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The Stokes flow around a rotating body in the whole space

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Abstract. We analyze in weighted L^q-spaces the linearized system of partial differential equations arising from the motion of a rotating obstacle in a fluid. We prove some existence, uniqueness and regularity results of decaying or growing weak solutions. Two auxiliary equations are also considered and treated.

1. Introduction.

The motion of a rotating rigid body in a viscous incompressible fluid occupying an unbounded domain $\Omega \subset \mathbb{R}^n$, $n = 3$ or 2, can be modelled by the modified Navier-Stokes equations

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} - \nu\Delta\boldsymbol{u} + \boldsymbol{u}.\nabla\boldsymbol{u} - (\boldsymbol{\omega}\times\boldsymbol{x}).\nabla\boldsymbol{u} + \boldsymbol{\omega}\times\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega\times(0,+\infty), \\ \operatorname{div}\boldsymbol{u} = 0, \end{aligned} \tag{1.1}$$

completed with initial and boundary conditions when $\Omega \neq \mathbb{R}^n$, and with the asymptotic condition

$$|\boldsymbol{u}| \longrightarrow 0 \quad \text{when} \quad |\boldsymbol{x}| \longrightarrow +\infty. \tag{1.2}$$

Here $\boldsymbol{u}$ and p denote the unknown velocity and the pressure, $\boldsymbol{f}$ is the prescribed external force and $\boldsymbol{\omega} = |\boldsymbol{\omega}|(0,0,1)^T \neq \boldsymbol{0}$ is the angular velocity. System (1.1) is obtained by rewriting usual Navier-Stokes equations in a rotating coordinate system attached to the rigid body. Neglecting the nonlinear terms and considering only the stationary problem yield the modified Stokes system

$$\begin{aligned} -\nu\Delta\boldsymbol{u} - (\boldsymbol{\omega}\times\boldsymbol{x}).\nabla\boldsymbol{u} + \boldsymbol{\omega}\times\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega. \end{aligned} \tag{1.3}$$

In the two dimensional case $(n = 2)$, the product $\boldsymbol{\omega}\times\boldsymbol{a}$, $\boldsymbol{a} = (a_1, a_2) \in \mathbb{R}^2$, must

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be understood in the following sense

$$\boldsymbol{\omega} \times \boldsymbol{a} = |\boldsymbol{\omega}|(-a_2, a_1)^T.$$

We henceforth suppose without loss that $\nu = 1$. We deal with the following system in $\mathbb{R}^n$, $n = 2$ or 3

$$\begin{aligned} -\Delta \boldsymbol{u} - (\boldsymbol{\omega} \times \boldsymbol{x}).\nabla \boldsymbol{u} + \boldsymbol{\omega} \times \boldsymbol{u} + \nabla p = \boldsymbol{f} \quad &\text{in } \mathbb{R}^n, \\ \operatorname{div} \boldsymbol{u} = g \quad &\text{in } \mathbb{R}^n. \end{aligned} \tag{1.4}$$

From a mathematical viewpoint, analyzing equations (1.4) is an important step in studying the system (1.1) which attracted the attention of several authors in the two last decades; see for example [**17**], [**18**], [**19**], [**10**], [**8**], [**21**], [**7**], [**9**], [**11**] and references therein.

In [**8**], authors considered equations (1.4) when $\boldsymbol{f}$ belongs to $L^q(\mathbb{R}^n)^n$ and ∇g, $(\boldsymbol{\omega} \times \boldsymbol{x})g \in L^q(\mathbb{R}^n)^n$, $1 < q < +\infty$. They prove the existence of a weak solution $(\boldsymbol{u}, p) \in L^1_{loc}(\mathbb{R}^n)^n \times L^1_{loc}(\mathbb{R}^n)$ satisfying the estimate

$$\|\nabla^2 \boldsymbol{u}\|_{L^q(\mathbb{R}^n)^{n^3}} + \|\nabla p\|_{L^q(\mathbb{R}^n)^n} \lesssim \|\boldsymbol{f}\|_{L^q(\mathbb{R}^n)^n} + \|\nabla g + (\boldsymbol{\omega} \times \boldsymbol{x})g\|_{L^q(\mathbb{R}^n)^n}, \tag{1.5}$$

where here and henceforth the notation $A \lesssim B$ means that there exists a constant c independent of the involved functions such that $A \leq cB$.

In [**20**], Hishida proved the existence, uniqueness and L^q estimates

$$\begin{aligned} &\|\nabla \boldsymbol{u}\|_{L^q(\mathbb{R}^n)^n} + \|p\|_{L^q(\mathbb{R}^n)} \\ &\qquad \lesssim \|f\|_{H^{-1,q}(\mathbb{R}^n)} + \|g\|_{L^q(\mathbb{R}^n)^n} + \|(\boldsymbol{\omega} \times \boldsymbol{x})g\|_{H^{-1,q}(\mathbb{R}^n)}, \end{aligned} \tag{1.6}$$

where $H^{-1,q}(\mathbb{R}^n)$ denotes the dual of the homgeneous space $H^{1,q'}(\mathbb{R}^n)$ (the completion of $\mathscr{C}_0^\infty(\mathbb{R}^n)$ with repsect to the norm $\|.\|_{L^q(\mathbb{R}^n)}$). This result is generalized in Farwig and Hishida [**7**] where existence is proved in Lorentz spaces.

In Farwig et al. [**9**], estimate (1.5) when $g = 0$ was extended to weighted L^q spaces. More precisely, given a weight function w belonging to a special class, and assuming that

$$\int_{\mathbb{R}^n} w(\boldsymbol{x})|\boldsymbol{f}|^q dx < +\infty,$$

the authors proved the existence of a solution $(\boldsymbol{u}, p) \in L^1_{loc}(\mathbb{R}^n)^n \times L^1_{loc}(\mathbb{R}^n)$ of (1.4) satisfying

$$\sum_{i,\,j\le n}\int_{\mathbb{R}^n} w(\boldsymbol{x})\left|\frac{\partial^2\boldsymbol{u}}{\partial x_i\partial x_j}\right|^q dx + \int_{\mathbb{R}^n} w(\boldsymbol{x})|\nabla p|^q dx \lesssim \int_{\mathbb{R}^n} w(\boldsymbol{x})|\boldsymbol{f}|^q dx.$$

In particular, this estimate holds when w is of the form $w(\boldsymbol{x}) = \langle x\rangle^\alpha$ with

$$-\min\left(\frac{1}{q},\frac{1}{2}\right) < \frac{\alpha}{nq} < \min\left(\frac{1}{q'},\frac{1}{2}\right) = 1 - \max\left(\frac{1}{2},\frac{1}{q}\right). \tag{1.7}$$

Notice that here and henceforth $\langle x\rangle$ stands for the basic weight

$$\langle x\rangle = (1+|\boldsymbol{x}|^2)^{1/2} \quad\text{and}\quad |\boldsymbol{x}| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}. \tag{1.8}$$

Unfortunately, this last result does not give any information on the behavior of u and its lower derivatives when $|\boldsymbol{x}|$ goes to infinity.

We finally mention the work of Galdi [**10**] in which the existence, uniqueness of a solution $(\boldsymbol{u}, p)$ satisfying the estimate

$$\langle x\rangle|\boldsymbol{u}(\boldsymbol{x})| + \langle x\rangle^2(|\nabla\boldsymbol{u}(\boldsymbol{x})| + |p(\boldsymbol{x})|) + \langle x\rangle^3|\nabla p(\boldsymbol{x})| \le C, \quad \text{a.e.}$$

was proved (here C denotes a constant).

In the present paper we propose a different approach for treating system (1.4). This approach, based on the use of a suitable family of weighted spaces, allows to extend some of the above results to a larger class of asymptotic behaviors. That is, we look for weak solutions satisfying conditions of the form

$$\langle x\rangle^{k-2}\boldsymbol{u} \in L^q(\mathbb{R}^n)^n, \qquad \langle x\rangle^{k-1}\nabla\boldsymbol{u} \in L^q(\mathbb{R}^n)^{n^2},$$
$$\langle x\rangle^{k}\nabla^2\boldsymbol{u} \in L^q(\mathbb{R}^n)^{n^3}, \qquad \langle x\rangle^{k-1}p \in L^q(\mathbb{R}^n),$$

for several values of $k\in\mathbb{Z}$. We deal by the way with the first equation of (1.4), forgetting the pressure and the condition $\operatorname{div}\boldsymbol{v} = 0$, that is

$$-\Delta\boldsymbol{v} - (\boldsymbol{\omega}\times\boldsymbol{x}).\nabla\boldsymbol{v} + \boldsymbol{\omega}\times\boldsymbol{v} = \boldsymbol{f} \quad\text{in}\quad \mathbb{R}^n, \tag{1.9}$$

and with the scalar equation

$$\Delta\phi + (\boldsymbol{\omega}\times\boldsymbol{x}).\nabla\phi = h \quad\text{in}\quad \mathbb{R}^n. \tag{1.10}$$

Equations (1.9) and (1.10) are in fact intimately linked; roughly speaking, one can observe that if $\boldsymbol{v}$ is solution of (1.9), then $\operatorname{div} \boldsymbol{v}$ and $\boldsymbol{v}.\boldsymbol{\omega}$, when $n = 3$, are solutions of (1.9) with $h = -\operatorname{div} \boldsymbol{f}$ and $h = -\boldsymbol{f}.\boldsymbol{\omega}$ respectively. Conversely, if ϕ is solution of (1.10), then $\nabla\phi$ is solution of (1.9) with $\boldsymbol{f} = -\nabla h$.

An auxiliary objective is to give a complete characterization of the null spaces associated to the systems (1.4) and (1.9) and to the scalar equation (1.10). The nonlinear case will be treated in a forthcoming paper.

It is worth noting that the case of the usual Stokes equation (when $\boldsymbol{\omega} = \mathbf{0}$), was treated with success using the same functional framework; see, e.g., [**13**], [**14**], [**12**], [**1**], [**3**], [**5**] and [**6**]. However, when $\boldsymbol{\omega} \neq 0$ the problem contains some additional technical difficulties, since the term $-(\boldsymbol{\omega}\times\boldsymbol{x})\nabla\boldsymbol{u}+\boldsymbol{\omega}\times\boldsymbol{u}$ has unbounded and non decreasing coefficients.

The outline of this paper is as follows. In the next section we display formal properties of the scalar and vectorial operators involved in the above systems. A short review of the weighted Sobolev spaces we use here is given. In Section 3, we state the main results. Sections 4 is devoted to proofs.

2. Preliminaries.

Throughout all the paper n belongs to $\{2, 3\}$.

2.1. Some operators and their formal properties.

Let us examine here the operators involved in equations (1.4) and some of their formal properties. In what follows, D_θ and $L_\pm$ stand for the scalar differential operators defined formally by

$$D_\theta\phi = (\widehat{\boldsymbol{\omega}} \times \boldsymbol{x}).\nabla\phi, \quad L_\pm = \Delta \pm |\boldsymbol{\omega}| D_\theta,$$

where ϕ is an arbitrary scalar function and $\widehat{\boldsymbol{\omega}} = \boldsymbol{\omega}/|\boldsymbol{\omega}|$. In terms of cylindrical coordinates (r, θ, x_3) when $n = 3$ or polar coordinates (r, θ) when $n = 2$, $D_\theta\phi$ is nothing but the angular derivative of ϕ, say $D_\theta\phi = \partial_\theta\phi$.

