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Eventual colorings of homeomorphisms

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Abstract. In this paper, we study some dynamical properties of fixed-point free homeomorphisms of separable metric spaces. For each natural number p, we define eventual colorings within p of homeomorphisms which are generalized notions of colorings of fixed-point free homeomorphisms, and we investigate the eventual coloring number $C(f,p)$ of a fixed-point free homeomorphism $f : X \to X$ with zero-dimensional set of periodic points. In particular, we show that if $\dim X < \infty$, then there is a natural number p, which depends on $\dim X$, and X can be divided into two closed regions C_1 and C_2 such that for each point $x \in X$, the orbit $\{f^k(x)\}_{k=0}^{\infty}$ of x goes back and forth between $C_1 - C_2$ and $C_2 - C_1$ within the time p.

1. Introduction.

In this paper, we assume that all spaces are nonempty separable metric spaces and maps are continuous functions. Let $\mathbb{N}$ be the set of all natural numbers, i.e., $\mathbb{N} = \{1, 2, 3, \dots\}$. For a (separable metric) space X, $\dim X$ denotes the topological dimension of X. For each map $f : X \to X$, let $P(f)$ be the set of all periodic points of f, i.e.,

$$P(f) = \{x \in X | \ f^j(x) = x \text{ for some } j \in \mathbb{N}\}.$$

Let $f : X \to X$ be a fixed-point free closed map of a separable metric space X, i.e., $f(x) \neq x$ for each $x \in X$. In this paper, we assume that all maps are closed maps, i.e., for any closed subset A of X, $f(A)$ is closed in X. A subset C of X is called a *color* (see [**9**]) of f if $f(C) \cap C = \emptyset$. Note that $f(C) \cap C = \emptyset$ if and only if $C \cap f^{-1}(C) = \emptyset$. We say that a cover $\mathcal{C}$ of X is a *coloring* of f if each element C of $\mathcal{C}$ is a color of f. The minimal cardinality $C(f)$ of closed (or open) colorings of f is called the *coloring number* of f. The coloring number $C(f)$ has been investigated by many mathematicians (see [**1**]–[**5**] and [**7**]–[**9**]).

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Theorem 1.1 (Lusternik and Schnirelman [**7**]). *Let $f : S^n \to S^n$ be the antipodal map of the n-dimensional sphere S^n. Then $C(f) = n + 2$.*

Theorem 1.2 (Aarts, Fokkink and Vermeer [**1**]). *Let $f : X \to X$ be a fixed-point free involution of a (separable) metric space X with $\dim X = n < \infty$. Then $C(f) \leq n + 2$.*

Theorem 1.3 (Aarts, Fokkink and Vermeer [**1**]). *Let $f : X \to X$ be a fixed-point free homeomorphism of a (separable) metric space X with $\dim X = n < \infty$. Then $C(f) \leq n + 3$.*

Now, similarly we will consider more general notion of color as follows: Let $f : X \to X$ be a fixed-point free map of a space X and $p \in \mathbb{N}$. A subset C of X is *eventually colored within* p of f if $\bigcap_{i=0}^{p} f^{-i}(C) = \emptyset$. Note that C is a color of f if and only if C is eventually colored within 1. Then we have the following simple proposition. For completeness, we give the proof.

Proposition 1.4. *Let $f : X \to X$ be a fixed-point free map of a separable metric space X and $p \in \mathbb{N}$. Then the followings hold.*

(1) *A subset C of X is eventually colored within p of f if and only if each point $x \in C$ wanders off C within p, i.e., for each $x \in C$, $f^i(x) \notin C$ with some $i \leq p$.*
(2) *If a subset C of X satisfies the condition $\bigcap_{i=0}^{p} f^{i}(C) = \emptyset$, then C is eventually colored within p of f.*
(3) *If f is an injective map, then a subset C of X is eventually colored within p of f if and only if C satisfies the condition $\bigcap_{i=0}^{p} f^{i}(C) = \emptyset$.*

Proof. We prove (1). In fact, it is easily seen that $\bigcap_{i=0}^{p} f^{-i}(C) \neq \emptyset$ if and only if there is an element $x \in C$ such that $f^i(x) \in C$ for any $0 \leq i \leq p$. We prove (2). Suppose, on the contrary, that there is a point $x \in C$ such that $f^i(x) \in C$ for each $0 \leq i \leq p$. Then $f^p(x) \in \bigcap_{i=0}^{p} f^{i}(C) = \emptyset$. This is a contradiction. Finally we prove (3). We suppose that f is injective. Let C be eventually colored within p of f. Suppose, on the contrary, that $\bigcap_{i=0}^{p} f^{i}(C) \neq \emptyset$. Take a point $y \in \bigcap_{i=0}^{p} f^{i}(C)$. Choose a point $x \in C$ such that $f^p(x) = y$. Since f is injective, we see that $f^i(x) \in C$ for each $0 \leq i \leq p$. This is a contradiction. $\square$

Remark. In general, the converse assertion of (2) in the proposition above is not true. Let $X = \{a, b, c\}$ be a set consisting three points and let $f : X \to X$ be the map defined by $f(a) = b$, $f(b) = c$, $f(c) = b$. Then $C = \{a, b\}$ is eventually colored within 2 of f, but $\bigcap_{i=0}^{p} f^{i}(C) \neq \emptyset$ ($p \in \mathbb{N}$).

