ with the symbol “ \blacklozenge ”. For instance, if $\pi = (\{1, 3, 6, 7\}, \{2\}, \{4\}, \{5\}, \{8\})$ and $\{b_p\}_{p=1}^3 = \{1, 4, 6\}$, then



Given $\hat{\pi} \in \hat{\Pi}_x$, we define a matrix $y_{\hat{\pi}} \in \mathcal{M}^+$ by

$$(y_{\hat{\pi}})_{s,t} = \begin{cases} 1 & \text{for } (s, t) \in \{(a_p(\hat{\pi}), a_{p+1}(\hat{\pi}))_{p=1}^{\ell-1}, (b_p(\hat{\pi}), a_{p+1}(\hat{\pi}))_{p=1}^{\ell-1}\}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{Q}_{\hat{\pi}} = B \cdot y_{\hat{\pi}}$.

PROPOSITION 5. Assume $\lambda(x) = (\ell, 1^k)$ with $\ell \geq 3, k \geq 0, n = \ell + k$.

- (a) The map $\hat{\pi} \mapsto \mathcal{Q}_{\hat{\pi}}$ establishes a bijection between $\hat{\Pi}_x$ and the set of B -orbits of $\mathcal{O}_x \cap \mathcal{M}^+$. In particular, $\mathcal{O}_x \cap \mathcal{M}^+$ contains a finite number of B -orbits.
- (b) $\dim \mathcal{Q}_{\hat{\pi}} = (1/2)n(n-1) - (1/2)k(k+1) - \sum_{p=1}^{\ell-1} (a_{p+1}(\hat{\pi}) - b_p(\hat{\pi}) - 1)$ for all $\hat{\pi} \in \hat{\Pi}_x$. Thus, the irreducible components of $\mathcal{O}_x \cap \mathcal{M}^+$ are obtained as the closures of $\mathcal{Q}_{\hat{\pi}}$ for $\hat{\pi} \in \hat{\Pi}_x$ such that $b_p(\hat{\pi}) = a_{p+1}(\hat{\pi}) - 1$ for all $p = 1, \dots, \ell - 1$.
- (c) Given $\hat{\pi}, \hat{\rho} \in \hat{\Pi}_x$, we have $\mathcal{Q}_{\hat{\pi}} \subset \overline{\mathcal{Q}_{\hat{\rho}}}$ if and only if $b_p(\hat{\pi}) \leq b_p(\hat{\rho})$ and $a_{p+1}(\hat{\pi}) \geq a_{p+1}(\hat{\rho})$ for all $p = 1, \dots, \ell - 1$.
- (d) The orbit $\mathcal{Q}_{\hat{\pi}}$ is closed in $\mathcal{O}_x \cap \mathcal{M}^+$ if and only if $a_1(\hat{\pi}) = 1, a_{\ell}(\hat{\pi}) = n$, and $b_p(\hat{\pi}) = a_p(\hat{\pi})$ for all $p = 1, \dots, \ell - 1$. In particular, such $\mathcal{Q}_{\hat{\pi}}$ is a Jordan B -orbit, and $\dim \mathcal{Q}_{\hat{\pi}} = (1/2)n(n-1) - (1/2)k(k+3)$.
- (e) Fix $\hat{\pi} \in \hat{\Pi}_x$. Let $\hat{\pi}_1, \hat{\pi}_2 \in \hat{\Pi}_x$ be defined by $b_p(\hat{\pi}_1) = a_{p+1}(\hat{\pi}_1) - 1 = b_p(\hat{\pi})$ and $b_p(\hat{\pi}_2) = a_{p+1}(\hat{\pi}_2) - 1 = a_{p+1}(\hat{\pi}) - 1$ for each $p = 1, \dots, \ell - 1$. Then, $\mathcal{V}_i := \overline{\mathcal{Q}_{\hat{\pi}_i}} \cap \mathcal{O}_x$ ($i = 1, 2$) are irreducible components of $\mathcal{O}_x \cap \mathcal{M}^+$, and $\mathcal{Q}_{\hat{\pi}}$ is dense in $\mathcal{V}_1 \cap \mathcal{V}_2$.

PROOF. Given a matrix $y \in \mathcal{M}^+$, we let $y_{[j,i]} := (y_{s,t})_{(s,t) \in \{j, \dots, i\}^2}$ be the submatrix contained between the j -th and i -th rows and columns. Part (a) of the proposition is proved in [4, Proposition 6.4], where it is shown in addition that $\mathcal{Q}_{\hat{\pi}}$ is the set of matrices $y \in \mathcal{O}_x \cap \mathcal{M}^+$ such that

$$\begin{cases} (y_{[b_p(\hat{\pi})+1, n]})^{\ell-p} = 0, & (y_{[1, a_{p+1}(\hat{\pi})-1]})^p = 0, \\ (y_{[b_p(\hat{\pi}), n]})^{\ell-p} \neq 0, & (y_{[1, a_{p+1}(\hat{\pi})]})^p \neq 0, \end{cases} \quad \forall p = 1, \dots, \ell - 1.$$

We now invoke results of F. Fung [5] and J.A. Vargas [19] (these results are formulated in the setting of Springer fibers, but they can be applied to the present situation, in view of the relation between Springer fibers and the variety $\mathcal{O}_x \cap \mathcal{M}^+$, cf. [15]): the irreducible components of $\mathcal{O}_x \cap \mathcal{M}^+$ are parametrized by the sequences of integers of the form $\underline{c} = (c_2, \dots, c_{\ell})$ with $1 < c_2 < \dots < c_{\ell}$.

Namely, $\mathcal{V}(\underline{c})$, defined as the set of elements $y \in \mathcal{O}_x \cap \mathcal{M}^+$ such that

$$(y_{[c_{p+1}, n]})^{\ell-p} = 0, \quad (y_{[1, c_{p+1}-1]})^p = 0, \quad \forall p = 1, \dots, \ell - 1,$$

is a component of $\mathcal{O}_x \cap \mathcal{M}^+$, and each component can be obtained in this way. Moreover, given two such sequences $\underline{c}, \underline{d}$, one has $\mathcal{V}(\underline{c}) \cap \mathcal{V}(\underline{d}) \neq \emptyset$ if and only if $\max\{c_p, d_p\} < \min\{c_{p+1}, d_{p+1}\}$ for all $p = 1, \dots, \ell - 1$; in this case, $\mathcal{V}(\underline{c}) \cap \mathcal{V}(\underline{d})$ is irreducible and

$$\dim \mathcal{V}(\underline{c}) \cap \mathcal{V}(\underline{d}) = \frac{1}{2}n(n-1) - \frac{1}{2}k(k+1) - \sum_{p=2}^{\ell} |d_p - c_p|.$$

Part (e) and, in turn, parts (b)–(d) of the proposition follow from this description of the components and from the above description of the orbits $\mathcal{Q}_{\hat{\pi}}$. \square

4.4. B-orbits in case (iv).

Now, let $\lambda(x) = (3, 3)$. The fact that $\mathcal{O}_x \cap \mathcal{M}^+$ contains a finite number of B -orbits has been shown by H. Bürgstein (see [7, Section 5.7]). The next proposition, whose proof (which we skip) is based on concrete calculations, gives a precise description of the orbits.

