

## On invariant Gibbs measures conditioned on mass and momentum

By Tadahiro OH and Jeremy QUASTEL

(Received Dec. 16, 2010)

(Revised Apr. 27, 2011)

**Abstract.** We construct a Gibbs measure for the nonlinear Schrödinger equation (NLS) on the circle, conditioned on prescribed mass and momentum:

$$d\mu_{a,b} = Z^{-1} 1_{\{\int_{\mathbb{T}} |u|^2 = a\}} 1_{\{i \int_{\mathbb{T}} u \bar{u}_x = b\}} e^{\pm 1/p \int_{\mathbb{T}} |u|^{p-1/2} \int_{\mathbb{T}} |u|^2 dP}$$

for  $a \in \mathbb{R}^+$  and  $b \in \mathbb{R}$ , where  $P$  is the complex-valued Wiener measure on the circle. We also show that  $\mu_{a,b}$  is invariant under the flow of NLS. We note that  $i \int_{\mathbb{T}} u \bar{u}_x$  is the Lévy stochastic area, and in particular that this is invariant under the flow of NLS.

### 1. Introduction.

We consider the periodic nonlinear Schrödinger equation (NLS) on the circle:

$$iu_t + u_{xx} = \pm |u|^{p-2} u, \quad (x, t) \in \mathbb{T} \times \mathbb{R} \quad (1.1)$$

where  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ . Recall that (1.1) is a Hamiltonian PDE with Hamiltonian:

$$H(u) = \frac{1}{2} \int_{\mathbb{T}} |u_x|^2 \pm \frac{1}{p} \int_{\mathbb{T}} |u|^p. \quad (1.2)$$

Indeed, (1.1) can be written as

$$u_t = i \frac{\partial H}{\partial \bar{u}}. \quad (1.3)$$

Recall that (1.1) also conserves the mass  $M(u) = \int |u|^2$  and the momentum  $P(u) = i \int u \bar{u}_x$ . Moreover, the cubic NLS ( $p = 4$ ) is known to be completely integrable

---

2010 *Mathematics Subject Classification.* Primary 60H40; Secondary 60H30, 35Q53, 35Q55.

*Key Words and Phrases.* Gibbs measure, Schrödinger equation, Kortweg-de Vries equation, Lévy area.

J. Quastel was partially supported by the Natural Sciences and Engineering Research Council of Canada.

[ZS], [GKP] in the sense that it enjoys the Lax pair structure and thus there exist infinitely many conservation laws for (1.1). For general  $p \neq 4$ , the mass  $M$ , the momentum  $P$ , and the Hamiltonian  $H$  are the only known conservation laws. Our main goal in this paper is to construct an invariant Gibbs measure conditioned on mass and momentum.

First, consider a Hamiltonian flow on  $\mathbb{R}^{2n}$ :

$$\dot{p}_i = \frac{\partial H}{\partial q_j}, \quad \dot{q}_i = -\frac{\partial H}{\partial p_j} \quad (1.4)$$

with Hamiltonian  $H(p, q) = H(p_1, \dots, p_n, q_1, \dots, q_n)$ . Then, Liouville's theorem states that the Lebesgue measure  $\prod_{j=1}^n dp_j dq_j$  on  $\mathbb{R}^{2n}$  is invariant under the flow. Then, it follows from the conservation of the Hamiltonian  $H$  that the Gibbs measure  $e^{-H(p, q)} \prod_{j=1}^n dp_j dq_j$  is invariant under the flow of (1.4). Now note that if  $F(p, q)$  is any (reasonable) function that is conserved under the flow of (1.4), then the measure  $d\mu_F = F(p, q) e^{-H(p, q)} \prod_{j=1}^n dp_j dq_j$  is also invariant.

By viewing (1.1) as an infinite dimensional Hamiltonian system, one can consider the issue of invariant Gibbs measures for (1.1). Lebowitz-Rose-Speer [LRS] constructed Gibbs measures of the form

$$d\mu = Z^{-1} e^{-H(u)} \prod_{x \in \mathbb{T}} du(x) = Z^{-1} e^{\mp 1/p \int_{\mathbb{T}} |u|^p} \underbrace{e^{-1/2 \int_{\mathbb{T}} |u_x|^2} \prod_{x \in \mathbb{T}} du(x)}_{= \text{Wiener measure } P} \quad (1.5)$$