We define the eventual coloring number $C(f, p)$ as follows. A cover $\mathcal{C}$ of

X is called an *eventual coloring within* p if each element C of $\mathcal{C}$ is eventually colored within p. The minimal cardinality $C(f,p)$ of all closed (or open) eventual colorings within p is called the *eventual coloring number* of f within p. Note that $C(f,1) = C(f)$. If there is some $p \in \mathbb{N}$ with $C(f,p) < \infty$, we say that f is eventually colored. Similarly, we can consider the index $C^+(f,p)$ defined by

$$\min\Big\{|\mathcal{C}|;\ \mathcal{C} \text{ is a closed (open) cover of } X \quad \text{such that for each } C \in \mathcal{C},\ \bigcap_{i=0}^{p} f^i(C) = \emptyset\Big\}.$$

By the definitions, we see that $C(f,p) \leq C^+(f,p)$. In section 3, we show that $C(f,p) = C^+(f,p)$ if X is compact.

In this paper, we need the following notions. A finite cover $\mathcal{C}$ of X is a *closed partition* of X provided that each element C of $\mathcal{C}$ is closed, $\operatorname{int}(C) \neq \emptyset$ and $C \cap C' = \operatorname{bd}(C) \cap \operatorname{bd}(C')$ for any $C, C' \in \mathcal{C}$. Let $\mathcal{B}$ be a collection of subsets of a space X with $\dim X = n < \infty$. Then we say that $\mathcal{B}$ is *in general position in* X provided that if $\mathcal{S} \subset \mathcal{B}$ with $|\mathcal{S}| = m$, then $\dim(\bigcap\{S \mid S \in \mathcal{S}\}) \leq \max\{-1, n-m\}$. By a *swelling* of a family $\{A_s\}_{s\in S}$ of subsets of a space X, we mean any family $\{B_s\}_{s\in S}$ of subsets of X such that $A_s \subset B_s$ $(s \in S)$ and for every finite set of indices $s_1, s_2, \ldots, s_m \in S$,

$$\bigcap_{i=1}^{m} A_{s_i} \neq \emptyset \text{ if and only if } \bigcap_{i=1}^{m} B_{s_i} \neq \emptyset.$$

Conversely, for any cover $\{B_s\}_{s\in S}$ of X, a cover $\{A_s\}_{s\in S}$ of X is a *shrinking* of $\{B_s\}_{s\in S}$ if $A_s \subset B_s$ $(s \in S)$. The following facts are well-known;

(1) for any locally finite collection $\mathcal{F}$ of closed subsets of a space X, $\mathcal{F}$ has a swelling consisting of open subsets of X (e.g., see [**9**, Proposition 3.2.1]) and
(2) for any open cover $\mathcal{U}$ of X, $\mathcal{U}$ has a closed shrinking cover of X (e.g., see [**9**, Proposition A.7.1]).

Hence we see that if $f : X \to X$ is a closed map and a closed finite cover $\mathcal{B}$ of X is an eventual coloring of f, then we can find an open swelling $\mathcal{C}$ of $\mathcal{B}$ which is an eventual coloring of f.

2. Eventual coloring numbers of fixed-point free homeomorphisms.

In this section, we will define an index $\varphi_n(k)$. For each $n = 0, 1, 2, \dots,$ and each $k = 0, 1, 2, \dots, n+1$, we define the index $\varphi_n(k)$ as follows: Put $\varphi_n(0) = 1$. For each $k = 1, 2, \dots, n+1$, by induction on k we define the index $\varphi_n(k)$ by

$$\varphi_n(k) = 2\varphi_n(k-1) + [n/(n+2-k)] \cdot (\varphi_n(k-1) + 1),$$

where $[x] = \max\{m \in \mathbb{N} \cup \{0\} |\ m \le x\}$ for $x \in [0, \infty)$. Note that $\varphi_n(1) = 2$ $(n \ge 0)$ and $\varphi_n(2) = 7$ $(n \ge 1)$. Also, note that $\varphi_2(3) = 30$, $\varphi_n(3) = 22$ $(n \ge 3)$, $\varphi_3(4) = 113$, $\varphi_4(4) = 90$ and $\varphi_4(5) = 544$.

In this paper, we need the following two lemmas whose proofs are some modifications of the proofs of Kulesza [**6**, Lemma 3.3 and Lemma 3.5].