PROPOSITION 6. *Assume $\lambda(x) = (3, 3)$. Then, the variety $\mathcal{O}_x \cap \mathcal{M}^+$ contains exactly twelve B -orbits, among which ten are Jordan B -orbits:*

- (a) *Exactly five B -orbits have maximal dimension: the Jordan B -orbits $\mathcal{Q}_i := \mathcal{Q}_{\pi_i}$ ($i = 1, 2, 3$) corresponding to the partitions $\pi_1 = (\{1, 2, 3\}, \{4, 5, 6\})$, $\pi_2 = (\{1, 2, 6\}, \{3, 4, 5\})$, $\pi_3 = (\{1, 5, 6\}, \{2, 3, 4\})$, and the non-Jordan B -orbits $\mathcal{Q}_4 := B \cdot z_4$ and $\mathcal{Q}_5 := B \cdot z_5$ of representatives*

$$z_4 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad z_5 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

- (b) *The Jordan B -orbits $\mathcal{Q}_i := \mathcal{Q}_{\pi_i}$ ($i = 6, 7, 8, 9, 10, 11$), corresponding to the partitions $\pi_6 = (\{1, 2, 4\}, \{3, 5, 6\})$, $\pi_7 = (\{1, 2, 5\}, \{3, 4, 6\})$, $\pi_8 = (\{1, 3, 4\}, \{2, 5, 6\})$, $\pi_9 = (\{1, 3, 6\}, \{2, 4, 5\})$, $\pi_{10} = (\{1, 4, 5\}, \{2, 3, 6\})$, $\pi_{11} = (\{1, 4, 6\}, \{2, 3, 5\})$, have codimension 1. Each one lies in the closure of exactly two maximal dimensional B -orbits, and namely we have $\overline{\mathcal{Q}_1} \supset \mathcal{Q}_6, \mathcal{Q}_{10}$;*

$\overline{\mathcal{Q}}_2 \supset \mathcal{Q}_7, \mathcal{Q}_9$; $\overline{\mathcal{Q}}_3 \supset \mathcal{Q}_8, \mathcal{Q}_{11}$; $\overline{\mathcal{Q}}_4 \supset \mathcal{Q}_6, \mathcal{Q}_7, \mathcal{Q}_8$; and $\overline{\mathcal{Q}}_5 \supset \mathcal{Q}_9, \mathcal{Q}_{10}, \mathcal{Q}_{11}$. Note that the non-Jordan maximal dimensional B -orbits contain three one-codimensional orbits each.

- (c) The Jordan B -orbit $\mathcal{Q}_{12} := \mathcal{Q}_{\pi_{12}}$, corresponding to $\pi_{12} = (\{1, 3, 5\}, \{2, 4, 6\})$, is the only closed B -orbit of $\mathcal{O}_x \cap \mathcal{M}^+$; it has codimension 2.

4.5. B -orbits in case (iii).

Hereafter, we assume that $x \in \mathcal{M}_n(\mathbb{K})$ is a nilpotent matrix of Jordan form $\lambda(x) = (3, 2^k, 1^\ell)$, with k, ℓ nonnegative integers such that $3 + 2k + \ell = n$. Our next goal is to define a set $\hat{\Pi}_x$ which will parametrize the B -orbits of the set $\mathcal{O}_x \cap \mathcal{M}^+$ in this case.

Note that an element $\pi \in \Pi_x$ can be written as a sequence $\pi = (I_1, \dots, I_{k+\ell+1})$ whose terms are subsets of $\{1, \dots, n\}$ that satisfy $|I_1| = 3, |I_2| = \dots = |I_{k+1}| = 2, |I_{k+2}| = \dots = |I_{k+\ell+1}| = 1$, and $\bigsqcup_{p=1}^{k+\ell+1} I_p = \{1, \dots, n\}$.

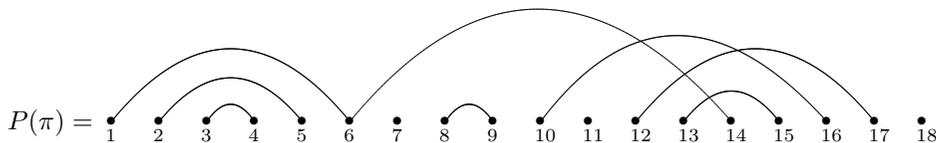
DEFINITION 1. Let $\hat{\Pi}_x$ be the set of tuples $\hat{\pi} = (\pi, b, \mathcal{E}_1, \mathcal{E}_2)$ consisting of the following data:

- $\pi \in \Pi_x$; we write $\pi = (I_1, \dots, I_{k+\ell+1})$ with $I_1 = \{a_1 < a_2 < a_3\}$ and $|I_2| = \dots = |I_{k+1}| = 2$ as above;
- $b \in \{a_2, a_2 + 1, \dots, a_3 - 1\}$ satisfying either $b = a_2$, or $\{b\} = I_p$ for some $p \in \{k + 2, \dots, k + \ell + 1\}$ (i.e., b is a fixed point of the graph $P(\pi)$);
- $\mathcal{E}_1, \mathcal{E}_2 \subset \{I_2, \dots, I_{k+1}\}$ two subsets (that can be empty) of the form $\mathcal{E}_1 = (\{g_p, h_p\}_{p=1}^{r_1})$, $\mathcal{E}_2 = (\{i_p, j_p\}_{p=1}^{r_2})$ (for some $r_1, r_2 \geq 0$) with

$$a_1 < g_1 < \dots < g_{r_1} < h_{r_1} < \dots < h_1 < a_2,$$

$$b < i_1 < \dots < i_{r_2} < a_3 < j_{r_2} < \dots < j_1 \leq n.$$

REMARK 4. The graph $P(\pi)$ representing a partition $\pi \in \Pi_x$ consists of a chain of two adjacent arcs $\{(a_1, a_2), (a_2, a_3)\}$, k other arcs joining pairwise disjoint pairs of vertices, and ℓ fixed points. E.g., for $\lambda(x) = (3, 2^6, 1^3)$ (i.e., $k = 6$ and $\ell = 3$), a possible graph is



Then, an element $\hat{\pi} = (\pi, b, \mathcal{E}_1, \mathcal{E}_2) \in \hat{\Pi}_x$ is in fact equivalent to the choice of

- a distinguished vertex b which is either a_2 or a fixed point under the arc (a_2, a_3) ;

- certain distinguished arcs $\mathcal{A}_1, \dots, \mathcal{A}_{r_1}$ (for some $r_1 \geq 0$) situated under (a_1, a_2) and forming a decreasing sequence (i.e., \mathcal{A}_{p+1} is under \mathcal{A}_p for each p);
- certain other distinguished arcs $\mathcal{B}_1, \dots, \mathcal{B}_{r_2}$ (for some $r_2 \geq 0$) situated on the right of b , all crossing (a_2, a_3) , and forming a decreasing sequence.

In the above example, the possible choices are $b \in \{6, 7, 11\}$, $(\mathcal{A}_1, \dots, \mathcal{A}_{r_1}) \in \{\emptyset, ((2, 5)), ((3, 4)), ((2, 5), (3, 4))\}$, $(\mathcal{B}_1, \dots, \mathcal{B}_{r_2}) \in \{\emptyset, ((10, 16)), ((12, 17)), ((13, 15)), ((10, 16), (13, 15)), ((12, 17), (13, 15))\}$ in the case $b \in \{6, 7\}$, and $(\mathcal{B}_1, \dots, \mathcal{B}_{r_2}) \in \{\emptyset, ((12, 17)), ((13, 15)), ((12, 17), (13, 15))\}$ in the case $b = 11$.

REMARK 5. In the case of $\lambda(x) = (3, 1^\ell)$, the notation is unambiguous towards the one introduced in Section 4.3. Indeed, in this case, an element $\hat{\pi} = (\pi, b, \mathcal{E}_1, \mathcal{E}_2) \in \hat{\Pi}_x$ satisfies $\mathcal{E}_1 = \mathcal{E}_2 = \emptyset$ and $b \in \{a_2, \dots, a_3 - 1\}$. Thus the sets $\hat{\Pi}_x$ of Sections 4.3 and 4.5 coincide up to identification of the tuples $(\pi, b, \emptyset, \emptyset)$ and $(\pi, (a_1, b))$.

DEFINITION 2. Let $\hat{\pi} = (\pi, b, \mathcal{E}_1, \mathcal{E}_2) \in \hat{\Pi}_x$ be like in Definition 1. We define a matrix $y_{\hat{\pi}} \in \mathcal{M}^+$ as follows:

$$(y_{\hat{\pi}})_{s,t} = \begin{cases} 1 & \text{if } s < t \text{ satisfy} \\ & (s, t) \in \{(a_1, a_2), (a_2, a_3), (b, a_3), (g_p, a_2)_{p=1}^{r_1}, (i_p, a_3)_{p=1}^{r_2}\} \\ & \text{or } \{s, t\} = I_p \text{ for some } p \in \{2, \dots, k+1\}, \\ 0 & \text{otherwise.} \end{cases}$$

We let $\mathcal{Q}_{\hat{\pi}} = B \cdot y_{\hat{\pi}}$.

PROPOSITION 7. Assume $\lambda(x) = (3, 2^k, 1^\ell)$ with $k, \ell \geq 0$ such that $3 + 2k + \ell = n$. Then, the map $\hat{\pi} \mapsto \mathcal{Q}_{\hat{\pi}}$ is a bijection from $\hat{\Pi}_x$ to the set of B -orbits of the variety $\mathcal{O}_x \cap \mathcal{M}^+$. In particular, $\mathcal{O}_x \cap \mathcal{M}^+$ has a finite number of B -orbits in this case.