as a weighted Wiener measure on  $\mathbb{T}$ . In the focusing case, i.e. with the plus sign in (1.5), the result holds only for  $p \leq 6$  with an  $L^2$ -cutoff  $1_{\{\int |u|^2 \leq B\}}$ , where  $B$  is any positive number when  $p < 6$  and  $B < \|Q\|_{L^2(\mathbb{R})}^2$  when  $p = 6$ . Here,  $Q$  is the ground state of the following elliptic equation:

$$(p-2)Q'' - (p+2)Q + Q^{p-1} = 0. \quad (1.6)$$

By analogy with the finite dimensional case, we expect such a Gibbs measure  $\mu$  is invariant under the flow of (1.1). (Recall that the  $L^2$ -norm is conserved.) In addressing the question of invariance of  $\mu$ , we need to have a well-defined flow on the support of  $\mu$ . However, as a weighted Wiener measure, the regularity of  $\mu$  is inherited from that of the Wiener measure, i.e.  $\mu$  is supported on  $H^s(\mathbb{T}) \setminus H^{1/2}(\mathbb{T})$ ,  $s < 1/2$ . In [B1], Bourgain proved local well-posedness of (1.1)

- in  $L^2(\mathbb{T})$  for (sub-) cubic NLS ( $p \leq 4$ ),
- in  $H^s(\mathbb{T})$ ,  $s > 0$ , for (sub-)quintic NLS ( $4 < p \leq 6$ ),
- in  $H^s(\mathbb{T})$ ,  $s > 1/2 - 1/p$ , for  $p > 6$ .

Using the Fourier analytic approach, he [B2] continued the study of Gibbs measures and proved the invariance of  $\mu$  under the flow of NLS.

Once the invariance of the Gibbs measure  $\mu$  is established, we can regard the flow map of (1.1) as a measure-preserving transformation on an (infinite-dimensional) phase space, say  $H^{1/2-\epsilon}$ , equipped with the Gibbs measure  $\mu$ . Then, it follows from Poincaré recurrence theorem that almost all the points of the phase space are stable according to Poisson [Z], i.e. if  $\mathcal{S}_t$  denotes a flow map of (1.1):  $u_0 \mapsto u(t) = \mathcal{S}_t u_0$ , then for almost all  $u_0$ , there exists a sequence  $\{t_n\}$  tending to  $\infty$  such that  $\mathcal{S}_{t_n} u_0 \rightarrow u_0$ . Moreover, such dynamics is also multiply recurrent in view of Furstenberg's multiple recurrence theorem [F]: let  $A$  be any measurable set with  $\mu(A) > 0$ . Then, for any integer  $k > 1$ , there exists  $n \neq 0$  such that  $\mu(A \cap \mathcal{S}_n A \cap \mathcal{S}_{2n} A \cap \cdots \cap \mathcal{S}_{(k-1)n} A) > 0$ . Note that this recurrence property is known to hold only in the support of the Gibbs measure, i.e. not for smooth functions.

Then, one of the natural questions, posed by Lebowitz-Rose-Speer [LRS] and Bourgain [B4], is the ergodicity of the invariant Gibbs measure  $\mu$ , i.e. is the phase space irreducible under the dynamics, or can it be decomposed into disjoint subsets, where the dynamics is recurrent within each disjoint component? In order to ask such a question, one needs to prescribe the  $L^2$ -norm since it is an integral of motion for (1.1). It is not difficult to see that the momentum is also finite almost surely on the support of the Gibbs measure. Indeed, if  $u$  is distributed according to the Wiener measure, then it can be represented as<sup>1</sup>

$$u(x) = \sum_{n \neq 0} \frac{g_n}{2\pi n} e^{2\pi i n x}, \quad (1.7)$$

where  $\{g_n\}_{n \neq 0}$  is a family of independent standard complex-valued Gaussian random variables, i.e. its real and imaginary parts are independent Gaussian random variables with mean zero and variance 1. Then, we can write the momentum as

$$P(u) = i \int u \bar{u}_x = \sum_{n \neq 0} \frac{|g_n(\omega)|^2}{2\pi n} = \sum_{n \geq 1} \frac{|g_n(\omega)|^2 - |g_{-n}(\omega)|^2}{2\pi n}.$$

Thus, we have  $\mathbb{E}[(P(u))^2] \lesssim \sum_{n \geq 1} n^{-2} < \infty$ .<sup>2</sup> Hence,  $|P(u)| < \infty$  a.s. In the following, we construct invariant Gibbs measures with prescribed  $L^2$ -norm and momentum as the first step in studying finer dynamical properties of the NLS

<sup>1</sup>We ignore the zero-frequency issue here. See (1.9) below.