Lemma 2.1 (cf. [**6**, Lemma 3.3]). *Let $\mathcal{C} = \{C_i |\ 1 \le i \le m\}$ be an open cover of a separable metric space X with $\dim X = n < \infty$ and let $\mathcal{B} = \{B_i |\ 1 \le i \le m\}$ be a closed shrinking of $\mathcal{C}$. Suppose that O is an open set in X and Z is a zero-dimensional subset of O. Then there is an open shrinking $\mathcal{C}' = \{C'_i |\ 1 \le i \le m\}$ of $\mathcal{C}$ such that for each $i \le m$,*

(0) $B_i \subset C'_i$,
(1) $C'_i = C_i$ *if* $\mathrm{bd}(C_i) \cap O = \emptyset$,
(2) $C'_i \cap (X - O) = C_i \cap (X - O)$,
(3) $\mathrm{bd}(C'_i) \cap (X - O) \subset \mathrm{bd}(C_i) \cap (X - O)$,
(4) $\mathrm{bd}(C'_i) \cap Z = \emptyset$, *and*
(5) $\{\mathrm{bd}(C') \cap O |\ C' \in \mathcal{C}'\}$ *is in general position.*

Proof. First, we will construct C'_1. Consider the subspace

$$Y_1 = \mathrm{cl}(C_1) \cap \mathrm{cl}(O) - (\mathrm{bd}(O) \cap \mathrm{bd}(C_1))$$

of X. Put $E_1 = Y_1 \cap (\mathrm{bd}(O) \cup B_1)$ and $F_1 = Y_1 \cap \mathrm{bd}(C_1)$. Then E_1 and F_1 are disjoint closed subsets of Y_1. Then we can take a closed separator (or partition) S_1 between E_1 and F_1 in Y_1 such that $\dim S_1 \le n-1$ and $S_1 \cap Z = \emptyset$ (e.g., see [**9**, Lemma 3.1.4]). Hence we have open subsets G_1 and H_1 of Y_1 such that $Y_1 - S_1 = G_1 \cup H_1$, $G_1 \cap H_1 = \emptyset$ and $G_1 \supset E_1, H_1 \supset F_1$. Put $C'_1 = (C_1 - O) \cup G_1$. Then C'_1 is an open set of X. By the construction, we see that C'_1 satisfies the conditions (0)–(4).

We proceed by induction on i. Now we suppose that there are C'_j $(j \le i-1)$ satisfying the conditions (0)–(4) and $\{\mathrm{bd}(C'_j) \cap O |\ 1 \le j \le i-1\}$ is in general position. Consider the subspace $Y_i = \mathrm{cl}(C_i \cap \mathrm{cl}(O)) - (\mathrm{bd}(O) \cap \mathrm{bd}(C_i))$ of X.

Put $E_i = Y_i \cap (\mathrm{bd}(O) \cup B_i)$ and $F_i = Y_i \cap \mathrm{bd}(C_i)$. Then E_i and F_i are disjoint closed subsets of Y_i. We can choose a zero-dimensional F_σ set Z' of O such that if $\mathcal{S} \subset \{O \cap \mathrm{bd}(C_j)|\ j \leq i-1\}$ with $|\mathcal{S}| = m$, then $\dim(\bigcap\{S|\ S \in \mathcal{S}\} - Z') \leq \max\{-1, n-m-1\}$ (e.g., see [**9**, Lemma 3.11.16]). Then we can take a closed separator S_i between E_i and F_i in Y_i such that $\dim S_i \leq n-1$ and $S_i \cap (Z \cup Z') = \emptyset$. Then we have open subsets G_i and H_i of Y_i such that $Y_i - S_i = G_i \cup H_i$, $G_i \cap H_i = \emptyset$ and $G_i \supset E_i, H_i \supset F_i$. Put $C'_i = (C_i - O) \cup G_i$. By the construction, we see that $\mathcal{C}' = \{C'_i|\ 1 \leq i \leq m\}$ satisfies the desired conditions. □

Lemma 2.2 (cf. [**6**, Lemma 3.5]). *Suppose that $f : X \to X$ is a fixed-point free homeomorphism of a separable metric space X such that $\dim X = n < \infty$ and $\dim P(f) \leq 0$. Let $\mathcal{C} = \{C_i|\ 1 \leq i \leq m\}$ be an open cover of X and let $\mathcal{B} = \{B_i|\ 1 \leq i \leq m\}$ be a closed shrinking of $\mathcal{C}$. Then for any $k \in \mathbb{N}$, there is an open shrinking $\mathcal{C}' = \{C'_i|\ 1 \leq i \leq m\}$ of $\mathcal{C}$ such that*

(0) $B_i \subset C'_i$,
(1) $\{f^j(\mathrm{bd}(C'))|\ C' \in \mathcal{C}', -k \leq j \leq k\}$ *is in general position,*
(2) $\mathrm{bd}(C') \cap P(f) = \emptyset$ *for each* $C' \in \mathcal{C}'$.