Consider also the vectorial operators $\mathscr{R}$, $\mathscr{L}_\pm$ defined formally by

$$\mathscr{R}\boldsymbol{v} = -\widehat{\boldsymbol{\omega}} \times \boldsymbol{v} + (\widehat{\boldsymbol{\omega}} \times \boldsymbol{x}).\nabla\boldsymbol{v}, \quad \mathscr{L}_\pm = \Delta \pm |\boldsymbol{\omega}|\mathscr{R},$$

where $\boldsymbol{v}$ is an arbitrary vector field. We can write when $n = 3$

$$\mathscr{R}\boldsymbol{v} = O(\theta)D_\theta(O(\theta)^t\boldsymbol{v}) \ \text{ with } \ O(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.1}$$

Here follows some formal observations concerning the operator $\mathscr{R}$:

(a) Firstly, when $n = 3$, the identity

$$\operatorname{curl}[(\widehat{\boldsymbol{\omega}} \times \boldsymbol{x}) \times \boldsymbol{v}] = -(\widehat{\boldsymbol{\omega}} \times \boldsymbol{x}).\nabla \boldsymbol{v} + \widehat{\boldsymbol{\omega}} \times \boldsymbol{v} + (\operatorname{div} \boldsymbol{v})(\widehat{\boldsymbol{\omega}} \times \boldsymbol{x})$$

gives a new expression of $\mathscr{R}$

$$\mathscr{R}\boldsymbol{v} = \operatorname{curl}[\boldsymbol{v} \times (\widehat{\boldsymbol{\omega}} \times \boldsymbol{x})] + (\operatorname{div} \boldsymbol{v})(\widehat{\boldsymbol{\omega}} \times \boldsymbol{x}).$$

When $n = 2$, one obtains the similar formula

$$\mathscr{R}\boldsymbol{v} = \operatorname{curl}(\boldsymbol{x}.\boldsymbol{v}) + (\operatorname{div} \boldsymbol{v})(\widehat{\boldsymbol{\omega}} \times \boldsymbol{x}).$$

Consequently,

$$\operatorname{div}(\mathscr{R}\boldsymbol{v}) = D_\theta(\operatorname{div} \boldsymbol{v}), \quad (\mathscr{R}\boldsymbol{v}).\widehat{\boldsymbol{\omega}} = D_\theta(\boldsymbol{v}.\widehat{\boldsymbol{\omega}}). \tag{2.2}$$

If ϕ is a scalar function, then

$$\mathscr{R}\nabla\phi = \nabla[D_\theta\phi], \quad \mathscr{R}(\phi\boldsymbol{v}) = \phi\mathscr{R}\boldsymbol{v} + (D_\theta\phi)\boldsymbol{v}. \tag{2.3}$$

The formal identities (2.2) and (2.3) illustrate the particular link between operators $\mathscr{R}$ and D_θ. Similarly, we have

$$\operatorname{div}(\mathscr{L}_\pm \boldsymbol{v}) = L_\pm \operatorname{div} \boldsymbol{v}, \quad \nabla(L_\pm\phi) = \mathscr{L}_\pm\nabla\phi, \quad \mathscr{L}_\pm(\phi\boldsymbol{\omega}) = (L_\pm\phi)\boldsymbol{\omega}. \tag{2.4}$$

Thus, if $\boldsymbol{v}$ is solenoidal, then so are $\mathscr{R}\boldsymbol{v}$ and $\mathscr{L}_+\boldsymbol{v}$.

(b) D_θ and $\mathscr{R}$ commute with the Laplacian and with Fourier transform $\mathscr{F}$

$$[\Delta, D_\theta] = 0, \quad [\mathscr{F}, D_\theta] = 0, \quad [\Delta, \mathscr{R}] = 0, \quad [\mathscr{F}, \mathscr{R}] = 0.$$

(c) $\mathscr{R}$ commutes with the curl, namely

$$[\operatorname{curl}, \mathscr{R}] = 0. \tag{2.5}$$

Notice by the way that $\mathscr{R}$ is a surfacic operator which involves only tangential derivative on the unit sphere. Moreover, one can prove that

$$\mathscr{L}_+\boldsymbol{I}_\ell^m = \mathscr{R}\boldsymbol{I}_\ell^m = im\boldsymbol{I}_\ell^m, \;\; \mathscr{L}_+\boldsymbol{T}_\ell^m = \mathscr{R}\boldsymbol{T}_\ell^m = im\boldsymbol{T}_\ell^m, \;\; \mathscr{L}_+\boldsymbol{N}_\ell^m = \mathscr{R}\boldsymbol{N}_\ell^m = im\boldsymbol{N}_\ell^m,$$

where $(\boldsymbol{I}_\ell^m)_{\ell\geq 0, |m|\leq \ell+1}$, $(\boldsymbol{T}_\ell^m)_{\ell\geq 1,|m|\leq \ell}$ and $(\boldsymbol{N}_\ell^m)_{\ell\geq 1,|m|\leq \ell-1}$ are vectorial spherical harmonics (see, e.g., [**22**]).

We now introduce some polynomial spaces which will be useful in studying the kernels of the operators $\mathscr{L}_\pm$ and $L_\pm$. For a given integer ℓ, we denote by $\mathbb{D}_\ell$ (resp. $\mathbb{D}_\ell^\Delta$) the subspace of $\mathbb{P}_\ell$ composed of those polynomials (resp. harmonic polynomials) whose angular derivative with respect to θ vanishes. Namely,

$$\mathbb{D}_\ell = \{\phi \in \mathbb{P}_\ell \mid D_\theta \phi = 0\}, \quad \mathbb{D}_\ell^\Delta = \{\phi \in \mathbb{P}_\ell^\Delta \mid D_\theta\phi = 0\} = \{\phi \in \mathbb{D}_\ell \mid \Delta\phi = 0\}.$$

In terms of cylindrical (or polar) coordinates (r, θ, x_3), $\mathbb{D}_\ell$ is composed of polynomials ϕ of $\mathbb{P}_\ell$ which do not depend on θ. When $n = 3$, a function ϕ belongs to $\mathbb{D}_\ell$ if and only if $\phi \in \mathbb{P}_\ell$ and $\phi(x_1, x_2, x_3) = \varphi(x_1^2 + x_2^2, x_3)$ for some convenient polynomial φ of two variables. When $n = 2$, the elements of $\mathbb{D}_\ell$ are radial polynomials, and $\mathbb{D}_\ell^\Delta$ is reduced to constant functions.

A closer investigation of the kernel of the operators $\mathscr{L}_\pm$ and $L_\pm$ reveals the utility of the following operator Π defined on the polynomial ring: given a polynomial P, set

$$\Pi P(x_1, x_2, x_3) = \frac{1}{2\pi} \int_0^{2\pi} P\Big(\sqrt{x_1^2 + x_2^2}\cos\theta, \sqrt{x_1^2 + x_2^2}\sin\theta, x_3\Big) d\theta. \tag{2.6}$$

(the variable x_3 is dropped when $n = 2$).

In Lemma 4.2 hereafter, we prove that ΠP is also a polynomial function. Since ΠP does not depend on θ, one has obviously $D_\theta \Pi = 0$, $\Pi D_\theta = 0$. A remarkable property of the operator Π is the commutation identity

$$[\Delta, \Pi] = 0.$$

In other words, Π can be considered as a linear map from $\mathbb{P}_k$ (resp. $\mathbb{P}_k^\Delta$), $k \in \mathbb{Z}$, into itself, or from $\mathbb{P}_k$ into $\mathbb{D}_k$ (resp. into $\mathbb{D}_k^\Delta$).

2.2. Weighted spaces.

Here, we introduce the spaces we need for treating equations (1.4), (1.9) and (1.10). In the sequel q stands for a real verifying $1 < q < +\infty$. We denote by q' its conjugate defined by $(1/q) + (1/q') = 1$. Given an integer $k \in \mathbb{Z}$, $\mathbb{P}_k$ (resp. $\mathbb{P}_k^\Delta$) stands for the space of polynomials (resp. harmonic polynomials) of degree less or equal to k. We denote by $\mathbb{H}_k$ the space of homogenous polynomials of degree equal to k.

Given two integers $m \geq 0$ and $k \in \mathbb{Z}$, define

$$W_k^{m,q}(\mathbb{R}^n) = \big\{u \in \mathscr{D}'(\mathbb{R}^n) \mid \forall \mu \in \mathbb{N}^n,\ |\mu| \leq m,\ \langle x\rangle^{|\mu|-m+k} D^\mu u \in L^q(\mathbb{R}^n)\big\}.$$

This space is equipped with the norm $\|u\|_{W_k^{m,q}(\mathbb{R}^n)} = (\sum_{|\mu|\leq m} \|\langle x\rangle^{|\mu|-m+k} \cdot D^\mu u\|^q_{L^q(\mathbb{R}^n)})^{1/q}$. This definition can be extended to negative values of m; for $m \leq 0$, $W_k^{m,q}(\mathbb{R}^n)$ stands for the dual space of $W_{-k}^{-m,q'}(\mathbb{R}^n)$. The following algebraic and topological inclusions hold true

$$\cdots \subset W_k^{m,q}(\mathbb{R}^n) \subset W_{k-1}^{m-1,q}(\mathbb{R}^n) \subset \cdots \subset W_{k-m}^{0,q}(\mathbb{R}^n) \subset W_{k-m-1}^{-1,q}(\mathbb{R}^n) \subset \cdots$$

Notice that elements of $W_k^{m,q}(\mathbb{R}^n)$, m, $k \in \mathbb{Z}$, are tempered distributions.

Spaces $W_k^{m,p}$ provide a valuable framework for describing the growth or the decay of functions at large distances. Giving a detailed account of their properties is beyond the scope of this paper. However, let us underline that they were employed in a host of papers for solving numerous problems in unbounded domains, especially in hydrodynamics. See, e.g., [**16**], [**15**], [**2**], [**3**], [**4**] and references therein.

We shall often use the following property: for all $j \in \mathbb{Z}$ and $P \in \mathbb{P}_j \backslash \mathbb{P}_{j-1}$,

$$P \in W_k^{m,q}(\mathbb{R}^n) \Longleftrightarrow j \leq \langle m-k\rangle, \tag{2.7}$$

where for each integer α, $\langle\alpha\rangle = \alpha - d$ with

$$d = \left[\frac{n}{q}\right] + 1. \tag{2.8}$$

In other words, $\langle m-k\rangle$ is the highest degree of the polynomials contained in $W_\alpha^{m,q}(\mathbb{R}^n)$. By the way, we set here and subsequently $\langle\alpha\rangle^\star = \alpha - d^\star$, with

$$d^\star = \left[\frac{n}{q'}\right] + 1. \tag{2.9}$$

Notice that $1 \leq d \leq n$ and $1 \leq d^\star \leq n$. Observe also that there is no reason to mix up the notations $\langle\alpha\rangle$ and $\langle\alpha\rangle^\star$, in which α is real, with the basic weight $\langle x\rangle$.

Finally, for all integers $m \in \mathbb{Z}$, we consider the space

$$V_k^{m,q}(\mathbb{R}^n) = \big\{\boldsymbol{v} \in W_k^{m,q}(\mathbb{R}^n)^n \mid \mathscr{R}\boldsymbol{v} \in W_k^{m-2,q}(\mathbb{R}^n)^n\big\},$$

equipped with the norm

$$\|\boldsymbol{v}\|_{V_k^{m,q}(\mathbb{R}^n)} = \Big\{\|\boldsymbol{v}\|^q_{W_k^{m,q}(\mathbb{R}^n)} + \|\mathscr{R}\boldsymbol{v}\|^q_{W_k^{m-2,q}(\mathbb{R}^n)}\Big\}^{1/q}.$$

When $m \geq 0$ and $k \in \mathbb{Z}$, define $(V_k^{m,q}(\mathbb{R}^n))^\star$ as the dual space of $V_k^{m,q}(\mathbb{R}^n)$. We have the imbeddings $V_{-k}^{-m,q'}(\mathbb{R}^n) \hookrightarrow W_{-k}^{-m,q'}(\mathbb{R}^n) \hookrightarrow (V_k^{m,q}(\mathbb{R}^n))^\star$.