The proof of this proposition is given in the next section. This result (combined with Corollary 1 and Propositions 4, 5, 6) completes the proof of Theorem 1.

5. Proof of Proposition 7.

We start with some notation. As in Section 4.5, we assume that $\lambda(x) = (3, 2^k, 1^\ell)$ with $k, \ell \geq 0$.

For $\hat{\pi} = (\pi, b, \mathcal{E}_1, \mathcal{E}_2) \in \hat{\Pi}_x$, we set $a_i(\hat{\pi}) = a_i$ (for $i = 1, 2, 3$), $b(\hat{\pi}) = b$, $\mathcal{E}_1(\hat{\pi}) = (\{g_p(\hat{\pi}), h_p(\hat{\pi})\}_{p=1}^{r_1(\hat{\pi})}) = \mathcal{E}_1$, and $\mathcal{E}_2(\hat{\pi}) = (\{i_p(\hat{\pi}), j_p(\hat{\pi})\}_{p=1}^{r_2(\hat{\pi})}) = \mathcal{E}_2$, where $a_1, a_2, a_3, \mathcal{E}_1, \mathcal{E}_2$ are like in Definition 1. The terms are thus organized as follows:

$$1 \leq a_1(\hat{\pi}) < g_1(\hat{\pi}) < \cdots < g_{r_1(\hat{\pi})}(\hat{\pi}) < h_{r_1(\hat{\pi})}(\hat{\pi}) < \cdots < h_1(\hat{\pi}) < a_2(\hat{\pi}),$$

$$a_2(\hat{\pi}) \leq b(\hat{\pi}) < i_1(\hat{\pi}) < \cdots < i_{r_2(\hat{\pi})}(\hat{\pi}) < a_3(\hat{\pi}) < j_{r_2(\hat{\pi})}(\hat{\pi}) < \cdots < j_1(\hat{\pi}) \leq n.$$

Moreover, we let $\alpha_{\hat{\pi}} = \alpha_{\pi}$, where $\alpha_{\pi} : \{1, \dots, n\} \rightarrow \{0, 1, \dots, n\}$ is the map described in Section 4.1. In particular

$$\alpha_{\hat{\pi}}(a_3(\hat{\pi})) = a_2(\hat{\pi}); \quad \alpha_{\hat{\pi}}(a_2(\hat{\pi})) = a_1(\hat{\pi}); \quad \alpha_{\hat{\pi}}(h_p(\hat{\pi})) = g_p(\hat{\pi}); \quad \alpha_{\hat{\pi}}(j_p(\hat{\pi})) = i_p(\hat{\pi});$$

$$\alpha_{\hat{\pi}}(a_1(\hat{\pi})) = \alpha_{\hat{\pi}}(g_p(\hat{\pi})) = \alpha_{\hat{\pi}}(i_p(\hat{\pi})) = 0.$$

Let (e_1, \dots, e_n) be the canonical basis of \mathbb{K}^n , and let $V_i = \langle e_1, \dots, e_i \rangle_{\mathbb{K}}$ for all i . In what follows, a linear endomorphism of \mathbb{K}^n will be identified with its matrix in the basis (e_1, \dots, e_n) .

Let $y \in \mathcal{O}_x \cap \mathcal{M}^+$, and so we see y as a linear endomorphism of \mathbb{K}^n . The fact that y is strictly upper triangular implies that $y(V_i) \subset V_{i-1}$ for all i (with $V_0 = 0$ by convention). The fact that $y \in \mathcal{O}_x$ implies that y has the same Jordan form as x , in particular $\dim \operatorname{Im} y^2 = 1$. We introduce the following numbers:

- $a_3(y) = \min\{i = 1, \dots, n : y^2(V_i) \neq 0\}$,
- $a_2(y) = \min\{i = 1, \dots, n : \operatorname{Im} y^2 \subset y(V_i)\}$,
- $a_1(y) = \min\{i = 1, \dots, n : y(V_{a_2(y)}) \subset y(V_{a_2(y)-1}) + V_i\}$,
- $a'_1(y) = \min\{i = 1, \dots, n : \operatorname{Im} y^2 \subset V_i\}$,
- $b(y) = \min\{i = 1, \dots, n : V_i \cap \operatorname{Im} y \not\subset \ker y\}$,
- $b'(y) = \min\{i = 1, \dots, n : y(V_{a_3(y)}) \subset y(V_{a_3(y)-1}) + V_i\}$.

LEMMA 2. *We have $a_1(y) \leq a'_1(y) < a_2(y) \leq b(y) \leq b'(y) < a_3(y)$.*

PROOF.

- By definition of $a_2(y)$, we have $y(V_{a_2(y)}) = y(V_{a_2(y)-1}) + \operatorname{Im} y^2$, and so $y(V_{a_2(y)}) \subset y(V_{a_2(y)-1}) + V_{a'_1(y)}$. Whence $a_1(y) \leq a'_1(y)$.
- By definition of $a_2(y)$, we have $\operatorname{Im} y^2 \subset y(V_{a_2(y)})$, thus $\operatorname{Im} y^2 \subset V_{a_2(y)-1}$. This implies $a'_1(y) < a_2(y)$.
- By definition of $b(y)$, we have $0 \neq y(V_{b(y)} \cap \operatorname{Im} y) \subset \operatorname{Im} y^2$. Since $\dim \operatorname{Im} y^2 = 1$, this implies $\operatorname{Im} y^2 \subset y(V_{b(y)})$. Whence $a_2(y) \leq b(y)$.
- By definition of $a_3(y)$, we have $y(V_{a_3(y)-1}) \subset \ker y$ and $y(V_{a_3(y)}) \not\subset \ker y$. Moreover by definition of $b'(y)$, we get $y(V_{a_3(y)}) \subset y(V_{a_3(y)-1}) + V_{b'(y)} \cap \operatorname{Im} y$. Thus, necessarily, $V_{b'(y)} \cap \operatorname{Im} y \not\subset \ker y$. Thereby, $b(y) \leq b'(y)$.
- Note that $y(V_{a_3(y)}) \subset V_{a_3(y)-1} = y(V_{a_3(y)-1}) + V_{a_3(y)-1}$. So, $b'(y) < a_3(y)$. □

Since by definition the flag (V_1, \dots, V_n) is fixed by B , the maps a_1, a_2, a_3 ,

$a'_1, b, b' : \mathcal{O}_x \cap \mathcal{M}^+ \rightarrow \mathbb{N}$ are constant on the B -orbits of $\mathcal{O}_x \cap \mathcal{M}^+$. Moreover, we have:

LEMMA 3. *Let $\hat{\pi} \in \hat{\Pi}_x$ and let $y \in \mathcal{Q}_{\hat{\pi}}$. Then, $y \in \mathcal{O}_x \cap \mathcal{M}^+$ and*

$$a_1(y) = a_1(\hat{\pi}); \quad a'_1(y) = \begin{cases} a_1(\hat{\pi}) & \text{if } r_1(\hat{\pi}) = 0 \\ g_{r_1(\hat{\pi})}(\hat{\pi}) & \text{if } r_1(\hat{\pi}) > 0 \end{cases}; \quad a_2(y) = a_2(\hat{\pi});$$

$$b(y) = b(\hat{\pi}); \quad b'(y) = \begin{cases} b(\hat{\pi}) & \text{if } r_2(\hat{\pi}) = 0 \\ i_{r_2(\hat{\pi})}(\hat{\pi}) & \text{if } r_2(\hat{\pi}) > 0 \end{cases}; \quad a_3(y) = a_3(\hat{\pi}).$$

PROOF. It is sufficient to check the lemma for $y = y_{\hat{\pi}}$, where $y_{\hat{\pi}}$ is the matrix defined in Definition 2. It follows from this definition that

$$\ker y = \langle e_i : \alpha_{\hat{\pi}}(i) = 0 \rangle_{\mathbb{K}}, \quad \ker y^2 = \langle e_i : \alpha_{\hat{\pi}}(i) = 0 \text{ or } \alpha_{\hat{\pi}} \circ \alpha_{\hat{\pi}}(i) = 0 \rangle_{\mathbb{K}}.$$

Since $\hat{\pi} \in \hat{\Pi}_x$, this yields $\dim \ker y = k + \ell + 1$ and $\dim \ker y^2 = n - 1$, which ensures that y has the same Jordan form as x . So, $y \in \mathcal{O}_x \cap \mathcal{M}^+$.