Proof. The proof is a modification of the proof of [**6**, Lemma 3.5]. We proceed by induction on k. First we will show that the case $k = 0$ is true. In fact, if we put $O = X$ and $Z = P(f)$, we see that the case $k = 0$ follows from Lemma 2.1. Now we suppose that the result for the case $k-1$ is true. We may assume that there is an open shrinking $\mathcal{D} = \{D_i|\ 1 \leq i \leq m\}$ of $\mathcal{C}$ such that

(0) $B_i \subset D_i$,
(1) $\{f^j(\mathrm{bd}(D))|\ D \in \mathcal{D}, -k+1 \leq j \leq k-1\}$ is in general position,
(2) $\mathrm{bd}(D) \cap P(f) = \emptyset$ for each $D \in \mathcal{D}$.

Put $F = \bigcup\{\mathrm{bd}(D)|\ D \in \mathcal{D}\}$. Since $F \cap P(f) = \emptyset$, we can choose a star finite open cover $\mathcal{O} = \{O_j|\ j \in \mathbb{N}\}$ of F such that $O_j \cap F \neq \emptyset$ and $f^p(O_j) \cap f^q(O_j) = \emptyset$ for each $j \in \mathbb{N}$ and for $p \neq q$, $-2k \leq p, q \leq 2k$. We will construct a sequence $\{\mathcal{D}(j)|\ j = 0, 1, 2, \dots\}$ of open shrinkings of $\mathcal{C} = \{C_i|\ 1 \leq i \leq m\}$ such that $\mathcal{D}(j+1)$ is a shrinking of $\mathcal{D}(j)$ for each j satisfying the following conditions:

(a) $\mathcal{D}(j) = \{D(j)_i|\ 1 \leq i \leq m\}$.
(b) $B_i \subset D(j)_i$.
(c) $\mathcal{D}(0) = \mathcal{D}$.
(d) $D(j-1)_i \cap (X - O_j) = D(j)_i \cap (X - O_j)$, $\mathrm{bd}(D(j-1)_i) \cap (X - O_j) \supset \mathrm{bd}(D(j)_i) \cap (X - O_j)$, and if $\mathrm{bd}(D(j-1)_i) \cap O_j = \emptyset$, then $\mathrm{bd}(D(j)_i) \cap O_j = \emptyset$.
(e) $\mathcal{B}_j = \{f^p(\mathrm{bd}(D))|\ D \in \mathcal{D}(j), -k+1 \leq p \leq k-1\} \cup \{f^{-k}(\mathrm{bd}(D)) \cap (\bigcup_{p=1}^{j} O_p)|\ D \in \mathcal{D}(j)\} \cup \{f^k(\mathrm{bd}(D)) \cap (\bigcup_{p=1}^{j} O_p)|\ D \in \mathcal{D}(j)\}$ is in general position.
(f) $\mathrm{bd}(D) \cap P(f) = \emptyset$ for $D \in \mathcal{D}(j)$.

Also, we proceed by induction on j. Suppose that we have $\mathcal{D}(j)$. We will construct $\mathcal{D}(j+1)$. For each p with $-k \leq p \leq k$, consider the collection $\mathcal{S}_p = \{B \cap f^p(O_{j+1}) | \ B \in \mathcal{B}_j\}$. Then there is a zero-dimensional F_σ-set Z_p of $f^p(O_{j+1})$ such that if $\mathcal{S} \subset \mathcal{S}_p, |\mathcal{S}| = m$, then $\dim(\bigcap \mathcal{S} - Z_p) \leq \max\{-1, n-m-1\}$. Let $Z = (\bigcup_{p=-k}^{k} f^{-p}(Z_p)) \cup (P(f) \cap O_{j+1})$. Note that Z is a zero-dimensional F_σ-set of O_{j+1}. Now, we use the same arguments as in the proof of Lemma 2.1. First, we construct $D(j+1)_1$ and by induction on i, we can construct $D(j+1)_i$ $(2 \leq i \leq m)$. Consequently we obtain $\mathcal{D}(j+1) = \{D(j+1)_i | \ 1 \leq i \leq m\}$. By the constructions and the similar arguments to the proof of [**6**, Lemma 3.5], we see that $\mathcal{D}(j+1)$ satisfies the conditions (a)–(f).

Now, we obtain the above $\{\mathcal{D}(j) | \ j = 0, 1, 2, \dots\}$ satisfying the conditions (a)–(f). Then we put $C'_i = \bigcap_{j=0}^{\infty} D(j)_i$ for each $i = 1, 2, \dots, m$. Since $\mathcal{O}$ is star finite and by the construction of $\{\mathcal{D}(j) | \ j = 0, 1, 2, \dots\}$, we see that $\mathcal{C}' = \{C'_i | \ 1 \leq i \leq m\}$ is an open cover of X. Also we see that $\mathcal{C}'$ satisfies the desired conditions. $\square$

The following result is the main theorem of this paper.

Theorem 2.3 (cf. [**1**]). *Let $f : X \to X$ be a fixed-point free homeomorphism of a separable metric space X with $\dim X = n < \infty$. If $\dim P(f) \leq 0$, then*

$$C(f, \varphi_n(k)) \leq n + 3 - k$$

for each $k = 0, 1, 2, \dots, n+1$.