3. The main results.

We expose in this section the main results of the paper. The first result (Theorem 3.1) gives a characterization of solutions to equations (1.9), (1.4) or (1.9), when the data is zero. The second and the third results (Theorems 3.3 and 3.5) concern the existence and uniqueness of solutions of the pressureless equation (1.9) and those of the original system (1.4). Corollary 3.4 is devoted to the scalar equation (1.9). The proofs of these results are given in Section 4.

Define the null spaces

$$\begin{aligned}
\mathbb{M}_k^q(\mathscr{L}_\pm) &= \big\{\boldsymbol{v} \in W_{m+k}^{m,q}(\mathbb{R}^n)^n \mid \mathscr{L}_\pm \boldsymbol{v} = \boldsymbol{0}\big\},\\
\mathbb{N}_k^q(\mathscr{L}_\pm) &= \big\{(\boldsymbol{v},\theta) \in W_{m+k}^{m,q}(\mathbb{R}^n)^n \times W_{m+k}^{m-1,q}(\mathbb{R}^n) \mid -\mathscr{L}_\pm \boldsymbol{v} + \nabla\theta = \boldsymbol{0},\ \operatorname{div} \boldsymbol{v} = 0\big\},\\
\mathbb{K}_k^q(L_\pm) &= \big\{\phi \in W_{m+k}^{m,q}(\mathbb{R}^n) \mid L_\pm \phi = 0\big\}.
\end{aligned}$$

In [**8**], in proof of Theorem 1.1, the authors proved that a tempered distribution $\boldsymbol{v}$ satisfying $\mathscr{L}_+\boldsymbol{v} = 0$ is necessarily polynomial. The same conclusion holds for a pair $(\boldsymbol{v},\theta)$ of tempered distributions satisfying $-\mathscr{L}_+\boldsymbol{v} + \nabla\theta = 0$ and $\operatorname{div}\boldsymbol{v} = 0$. Since elements of $W_k^{m,q}(\mathbb{R}^n)$ are tempered distributions for $m,\, k \in \mathbb{Z}$, the spaces $\mathbb{M}_k^q(\mathscr{L}_\pm)$, $\mathbb{N}_k^q(\mathscr{L}_\pm)$ and $\mathbb{K}_k^q(L_\pm)$ are composed of polynomial functions. With property (2.7) we easily get

$$\begin{aligned}
\mathbb{M}_k^q(\mathscr{L}_\pm) &= \{\boldsymbol{v} \in (\mathbb{P}_\ell)^n \mid \mathscr{L}_\pm \boldsymbol{v} = \boldsymbol{0}\},\\
\mathbb{N}_k^q(\mathscr{L}_\pm) &= \{(\boldsymbol{v},\theta) \in (\mathbb{P}_\ell)^n \times \mathbb{P}_{\ell-1} \mid -\mathscr{L}_\pm \boldsymbol{v} + \nabla\theta = \boldsymbol{0},\ \operatorname{div}\boldsymbol{v} = 0\},\\
\mathbb{K}_k^q(L_\pm) &= \{\phi \in \mathbb{P}_\ell \mid L_\pm\phi = 0\}.
\end{aligned}$$

It follows that these spaces are finite dimensional and independent of the parameter m. This parameter m is dropped in the notations $\mathbb{M}_k^q(\mathscr{L}_\pm)$, $\mathbb{N}_k^q(\mathscr{L}_\pm)$ and $\mathbb{K}_k^q(\mathscr{L}_\pm)$.

The following result gives a complete characterization of the space $\mathbb{M}_k^q(\mathscr{L}_\pm)$, $\mathbb{N}_k^q(\mathscr{L}_\pm)$ and $\mathbb{K}_k^q(L_\pm)$.

THEOREM 3.1. *Let $k \in \mathbb{Z}$ be an integer and set $\ell = \langle -k\rangle$. Then,*

- $\mathbb{K}_k^q(L_-) = \mathbb{K}_k^q(L_+) = \mathbb{D}_\ell^\Delta$ *and* $\mathbb{M}_k^q(\mathscr{L}_+) = \mathbb{M}_k^q(\mathscr{L}_-)$.
- *When $n = 3$, $\mathbb{M}_k^q(\mathscr{L}_+)$ is composed of the polynomial vector functions of the form*

$$\boldsymbol{\varphi} = \nabla\alpha + \widehat{\boldsymbol{\omega}} \times \nabla\beta + \gamma\widehat{\boldsymbol{\omega}}, \tag{3.1}$$

with α, $\beta \in \mathbb{D}^{\Delta}_{\ell+1}$ *and* $\gamma \in \mathbb{D}^{\Delta}_{\ell}$. *Similarly,* $\mathbb{N}^q_k(\mathscr{L}_\pm)$ *is composed of the pairs of the form*

$$\begin{aligned} \boldsymbol{v} &= \nabla\lambda + \widehat{\boldsymbol{\omega}} \times \nabla\beta + 2\gamma\widehat{\boldsymbol{\omega}} - \nabla[(\widehat{\boldsymbol{\omega}}.\boldsymbol{x})\gamma], \\ \pi &= -2\widehat{\boldsymbol{\omega}}.\nabla\gamma \pm |\boldsymbol{\omega}|D_\theta\lambda. \end{aligned} \tag{3.2}$$

with $\lambda \in \mathbb{D}^{\Delta}_{\ell+1} + \mathbb{P}^{\Delta}_{\ell-1}$, α, $\beta \in \mathbb{D}^{\Delta}_{\ell+1}$ *and* $\gamma \in \mathbb{D}^{\Delta}_{\ell}$.

- *When* $n = 2$, $\mathbb{M}^q_k(\mathscr{L}_+) = \{\mathbf{0}\}$ *while* $\mathbb{N}^q_k(\mathscr{L}_\pm)$ *is composed of the pairs of the form*

$$\boldsymbol{v} = \nabla\lambda, \quad \pi = \pm|\boldsymbol{\omega}|D_\theta\lambda, \tag{3.3}$$

where $\lambda \in \mathbb{P}^{\Delta}_{\ell-1}$.

Remark 3.2. When $n = 2$, $\mathbb{D}^{\Delta}_k \subset \mathbb{P}_0$ for all $k \in \mathbb{Z}$.

Theorem 3.3. *Let* k *be an integer such that* $k \in \{-1, 0, 1, 2, 3\}$ *if* $n = 3$ *and* $k \in \{0, 2\}$ *if* $n = 2$. *Suppose that* $q \notin \{n, n/(n-1)\}$ *and*

(a) $1 < q < 3$ *if* $k = -1$ $(n = 3)$,
(b) $q > 3/2$ *if* $k = 3$ $(n = 3)$.

Let $\boldsymbol{f} \in W^{0,q}_k(\mathbb{R}^n)^n$. *Then, equation* (1.9) *has at least one solution* $\boldsymbol{u} \in V^{2,q}_k(\mathbb{R}^n)$ *if and only if*

$$\forall \boldsymbol{p} \in \mathbb{M}^{q'}_{-k}(\mathscr{L}_-), \quad \langle \boldsymbol{f}, \boldsymbol{p}\rangle = 0. \tag{3.4}$$

This solution is unique up to elements of $\mathbb{M}^q_{k-2}(\mathscr{L}_+)$ *and*

$$\inf_{\boldsymbol{\varphi} \in \mathbb{M}^q_{k-2}(\mathscr{L}_+)} \|\boldsymbol{u} - \boldsymbol{\varphi}\|_{V^{2,q}_k(\mathbb{R}^n)} \lesssim \|\boldsymbol{f}\|_{W^{0,q}_k(\mathbb{R}^n)^n}.$$

Moreover, $\operatorname{div}\boldsymbol{u} = 0$ *if* $\operatorname{div}\boldsymbol{f} = 0$.

Corollary 3.4. *Suppose that* k *and* q *satisfy assumptions of Theorem* 3.3 *with* $n = 3$. *Let* $h \in W^{0,q}_k(\mathbb{R}^3)$. *Then, equation* (1.10) *has at least one solution* $\phi \in W^{2,q}_k(\mathbb{R}^3)$ *if and only if*

$$\forall \theta \in \mathbb{D}^{\Delta}_{\langle k\rangle^\star}, \quad \langle h, \theta\rangle = 0. \tag{3.5}$$

Moreover, this solution is unique up to elements of $\mathbb{D}^{\Delta}_{2+\langle -k\rangle}$ *and*

$$\inf_{\rho\in\mathbb{D}^{\Delta}_{2+\langle -k\rangle}} \left\{\|\phi-\rho\|_{W^{2,q}_k(\mathbb{R}^3)} + \|D_\theta(\phi-\rho)\|_{W^{0,q}_k(\mathbb{R}^3)}\right\} \lesssim \|h\|_{W^{0,q}_k(\mathbb{R}^3)^3}.$$

Concerning the smoothness of solutions, we state this

Theorem 3.5. *Let $k\in\mathbb{Z}$ and $m\geq 0$ be two integers, with k satisfying assumptions of Theorem* 3.3. *Let $\boldsymbol{f}\in W^{m,q}_k(\mathbb{R}^n)^n$ and $g\in W^{m+1,q}_k(\mathbb{R}^n)$ such that $D_\theta g\in W^{m-1,q}_k(\mathbb{R}^n)$ and*

$$\forall(\boldsymbol{p},\pi)\in\mathbb{N}^{q'}_{m-k}(\mathscr{L}_-),\quad (\boldsymbol{f},\boldsymbol{p})-(g,\pi)=0. \tag{3.6}$$

$$\forall\lambda\in\mathbb{P}^{\Delta}_{\langle k\rangle^\star+1-m},\quad \langle\operatorname{div}\boldsymbol{f}+D_\theta g,\lambda\rangle=0. \tag{3.7}$$

Then, there exists a pair $(\boldsymbol{u},p)\in V^{m+2,q}_k(\mathbb{R}^n)^n\times W^{m+1,q}_k(\mathbb{R}^n)$ solution of (1.4). *Moreover, this solution is unique up to elements of $\mathbb{N}^{q}_{k-m-2}(\mathscr{L}_+)$ and the following estimate holds*

$$\begin{aligned}\inf_{(\boldsymbol{\lambda},\mu)\in\mathbb{N}^{q}_{k-m-2}(\mathscr{L}_+)} &\left\{\|\boldsymbol{u}-\boldsymbol{\lambda}\|_{V^{m+2,q}_k(\mathbb{R}^n)^n} + \|p-\mu\|_{W^{m+1,q}_k(\mathbb{R}^n)}\right\}\\ &\lesssim \|\boldsymbol{f}\|_{W^{m,q}_k(\mathbb{R}^n)^n} + \|g\|_{W^{m+1,q}_k(\mathbb{R}^n)} + \|D_\theta g\|_{W^{m-1,q}_k(\mathbb{R}^n)}.\end{aligned} \tag{3.8}$$

Moreover, this result remains valid if $m=-1$ and $g=0$.