Again assuming that $y = y_{\hat{\pi}}$, we have:

- $y^2(e_i) = 0$ for all $i \neq a_3(\hat{\pi})$ and $y^2(e_{a_3(\hat{\pi})}) = e_{a_1(\hat{\pi})} + \sum_{p=1}^{r_1(\hat{\pi})} e_{g_p(\hat{\pi})} \neq 0$. This implies $a_3(y) = a_3(\hat{\pi})$.
- The same calculation shows that if $r_1(\hat{\pi}) = 0$ then $\text{Im } y^2 \subset V_{a_1(\hat{\pi})}$, $\text{Im } y^2 \not\subset V_{a_1(\hat{\pi})-1}$, and if $r_1(\hat{\pi}) > 0$ then $\text{Im } y^2 \subset V_{g_{r_1(\hat{\pi})}(\hat{\pi})}$, $\text{Im } y^2 \not\subset V_{g_{r_1(\hat{\pi})}(\hat{\pi})-1}$. Whence the formula for $a'_1(y)$.
- From the definition of $y_{\hat{\pi}}$, we get $\text{Im } y^2 = \langle y^2(e_{a_3(\hat{\pi})}) \rangle_{\mathbb{K}} = \langle y(e_{a_2(\hat{\pi})}) \rangle_{\mathbb{K}} \subset y(V_{a_2(\hat{\pi})})$ and $\text{Im } y^2 \not\subset y(V_{a_2(\hat{\pi})-1})$. Thus, $a_2(y) = a_2(\hat{\pi})$.
- Note that

$$y(e_{a_2(\hat{\pi})}) = e_{a_1(\hat{\pi})} + \sum_{p=1}^{r_1(\hat{\pi})} e_{g_p(\hat{\pi})} = e_{a_1(\hat{\pi})} + \sum_{p=1}^{r_1(\hat{\pi})} y(e_{h_p(\hat{\pi})})$$

$$\in (y(V_{a_2(\hat{\pi})-1}) + V_{a_1(\hat{\pi})}) \setminus (y(V_{a_2(\hat{\pi})-1}) + V_{a_1(\hat{\pi})-1}).$$

Thus $a_1(y) = a_1(\hat{\pi})$.

- Note that $\dim \text{Im } y / (\text{Im } y \cap \ker y) = 1$. Set $\varepsilon = 0$ if $b(\hat{\pi}) = a_2(\hat{\pi})$, and $\varepsilon = 1$ otherwise. We have

$$e_{a_2(\hat{\pi})} + \varepsilon e_{b(\hat{\pi})} = y(e_{a_3(\hat{\pi})}) - \sum_{p=1}^{r_2(\hat{\pi})} y(e_{j_p(\hat{\pi})}) \in \text{Im } y \setminus \ker y,$$

hence $\text{Im } y = (\text{Im } y \cap \ker y) \oplus \langle e_{a_2(\hat{\pi})} + \varepsilon e_{b(\hat{\pi})} \rangle_{\mathbb{K}}$. This clearly implies that $b(y) = b(\hat{\pi})$.

- Let ε be as above. Set $c = b(\hat{\pi})$ if $r_2(\hat{\pi}) = 0$, and $c = i_{r_2(\hat{\pi})}(\hat{\pi})$ otherwise. Since $y(e_{a_3(\hat{\pi})}) = e_{a_2(\hat{\pi})} + \varepsilon e_{b(\hat{\pi})} + \sum_{p=1}^{r_2(\hat{\pi})} e_{i_p(\hat{\pi})}$, we obtain that $y(e_{a_3(\hat{\pi})}) \in (y(V_{a_3(\hat{\pi})-1}) + V_c) \setminus (y(V_{a_3(\hat{\pi})-1}) + V_{c-1})$. This yields $b'(y) = c$. \square

The previous lemma shows in particular that the map $\hat{\pi} \mapsto \mathcal{Q}_{\hat{\pi}}$ is well defined from $\hat{\Pi}_x$ to the set of B -orbits of $\mathcal{O}_x \cap \mathcal{M}^+$. Our first purpose is to show the injectivity of this map.

PROPOSITION 8. *Let $\hat{\pi}, \hat{\rho} \in \hat{\Pi}_x$. If $\mathcal{Q}_{\hat{\pi}} = \mathcal{Q}_{\hat{\rho}}$, then $\hat{\pi} = \hat{\rho}$.*

PROOF. We have to check that

$$\alpha_{\hat{\pi}} = \alpha_{\hat{\rho}}; \quad b(\hat{\pi}) = b(\hat{\rho}); \quad \mathcal{E}_i(\hat{\pi}) = \mathcal{E}_i(\hat{\rho}) \quad (i = 1, 2).$$

To simplify the notation, we set

$$a_i = a_i(\hat{\pi}) \quad (i = 1, 2, 3), \quad b = b(\hat{\pi}), \quad \mathcal{E}_1(\hat{\pi}) = (\{g_p < h_p\})_{p=1}^{r_1}, \quad \mathcal{E}_2(\hat{\pi}) = (\{i_p < j_p\})_{p=1}^{r_2};$$

$$a'_i = a_i(\hat{\rho}) \quad (i = 1, 2, 3), \quad b' = b(\hat{\rho}), \quad \mathcal{E}_1(\hat{\rho}) = (\{g'_p < h'_p\})_{p=1}^{r'_1}, \quad \mathcal{E}_2(\hat{\rho}) = (\{i'_p < j'_p\})_{p=1}^{r'_2}.$$

By Lemma 3, we already have

$$a_i = a'_i \quad \text{for all } i = 1, 2, 3; \quad b = b'; \tag{1}$$

$$r_1 > 0 \Leftrightarrow r'_1 > 0, \quad \text{and in this case, } g_{r_1} = g'_{r'_1}; \tag{2}$$

$$r_2 > 0 \Leftrightarrow r'_2 > 0, \quad \text{and in this case, } i_{r_2} = i'_{r'_2}. \tag{3}$$

Now, we check that $\alpha_{\hat{\pi}} = \alpha_{\hat{\rho}}$. We argue by contradiction. So, assume that there is $j \in \{1, \dots, n\}$ such that $\alpha_{\hat{\pi}}(j) \neq \alpha_{\hat{\rho}}(j)$, say $\alpha_{\hat{\pi}}(j) < \alpha_{\hat{\rho}}(j)$. By (1), we have $j \notin \{a_i : i = 1, 2, 3\}$. Let $y = y_{\hat{\pi}}$. Since we assume that $\mathcal{Q}_{\hat{\pi}} = \mathcal{Q}_{\hat{\rho}}$, there is a basis (f_1, \dots, f_n) of \mathbb{K}^n such that $\langle f_1, \dots, f_i \rangle_{\mathbb{K}} = \langle e_1, \dots, e_i \rangle_{\mathbb{K}} = V_i$ for all i , and such that the matrix of y in (f_1, \dots, f_n) is $y_{\hat{\rho}}$. We can write

$$e_j = \sum_{i=1}^j \lambda_i f_i \tag{4}$$

with $\lambda_1, \dots, \lambda_j \in \mathbb{K}$, $\lambda_j \neq 0$. Let $I = \{i = 1, \dots, j : \lambda_i \neq 0 \text{ and } \alpha_{\hat{\rho}}(i) \neq 0\}$. Note that $y^2(e_j) = 0$ (since $j \neq a_3$), hence $a'_3 \notin I$. Applying y to equality (4), we get

$$e_{\alpha_{\hat{\pi}}(j)} = \lambda_j f_{\alpha_{\hat{\rho}}(j)} + \sum_{i \in I \setminus \{j, a'_2\}} \lambda_i f_{\alpha_{\hat{\rho}}(i)} + \lambda_{a'_2} \left(f_{a'_1} + \sum_{p=1}^{r'_1} f_{g'_p} \right)$$

where $\lambda_{a'_2} := 0$ if $j < a'_2$. In the case where $\lambda_{a'_2} \neq 0$, we then have $j > a'_2 > h'_1 > \dots > h'_{r'_1}$, hence $\alpha_{\hat{\rho}}(j) \notin \{g'_p : p = 1, \dots, r'_1\}$. This yields in all cases

$$e_{\alpha_{\hat{\pi}}(j)} - \lambda_j f_{\alpha_{\hat{\rho}}(j)} \in \langle f_i : i \neq \alpha_{\hat{\rho}}(j) \rangle_{\mathbb{K}}, \text{ hence } e_{\alpha_{\hat{\pi}}(j)} \notin V_{\alpha_{\hat{\rho}}(j)-1}.$$