Remark. If we do not assume $\dim P(f) \leq 0$, the above theorem is not true. Let $f : S^n \to S^n$ be the antipodal map of the n-dimensional sphere S^n. Note that $P(f) = S^n$ and $C(f, p) = C(f, 1) = n + 2$ for any $p \in \mathbb{N}$.

Proof of Theorem 2.3. We proceed by induction on k. In the case $k = 0$, Theorem 2.3 follows from Theorem 1.3. Now we suppose that Theorem 2.3 holds for $k-1$. We have an open cover $\mathcal{C} = \{C_i | \ 1 \leq i \leq n+3-(k-1)\}$ of X which is an eventual coloring within $\varphi_n(k-1)$. Take a closed shrinking $\mathcal{B} = \{B_i | \ 1 \leq i \leq n+3-(k-1)\}$ of $\mathcal{C}$. By use of Lemma 2.2, we have an open cover $\mathcal{C}' = \{C'_i | \ 1 \leq i \leq n+3-(k-1)\}$ such that

(0) $B_i \subset C'_i$,
(1) $\{f^j(\mathrm{bd}(C')) | \ C' \in \mathcal{C}', 0 \leq j \leq \varphi_n(k-1) + [n/(n+2-k)] \cdot (\varphi_n(k-1)+1)\}$ is in general position,
(2) $\mathrm{bd}(C') \cap P(f) = \emptyset$ for each $C' \in \mathcal{C}'$.

Put $K_i = \mathrm{cl}(C'_i)$ for $1 \leq i \leq n+3-(k-1)$ and let $\mathcal{K} = \{K_i | \ 1 \leq i \leq n+3-(k-1)\}$. Put

$$L_1 = K_1,\ L_i = \mathrm{cl}(K_i - (K_1 \cup K_2 \cup \cdots \cup K_{i-1}))\ (i \geq 2).$$

Then the collection $\mathcal{L} = \{L_i |\ 1 \leq i \leq n+3-(k-1)\}$ is a closed partition of X and $\mathcal{L}$ satisfies the condition; for $1 \leq i_1 < i_2 < \cdots < i_m \leq n+3-(k-1)$,

$$\mathrm{bd}(L_{i_1}) \cap \mathrm{bd}(L_{i_2}) \cap \cdots \mathrm{bd}(L_{i_m}) \subset \mathrm{bd}(K_{i_1}) \cap \mathrm{bd}(K_{i_2}) \cap \cdots \cap \mathrm{bd}(K_{i_{m-1}}).$$

Put $D = L_{n+3-(k-1)} \in \mathcal{L}$. Let $x \in D$. Since D is eventually colored within $\varphi_n(k-1)$, we see that $|J_x| \geq [n/(n+2-k)]+1$, where

$$J_x = \{j |\ 0 \leq j \leq \varphi_n(k-1) + [n/(n+2-k)] \cdot (\varphi_n(k-1)+1) \text{ and } f^j(x) \notin D\}.$$

For each $j \in J_x$, put

$$I(j) = \{i \in \{1,2,\ldots,n+3-k\} |\ f^j(x) \in L_i\}.$$

Suppose, on the contrary, that $|I(j)| = n+3-k$ for all $j \in J_x$. Then

$$f^j(x) \in \bigcap_{i=1}^{n+3-k} L_i = \bigcap_{i=1}^{n+3-k} \mathrm{bd}(L_i) \subset \bigcap_{i=1}^{n+2-k} \mathrm{bd}(K_i) \subset \bigcap_{i=1}^{n+2-k} \mathrm{bd}(C_i').$$

Since $\{f^j(\mathrm{bd}(C'))|\ C' \in \mathcal{C}', 0 \leq j \leq \varphi_n(k-1) + [n/(n+2-k)] \cdot (\varphi_n(k-1)+1)\}$ is in general position, we see that $([n/(n+2-k)]+1)(n+2-k) \leq n$. However, we have the following inequality

$$([n/(n+2-k)]+1)(n+2-k) \geq n+1.$$

This is a contradiction. Hence there is some $j(x) \in J_x$ such that $|I(j(x))| < n+3-k$. We choose $L_{i(x)}$ such that $f^{j(x)}(x) \notin L_{i(x)}$. Take an open neighborhood $U(x)$ of x in D such that $f^{j(x)}(\mathrm{cl}(U(x))) \cap (D \cup L_{i(x)}) = \emptyset$. Consider the collection $\mathcal{U} = \{U(x) | x \in D\}$ and take a locally finite closed refinement $\mathcal{W}$ of $\mathcal{U}$. For each $W \in \mathcal{W}$, we can choose $U(x)$ such that $W \subset U(x)$. Put $j(W) = i(x)$. For each $1 \leq j \leq n+3-k$, put $E_j = \bigcup\{W \in \mathcal{W}|\ j(W) = j\}$ and define $F_j = L_j \cup E_j$. We will show that F_j is eventually colored within $\varphi_n(k)$.