Remark 3.6. Condition (3.6) is automatically fulfilled if $m\geq 2$, since $\mathbb{N}^{q'}_{m-k}(\mathscr{L}_-)=\{(\boldsymbol{0},0)\}$. If $\langle -k\rangle\leq -1$, then $\mathbb{M}^q_k(\mathscr{L}_+)=\{\boldsymbol{0}\}$, $\mathbb{N}^q_k(\mathscr{L}_+)=\{(\boldsymbol{0},0)\}$ and $\mathbb{K}^q_k(L_+)=\{0\}$.

If $\langle -k\rangle=0$, then $\mathbb{M}^q_k(\mathscr{L}_+)=\{\boldsymbol{0}\}$ when $n=2$ and $\mathbb{M}^q_k(\mathscr{L}_+)=\operatorname{span}(\widehat{\boldsymbol{\omega}})$ when $n=3$. In both the cases, $\mathbb{N}^q_k(\mathscr{L}_+)=\mathbb{M}^q_k(\mathscr{L}_+)\times\{0\}$ and $\mathbb{K}^q_k(L_+)=\mathbb{R}$.

If $\langle -k\rangle=1$ and $n=3$, then $\mathbb{M}^q_k(\mathscr{L}_+)$ is composed of all the polynomial vector functions of the form $\boldsymbol{v}=a\widehat{\boldsymbol{\omega}}+b\widehat{\boldsymbol{\omega}}\times\boldsymbol{x}+c\boldsymbol{x}+d(\boldsymbol{x}.\widehat{\boldsymbol{\omega}})\widehat{\boldsymbol{\omega}}$, $\mathbb{K}^q_k(L_+)=\operatorname{span}(\widehat{\boldsymbol{\omega}}.\boldsymbol{x})\oplus\mathbb{R}$ and $\mathbb{N}^q_k(\mathscr{L}_+)=\mathbb{M}^{q,0}_k(\mathscr{L}_+)\times\mathbb{R}$, with $\mathbb{M}^{q,0}_k(\mathscr{L}_+)=\{\boldsymbol{v}\in\mathbb{M}^q_k(\mathscr{L}_+)\mid\operatorname{div}\boldsymbol{v}=0\}$. We retrieve the result of Farwig and al. [**8**].

The following tables summarize the possible values of $\langle -k\rangle$ and $\langle k\rangle^\star$, when k and q are satisfying assumptions of Theorems 3.3 and 3.5.

Table 1. Possible values of $\langle -k\rangle$ and $\langle k\rangle^\star$ when $n=2$.

k	0	2
q	$\neq 2$	$\neq 2$
$\langle -k\rangle$	$\{-2,-1\}$	$\{-3,-4\}$
$\langle k\rangle^\star$	$\{-2,-1\}$	$\{0,1\}$

Table 2. Possible values of $\langle -k\rangle$ and $\langle k\rangle^\star$ when $n=3$.

k	-1	0	1	2	3
q	<3				$>3/2$
$\langle -k\rangle$	-1	$\{-2,-1\}$	$\{-3,-2\}$	$\{-4,-3\}$	-4
$\langle k\rangle^\star$	$\{-3,-2\}$	$\{-3,-2,-1\}$	$\{-2,-1,0\}$	$\{-1,0,1\}$	$\{0,1\}$

4. The proofs.

4.1. Description of the nullspaces. Proof of Theorem 3.1.

The main objective of this section is to give an explicit characterization of the nullspaces $\mathbb{M}_k^q(\mathscr{L}_\pm)$, $\mathbb{N}_k^q(\mathscr{L}_\pm)$ and $\mathbb{K}_k^q(L_\pm)$. We start with

Lemma 4.1. *Let A and B be two linear maps from the space $(\mathbb{P}_\ell)^s$, $\ell, s\geq 1$, into itself such that*

- *A and B commute,*
- $\ker B^2 = \ker B$,
- *For each $k\in [\![2,\ell]\!]$, $A(\mathbb{H}_k)^s\subset(\mathbb{H}_{k-2})^s$ and $B(\mathbb{H}_k)^s\subset(\mathbb{H}_k)^s$.*

Then, $\ker(A+B)=\ker A\cap\ker B$.

Proof of Lemma 4.1. Let $\Psi\in\ker(A+B)$. Since $(\mathbb{P}_\ell)^s=(\mathbb{H}_0)^s+\cdots+(\mathbb{H}_\ell)^s$, one can decompose Ψ into the form: $\Psi=\sum_{k=0}^{\ell}\Psi_k$, where $\Psi_k\in(\mathbb{H}_k)^s$ for each $k\leq\ell$. The identity $(A+B)\Psi=0$ becomes

$$B\Psi_\ell=0,\quad B\Psi_{\ell-1}=0,\quad \text{and } A\Psi_{k+2}+B\Psi_k=0 \text{ for each } k\leq\ell-2.$$

Thus, $B^2\Psi_{\ell-2}=-BA\Psi_\ell=-A(B\Psi_\ell)=0$. Hence $\Psi_{\ell-2}\in\ker B^2=\ker B$ and $B\Psi_{\ell-2}=0$. Similarly $B\Psi_{\ell-3}=0$. Going down step by step, one deduces easily that $B\Psi_k=0$ for each $k\leq\ell$. Thus, $B\Psi=0$ and $A\Psi=0$. $\square$

Lemma 4.2. *Let $\ell\in\mathbb{Z}$. Then, Π is a projection of $\mathbb{P}_\ell$ and*

$$\begin{aligned} &\Pi(\mathbb{P}_\ell) = \mathbb{D}_\ell, \quad (I - \Pi)(\mathbb{P}_\ell) = D_\theta(\mathbb{P}_\ell), \\ &\Pi(\mathbb{P}_\ell^\Delta) = \mathbb{D}_\ell^\Delta, \quad (I - \Pi)(\mathbb{P}_\ell^\Delta) = D_\theta(\mathbb{P}_\ell^\Delta). \end{aligned} \tag{4.1}$$

Proof of Lemma 4.2. Let $P = X_1^k X_2^j X_3^m$ with $k + j + m \leq \ell$. In terms of cylindrical coordinates (r, θ, x_3) one can write

$$P(\boldsymbol{x}) = x_3^m r^{k+j} (\cos\theta)^k (\sin\theta)^j, \quad \boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3.$$

Using this form, one can prove easily that ΠP is a polynomial.

Now, observe first that $\Pi Q = Q$ when $Q \in \ker D_\theta$. It follows that Π and D_θ are two endomorphisms of $\mathbb{P}_\ell$ satisfying

$$\mathrm{R}(D_\theta) \subset \ker \Pi, \quad \ker D_\theta \subset \mathrm{R}(\Pi).$$

The rank-nullity theorem implies that these inclusions are equalities. The same argument remains valid with $\mathbb{P}_\ell^\Delta$ instead of $\mathbb{P}_\ell$ (recall that Π and D_θ commute with Δ). □

The following lemma is well known

Lemma 4.3. *Let $\theta \in \mathbb{P}_\ell$, $\ell \in \mathbb{Z}$. Then, there exists $\Psi \in \mathbb{P}_{\ell+2}$ such that $\Delta\Psi = \theta$.*

Lemma 4.4. *$\Delta(\mathbb{D}_{\ell+2}) = \mathbb{D}_\ell$ for $\ell \in \mathbb{Z}$.*

Proof of Lemma 4.4. Let $\phi \in \mathbb{D}_\ell$ and let $\Psi_1 \in \mathbb{P}_{\ell+2}$ such that $\Delta\Psi_1 = \phi$. With $\Psi = \Pi\Psi_1 \in \mathbb{D}_{\ell+2}$ one has $\Delta\Psi = \phi$. □

Lemma 4.5. *Let $\ell \in \mathbb{Z}$. Then, $\{P \in \mathbb{P}_\ell \mid D_\theta^2 P = 0\} = \mathbb{D}_\ell$.*

Lemma 4.6. *Let $\boldsymbol{v} \in (\mathbb{P}_\ell)^n$, $\ell \in \mathbb{Z}$. The following assertions are equivalent*

(i) $\mathscr{R}\boldsymbol{v} = \mathbf{0}$.
(ii) $\mathscr{R}^2\boldsymbol{v} = \mathbf{0}$.
(iii) *There exist two functions a, $b \in \mathbb{D}_{\ell-1}$ and a function $c \in \mathbb{D}_\ell$ such that*

$$\boldsymbol{v} = a\boldsymbol{x} + b\widehat{\boldsymbol{\omega}} \times \boldsymbol{x} + c\widehat{\boldsymbol{\omega}}, \tag{4.2}$$

(the term $c\widehat{\boldsymbol{\omega}}$ is dropped when $n = 2$).

Proof of Lemma 4.6. We give the proof only in the case $n = 3$.

(i) $\Longrightarrow$ (ii) obvious.

(ii) $\Longrightarrow$ (i) Suppose that $\mathscr{R}^2\boldsymbol{v} = \mathbf{0}$. According to formula (2.1), one has $D_\theta^2(O^t\boldsymbol{v}) = \mathbf{0}$. Let $M = \operatorname{diag}(r,r,1)O^t\boldsymbol{v}$, with $r = (x_1^2 + x_2^2)^{1/2}$. Then, $M\boldsymbol{v} \in \mathbb{P}_{\ell+1} \times \mathbb{P}_{\ell+1} \times \mathbb{P}_\ell$ and $D_\theta^2(M\boldsymbol{v}) = 0$. Thus, $D_\theta(M\boldsymbol{v}) = 0$, thanks to Lemma 4.5. It follows that $D_\theta(O^t\boldsymbol{v}) = \mathbf{0}$.

(iii) $\Longrightarrow$ (i) a direct calculus.

(i) $\Longrightarrow$ (iii) The equation $\mathscr{R}\boldsymbol{v} = 0$ writes

$$D_\theta v_1 = -v_2, \quad D_\theta v_2 = v_1, \quad D_\theta v_3 = 0.$$

It follows that $v_3 \in \mathbb{D}_\ell$ and $D_\theta^2 v_1 + v_1 = 0$. Using Fourier series in terms of θ, one deduces that $\boldsymbol{v}_1$ is of the form

$$v_1(x_1, x_2, x_3) = ra\cos\theta + rb\sin\theta = ax + by,$$

with $a \in \mathbb{D}_{\ell-1}$ and $b \in \mathbb{D}_{\ell-1}$. Thus,

$$v_2(x_1, x_2, x_3) = -rb\cos\theta + ar\sin\theta = -bx + ay.$$

The formula follows by setting $c = v_3 - ax_3$. $\square$

Lemma 4.7. *Let $\pi \in \mathbb{P}_\ell^\Delta$. Then, there exists $\gamma \in \mathbb{D}_{\ell+2} + \mathbb{P}_\ell^\Delta$ such that $L_+\gamma = \pi$ (respectively $L_-\gamma = \pi$).*

Proof of Lemma 4.7. In view of Lemma 4.2, we can decompose π into the form $\pi = \pi_0 + \pi_1$, with $\pi_0 \in \mathbb{D}_\ell^\Delta$ and $\pi_1 \in D_\theta(\mathbb{P}_\ell^\Delta)$. There exists $\phi_1 \in \mathbb{P}_\ell^\Delta$ such that $D_\theta\phi_1 = \pi_1$. On the other hand, according to Lemma 4.4, we can introduce yet another function $\phi_0 \in \mathbb{D}_{\ell+2}$ picked so that $\Delta\phi_0 = \pi_0$. The function $\phi = \phi_0 \pm \phi_1$ satisfies

$$L_\pm\phi = L_\pm\phi_0 \pm L_\pm\phi_1 = \Delta\phi_0 + D_\theta\phi_1 = \pi. \qquad \square$$

Proof of Theorem 3.1. We assume $n = 3$. The proof when $n = 2$ is quite similar.