But this is contradictory, since $e_{\alpha_{\hat{\pi}}(j)} \in V_{\alpha_{\hat{\pi}}(j)} \subset V_{\alpha_{\hat{\rho}}(j)-1}$, where we use the assumption made that $\alpha_{\hat{\pi}}(j) < \alpha_{\hat{\rho}}(j)$. Finally, we have shown:

$$\alpha_{\hat{\pi}} = \alpha_{\hat{\rho}}. \tag{5}$$

Our next purpose is to establish the relation $\mathcal{E}_1(\hat{\pi}) = \mathcal{E}_1(\hat{\rho})$. To this end, we need to show that

$$r_1 = r'_1 \text{ and } (g_p, h_p) = (g'_p, h'_p) \quad \forall p = 1, \dots, r_1. \tag{6}$$

By (2), we may assume that $r_1, r'_1 > 0$ (otherwise there is nothing to prove). Fix $y \in \mathcal{Q}_{\hat{\pi}} = \mathcal{Q}_{\hat{\rho}}$. We claim that

$$g_p = \min \{ i = 1, \dots, n : \text{Im } y^2 \subset y(V_{h_{p+1}}) + V_i \} \quad \forall p = 1, \dots, r_1 - 1, \tag{7}$$

$$a_1(y) = a_1 = \min \{ i = 1, \dots, n : \text{Im } y^2 \subset y(V_{h_1}) + V_i \}. \tag{8}$$

Since the right-hand sides in (7), (8) are clearly independent of $y \in \mathcal{Q}_{\hat{\pi}}$, it is sufficient to establish the relations for $y = y_{\hat{\pi}}$. Then, (7), (8) follow from Definition 2 and straightforward calculations. Similarly, we have

$$g'_p = \min \{ i = 1, \dots, n : \text{Im } y^2 \subset y(V_{h'_{p+1}}) + V_i \} \quad \forall p = 1, \dots, r'_1 - 1, \tag{9}$$

$$a_1(y) = a'_1 = \min \{ i = 1, \dots, n : \text{Im } y^2 \subset y(V_{h'_1}) + V_i \}. \tag{10}$$

Let us use (7)–(10) to establish (6). We first show by induction on $p \geq 0$ that

$$(g_{r_1-p}, h_{r_1-p}) = (g'_{r'_1-p}, h'_{r'_1-p}) \quad \forall p \in \{0, \dots, \min\{r_1 - 1, r'_1 - 1\}\}. \tag{11}$$

By (2), we have $g_{r_1} = g'_{r'_1}$. As we know $\alpha_{\hat{\pi}} = \alpha_{\hat{\rho}}$ and $g_{r_1} = \alpha_{\hat{\pi}}(h_{r_1})$, $g'_{r'_1} = \alpha_{\hat{\rho}}(h'_{r'_1})$, we get $h_{r_1} = h'_{r'_1}$. Hence, the desired property holds for $p = 0$. Assuming that the

formula holds until rank $p - 1 \in \{0, \dots, \min\{r_1 - 2, r'_1 - 2\}\}$, by (7), (9), we get

$$\begin{aligned} g_{r_1-p} &= \min \{i = 1, \dots, n : \text{Im } y^2 \subset y(V_{h_{r_1-p+1}}) + V_i\} \\ &= \min \{i = 1, \dots, n : \text{Im } y^2 \subset y(V_{h'_{r'_1-p+1}}) + V_i\} = g'_{r'_1-p}. \end{aligned}$$

As $\alpha_{\hat{\pi}} = \alpha_{\hat{\rho}}$ and $g_{r_1-p} = \alpha_{\hat{\pi}}(h_{r_1-p})$, $g'_{r'_1-p} = \alpha_{\hat{\rho}}(h'_{r'_1-p})$, this yields $h_{r_1-p} = h'_{r'_1-p}$, and we have shown the property at rank p . By induction, this finishes the proof of (11). To deduce (6), it remains to show that $r_1 = r'_1$. Assume that $r_1 \neq r'_1$, and say, for instance, $r_1 < r'_1$. Then (8), (9) imply

$$\begin{aligned} a_1(y) &= \min \{i = 1, \dots, n : \text{Im } y^2 \subset y(V_{h_1}) + V_i\} \\ &= \min \{i = 1, \dots, n : \text{Im } y^2 \subset y(V_{h'_{r'_1-r_1+1}}) + V_i\} = g'_{r'_1-r_1}. \end{aligned}$$

However we have $g'_{r'_1-r_1} \geq g'_1 > a'_1 = a_1(y)$. This is a contradiction. Therefore, $r_1 = r'_1$. This finishes the proof of (6).

To finish the proof of $\hat{\pi} = \hat{\rho}$ (and thus to complete the proof of the proposition), it remains to show the relation $\mathcal{E}_2(\hat{\pi}) = \mathcal{E}_2(\hat{\rho})$, which reduces to show that

$$r_2 = r'_2 \quad \text{and} \quad (i_p, j_p) = (i'_p, j'_p) \quad \forall p = 1, \dots, r_2. \tag{12}$$

By (3), we may assume that $r_2, r'_2 > 0$ (otherwise there is nothing to prove). Fixing $y \in \mathcal{Q}_{\hat{\pi}} = \mathcal{Q}_{\hat{\rho}}$, we claim that

$$\begin{aligned} i_p &= \min \{i = 1, \dots, n : y(V_{j_{p+1}}) \subset y(V_{j_{p+1}-1}) + V_i\} \\ &\quad \forall p = 1, \dots, r_2 - 1, \end{aligned} \tag{13}$$

$$b(y) = b = \min \{i = 1, \dots, n : y(V_{j_1}) \subset y(V_{j_1-1}) + V_i\}. \tag{14}$$

Like for (7), (8) above, it is sufficient to check these relations for $y = y_{\hat{\pi}}$. Then, (13), (14) easily follow from Definition 2. Similarly, we have

$$\begin{aligned} i'_p &= \min \{i = 1, \dots, n : y(V_{j'_{p+1}}) \subset y(V_{j'_{p+1}-1}) + V_i\} \\ &\quad \forall p = 1, \dots, r'_2 - 1, \end{aligned} \tag{15}$$

$$b(y) = b' = \min \{i = 1, \dots, n : y(V_{j'_1}) \subset y(V_{j'_1-1}) + V_i\}. \tag{16}$$

Exactly like for (6) before, (12) is implied by (13)–(16), after easy induction. \square

Finally, we prove the surjectivity of the map $\hat{\pi} \mapsto \mathcal{Q}_{\hat{\pi}}$ ($\hat{\pi} \in \hat{\Pi}_x$):

PROPOSITION 9. *Each B-orbit of $\mathcal{O}_x \cap \mathcal{M}^+$ is of the form $\mathcal{Q}_{\hat{\pi}}$ for some $\hat{\pi} \in \hat{\Pi}_x$.*

PROOF. Fix an arbitrary element $y \in \mathcal{O}_x \cap \mathcal{M}^+$. We need to check that y lies in an orbit of the form $\mathcal{Q}_{\hat{\pi}}$ (for $\hat{\pi} \in \hat{\Pi}_x$). As before, (e_1, \dots, e_n) is the canonical basis of \mathbb{K}^n and we let $V_i = \langle e_1, \dots, e_i \rangle_{\mathbb{K}}$ for all i . To prove the proposition, it is sufficient to provide a basis (e'_1, \dots, e'_n) of \mathbb{K}^n with the properties

- $V_i = \langle e'_1, \dots, e'_i \rangle_{\mathbb{K}}$ for each i ,
- the matrix of y in (e'_1, \dots, e'_n) is $y_{\hat{\pi}}$ for some $\hat{\pi} \in \hat{\Pi}_x$.