<sup>2</sup>We use  $A \lesssim B$  to denote an estimate of the form  $A \leq CB$  for some  $C > 0$ . Similarly, we use  $A \sim B$  to denote  $A \lesssim B$  and  $B \lesssim A$ .

flow equipped with the invariant Gibbs measure, viewed as an infinite-dimensional dynamical system with a measure-preserving transformation.

REMARK 1.1. Recall that the cubic NLS ( $p = 4$ ) is completely integrable. Hence, it makes sense to pose a question of ergodicity only for  $p \neq 4$ . See [LRS].

There are infinitely many conservation laws for the cubic NLS, with the leading term of the form  $\int_{\mathbb{T}} |\partial_x^k u|^2 dx$ , roughly corresponding to the  $H^k$ -norm, and of the form  $\int_{\mathbb{T}} u \partial_x^{2k+1} \bar{u} dx$ ,  $k \in \mathbb{N} \cup \{0\}$ . See [FT], [ZM]. By (1.7), we can easily see that all these conservation laws, except for the  $L^2$ -norm and momentum, are almost surely divergent under the Gibbs measure. Thus, it may seem that the  $L^2$ -norm and momentum are the only conserved quantities which are finite a.s. in the support of the Gibbs measure. However, from a different perspective, we have a different set of infinitely many conserved quantities for (1.1), namely the spectrum of the Zakharov-Shabat operator  $L$  (also called the Dirac operator) appearing in the Lax pair formulation of (1.1):  $\partial_t L = [B, L]$  (with some appropriate  $B$ .) These are finite under the Gibbs measure. Expressing the flow of (1.1) in the Liouville coordinates (or rather in the Birkhoff coordinates) with actions and angles (which are determined in terms of the spectral data), the flow basically becomes trivial. See [GKP].

In constructing a Gibbs measure conditioned on mass and momentum, we first condition the Wiener measure on mass and momentum. Recall that if  $u$  is distributed according to the Wiener measure  $P$  given by<sup>3</sup>

$$dP = Z^{-1} e^{-1/2 \int_{\mathbb{T}} |u|^2 - 1/2 \int_{\mathbb{T}} |u_x|^2} \prod_{x \in \mathbb{T}} du(x), \quad (1.8)$$

then it can be represented as

$$u(x) = \sum_{n \in \mathbb{Z}} \frac{g_n}{\sqrt{1 + 4\pi^2 n^2}} e^{2\pi i n x}, \quad (1.9)$$

where  $\{g_n\}_{n \in \mathbb{Z}}$  is a family of independent standard complex-valued Gaussian random variables. Note that (1.9) is basically the Fourier-Wiener series for the Brownian motion (except for the zeroth mode.) Given  $a > 0$  and  $b \in \mathbb{R}$ , define the conditional Wiener measures  $P_\varepsilon = P_{\varepsilon, a, b}$ ,  $\varepsilon > 0$ , as follows. Given a measurable set  $E$ , we define  $P_\varepsilon(E)$  by

---

<sup>3</sup>The mass is added to take care of the zeroth frequency. We still refer to  $P$  in (1.8) and  $u$  in (1.9) as the Wiener measure and the Brownian motion, respectively.

$$P_\varepsilon(E) = P\left(E \mid \int_{\mathbb{T}} |u|^2 \in A_\varepsilon(a), i \int_{\mathbb{T}} u \bar{u}_x \in B_\varepsilon(b)\right), \quad (1.10)$$

where  $A_\varepsilon(a)$  and  $B_\varepsilon(b)$  are neighborhoods shrinking nicely<sup>4</sup> to  $a$  and  $b$  as  $\varepsilon \rightarrow 0$ . Here  $P(C \mid D) = P(C \cap D)/P(D)$  is the standard, naive, conditional probability given by Bayes' rule. In terms of the density, we have

$$dP_\varepsilon = \hat{Z}_\varepsilon^{-1} \mathbf{1}_{\{\int_{\mathbb{T}} |u|^2 \in A_\varepsilon(a)\}} \mathbf{1}_{\{i \int_{\mathbb{T}} u \bar{u}_x \in B_\varepsilon(b)\}} dP. \quad (1.11)$$

Now, we would like to define the conditioned measure

$$P_0(E) = P_{0,a,b}(E) = P\left(E \mid \int_{\mathbb{T}} |u|^2 = a, i \int_{\mathbb{T}} u \bar{u}_x = b\right)$$

by  $P_0 = \lim_{\varepsilon \rightarrow 0} P_\varepsilon$ . Namely, we define  $P_0$  by

$$P_0(E) := \lim_{\varepsilon \rightarrow 0} P\left(E \mid \int_{\mathbb{T}} |u|^2 \in A_\varepsilon(a), i \int_{\mathbb{T}} u \bar{u}_x \in B_\varepsilon(b)\right). \quad (1.12)$$

Note that the normalization constant  $\hat{Z}_\varepsilon$  in (1.11) tends to 0 as  $\varepsilon \rightarrow 0$ . Hence, some care is needed. We discuss details in Subsection 2.1.