1. We know that

$$\mathbb{M}_k^q(\mathscr{L}_\pm) = \{\boldsymbol{v} \in (\mathbb{P}_\ell)^n \mid \mathscr{L}_\pm\boldsymbol{v} = \mathbf{0}\}.$$

Consider the linear operators $A = \Delta$, $B = \pm\mathscr{R}$ defined on $(\mathbb{P}_\ell)^n$. We know that $\ker B^2 = \ker B$, thanks to Lemma 4.6.

Let $\boldsymbol{\varphi} \in (\mathbb{P}_\ell)^n$ such that $\mathscr{L}_\pm \boldsymbol{\varphi} = \mathbf{0}$. According to Lemma 4.1, we deduce that $\Delta \boldsymbol{\varphi} = 0$ and $\mathscr{R}\boldsymbol{\varphi} = \mathbf{0}$. Moreover, in view of Lemma 4.6 there exist three polynomial functions a, $b \in \mathbb{D}_{\ell-1}$ and $c \in \mathbb{D}_\ell$ such that

$$\boldsymbol{\varphi} = a\boldsymbol{x} + b\widehat{\boldsymbol{\omega}} \times \boldsymbol{x} + c\widehat{\boldsymbol{\omega}}.$$

The condition $\Delta\boldsymbol{\varphi} = \mathbf{0}$ writes in cylindrical coordinates

$$\Delta(ra) = \frac{a}{r}, \quad \Delta(rb) = \frac{b}{r}, \quad \Delta(c + ax_3) = 0.$$

Define in terms of cylindrical coordinates the functions

$$\alpha(\boldsymbol{x}) = \int_0^r ta(t, x_3)dt - 2\int_0^{x_3}\int_0^u a(0, v)dvdu,$$
$$\beta(\boldsymbol{x}) = \int_0^r tb(t, x_3)dt - 2\int_0^{x_3}\int_0^u b(0, v)dvdu,$$

with $r = (x_1^2 + x_2^2)^{1/2}$. One can prove easily that α, $\beta \in \mathbb{D}_{\ell+1}^{\Delta}$. Moreover,

$$\boldsymbol{\varphi} = \nabla\alpha + \widehat{\boldsymbol{\omega}} \times \nabla\beta + \Big(c - \frac{\partial\alpha}{\partial x_3} + ax_3\Big)\widehat{\boldsymbol{\omega}}.$$

Setting $\gamma = c - (\partial\alpha/\partial x_3) + ax_3$ ends the proof of the formula (3.1) in Theorem 3.1.

2. Let $(\boldsymbol{v}, \pi) \in \mathbb{N}_q^k(\mathscr{L}_+)$. Lemma 4.7 asserts that there exists two functions $\Psi_0 \in \mathbb{D}_{\ell+1}$ and $\Psi_1 \in \mathbb{P}_{\ell-1}^{\Delta}$ such that $L_\pm(\Psi_0 + \Psi_1) = \pi$. We set $\boldsymbol{\varphi} = \boldsymbol{v} - \nabla(\Psi_0 + \Psi_1) \in (\mathbb{P}_\ell)^n$. Then,

$$\mathscr{L}_\pm\boldsymbol{\varphi} = \mathscr{L}_\pm\boldsymbol{v} - \nabla(L_\pm(\Psi_0 + \Psi_1)) = \mathscr{L}_\pm\boldsymbol{v} - \nabla\pi = \mathbf{0}.$$

Thus, $\boldsymbol{\varphi} \in \mathbb{M}_q^k(\mathscr{L}_\pm)$. Hence, there exists α, $\beta \in \mathbb{D}_{\ell+1}^{\Delta}$ and $\gamma \in \mathbb{D}_\ell^{\Delta}$ such that

$$\boldsymbol{\varphi} = \nabla\alpha + \widehat{\boldsymbol{\omega}} \times \nabla\beta + \gamma\widehat{\boldsymbol{\omega}}.$$

On the other hand, $\operatorname{div}\boldsymbol{\varphi} = -\Delta\Psi_0$. Hence

$$\Delta\Psi_0 = -\frac{\partial\gamma}{\partial x_3} = -\Delta\Big(\frac{x_3\gamma}{2}\Big).$$

Set $\lambda = \alpha + (x_3\gamma/2) + \Psi_0 + \Psi_1$. Then, $\lambda \in \mathbb{D}^{\Delta}_{\ell+1} + \mathbb{P}^{\Delta}_{\ell-1}$ and

$$\boldsymbol{v} = \nabla\lambda + \widehat{\boldsymbol{\omega}} \times \nabla\beta + \gamma\widehat{\boldsymbol{\omega}} - \nabla\Big(\frac{x_3\gamma}{2}\Big),$$

$$\pi = L_{\pm}(\Psi_0 + \Psi_1) = \Delta\Psi_0 \pm D_\theta\Psi_1 = -\frac{\partial\gamma}{\partial x_3} \pm D_\theta\lambda.$$

This ends the proof. □

4.2. The equation without pressure. Proof of Theorem 3.3.

We need the following result concerning Poisson's equation in $\mathbb{R}^n$ (see Giroire [**15**] when $q = 2$ and $n = 2$ or 3, and Amrouche et al. [**2**] for the other cases).

PROPOSITION 4.8. *Let $m \in \mathbb{Z}$ and k be two integers and $q > 1$ a real such that*

$$\begin{aligned} &\frac{n}{q'} \notin \{1, \dots, k+1\} \quad \text{if } k \geq 0,\\ &\frac{n}{q} \notin \{1, \dots, -k+1\} \quad \text{if } k \leq 0, \end{aligned} \tag{4.3}$$

then the following operator is an isomorphism

$$\Delta \,:\, W^{m+1,q}_{m+k}(\mathbb{R}^n)/\mathbb{P}^{\Delta}_{\langle 1-k\rangle} \mapsto W^{m-1,q}_{m+k}(\mathbb{R}^n) \perp \mathbb{P}^{\Delta}_{\langle 1+k\rangle^\star}.$$

4.2.1. Some preliminary lemma.

LEMMA 4.9. *Suppose that $n = 3$ (resp. $n = 2$). Let $k \in \mathbb{Z}$ and $\boldsymbol{p} \in (\mathbb{P}_k)^3$ (resp. $\boldsymbol{p} \in (\mathbb{P}_k)^2$) such that* $\operatorname{div}\boldsymbol{p} = 0$. *Then, there exists a vector field $\boldsymbol{m}$ in $(\mathbb{P}_{k+1})^3$ (resp. a function φ in $\mathbb{P}_{k+1}$) such that* $\operatorname{curl}\boldsymbol{m} = \boldsymbol{p}$ *(resp.* $\operatorname{curl}\varphi = \boldsymbol{p}$*).*

PROOF OF LEMMA 4.9. When $n = 3$, one can take

$$\boldsymbol{m} = \Big(\int_0^{x_3} p_2(x_1, x_2, s)ds, \int_0^{x_1} p_3(t, x_2, 0)dt - \int_0^{x_3} p_1(x_1, x_2, s)ds, 0\Big)^t.$$

When $n = 2$, one can take $\varphi = -\int_0^{x_1} p_2(t, 0)dt + \int_0^{x_2} p_1(x_1, s)ds$. □

LEMMA 4.10. *Let $k \in \mathbb{Z}$ satisfying* (4.3). *Let $\boldsymbol{z} \in W^{-1,q}_k(\mathbb{R}^3)^2$ (resp. $z \in W^{-1,q}_k(\mathbb{R}^2)^2$) such that* $\operatorname{div}\boldsymbol{z} = 0$. *Then, there exists $\boldsymbol{v} \in W^{0,q}_k(\mathbb{R}^3)^3$ (resp. $\theta \in W^{0,q}_k(\mathbb{R}^2)$) such that* $\operatorname{curl}\boldsymbol{v} = \boldsymbol{z}$ *(resp.* $\operatorname{curl}\theta = \boldsymbol{z}$*).*

Proof of Lemma 4.10. We give the proof only in the case $n = 3$. Suppose first that $\boldsymbol{z}$ satisfies the following condition

$$\forall \boldsymbol{h} \in (\mathbb{P}_\ell^\Delta)^n,\ \langle \boldsymbol{z}, \boldsymbol{h} \rangle_{W_k^{-1,q}(\mathbb{R}^n)\times W_{-k}^{1,q'}(\mathbb{R}^n)} = 0, \quad \text{with } \ell = \langle k+1 \rangle^\star. \tag{4.4}$$

According to Proposition 4.8, there exists a vectorial function $\boldsymbol{u} \in W_k^{1,q}(\mathbb{R}^n)^n$ such that $-\Delta \boldsymbol{u} = \boldsymbol{z}$ in $\mathbb{R}^n$. Since, $\Delta(\operatorname{div} \boldsymbol{u}) = \operatorname{div}(\Delta \boldsymbol{u}) = \operatorname{div} \boldsymbol{z} = \boldsymbol{0}$, one has $\operatorname{div} \boldsymbol{u} \in (\mathbb{P}_s^\Delta)^3$ with $s = \langle -k \rangle$. Hence, by Lemma 4.9, there exists $\boldsymbol{\theta} \in (\mathbb{P}_{\ell+1})^3$ such that

$$\operatorname{curl} \boldsymbol{\theta} = \nabla \operatorname{div} \boldsymbol{u}.$$

Set $\boldsymbol{v} = \operatorname{curl} \boldsymbol{u} - \boldsymbol{\theta}$. Then, $\operatorname{curl} \boldsymbol{v} = \operatorname{curl}(\operatorname{curl} \boldsymbol{u} - \boldsymbol{\theta}) = -\Delta \boldsymbol{u} = \boldsymbol{z}$. Now, let us come back to the more general case in which $\boldsymbol{z}$ is not supposed satisfying (4.4). Let ν a cut-off function verifying

$$\nu \in \mathscr{D}(\mathbb{R}^n), \quad 0 \le \nu \le 1, \quad \nu = 1 \text{ in } B_1,\ \operatorname{supp}(\nu) \subset B_2,$$

where B_1 and B_2 are two balls centered at the origin such that $|B_1| \neq 0$ and $B_1 \subset B_2 \subset \mathbb{R}^n$. Consider the weighted bilinear form

$$(\theta_1, \theta_2)_\nu = \int_{\mathbb{R}^n} \nu(\boldsymbol{x}) \theta_1(\boldsymbol{x}).\theta_2(\boldsymbol{x}) d\boldsymbol{x}.$$

This bilinear form defines an inner product on $\mathbb{P}_\ell$. We set $\mathbb{G}_\ell = \{\nabla q \mid q \in \mathbb{P}_{\ell+1}\}$ and

$$\mathbb{G}_\ell^\perp = \{\boldsymbol{\gamma} \in (\mathbb{P}_\ell)^n \mid \forall q \in \mathbb{P}_{\ell+1},\ (\boldsymbol{\gamma}, \nabla q)_\nu = 0\}.$$

Now, let $\boldsymbol{p}_0$ be the unique element of $(\mathbb{P}_\ell)^n$ verifying

$$\forall \boldsymbol{p} \in \mathbb{G}_\ell^\perp, \quad \int_{\mathbb{R}^n} \nu(\boldsymbol{x}) \operatorname{curl} \boldsymbol{p}_0 . \operatorname{curl} \boldsymbol{p} dx = \langle \boldsymbol{z}, \boldsymbol{p} \rangle. \tag{4.5}$$