Let $a_3 = a_3(y)$. By the definition of $a_3(y)$ given in the beginning of Section 5, we have

$$V_i \subset V_{i-1} + \ker y^2 \quad \forall i \in \{1, \dots, n\} \setminus \{a_3\}.$$

Thus we can find $f_i \in V_i \setminus V_{i-1}$ ($i = 1, \dots, n$) such that $y^2(f_i) = 0$ for all $i \neq a_3$. Thus, $\ker y^2 = \langle f_i : i \in \{1, \dots, n\} \setminus \{a_3\} \rangle_{\mathbb{K}}$, and the restriction $y' := y|_{\ker y^2}$ has Jordan form $\lambda(y') = (2^{k+1}, 1^\ell)$. By virtue of Proposition 4 (a), we may assume that the basis (f_1, \dots, f_n) has been chosen so that

$$y(f_i) = f_{\alpha(i)} \quad \forall i \in \{1, \dots, n\} \setminus \{a_3\},$$

where $\alpha : \{1, \dots, n\} \setminus \{a_3\} \rightarrow \{0, 1, \dots, n\} \setminus \{a_3\}$ is a map satisfying the following properties

- $\alpha(i) < i$ for each i ,
- if $\alpha(i) = \alpha(j) \neq 0$, then $i = j$,
- $|\{i : \alpha(i) = 0\}| = k + \ell + 1$, $|\{i : \alpha(i) \neq 0\}| = k + 1$,

and where $f_0 = 0$ by convention. Moreover, we have

$$y(f_{a_3}) = \sum_{i=1}^{a_3-1} \lambda_i f_i \tag{17}$$

where $\lambda_i \in \mathbb{K}$. By replacing f_{a_3} by $f_{a_3} - \sum_{i=1}^{a_3-1} \lambda_{\alpha(i)} f_i$ (where $\lambda_0 = 0$ by convention), we may suppose that $\lambda_{\alpha(i)} = 0$ for each $i \in \{1, \dots, a_3 - 1\}$. Then, let $I = \{i = 1, \dots, a_3 - 1 : \lambda_i \neq 0\}$. By replacing f_i by $\lambda_i f_i$ for each $i \in I$, $f_{\alpha(i)}$ by $\lambda_i f_{\alpha(i)}$ for each $i \in I$ such that $\alpha(i) \neq 0$, and f_i by $\lambda_{\alpha(i)} f_i$ for each $i > a_3$ such that $\alpha(i) \in I$, we may suppose that $\lambda_i = 1$ for all $i \in I$. Thus, (17) becomes

$$y(f_{a_3}) = \sum_{i \in I} f_i. \tag{18}$$

We decompose the set I as $I = I_1 \sqcup I_2 \sqcup I_3$, where

$$\begin{aligned} I_1 &= \{i \in I : \alpha(i) \neq 0\}, \\ I_2 &= \{i \in I : \alpha(i) = 0, \text{ and } i \neq \alpha(j) \text{ for all } j\}, \\ I_3 &= \{i \in I : \alpha(i) = 0, \text{ and } i = \alpha(j) \text{ for some } j > a_3\}. \end{aligned}$$

We then emphasize some indices as follows:

- First, we notice that $I_1 \neq \emptyset$ (since $\sum_{i \in I_1} f_{\alpha(i)} = y^2(f_{a_3}) \neq 0$). Let

$$a_2 = \max I_1 \quad \text{and} \quad a_1 = \alpha(a_2).$$

Thereby, we have $0 < a_1 < a_2 < a_3$. We denote $I'_1 = \{i \in I_1 : \alpha(i) \leq a_1\}$ and $I''_1 = \{i \in I_1 : \alpha(i) > a_1\} =: \{h_1 > h_2 > \dots > h_{r_1}\}$. Set $g_p = \alpha(h_p)$ for each $p \in \{1, \dots, r_1\}$. Finally, we let

$$\mathcal{E}_1 = (\{g_p, h_p\})_{p=1}^{r_1}.$$

- Set

$$b = \max(I_1 \cup I_2).$$

Moreover, let $I'_2 = \{i \in I_2 : i < a_2\}$ and $I''_2 = \{i \in I_2 : i > a_2\}$. In particular, we see that $a_2 \leq b < a_3$, with $a_2 < b$ if and only if $I''_2 \neq \emptyset$ (and in this case, $b \in I''_2$).

- We write $I_3 = I'_3 \sqcup I''_3 \sqcup I'''_3$ where $I'_3 = \{i \in I_3 : i < a_2\}$, $I''_3 = \{i \in I_3 : a_2 < i < b\}$ and $I'''_3 = \{i \in I_3 : i > b\} =: \{i_1 < i_2 < \dots < i_{r_2}\}$. For each $p \in \{1, \dots, r_2\}$, let $j_p > a_3$ be such that $i_p = \alpha(j_p)$. We denote

$$\mathcal{E}_2 = (\{i_p, j_p\})_{p=1}^{r_2}.$$

Now, we introduce a new basis (e'_1, \dots, e'_n) as follows. Set

$$e'_{a_2} = \sum_{i \in I_1 \cup I'_2 \cup I'_3} f_i \quad \text{and} \quad e'_{a_1} = \sum_{i \in I'_1} f_{\alpha(i)}.$$

If $b > a_2$, then we set in addition

$$e'_b = \sum_{i \in I''_2 \cup I''_3} f_i.$$

Finally, let $e'_i = f_i$ for each $i \in \{1, \dots, n\} \setminus \{a_1, a_2, b\}$. We also extend the map α into $\alpha : \{1, \dots, n\} \rightarrow \{0, 1, \dots, n\}$ by letting $\alpha(a_3) = a_2$. There is $\pi \in \Pi_x$ such that $\alpha = \alpha_\pi$. Finally, we have shown that there exist a tuple $(\pi, b, \mathcal{E}_1, \mathcal{E}_2)$ and a basis (e'_1, \dots, e'_n) such that

$$\left\{ \begin{array}{l} \bullet \pi \in \Pi_x, \text{ containing a unique subset with 3 elements } \{a_1 < a_2 < a_3\}, \\ \bullet a_2 \leq b < a_3 \text{ and either } b = a_2 \text{ or } (\alpha_\pi(b) = 0 \text{ and } b \neq \alpha_\pi(j) \text{ for all } j), \\ \bullet \mathcal{E}_1 = (\{g_p, h_p\})_{p=1}^{r_1}, \mathcal{E}_2 = (\{i_p, j_p\})_{p=1}^{r_2} \text{ with} \\ \quad h_{r_1} < \dots < h_2 < h_1 < a_2, \text{ and } a_1 < g_p = \alpha_\pi(h_p) < h_p \text{ for all } p, \\ \quad b < i_1 < i_2 < \dots < i_{r_2}, \text{ and } i_p = \alpha_\pi(j_p) < a_3 < j_p \text{ for all } p, \end{array} \right. \quad (19)$$

and

$$\left\{ \begin{array}{l} V_i = \langle e'_1, \dots, e'_i \rangle_{\mathbb{K}} \text{ for each } i \in \{1, \dots, n\}, \\ y(e'_i) = e'_{\alpha_\pi(i)} \quad \forall i \in \{1, \dots, n\} \setminus \{a_2, a_3\} \text{ (where } e'_0 = 0), \\ y(e'_{a_2}) = e'_{a_1} + \sum_{p=1}^{r_1} e'_{g_p} \quad \text{and} \quad y(e'_{a_3}) = e'_{a_2} + \varepsilon e'_b + \sum_{p=1}^{r_2} e'_{i_p} \end{array} \right. \quad (20)$$

with $\varepsilon = 1$ if $b > a_2$ and $\varepsilon = 0$ otherwise. Choose a tuple $(\pi, b, \mathcal{E}_1, \mathcal{E}_2)$ and a basis (e'_1, \dots, e'_n) satisfying (19)–(20) such that $r_1 + r_2$ is minimal. Then, we claim that

$$a_1 < g_1 < \dots < g_{r_1} < h_{r_1} < \dots < h_1 < a_2, \quad (21)$$

$$b < i_1 < \dots < i_{r_2} < a_3 < j_{r_2} < \dots < j_1. \quad (22)$$

Note that (19), (21), (22) imply that $\hat{\pi} := (\pi, b, \mathcal{E}_1, \mathcal{E}_2) \in \hat{\Pi}_x$. Then, (20) shows that the matrix of y in the basis (e'_1, \dots, e'_n) is $y_{\hat{\pi}}$. In other words, it is sufficient to establish (21), (22) in order to complete the proof of the proposition.