Let $y \in F_j (= L_j \cup E_j)$. If $y \in E_j$, then we can choose $W \in \mathcal{W}$ and $U(x) \in \mathcal{U}$ such that $y \in W \subset U(x)$. Then $j(x) \leq \varphi_n(k-1) + [n/(n+2-k)] \cdot (\varphi_n(k-1)+1)$ and $f^{j(x)}(y) \notin (L_j \cup D)$. If $y \in L_j$, we can choose $p \leq \varphi_n(k-1)$ such that $y' = f^p(x) \notin L_j$. If $y' \notin E_j$, then $f^p(x) \notin F_j$. Finally, if $y' \in E_j$, the previous argument shows that there is $q \leq \varphi_n(k-1) + [n/(n+2-k)] \cdot (\varphi_n(k-1)+1)$ such that $f^q(y') \notin F_j$. Hence $f^{p+q}(y) \notin F_j$ and $p+q \le \varphi_n(k)$. Then the closed cover $\mathcal{F} = \{F_j|\ 1 \le j \le n+3-k\}$ of X is an eventual coloring within $\varphi_n(k)$. This implies that $C(f, \varphi_n(k)) \le n+3-k$. This completes the proof. $\square$

Corollary 2.4. *Let $f: X \to X$ be a fixed-point free homeomorphism of a separable metric space X with $\dim X = n < \infty$. If $\dim P(f) \le 0$, then $C(f,2) \le n+2$ $(n \ge 0)$ and $C(f,7) \le n+1$ $(n \ge 1)$.*

Now we have the following general problem for eventual coloring numbers.

Problem 2.5. For each $n \ge 0$ and each $1 \le k \le n+1$, determine the minimal number $m_n(k)$ of natural numbers p satisfying the condition; if $f: X \to X$ is any fixed-point free homeomorphism of a separable metric space X such that $\dim X = n$ and $\dim P(f) \le 0$, then $C(f,p) \le n+3-k$.

Next, we will consider another index $\tau_n(k)$ defined by $\tau_n(k) = k(2n+1)+1$ for each $n = 0,1,2,\ldots,$ and each $k = 0,1,2,\ldots,n+1$.

Theorem 2.6. *Let $f: X \to X$ be a fixed-point free homeomorphism of a separable metric space X with $\dim X = n < \infty$. If $\dim P(f) \le 0$, then*

$$C(f, \tau_n(k)) \le n+3-k$$

for each $k = 0,1,2,\ldots,n+1$.

Proof. The proof is similar to the proof of Theorem 2.3. We proceed by induction on k. In the case $k=0$, Theorem 2.6 follows from Theorem 1.3. Now we suppose that $k \ge 1$ and there is an open cover $\mathcal{C} = \{C_i|\ 1 \le i \le n+3-(k-1)\}$ of X such that $\mathrm{cl}(C_1), \mathrm{cl}(C_2)$ are eventually colored within $\tau_n(k-1) = (k-1)(2n+1)+1$ and $\mathrm{cl}(C_i)$ $(3 \le i \le n+3-(k-1))$ are colored (=eventually colored within 1). By use of Lemma 2.2, we may assume that

$$\{f^j(\mathrm{bd}(C))|\ C \in \mathcal{C}, 0 \le j \le 2n+1\}$$

is in general position. In particular, $\{f^j(\mathrm{bd}(C_1))|\ 0 \le j \le 2n+1\}$ is in general position.

Put $K_i = \mathrm{cl}(C_i)$ for $1 \le i \le n+3-(k-1)$ and let $\mathcal{K} = \{K_i|\ 1 \le i \le n+3-(k-1)\}$. Put

$$L_1 = K_1,\ L_i = \mathrm{cl}(K_i - (K_1 \cup K_2 \cup \cdots \cup K_{i-1}))\ (i \ge 2).$$

Then the collection $\mathcal{L} = \{L_i|\ 1 \le i \le n+3-(k-1)\}$ is a closed partition of X.

Note that $L_1 \cap L_2 \subset \mathrm{bd}(C_1)$. Let $x \in L_3$. Since L_3 is colored, $|J_3(x)| \geq n+1$, where $J_3(x) = \{j|\ 0 \leq j \leq 2n+1 \text{ and } f^j(x) \notin L_3\}$. By the similar argument to the proof of Theorem 2.3, we see that there is some $j(x) \in J_3(x)$ such that $f^{j(x)}(x) \notin L_1$ or $f^{j(x)}(x) \notin L_2$. Also, by the similar argument to the proof of Theorem 2.3, we have a closed cover

$$\mathcal{F} = \{F_i|\ 1 \leq i \leq n+3-k\}$$

of X such that F_1, F_2 are eventually colored within $\tau_n(k) = k(2n+1)+1$ and F_i $(3 \leq i \leq n+3-k)$ are colored. $\square$

We have the following result which is the case $C(f,p) = 2$.