This is a finite dimensional problem which can be reduced to a linear system. Moreover, let $\boldsymbol{p} \in \mathbb{G}_\ell$ such that $\nu \operatorname{curl} \boldsymbol{p} = \boldsymbol{0}$. Necessarily $\operatorname{curl} \boldsymbol{p} = \boldsymbol{0}$ everywhere in $\mathbb{R}^3$ and $\boldsymbol{p} = \nabla \phi$ for some $\phi \in \mathbb{P}_{\ell+1}$. The condition $(\boldsymbol{p}, \nabla \phi)_\nu = 0$ implies that ϕ is constant in B_1. Moreover, ϕ is constant everywhere since it is polynomial, and $\nabla p = \boldsymbol{0}$. In other words, the linear system (4.5) is invertible and has one and only one solution $\boldsymbol{p}_0$. We set

$$\boldsymbol{\lambda} = \nu \operatorname{curl} \boldsymbol{p}_0.$$

Obviously $\boldsymbol{\lambda} \in W_k^{0,q}(\mathbb{R}^n)^3$ (since $\boldsymbol{\lambda}(\boldsymbol{x}) = 0$ if $\boldsymbol{x} \in \mathbb{R}^n \backslash B_2$). Consider now a vector function $\boldsymbol{r} \in (\mathbb{P}_\ell^\Delta)^n$. Since $(\mathbb{P}_\ell)^n = \mathbb{G}_\ell \oplus \mathbb{G}_\ell^\perp$, there exists a function $\alpha \in \mathbb{P}_{\ell+1}$ and a vector function $\boldsymbol{p} \in \mathbb{G}_\ell^\perp$ such that

$$\boldsymbol{r} = \boldsymbol{p} + \nabla\alpha.$$

Thus,

$$\begin{aligned}
\langle \boldsymbol{z} - \operatorname{curl} \boldsymbol{\lambda}, \boldsymbol{r} \rangle_{W_k^{-1,q}, W_{-k}^{1,q'}} &= \langle \boldsymbol{z} - \operatorname{curl} \boldsymbol{\lambda}, \boldsymbol{p} + \nabla\alpha \rangle \\
&= \langle \boldsymbol{z}, \boldsymbol{p} \rangle - \int_{\mathbb{R}^n} \boldsymbol{\lambda}. \operatorname{curl} \boldsymbol{p} dx - \langle \operatorname{div} (\boldsymbol{z} - \operatorname{curl} \boldsymbol{\lambda}), \alpha \rangle \\
&= \langle \boldsymbol{z}, \boldsymbol{p} \rangle - (\nu \operatorname{curl} \boldsymbol{p}_0, \operatorname{curl} \boldsymbol{p})_\nu = 0.
\end{aligned}$$

The modified function $\boldsymbol{z}^\star = \boldsymbol{z} - \operatorname{curl} \boldsymbol{\lambda}$ verifies condition (4.4). It follows that there exists $\boldsymbol{v}^\star$ in $W_k^{0,q}(\mathbb{R}^n)^n$ such that $\operatorname{curl} \boldsymbol{v}^\star = \boldsymbol{z}^\star$. The vector field $\boldsymbol{v} = \boldsymbol{v}^\star + \boldsymbol{\lambda}$ verifies $\operatorname{curl} \boldsymbol{v} = \boldsymbol{z}$. □

LEMMA 4.11. *Let m and k be two integers in $\mathbb{Z}$ satisfying condition* (4.3). *Let $\boldsymbol{z}$ be a solenoidal vector field in $W_{m+k}^{m,q}(\mathbb{R}^n)^n$. Then, the problem*

$$\operatorname{curl} \boldsymbol{v} = \boldsymbol{z} \text{ in } \mathbb{R}^n, \quad \operatorname{div} \boldsymbol{v} = \boldsymbol{0}, \tag{4.6}$$

has a solution $\boldsymbol{v} \in W_{m+k}^{m+1,q}(\mathbb{R}^n)^n$ if and only if $\boldsymbol{z}$ satisfies

$$\forall \boldsymbol{p} \in \left(\mathbb{P}_{\langle k \rangle^\star + 1}^\Delta\right)^3, \quad \langle \operatorname{curl} \boldsymbol{z}, \boldsymbol{p} \rangle = 0. \tag{4.7}$$

Moreover, if $\langle k \rangle^\star \leq 0$, then (4.7) *is automatically fulfilled.*

PROOF. Suppose first that (4.6) admits at least one solution $\boldsymbol{v} \in W_{m+k}^{m+1,q}(\mathbb{R}^n)^n$. Then, applying the curl operator to the identity $\operatorname{curl} \boldsymbol{v} = \boldsymbol{z}$ gives

$$-\Delta \boldsymbol{v} = \operatorname{curl} \boldsymbol{z}. \tag{4.8}$$

Condition (4.7) follows immediately from Proposition 4.8.

Conversely, suppose that $\boldsymbol{z}$ satisfies (4.7). Then, from Proposition 4.8, there exists at least one vector function $\boldsymbol{v}^\star \in W_k^{m+1,q}(\mathbb{R}^n)^n$ satisfying $-\Delta \boldsymbol{v}^\star = \operatorname{curl} \boldsymbol{z}$.

Necessarily $\Delta(\operatorname{div} \boldsymbol{v}^\star) = 0$ and, thus, $\operatorname{div} \boldsymbol{v}^\star \in \mathbb{P}^{\Delta}_{\langle -k\rangle}$. From lemma 4.3, there exists a function $\theta \in \mathbb{P}_{\langle -k\rangle+2}$ such that $\Delta\Psi = \operatorname{div} \boldsymbol{v}^\star$. The function $\boldsymbol{v} = \boldsymbol{v}^\star - \nabla\Psi$ is solution of (4.6). □

4.2.2. Proof of Theorem 3.3.

• Suppose first that the following assumption holds

$$-\frac{n}{2} < k < \frac{n}{2} \quad \text{and} \quad -k < \frac{n}{q} < -k+n. \tag{4.9}$$

This assumption is equivalent to condition (1.7) with $\alpha = qk$. It implies

$$\langle -k\rangle \leq -1, \quad \langle k\rangle^\star \leq -1.$$

Let us first prove the existence of a solution $\boldsymbol{u} \in V^{2,q}_k(\mathbb{R}^n)$ of (1.4). According to Farwig et al. [**9**] problem (1.4) has at least one solution $\boldsymbol{u}_1 \in L^1_{loc}(\mathbb{R}^n)^n$ satisfying

$$\|\nabla^2 \boldsymbol{u}_1\|_{W^{0,q}_k(\mathbb{R}^n)^n} + \|\mathscr{R}\boldsymbol{u}_1\|_{W^{0,q}_k(\mathbb{R}^n)^n} \lesssim \|\boldsymbol{f}\|_{W^{0,q}_k(\mathbb{R}^n)^n}. \tag{4.10}$$

and such that $\operatorname{div} \boldsymbol{u}_1 = 0$ if $\operatorname{div} \boldsymbol{f} = 0$. Unfortunately, there is no reason that $\boldsymbol{u}_1$ belongs to $W^{2,q}_0(\mathbb{R}^n)^n$. Nevertheless, we shall use $\boldsymbol{u}_1$ to get a solution in $V^{2,q}_0(\mathbb{R}^n)$. The starting point is the estimate (4.10). Since $\Delta\boldsymbol{u}_1 \in W^{0,q}_k(\mathbb{R}^n)$, and since the operator

$$\Delta \;:\; W^{2,q}_k(\mathbb{R}^n)/\mathbb{P}^{\Delta}_{2+\langle -k\rangle} \mapsto W^{0,q}_k(\mathbb{R}^n),$$

is an isomorphism, there exists a function $\boldsymbol{z}$ in $(W^{2,q}_k(\mathbb{R}^n))^n$ such that

$$\Delta\boldsymbol{z} = \Delta\boldsymbol{u}_1,$$

and subject to the estimate

$$\inf_{\boldsymbol{s}\in\mathbb{P}_{2+\langle -k\rangle}} \|\boldsymbol{z} - \boldsymbol{s}\|_{W^{2,q}_0(\mathbb{R}^n)} \lesssim \|\Delta\boldsymbol{u}_1\|_{W^{0,q}_k(\mathbb{R}^n)^n} \lesssim \|\boldsymbol{f}\|_{W^{0,q}_k(\mathbb{R}^n)}. \tag{4.11}$$

Let $\boldsymbol{m} = \boldsymbol{z} - \boldsymbol{u}_1$. Then, $\boldsymbol{m}$ is a harmonic tempered distribution. It follows that $\boldsymbol{m}$ is polynomial. Estimates (4.10) and (4.11) give $\partial^2_{i,j}\boldsymbol{m} \in W^{0,q}_k(\mathbb{R}^n)^n$, $1 \leq i,\ j \leq n$. Since $W^{0,q}_k(\mathbb{R}^n)$ does not contain non vanishing polynomial, we deduce that $\partial^2_{i,j}\boldsymbol{m} = \boldsymbol{0}$, $1 \leq i,\ j \leq n$, and $\boldsymbol{m} \in (\mathbb{P}_1)^n$. On the other hand, since $\mathscr{R}\boldsymbol{u}_1 \in W^{0,q}_k(\mathbb{R}^n)^n$, $\mathscr{R}\boldsymbol{z} \in W^{0,q}_{k-2}(\mathbb{R}^n)^n$, one deduces that $\mathscr{R}\boldsymbol{m} \in W^{0,q}_{k-2}(\mathbb{R}^n)^n$. By virtue of (2.7), it results that

$$\mathscr{R}\boldsymbol{m} \in \mathbb{P}_{[2-k-(d/q)]} = \mathbb{P}_{2+\langle -k\rangle}. \tag{4.12}$$

At this stage, three cases are distinguished

(1) If $\langle -k\rangle = -1$. Then, $\mathbb{P}_1 \subset W^{2,q}_k(\mathbb{R}^n)$ and $\boldsymbol{m} \in W^{2,q}_k(\mathbb{R}^n)^n$. It follows that $\boldsymbol{u}_1 = \boldsymbol{z} - \boldsymbol{m}$ belongs to $W^{2,q}_k(\mathbb{R}^n)^n$ and is a solution of (1.9).

(2) If $\langle -k\rangle = -2$. Then, $\mathbb{P}_{2+\langle -k\rangle} = \mathbb{P}_0$. From (4.12) we deduce that there exists a constant vector $\boldsymbol{c}_0$ such that

$$\forall \boldsymbol{x} \in \mathbb{R}^n, \quad (\mathscr{R}\boldsymbol{m})(\boldsymbol{x}) = \boldsymbol{c}_0.$$

Taking $\boldsymbol{x} = \boldsymbol{0}$ on the left hand side gives

$$\boldsymbol{c}_0 = \widehat{\boldsymbol{\omega}} \times \boldsymbol{m}(\boldsymbol{0}).$$

Let $\boldsymbol{d}_0 = \boldsymbol{m}(\boldsymbol{0})$. Then, $\boldsymbol{d}_0 \in W^{2,q}_k(\mathbb{R}^n)$ and $\mathscr{L}_+\boldsymbol{d}_0 = \boldsymbol{c}_0 = \mathscr{L}_+\boldsymbol{m}$. Set $\boldsymbol{u} = \boldsymbol{z} - \boldsymbol{d}_0 \in W^{2,q}_k(\mathbb{R}^n)$ gives

$$\mathscr{L}_+\boldsymbol{u} = \mathscr{L}_+\boldsymbol{u}_1 + \mathscr{L}_+\boldsymbol{m} - \mathscr{L}_+\boldsymbol{d}_0 = \mathscr{L}_+\boldsymbol{u}_1 = \boldsymbol{f},$$

and

$$\operatorname{div} \boldsymbol{u} = \operatorname{div} \boldsymbol{u}_1 = 0.$$

Hence, $\boldsymbol{u}$ is a solution of the problem (1.9).