First, we show (21). Arguing by contradiction, we assume that (21) does not hold. Hence there are $p, q \in \{1, \dots, r_1\}$ with $p < q$ such that

$$g_q < h_q < g_p < h_p \quad \text{or} \quad g_q < g_p < h_q < h_p.$$

In both cases, we let $e''_{h_p} = e'_{h_p} + e'_{h_q}$, $e''_{g_p} = e'_{g_p} + e'_{g_q}$, and $e''_i = e'_i$ for each $i \notin \{g_p, h_p\}$. The new basis satisfies

$$y(e''_{a_2}) = e''_{a_1} + \sum_{\substack{m=1 \\ m \neq q}}^{r_1} e''_{g_m}, \quad y(e''_{a_3}) = e''_{a_2} + \varepsilon e''_b + \sum_{m=1}^{r_2} e''_{i_m}$$

and $y(e''_i) = e''_{\alpha_\pi(i)}$ for $i \notin \{a_2, a_3\}$ (with $e''_0 := 0$). Hence, the new tuple $(\pi, b, \mathcal{E}'_1, \mathcal{E}_2)$, where $\mathcal{E}'_1 := (\{g_m, h_m\})_{m \in \{1, \dots, r_1\} \setminus \{q\}}$, and the basis (e''_1, \dots, e''_n) satisfy (19)–(20). This contradicts the minimality of $r_1 + r_2$. Whence (21).

Next, we prove (22). Again arguing by contradiction, we assume that there are $p, q \in \{1, \dots, r_2\}$ with $p < q$ such that

$$i_p < i_q < a_3 < j_p < j_q.$$

Similarly, we construct a new basis (e''_1, \dots, e''_n) by letting $e''_{i_q} = e'_{i_p} + e'_{i_q}$, $e''_{j_q} = e'_{j_p} + e'_{j_q}$, and $e''_i = e'_i$ for each $i \notin \{i_q, j_q\}$. Again, the new basis satisfies

$$y(e''_{a_2}) = e''_{a_1} + \sum_{m=1}^{r_1} e''_{g_m}, \quad y(e''_{a_3}) = e''_{a_2} + \varepsilon e''_b + \sum_{\substack{m=1 \\ m \neq p}}^{r_2} e''_{i_m}$$

and $y(e''_i) = e''_{\alpha_\pi(i)}$ for all $i \notin \{a_2, a_3\}$ (with $e''_0 := 0$). As above, this contradicts the minimality of $r_1 + r_2$. Finally, we obtain that (22) holds. The proof of the proposition is complete. □

6. Further remarks.

In this last section, we formulate few remarks relative to Theorem 1, and relative to the properties of the B -orbits in cases (i)–(iv) of Theorem 1 (c).

6.1.

We may notice that Corollary 1 yields a refined version of Theorem 1. Namely, we get that the conditions (a), (b), (c) of Theorem 1 are equivalent with

(b') *Each Richardson orbital variety $\mathcal{V}^R(\eta) \subset \mathcal{O}_x \cap \mathcal{M}^+$ admits a dense B -orbit.*

This observation testifies that the Richardson orbital varieties form the family of orbital varieties which are the most susceptible to admit no dense B -orbit.

6.2.

The equivalence between (a) and (b) in Theorem 1 hides that a given orbital

variety may contain an infinite number of B -orbits even when it has a dense B -orbit. To illustrate this fact, let us emphasize another particular family of orbital varieties, dual to the family of Richardson orbital varieties.

We start with a definition in the general case of G being reductive. The notation is as in Sections 2–3. Like in the definition of Richardson orbital varieties, we consider the situation of a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ such that $\mathfrak{p} \supset \mathfrak{b}$. Let $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}_P$ be a Levi decomposition, with Levi factor \mathfrak{l} and nilradical \mathfrak{u}_P . As seen in Section 3, a Richardson orbital variety arises from this decomposition after focusing on the nilradical \mathfrak{u}_P . Here, we rather focus on the Levi factor \mathfrak{l} . Its regular nilpotent orbit $\mathcal{O}_{\mathfrak{l}, \text{reg}} \subset \mathfrak{l}$ lies in a unique nilpotent orbit (for G) $\mathcal{O}' \subset \mathfrak{g}$. If we pick up an element $y \in \mathcal{O}_{\mathfrak{l}, \text{reg}} \cap \mathfrak{n}$, then, by [18, Section 4], $\overline{B \cdot y} \cap \mathcal{O}'$ turns out to be an irreducible component of $\mathcal{O}' \cap \mathfrak{n}$ (independent of the choice of \mathfrak{l} and y , cf. [4, Section 2]). Hence, it is an orbital variety attached to the nilpotent orbit \mathcal{O}' . An orbital variety obtained in this way is called a *Bala-Carter orbital variety*. It contains a dense B -orbit by construction.

In the case of $G = GL_n(\mathbb{K})$, the Bala-Carter orbital varieties are parametrized by the compositions of n : let $\eta = (\eta_1, \dots, \eta_r)$ be a composition of n , and let $\lambda(\eta)$ be the corresponding partition of n (i.e., the partition obtained by arranging the terms in the sequence η in nonincreasing order). As in Section 3, let $\mathfrak{p}(\eta)$, $\mathfrak{l}(\eta)$ be respectively the standard parabolic subalgebra of \mathfrak{g} corresponding to η , and its standard Levi factor. Thus, a regular nilpotent element $y_\eta \in \mathfrak{l}(\eta)$ has Jordan form $\lambda(y_\eta) = \lambda(\eta)$. The intersection $\mathcal{V}^{BC}(\eta) := \overline{B \cdot y_\eta} \cap \mathcal{O}_{y_\eta}$ is the Bala-Carter orbital variety associated to the Levi factor $\mathfrak{l}(\eta)$ in the above sense. Each Bala-Carter orbital variety of $\mathcal{M}_n(\mathbb{K})$ is of the form $\mathcal{V}^{BC}(\eta)$ for a unique composition η , and the Bala-Carter orbital varieties which lie in a prescribed nilpotent conjugacy class \mathcal{O}_x (for $x \in \mathcal{M}_n(\mathbb{K})$) correspond to the compositions of n such that $\lambda(\eta) = \lambda(x)$.

Thus, the Bala-Carter orbital varieties form a family of orbital varieties admitting dense B -orbits. In the case of $GL_n(\mathbb{K})$, they occur in all nilpotent conjugacy classes.

However, not all the Bala-Carter orbital varieties consist of a finite number of B -orbits. For instance, if $\eta = (6, 4)$, then it can be seen that $\mathcal{V}^{BC}(\eta)$ contains an infinite number of B -orbits (see [4, Section 6.4] for the detail of the computations).

The question to determine whether a given orbital variety consists of a finite number of B -orbits seems to be difficult. Until now, the only cases where this question is answered are cases (i)–(iv) of Theorem 1 (c) and the case of Richardson orbital varieties (see Proposition 2). Moreover, we do not know whether in every nilpotent conjugacy class $\mathcal{O}_x \subset \mathcal{M}_n(\mathbb{K})$ we can find at least one orbital variety which comprises a finite number of B -orbits.

6.3.

The fact that an orbital variety $\mathcal{V} \subset \mathcal{O}_x \cap \mathcal{M}^+$ admits a finite number of B -orbits does not imply that its closure $\overline{\mathcal{V}} \subset \mathcal{M}^+$ (in Zariski topology) admits finitely many B -orbits. More particularly, given η a composition of n , the corresponding nilradical $\mathfrak{u}(\eta) = \overline{\mathcal{V}^R(\eta)} \subset \mathcal{M}^+$ may simultaneously admit a dense B -orbit and an infinite number of B -orbits. A trivial example is $\eta = (1, \dots, 1) = (1^r)$ with $r \geq 6$: in this case, $\mathfrak{u}(\eta) = \mathcal{M}^+$ admits a dense B -orbit but an infinite number of B -orbits. More generally, if $\eta = (\eta_1, \dots, \eta_r)$ with $\eta_1, \dots, \eta_r \geq 1$ and $r \geq 6$, then it follows from [8] that $\mathfrak{u}(\eta)$ has an infinite number of B -orbits. This observation illustrates that, in Lemma 1, the assumption that the parabolic subgroup P acts transitively is indispensable.

6.4.

For $x \in \mathcal{M}_n(\mathbb{K})$ a nilpotent matrix, we may notice that $\overline{\mathcal{O}_x \cap \mathcal{M}^+} = \overline{\mathcal{O}_x} \cap \mathcal{M}^+$ (cf. [12]).