Finally, we define the conditioned Gibbs measure  $\mu_0 = \mu_{a,b}$  in terms of the Wiener measure  $P_0 = P_{0,a,b}$  conditioned on mass and momentum, by setting

$$d\mu_0 = Z_0^{-1} e^{\mp 1/p \int_{\mathbb{T}} |u|^p} dP_0. \quad (1.13)$$

In the defocusing case, this clearly defines a probability measure since  $e^{-1/p \int_{\mathbb{T}} |u|^p} \leq 1$ . In the focusing case, we need to show that

$$e^{1/p \int_{\mathbb{T}} |u|^p} \in L^1(dP_0). \quad (1.14)$$

Lebowitz-Rose-Speer [**LRS**] and Bourgain [**B2**] proved a similar integrability result of the weight  $e^{1/p \int_{\mathbb{T}} |u|^p}$  with respect to the (unconditioned) Wiener measure  $P$  defined in (1.8). Bourgain's argument was based on dyadic pigeonhole principle and a large deviation estimate (see Lemma 4.2 in [**OQV**].) In Subsection 2.2, we follow Bourgain's argument and prove (1.14) by dyadic pigeonhole principle and a large deviation estimate for  $P_0$ . This large deviation estimate for  $P_0$  is by

---

<sup>4</sup>See Subsection 2.1 for the definition.

no means automatic, and we need to deduce it by establishing a *uniform* large deviation estimate for the conditioned Wiener measures  $P_\varepsilon$ ,  $\varepsilon > 0$  (see Lemma 2.4 below.) As a result, we obtain the  $L^1$ -boundedness result

$$\mathbb{E}_{P_\varepsilon} [e^{1/p \int_{\mathbb{T}} |u|^p}] \leq C_p < \infty$$

for all sufficiently small  $\varepsilon \geq 0$ . We point out that the proof of Lemma 2.4 (and hence the argument in Subsection 2.1) is the heart of this paper.

We state the main theorem. The proof is presented in in the next section.

**THEOREM 1.** *Let  $a > 0$  and  $b \in \mathbb{R}$ . For  $p > 2$ , let  $\mu_0$  be the Gibbs measure  $\mu_0 = \mu_{a,b}$  conditioned on mass and momentum defined in (1.13). Also, assume that  $p \leq 6$  in the focusing case. Then,  $\mu_0$  is a well-defined probability measure (with sufficiently small mass  $a$  when  $p = 6$  in the focusing case), absolutely continuous to the conditioned Wiener measure  $P_0$ . Moreover,  $\mu_\varepsilon$  converges weakly to  $\mu_0$  as  $\varepsilon \rightarrow 0$ , where  $\mu_\varepsilon$  is defined by*

$$d\mu_\varepsilon := Z_\varepsilon^{-1} e^{\mp 1/p \int_{\mathbb{T}} |u|^p} dP_\varepsilon. \quad (1.15)$$

**REMARK 1.2.** In the critical case, i.e. focusing with  $p = 6$ , Lebowitz-Rose-Speer [**LRS**] proved that the weight  $1_{\{\int_{\mathbb{T}} |u|^2 \leq B\}} e^{1/p \int_{\mathbb{T}} |u|^p}$  is integrable with respect to the (unconditioned) Wiener measure  $P$  in (1.8) as long as  $B < \|Q\|_{L^2(\mathbb{R})}^2$ , where  $Q$  is the ground state for (1.6). Indeed, this is sharp (except for the endpoint  $B = \|Q\|_{L^2(\mathbb{R})}^2$ .) By Fourier analytic techniques, Bourgain [**B2**] provided another proof of this  $L^1$ -boundedness result. However, his argument does not allow us to determine the (sharp) upperbound on the size  $B$  of the  $L^2$ -cutoff in the critical case. We believe that, in the critical case, the upperbound on  $a = \int_{\mathbb{T}} |u|^2 dx$  in Theorem 1 is also given by  $\|Q\|_{L^2(\mathbb{R})}^2$ . Unfortunately, our proof of Theorem 1, following Bourgain's idea, does not provides such a quantitative bound.