(3) $\langle -k\rangle < -2$. Then, $\mathbb{P}_{1-\langle -k\rangle} = \{0\}$. From (2.7), we deduce that

$$\mathscr{R}\boldsymbol{m} = 0.$$

Thus, $\mathscr{L}_+\boldsymbol{z} = \mathscr{L}_+\boldsymbol{u}_1 + \mathscr{L}_+\boldsymbol{m} = \boldsymbol{f}$.

On the other hand, $\operatorname{div} \boldsymbol{z} \in W^{1,q}_k(\mathbb{R}^n)$ and $\Delta(\operatorname{div} \boldsymbol{z}) = \Delta(\operatorname{div} \boldsymbol{u}_1) = 0$. Hence, $\operatorname{div} \boldsymbol{z} \in \mathbb{P}_{\langle 1-k\rangle} = \{0\}$. We conclude that $\boldsymbol{z}$ is a solution of (1.9) in $W^{2,q}_k(\mathbb{R}^n)$.

- Consider now the second case in which k satisfies condition

$$2 - \frac{n}{2} < k < 2 + \frac{n}{2} \quad \text{and} \quad -k + 2 < \frac{n}{q} < -k + 2 + n. \tag{4.13}$$

This condition means that $k = 2$ if $n = 2$ and $k \in \{1, 2, 3\}$ if $n = 3$ with $q > 3/2$ if $k = 3$ and $q < 3$ if $k = 1$.

Let $\ell = 2 - k$. Then, ℓ satisfies assumption (4.9). Consider the operator $\mathscr{L}_+$ defined from $D(\mathscr{L}_+) = V_k^{2,q}(\mathbb{R}^n) \subset W_{-\ell}^{0,q}(\mathbb{R}^n)^n$ into $W_k^{0,q}(\mathbb{R}^n)$. Let $\mathscr{L}_+^\star$ be its dual operator defined from $W_{\ell-2}^{0,q'}(\mathbb{R}^n)^n$ into $W_\ell^{0,q'}(\mathbb{R}^n)^n$ (see, e.g., Yosida [**23**]). From the formula

$$\langle \boldsymbol{v}, \mathscr{L}_+ \boldsymbol{z} \rangle_{W_{\ell-2}^{0,q'}(\mathbb{R}^n)^n, W_k^{0,q}(\mathbb{R}^n)^n} = \langle \mathscr{L}_- \boldsymbol{v}, \boldsymbol{z} \rangle_{W_\ell^{0,q'}(\mathbb{R}^n)^n, W_{k-2}^{0,q}(\mathbb{R}^n)^n}$$

which is valid for $\boldsymbol{v} \in V_\ell^{2,q'}(\mathbb{R}^n)$ and $\boldsymbol{z} \in V_k^{2,q}(\mathbb{R}^n)$, one deduces that $V_\ell^{2,q'}(\mathbb{R}^n) \subset D(\mathscr{L}_+^\star)$ and

$$\forall \boldsymbol{v} \in V_\ell^{2,q'}(\mathbb{R}^n), \quad \mathscr{L}_+^\star \boldsymbol{v} = \mathscr{L}_- \boldsymbol{v}. \tag{4.14}$$

From the first part of the proof we know that $\mathscr{L}_-$, considered as an operator from $V_\ell^{2,q'}(\mathbb{R}^n)$ into $W_\ell^{0,q'}(\mathbb{R}^n)^n$, is onto. It follows that $R(\mathscr{L}_+^\star) = W_\ell^{0,q'}(\mathbb{R}^n)^n$. The Closed Range Theorem of Banach implies that

$$R(\mathscr{L}_+) = W_k^{0,q}(\mathbb{R}^n)^n \perp \mathbb{M}_{-k}^{q'}(\mathscr{L}_-).$$

Consider now a vector function $\boldsymbol{f} \in W_k^{0,q}(\mathbb{R}^n)^n \perp \mathbb{M}_{1-k}^{q'}(\mathscr{L}_-)$ and let $\boldsymbol{v} \in V_k^{2,q}(\mathbb{R}^n)^n$ such that $\mathscr{L}_+ \boldsymbol{v} = \boldsymbol{f}$. Suppose that $\operatorname{div} \boldsymbol{f} = 0$. Applying the divergence operator to the identity $\mathscr{L}_+ \boldsymbol{v} = \boldsymbol{f}$ gives

$$L_+ \operatorname{div} \boldsymbol{v} = 0. \tag{4.15}$$

Thus, $\operatorname{div} \boldsymbol{v} \in \mathbb{K}_{k-1}^q(L_+) \subset \mathbb{P}_{\langle 1-k \rangle} = \{0\}$. Hence, $\operatorname{div} \boldsymbol{v} = 0$.

4.3. Application to the operator $\boldsymbol{\Delta} + (\boldsymbol{\omega} \times \boldsymbol{x}).\boldsymbol{\nabla}$. Proof of Corollary 3.4.

Assume that $n = 3$. Let $h \in W_k^0(\mathbb{R}^3)$ and suppose that equation (1.10) admits at least one solution $u \in W_k^2(\mathbb{R}^3)$. Multiplying by a function $\phi \in \mathbb{D}_{\langle k \rangle^\star}^{\Delta}$ and integrating over $\mathbb{R}^3$ gives conditions (3.5).

Conversely, suppose that (3.5) is fulfilled and let us prove that equation (1.10) possesses at least one solution u depending continuously on h. Set $\boldsymbol{f} = h\widehat{\boldsymbol{\omega}}$. Using the characterization (3.1), one can prove easily that $\boldsymbol{f}$ satisfies (3.4). Theorem 3.3 asserts that there exists a vector function $\boldsymbol{v}$ solution of the equation $\mathscr{L}_+ \boldsymbol{v} = \boldsymbol{f}$. The function $u = \boldsymbol{v}.\widehat{\boldsymbol{\omega}}$ ($\widehat{\boldsymbol{\omega}} = \boldsymbol{e}_3$) satisfies $L_+ u = h$ and

$$\inf_{\gamma\in\mathbb{D}^{\Delta}_{\ell}} \|u-\gamma\|_{W^{2,q}_k(\mathbb{R}^3)} + \|D_\theta(u-\gamma)\|_{W^{0,q}_k(\mathbb{R}^3)} \le \inf_{\gamma\in\mathbb{D}^{\Delta}_{\ell}} \|\boldsymbol{v}-\gamma\widehat{\boldsymbol{\omega}}\|_{V^{2,q}_k(\mathbb{R}^3)}$$
$$\le \inf_{\boldsymbol{z}\in\mathbb{M}^{q}_{k-2}(\mathscr{L}_+)} \|\boldsymbol{v}-\boldsymbol{z}\|_{V^{2,q}_k(\mathbb{R}^3)},$$

with $\ell = 2 + \langle -k\rangle$. This ends the proof of corollary.

4.4. Proof of Theorem 3.5.

In view of Proposition 4.8 and condition (3.7), equation

$$\Delta\Psi = \operatorname{div}\boldsymbol{f} + \Delta g + D_\theta g, \quad \text{in } \mathbb{R}^n,$$

admits at least one solution $\Psi \in W^{m+1,q}_k(\mathbb{R}^n)$. So we henceforth define

$$\boldsymbol{f}^\star = \boldsymbol{f} - \nabla\Psi.$$

Let $\boldsymbol{r} \in \mathbb{M}^{q'}_{m-k}(\mathscr{L}_-)$. From Theorem 3.1, we know that there exists three functions $\lambda \in \mathbb{D}^{\Delta}_{\ell+1} + \mathbb{P}^{\Delta}_{\ell-1}$, $\beta \in \mathbb{D}^{\Delta}_{\ell+1}$ and $\gamma \in \mathbb{D}^{\Delta}_{\ell}$, with $\ell = \langle k-m\rangle^\star$, such that

$$\boldsymbol{r} = \nabla\lambda + \widehat{\boldsymbol{\omega}} \times \nabla\beta + \gamma\widehat{\boldsymbol{\omega}}.$$

Then, $\operatorname{div}\boldsymbol{r} = \partial\gamma/\partial x_3 = (1/2)\Delta(x_3\gamma)$. Thus,

$$\begin{aligned}
\langle \boldsymbol{f}^\star, \boldsymbol{r}\rangle &= \langle \boldsymbol{f}, \boldsymbol{r}\rangle + \langle \Psi,\ \operatorname{div}\boldsymbol{r}\rangle \\
&= \langle \boldsymbol{f}, \boldsymbol{r}\rangle + \frac{1}{2}\langle \Psi,\ \Delta(x_3\gamma)\rangle \\
&= \langle \boldsymbol{f}, \boldsymbol{r}\rangle + \frac{1}{2}\langle \Delta\Psi,\ x_3\gamma\rangle \\
&= \langle \boldsymbol{f}, \boldsymbol{r}\rangle + \frac{1}{2}\langle \operatorname{div}\boldsymbol{f} + \Delta g + D_\theta g,\ x_3\gamma\rangle \\
&= \left\langle \boldsymbol{f}, \boldsymbol{r} - \frac{1}{2}\nabla(x_3\gamma)\right\rangle + \left\langle g, \frac{\partial\gamma}{\partial x_3}\right\rangle + \frac{1}{2}\langle D_\theta g,\ x_3\gamma\rangle = 0,
\end{aligned}$$

since the pair $(\boldsymbol{r} - (1/2)\nabla(x_3\gamma), -\partial\gamma/\partial x_3)$ belongs to $\mathbb{N}^{q'}_{m-k}(\mathscr{L}_-)$.

Notice also that

$$\operatorname{div}\boldsymbol{f}^\star = -L_+ g. \tag{4.16}$$

At this stage, four cases are distinguished

Case 1: $m = 0$. Suppose that $\boldsymbol{f}$ belongs to $W_k^{0,q}(\mathbb{R}^n)^n$. Theorem 3.3 asserts that there exists a solenoidal vector field $\boldsymbol{u}$ satisfying

$$-\mathscr{L}_+\boldsymbol{u} = \boldsymbol{f}^\star.$$

Taking the divergence of both the sides gives

$$L_+(\operatorname{div}\boldsymbol{u} - g) = 0.$$

It follows that $\operatorname{div}\boldsymbol{u} - g \in \mathbb{D}^\Delta_{1+\langle -k\rangle}$, thanks to theorem 3.1. From Lemma 4.4, there exists $s \in \mathbb{D}^\Delta_{3+\langle -k\rangle}$ such that $\Delta s = \operatorname{div}\boldsymbol{u} - g$. The pair $(\boldsymbol{u} - \nabla s, \Psi)$ is clearly solution of (1.4) in $V_k^2(\mathbb{R}^n)$.