Corollary 2.7. *Let $f : X \to X$ be a fixed-point free homeomorphism of a separable metric space X with $\dim X = n < \infty$. If $\dim P(f) \leq 0$, then there is some $p \in \mathbb{N}$ with $p \leq \min\{\varphi_n(n+1), \tau_n(n+1)\}$ such that*

$$C(f,p) = 2.$$

In other words, X can be divided into two closed subsets C_1, C_2 (i.e., $X = C_1 \cup C_2$) and there is some $p \in \mathbb{N}$ such that if $x \in C_i$ ($i \in \{1,2\}$), there is a strictly increasing sequence $\{n_x(k)\}_{k=1}^{\infty}$ of natural numbers such that $1 \leq n_x(1) \leq p$, $n_x(k+1) - n_x(k) \leq p$ and if $j \in \{1,2\}$ with $j \neq i$, then

$$f^{n_x(k)}(x) \in C_j - C_i \ (k : \textit{odd}), \quad f^{n_x(k)}(x) \in C_i - C_j \ (k : \textit{even}).$$

By the above corollary, we see that $m_n(k) \leq \min\{\varphi_n(k), \tau_n(k)\}$. We see that $\varphi_0(1) = 2$, $\varphi_1(2) = 7$, $\varphi_2(3) = 30$, $\varphi_3(4) = 113$ and $\varphi_4(5) = 544$, Also, $\tau_0(1) = 2$, $\tau_1(2) = 7$, $\tau_2(3) = 16$, $\tau_3(4) = 29$ and $\tau_4(5) = 46$. Hence we have the partial answer to the above problem.

Corollary 2.8. *Suppose that $f : X \to X$ is a fixed-point free homeomorphism of a separable metric space X and $\dim P(f) \leq 0$.*

(1) *If $\dim X = 0$, then $C(f,2) = 2$.*
(2) *If $\dim X = 1$, then $C(f,7) = 2$.*
(3) *If $\dim X = 2$, then $C(f,16) = 2$.*
(4) *If $\dim X = 3$, then $C(f,29) = 2$.*
(5) *If $\dim X = 4$, then $C(f,46) = 2$.*

In other words, $m_0(1) = 2$, $m_1(2) \leq 7$, $m_2(3) \leq 16$, $m_3(4) \leq 29$ and $m_4(5) \leq 46$.

3. Eventual coloring numbers of fixed-point free maps of compact metric spaces.

In this section, we consider eventual coloring numbers of fixed-point free maps of compact metric spaces. Let X be a compact metric space and let $f : X \to X$ be a map. Consider the inverse limit (X, f) of f, i.e.

$$(X, f) = \left\{(x_i)_{i=0}^{\infty} \mid x_i \in X, f(x_i) = x_{i-1} \text{ for } i \in \mathbb{N}\right\} \subset X^{\infty} = \prod_{j=0}^{\infty} X_j.$$

Then we have the shift homeomorphism $\widetilde{f} : (X, f) \to (X, f)$ of f and the natural projection $p_j : (X, f) \to X_j = X$ $(j \geq 0)$ defined by

$$\widetilde{f}((x_i)_{i=0}^{\infty}) = (f(x_i))_{i=0}^{\infty}, \quad p_j((x_i)_{i=0}^{\infty}) = x_j.$$

Note that $p_j \cdot \widetilde{f} = f \cdot p_j$. We see that if $f : X \to X$ is a fixed-point free map of a compact metric space X, then $\widetilde{f} : (X, f) \to (X, f)$ is a fixed-point free homeomorphism. By a modification of the proof of [**1**, Theorem 6], we have the following theorem which is a more precise result than [**1**, Theorem 6].

Theorem 3.1. *Let $f : X \to X$ be a fixed-point free map of a compact metric space X and let $\widetilde{f} : (X, f) \to (X, f)$ be the shift homeomorphism of f. Then for $p \in \mathbb{N}$,*

$$C(f, p) = C^+(f, p) = C(\widetilde{f}, p).$$

Proof. Since $\widetilde{f}$ is a homeomorphism, we see that $C(\widetilde{f}, p) = C^+(\widetilde{f}, p)$. Also, note that $C^+(f, p) \geq C(f, p)$. First, we suppose that the map $f : X \to X$ is surjective. We show that $C^+(f, p) \leq C(\widetilde{f}, p) \leq C(f, p)$. We will prove $C(\widetilde{f}, p) \leq C(f, p)$. Let $\mathcal{C}$ be an eventual (closed) coloring within p of f. Since $p_0 \cdot \widetilde{f} = f \cdot p_0$, we see that $p_0^{-1}(\mathcal{C})$ is a closed cover of (X, f) and

$$\bigcap_{i=0}^{p} \widetilde{f}^{-i}(p_0^{-1}(C)) = p_0^{-1}\left(\bigcap_{i=0}^{p} f^{-i}(C)\right) = \emptyset$$

for each $C \in \mathcal{C}$. This implies that $C(\widetilde{f}, p) \leq C(f, p)$. We prove that $C^+(f, p) \leq C(\widetilde{f}, p) = C^+(\widetilde{f}, p)$. Let $\mathcal{C}$ be an eventual (closed) coloring within p of $\widetilde{f}$. Let $C \in \mathcal{C}$. Then we see that $\bigcap_{i=0}^{p} \widetilde{f}^{i}(C) = \emptyset$. Take an open neighborhood $U(C)$ of