Now, recall the dominance order between partitions. Given $\lambda = (\lambda_1, \dots, \lambda_r)$ and $\mu = (\mu_1, \dots, \mu_s)$ two partitions of n , we write $\lambda \preceq \mu$ if we have $r \geq s$ and $\sum_{i=1}^j \lambda_i \leq \sum_{i=1}^j \mu_i$ for all $j = 1, \dots, s$. By a result of M. Gerstenhaber [6], for $x, y \in \mathcal{M}_n(\mathbb{K})$ being nilpotent matrices, we have $\mathcal{O}_y \subset \overline{\mathcal{O}_x}$ if and only if $\lambda(y) \preceq \lambda(x)$.

Combining these observations, we deduce from Theorem 1 the following

COROLLARY 2. *Let $x \in \mathcal{M}_n(\mathbb{K})$ be a nilpotent matrix. The following conditions are equivalent.*

- (a) $\overline{\mathcal{O}_x \cap \mathcal{M}^+}$ consists of a finite number of B -orbits;
- (b) the Jordan form $\lambda(x)$ is of one of the following types:
 - (i') $\lambda(x) = (\lambda_1, 1^k)$ with either $\lambda_1 \in \{1, 2, 3, 4\}$ and $k \geq 0$, or $\lambda_1 = 5$ and $k = 0$;
 - (ii) $\lambda(x) = (2^k, 1^\ell)$ with $k \geq 2, \ell \geq 0$;
 - (iii) $\lambda(x) = (3, 2^k, 1^\ell)$ with $k \geq 1, \ell \geq 0$;
 - (iv) $\lambda(x) = (3, 3)$.

Only case (i') is different from its homologue (case (i)) of Theorem 1 (c).

6.5.

The main question which remains is to determine the degeneracy order for the B -orbits in case (iii) of Theorem 1 (c).

It is noticeable that the inclusion relations between B -orbit closures can be described by a common criterion in cases (i), (ii), (iv) of Theorem 1 (c). For $y \in \mathcal{M}^+$ and $1 \leq j < i \leq n$, recall that $y_{[j,i]} := (y_{k,\ell})_{j \leq k, \ell \leq i}$ is the submatrix contained between the j -th and i -th rows and columns. The following proposition

can be deduced from the description of the B -orbits given in Sections 4.2–4.4.

PROPOSITION 10. *Let $x \in \mathcal{M}_n(\mathbb{K})$ be a nilpotent matrix whose Jordan form $\lambda(x)$ is like in case (i), (ii) or (iv) of Theorem 1(c). Let $y, z \in \mathcal{O}_x \cap \mathcal{M}^+$. The following conditions are equivalent:*

- (a) $B \cdot y \subset \overline{B \cdot z}$;
- (b) $\text{rk } y_{[j,i]}^k \leq \text{rk } z_{[j,i]}^k$ for all $1 \leq j < i \leq n$, $k \geq 1$.

In case (iii) of Theorem 1(c), the criterion given in the proposition is not sufficient to describe the B -orbit closures. For instance, let

$$y = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad z = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The matrices y, z lie in $\mathcal{O}_x \cap \mathcal{M}^+$ for $\lambda(x) = (3, 2, 1)$. It can be checked that they satisfy $\text{rk } y_{[j,i]}^k \leq \text{rk } z_{[j,i]}^k$ for all $1 \leq j < i \leq 6$, $k \geq 1$. But $B \cdot y \not\subset \overline{B \cdot z}$. Indeed, letting (e_1, \dots, e_6) be the canonical basis of \mathbb{K}^6 , we see that each element $z' \in \overline{B \cdot z}$ satisfies $\ker z' \cap \text{Im } z' \subset \langle e_1, e_2, e_3 \rangle_{\mathbb{K}}$, whereas y does not satisfy this property.

6.6.

Comparing Theorem 1 with the main result of [3] (and invoking for instance [4, Proposition 3.2]) yields the following coincidence: given a nilpotent matrix $x \in \mathcal{M}_n(\mathbb{K})$, let $x^* \in \mathcal{M}_n(\mathbb{K})$ be a nilpotent matrix whose Jordan form is conjugate (i.e., if $\lambda(x) = (\lambda_1, \dots, \lambda_r)$, then $\lambda(x^*) = (\lambda_1^*, \dots, \lambda_{\lambda_1}^*)$ with $\lambda_j^* = |\{i = 1, \dots, r : \lambda_i \geq j\}|$). Then, it holds that every irreducible component of $\mathcal{O}_x \cap \mathcal{M}^+$ has a dense B -orbit (resp. $\mathcal{O}_x \cap \mathcal{M}^+$ has a finite number of B -orbits) if and only if every component of $\mathcal{O}_{x^*} \cap \mathcal{M}^+$ is smooth. This unexpected relation between smoothness and existence of dense B -orbits for orbital varieties is underlined in [4]. Another aspect of this relation, shown in [4, Theorem 1.2], is for instance that a Richardson orbital variety $\mathcal{V}^R(\eta)$ has a dense B -orbit if and only if the Bala-Carter orbital variety $\mathcal{V}^{BC}(\eta)$ (corresponding to the same composition η) is smooth.

ACKNOWLEDGEMENTS. I am grateful to William M. McGovern and to Anna Melnikov for discussions which have motivated this work.

References

- [1] M. Brion, Quelques propriétés des espaces homogènes sphériques, *Manuscripta Math.*, **55** (1986), 191–198.
- [2] L. Fresse, On the singularity of some special components of Springer fibers, *J. Lie Theory*, **21** (2011), 205–242.
- [3] L. Fresse and A. Melnikov, On the singularity of the irreducible components of a Springer fiber in \mathfrak{sl}_n , *Selecta Math. (N.S.)*, **16** (2010), 393–418.
- [4] L. Fresse and A. Melnikov, Smooth orbital varieties and orbital varieties with a dense B -orbit, *Int. Math. Res. Not.*, **2013** (2013), 1122–1203.
- [5] F. Y. C. Fung, On the topology of components of some Springer fibers and their relation to Kazhdan-Lusztig theory, *Adv. Math.*, **178** (2003), 244–276.
- [6] M. Gerstenhaber, On dominance and varieties of commuting matrices, *Ann. of Math. (2)*, **73** (1961), 324–348.
- [7] W. H. Hesselink, A classification of the nilpotent triangular matrices, *Compositio Math.*, **55** (1985), 89–133.
- [8] L. Hille and G. Röhrle, A classification of parabolic subgroups of classical groups with a finite number of orbits on the unipotent radical, *Transform. Groups*, **4** (1999), 35–52.
- [9] A. Joseph, On the variety of a highest weight module, *J. Algebra*, **88** (1984), 238–278.
- [10] A. Melnikov, Description of B -orbit closures of order 2 in upper-triangular matrices, *Transform. Groups*, **11** (2006), 217–247.
- [11] A. Melnikov, B -orbits of nilpotent order 2 and link patterns, preprint, arxiv: math/0703371.
- [12] D. Mertens, Über einen Satz von Spaltenstein, University of Wuppertal, preprint.
- [13] D. I. Panyushev, Complexity and nilpotent orbits, *Manuscripta Math.*, **83** (1994), 223–237.
- [14] R. W. Richardson, Jr., Conjugacy classes in parabolic subgroups of semisimple algebraic groups, *Bull. London Math. Soc.*, **6** (1974), 21–24.
- [15] N. Spaltenstein, On the fixed point set of a unipotent element on the variety of Borel subgroups, *Topology*, **16** (1977), 203–204.
- [16] T. A. Springer, Trigonometric sums, Green functions of finite groups and representations of Weyl groups, *Invent. Math.*, **36** (1976), 173–207.
- [17] T. A. Springer, A construction of representations of Weyl groups, *Invent. Math.*, **44** (1978), 279–293.
- [18] R. Steinberg, On the desingularization of the unipotent variety, *Invent. Math.*, **36** (1976), 209–224.
- [19] J. A. Vargas, Fixed points under the action of unipotent elements of SL_n in the flag variety, *Bol. Soc. Mat. Mexicana (2)*, **24** (1979), 1–14.
- [20] È. B. Vinberg, Complexity of actions of reductive groups, *Funktional. Anal. i Prilozhen.*, **20** (1986), 1–13. In Russian; English translation in *Funct. Anal. Appl.*, **20** (1986), 1–11.

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