Case 2: $m = -1$ and $g = 0$. Suppose that $\boldsymbol{f}$ belongs to $W_k^{-1,q}(\mathbb{R}^n)^n$. Since $\operatorname{div}\boldsymbol{f}^\star = 0$, it follows from Lemma 4.10 that there exists a vector function $\boldsymbol{F}$ in $W_k^{0,q}(\mathbb{R}^n)^n$ such that $\operatorname{curl}\boldsymbol{F} = \boldsymbol{f}^\star$. We need the lemma

Lemma 4.12. *Assume that $\langle k\rangle^\star \leq 1$. Then, for each $(\boldsymbol{v}, \pi) \in \mathbb{N}^{q'}_{-k}(\mathscr{L}_\pm)$, there exists a vector field $\boldsymbol{z} \in \mathbb{M}^{q'}_{-k-1}(\mathscr{L}_\pm)$ such that $\boldsymbol{v} = \operatorname{curl}\boldsymbol{z}$.*

Proof of Lemma 4.12. Let $\ell = \langle k\rangle^\star$ and $(\boldsymbol{v}, \pi) \in \mathbb{N}^{q'}_{-k}(\mathscr{L}_\pm)$. We know that $\boldsymbol{v}$ can be written into the form

$$\boldsymbol{v} = \nabla\lambda + \widehat{\boldsymbol{\omega}} \times \nabla\beta + 2\gamma\widehat{\boldsymbol{\omega}} - \nabla[(\widehat{\boldsymbol{\omega}}.\boldsymbol{x})\gamma],$$

with $\lambda \in \mathbb{D}^\Delta_{\ell+1} + \mathbb{P}^\Delta_{\ell-1}$, $\beta \in \mathbb{D}^\Delta_{\ell+1}$ and $\gamma \in \mathbb{D}^\Delta_\ell$. Since $\ell \leq 1$, λ and γ are of the form

$$\lambda(x_1, x_2, x_3) = a_1(r^2 - 2x_3^2) + b_1 x_3 + c_1,$$
$$\gamma(x_1, x_2, x_3) = a_2 x_3 + b_2,$$

with a_1, b_1, c_1, a_2 and b_2 real coefficients. We set

$$\chi(x_1, x_2, x_3) = -\frac{b_1 + b_2}{4}(2x_3^2 - r^2) - a_1\left(x_3 r^2 - \frac{2}{3}x_3^3\right) - c_1 x_3.$$

We have $\chi \in \mathbb{D}^\Delta_{\ell+2}$. Let

$$\boldsymbol{z} = -\beta\widehat{\boldsymbol{\omega}} + \widehat{\boldsymbol{\omega}} \times \nabla\chi.$$

Then, $\boldsymbol{z} \in \mathbb{M}^{q'}_{-k-1}(\mathscr{L}_\pm)$ and $\operatorname{curl}\boldsymbol{z} = \boldsymbol{v}$. $\square$

Consider now a pair $(\boldsymbol{v},\pi) \in \mathbb{N}^{q'}_{-k}(\mathscr{L}_-)$. From Lemma 4.12, there exists a vector field $\boldsymbol{w} \in \mathbb{M}^{q'}_{-k-1}(\mathscr{L}_\pm)$ such that $\operatorname{curl} \boldsymbol{w} = \boldsymbol{v}$. One has

$$\begin{aligned}\langle \boldsymbol{F}, \boldsymbol{v}\rangle &= \langle \boldsymbol{F}, \operatorname{curl} \boldsymbol{w}\rangle \\ &= \langle \operatorname{curl} \boldsymbol{F}, \boldsymbol{w}\rangle \\ &= \langle \boldsymbol{f}^\star, \boldsymbol{w}\rangle = 0.\end{aligned}$$

According to the first part of the proof, there exists a pair $(\boldsymbol{v},\pi) \in W^{2,q}_k(\mathbb{R}^n)^n \times W^{0,q}_k(\mathbb{R}^n)$ verifying

$$-\mathscr{L}_+\boldsymbol{v} + \nabla\pi = \boldsymbol{F} \quad \text{in } \mathbb{R}^n,\ \operatorname{div} \boldsymbol{v} = 0. \tag{4.17}$$

We set $\boldsymbol{u} = \operatorname{curl} \boldsymbol{v} \in W^{1,q}_k(\mathbb{R}^n)^n$. Applying the curl operator to (4.17) gives $\mathscr{L}_+\boldsymbol{u} = \boldsymbol{f} - \nabla p$. We deduce that the pair $(\boldsymbol{u}, p)$ is a solution of the original problem (1.4).

Case 3: $m \geq 1$ and $g = 0$. Let us prove by induction on $m \geq 0$ the following proposition

($\mathscr{P}_m$) *For each* $\boldsymbol{f} \in W^{m,q}_k(\mathbb{R}^n)$, *satisfying* (3.6) *and* (3.7) *with* $g = 0$, *problem* (1.4) *admits at least one solution* $(\boldsymbol{u}, p) \in W^{m+2,q}_k(\mathbb{R}^n)^n \times W^{m+1,q}_k(\mathbb{R}^n)$ *with* $p = 0$ *if* $\operatorname{div} \boldsymbol{f} = 0$.

We know that ($\mathscr{P}_0$) is true. Suppose that ($\mathscr{P}_m$) is true and let us prove ($\mathscr{P}_{m+1}$). Let $\boldsymbol{f} \in W^{m+1,q}_k(\mathbb{R}^n)$ and set $\boldsymbol{h} = \operatorname{curl} \boldsymbol{f} \in W^{m,q}_k(\mathbb{R}^n)$. This function $\boldsymbol{h}$ satisfies conditions (3.6) and (3.7) (with $g = 0$) since for all $(\boldsymbol{v},\pi) \in \mathbb{N}^{q'}_{m-k}(\mathscr{L}_-)$, $(\operatorname{curl} \boldsymbol{v}, 0) \in \mathbb{N}^{q'}_{m+1-k}(\mathscr{L}_-)$. Induction hypothesis implies that there exists a pair $(\boldsymbol{z},\mu) \in W^{m+2,q}_k(\mathbb{R}^n) \times W^{m+1,q}_k(\mathbb{R}^n)$ satisfying $-\mathscr{L}_+\boldsymbol{z} + \nabla\mu = \boldsymbol{h}$. Necessarily, $\mu = 0$ since $\operatorname{div} \boldsymbol{h} = 0$. By Lemma 4.10, there exists $\boldsymbol{v} \in W^{m+3,q}_k(\mathbb{R}^n)$ such that $\operatorname{curl} \boldsymbol{v} = \boldsymbol{z}$ and $\operatorname{div} \boldsymbol{v} = 0$. Let

$$\boldsymbol{r} = \mathscr{L}_+\boldsymbol{v} + \boldsymbol{f}.$$

We have

$$\operatorname{curl} \boldsymbol{r} = \mathscr{L}_+\boldsymbol{z} + \boldsymbol{h} = \boldsymbol{0}, \quad \operatorname{div} \boldsymbol{r} = \operatorname{div} \boldsymbol{f}.$$

Let $\chi_1 \in W^{m+1,q}_k(\mathbb{R}^n)^n$ solution of the equation $\Delta\chi_1 = \operatorname{div} \boldsymbol{f}$. This equation admits a solution, thanks to Proposition 4.8 and condition (3.7). We set $\boldsymbol{p} = \boldsymbol{r} - \nabla\chi_1$. Then $\operatorname{curl} \boldsymbol{p} = \boldsymbol{0}$ and $\operatorname{div} \boldsymbol{p} = 0$. It follows that there exists a polynomial

$\chi_2 \in \mathbb{P}^{\Delta}_{m+2+\langle -k\rangle}$ such that $\boldsymbol{p} = \nabla\chi_2$. The pair $(\boldsymbol{v}, \pi)$, with $\pi = \chi_1 + \chi_2$, is a solution of (1.4) in $W^{m+3,q}_k(\mathbb{R}^n) \times W^{m+2,q}_k(\mathbb{R}^n)$.

Suppose now that $\operatorname{div} \boldsymbol{f} = 0$. Then, π is a polynomial and $\pi \in \mathbb{P}^{\Delta}_{m+2+\langle -k\rangle}$. According to Lemma 4.7, there exists $\Psi \in \mathbb{P}_{m+4+\langle -k\rangle}$ such that $L_+\Psi = \pi$. Thus

$$\mathscr{L}_+\boldsymbol{v} - \boldsymbol{f} = \nabla(L_+\Psi) = \mathscr{L}_+\nabla\Psi.$$

It follows that $(\boldsymbol{v}' = \boldsymbol{v} - \nabla\Psi, 0)$ is also a solution of (1.4). This ends the proof of $(\mathscr{P}_m)$ by induction.

Case 4: $m \geq 1$ and $g \neq 0$. Assume that $m \geq 3$. Let $\phi \in W^{m+3,q}_k(\mathbb{R}^n)$ and $\chi \in W^{m+1}_k(\mathbb{R}^n)$ be solutions to the Poisson equations

$$\Delta\phi = g, \quad \Delta\chi = D_\theta g. \tag{4.18}$$

Both the problems admit a solution since $\langle k\rangle^\star - (m+1) < 0$ and $\langle k\rangle^\star - (m-1) < 0$ (see Tables 1 and 2). Since $[D_\theta, \Delta] = 0$, the function $\epsilon = D_\theta\phi - \chi$, which belongs to $W^{m+1}_{k-2}(\mathbb{R}^n)$, is harmonic. It follows that $\epsilon \in \mathbb{P}^{\Delta}_{\ell}$ with $\ell = m + 3 + \langle -k\rangle$, thanks to (2.7). Let $\epsilon_0 = \Pi\pi \in \mathbb{D}^{\Delta}_{\ell+1}$ and $\epsilon_1 = \epsilon - \epsilon_0$. According to Lemma 4.2, we know that there exists a function $\lambda \in \mathbb{P}^{\Delta}_{\ell}$ such that $\epsilon_1 = D_\theta\lambda$. Thus,

$$\epsilon_0 + \chi = D_\theta(\phi - \lambda).$$

Integration with respect to the variable θ between 0 and 2π, gives

$$2\pi\epsilon_0(r, x_3) = -\int_0^{2\pi} \chi(r\cos\theta, r\sin\theta, x_3)d\theta.$$

Thus, by Hölder inequality we get

$$\int_0^{\infty}\int_{-\infty}^{+\infty} \langle x\rangle^{qs}|\epsilon_0|^q r dx_3 dr \lesssim \int_0^{+\infty}\int_{-\infty}^{+\infty}\int_0^{2\pi} \langle x\rangle^{qs}|\chi|^q r d\theta dx_3 dr < +\infty,$$

with $s = k - m - 1$. It follows that $\epsilon_0 \in W^0_{k-m-1}(\mathbb{R}^n)$. From (2.7), we deduce that $\epsilon_0 \in \mathbb{D}^{\Delta}_{\ell-2}$. Now, we put $\boldsymbol{v}_0 = \nabla(\phi - \lambda)$. Then, $\boldsymbol{v}_0 \in W^{m+2,q}_k(\mathbb{R}^n)$ and $\mathscr{R}\boldsymbol{v}_0 = \nabla D_\theta(\phi - \lambda) = \nabla\epsilon_0 + \nabla\chi \in W^m_k(\mathbb{R}^n)$. Moreover, $\operatorname{div} \boldsymbol{v}_0 = g$. Setting $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{v}_0$ in (1.4) leads to a similar problem with $g = 0$.

Suppose now that $1 \leq m \leq 2$. Then, $\langle k\rangle^\star + 1 - m \leq 1$ and $\mathbb{P}^{\Delta}_{\langle k\rangle^\star+1-m} \subset \mathbb{P}_1$. For each $\lambda \in \mathbb{P}^{\Delta}_{\langle k\rangle^\star+1-m}$, the pair $(\nabla\lambda, 0)$ belongs to $\mathbb{N}^{q'}_k(\mathscr{L}_-)$. In view of condition (3.6), we deduce that $\langle \boldsymbol{f}, \nabla\lambda\rangle = 0$. Replacing in (3.7) gives $\langle D_\theta g, \lambda\rangle = 0$. It follows that the RHS in the second equation (4.18) satisfies the compatibility condition which ensures for the existence of a solution. The above argument remains valid in this case.

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