C in (X, f) such that $\bigcap_{i=0}^{p} \widetilde{f}^i(U(C)) = \emptyset$. Note that for any $\epsilon > 0$, there is a sufficiently large $j \in \mathbb{N}$ such that p_j is an ϵ-map, i.e., diam $p_j^{-1}(x) < \epsilon$ for $x \in X$. Hence we see that there is $j \in \mathbb{N}$ such that $p_j^{-1}(f^i(p_j(C))) \subset \widetilde{f}^i(U(C))$. Hence $\bigcap_{i=0}^{p} f^i(p_j(C)) = \emptyset$. Since $p_j : (X, f) \to X$ is surjective, the family $p_j(\mathcal{C})$ is a closed cover of X. This implies that $C^+(f, p) \leq C(\widetilde{f}, p)$ and hence $C(f, p) = C^+(f, p) = C(\widetilde{f}, p)$. Next, we consider the general case in which f is any map. Put

$$K = \bigcap \{f^j(X) |\ j \in \mathbb{N}\}.$$

Consider the map $g = f|K : K \to K$. Note that g is surjective. We prove that $C^+(g, p) \geq C^+(f, p)$. Consider any closed cover $\mathcal{C}$ of K such that for each $C \in \mathcal{C}$, $\bigcap_{i=0}^{p} f^i(C) = \emptyset$. Take an open swelling $\mathcal{C}'$ of $\mathcal{C}$ in X. We may assume that for each $C' \in \mathcal{C}'$, $\bigcap_{i=0}^{p} f^i(C') = \emptyset$. Note that K is an attractor of f, i.e., there is $q \in \mathbb{N}$ such that $f^q(X) \subset U = \bigcup \mathcal{C}'$. Then $f^{-q}(\mathcal{C}')$ is an open cover of X and for each $C' \in \mathcal{C}'$,

$$\bigcap_{i=0}^{p} f^i(f^{-q}(C')) \subset \bigcap_{i=0}^{p} f^{-q}(f^i(C')) = f^{-q}\left(\bigcap_{i=0}^{p} f^i(C')\right) = \emptyset.$$

This implies that $C^+(g, p) \geq C^+(f, p)$. Note that $(X, f) = (K, g)$ and $\widetilde{f} = \widetilde{g}$. Since g is surjective, by the above arguments we see that $C(g, p) = C^+(g, p) = C(\widetilde{f}, p)$. Note that $C(g, p) \leq C(f, p) \leq C^+(f, p) \leq C^+(g, p)$. Hence $C(g, p) = C(f, p) = C^+(f, p) = C^+(g, p)$. Consequently, $C(f, p) = C^+(f, p) = C(\widetilde{f}, p)$. □

Corollary 3.2 (cf. [**1**, Theorem 6]). *Let $f : X \to X$ be a fixed-point free map of a compact metric space X with $\dim X = n < \infty$. If $\dim P(f) \leq 0$, then there is $p \in \mathbb{N}$ with $p \leq \min\{\varphi_n(k), \tau_n(k)\}$ such that*

$$C(f, p) \leq n + 3 - k$$

for each $k = 0, 1, 2, \ldots, n + 1$.

Proof. Let $0 \leq k \leq n + 1$. Since $\dim P(f) \leq 0$, we see that $\dim P(\widetilde{f}) \leq 0$. Note that $\widetilde{f}$ is a fixed-point free homeomorphism and $\dim(X, f) \leq n$. By Theorems 2.3 and 2.6, $C(\widetilde{f}, p) \leq n + 3 - k$ some $p \leq \min\{\varphi_n(k), \tau_n(k)\}$. By Theorem 3.1, we see that

$$C(f, p) = C(\widetilde{f}, p) \leq n + 3 - k.$$ □

Example. There are a (zero-dimensional) separable metric space X and a fixed-point free map $f : X \to X$ such that $\dim P(f) \leq 0$ and

(1) f is closed,
(2) f is finite-to-one, and
(3) f cannot be eventually colored within any $p \in \mathbb{N}$.

In fact, let $f : X \to X$ be the map as in [**9**, Theorem 3.12.7]. Then we see that $P(f) = \emptyset$ and f cannot be colored and satisfies the conditions (1), (2) (see [**9**, Theorem 3.12.7]). Let $\mathcal{U}$ be any finite open cover of X. Then there exist some $U \in \mathcal{U}$ and a point $x \in U$ such that $f^p(x) \in U$ for any $p \in \mathbb{N}$ (see [**9**, Corollary 3.12.6] and the proof of [**9**, Theorem 3.12.7]). This implies that $\mathcal{U}$ is not an eventual coloring within any $p \in \mathbb{N}$.

Remark. In the statement of Theorem 1.3, "a separable metric space X" can be replaced with "a paracompact space X" (see [M. A. van Hartskamp and J. Vermeer, On colorings of maps, Topology and its Applications 73 (1996), 181–190]). Hence Theorem 2.3 is also true for the case that X is a paracompact space.

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