It follows from invariance of the Gibbs measure  $\mu$  in (1.5) (with an  $L^2$ -cutoff in the focusing case) and the conservation of mass and momentum that  $\mu_\varepsilon$  is invariant under the flow of (1.1) for each *fixed*  $\varepsilon > 0$ . As a corollary to Theorem 1, we obtain invariance of the conditioned Gibbs measure  $\mu_0$ .

**THEOREM 2.** *Let  $a > 0$ ,  $b \in \mathbb{R}$ , and  $p > 2$  be as in Theorem 1. Then, the conditioned Gibbs measure  $\mu_0 = \mu_{a,b}$  defined in (1.13) is invariant under the flow of NLS (1.1).*

We conclude this introduction with several remarks. The first is about conditional probabilities.

REMARK 1.3. A natural way to proceed with this construction is to start with the (unconditioned) Gibbs measure  $\mu$  in (1.5) on the space  $\Omega$ , which is the space of continuous complex-valued functions on the circle, with the topology of uniform convergence and the Borel  $\sigma$ -field  $\mathcal{F}$ . This is a complete separable metric space. Let  $\mathcal{G}$  be the sub  $\sigma$ -field generated by the measurable maps  $\int_{\mathbb{T}} |u|^2$  and  $i \int_{\mathbb{T}} u \bar{u}_x$ . There is a general theorem which guarantees the existence of a conditional probability, i.e. a family of measures  $\mu_u$ ,  $u \in \Omega$  such that (i) for any  $A \in \mathcal{F}$ ,  $\mu_u(A)$  is measurable with respect to  $\mathcal{G}$  as a function of  $u$ ; (ii) for any  $A \in \mathcal{G}$  and  $B \in \mathcal{F}$ ,  $\mu(A \cap B) = E_\mu[\mathbf{1}_A \mu_u(B)]$ . It follows from (i) and (ii) that given  $B \in \mathcal{F}$ , we have

$$\mu_u(B) = \mu_{\int_{\mathbb{T}} |u|^2, i \int_{\mathbb{T}} u \bar{u}_x}(B) \quad (1.16)$$

for  $\mu$ -almost every  $u$ . The sets of measure zero, on which (1.16) fails, depend on  $B \in \mathcal{F}$ , and thus their union could be a set of nontrivial measure. Hence, one needs some regularity. The best that can be said in such a general context is that if  $\mathcal{G}$  is countably generated (and one can check that ours is), then  $\mu_u$  is a *regular* conditional probability in the sense that (iii)  $\mu_u(A) = \mathbf{1}_A(u)$  for  $A \in \mathcal{G}$ . In our context, this reassures us that our conditioned Gibbs measure  $\mu_0 = \mu_{a,b}$  gives mass one to  $u$  with  $\int_{\mathbb{T}} |u|^2 = a$  and  $i \int_{\mathbb{T}} u \bar{u}_x = b$ . However, we only know that this property holds for *almost every*  $a$  and  $b$ , and there is no soft way out to obtain the same for *all*  $a$  and  $b$ . (Another way to think of this is that applying the Lebesgue differentiation theorem to (ii) gives Theorem 1 for almost every  $a$  and  $b$ .) Since we want our conditioned measures to be defined for every value of  $a$  and  $b$ , we have to define them directly. For the conditioned Wiener measure  $P_0$ , which is just a Gaussian measure, this is straightforward. In this case, we can even use the fact that the distributions of  $a$  and  $b$  are basically explicit. However, for the Gibbs measure  $\mu_{a,b}$ , it requires hard analysis.

REMARK 1.4. Consider the (generalized) Korteweg-de Vries equation (gKdV):

$$u_t + u_{xxx} = \pm u^{p-2} u_x. \quad (1.17)$$

For an integer  $p \geq 3$ , (1.1) is a Hamiltonian PDE with Hamiltonian:

$$H(u) = \frac{1}{2} \int_{\mathbb{T}} u_x^2 \pm \frac{1}{p(p-1)} \int_{\mathbb{T}} u^p, \quad (1.18)$$

and (1.17) can be written as  $u_t = \partial_x(dH/du)$ . Also recall that (1.17) preserves the

mean  $\int_{\mathbb{T}} u$  and the  $L^2$ -norm. Bourgain [B2] constructed Gibbs measures of the form (1.5) (with an appropriate  $L^2$ -cutoff  $1_{\{|u|^2 \leq B\}}$  unless it is defocusing when  $p$  is even) for (1.17), and proved its invariance under the flow for  $p = 3, 4$ . Recently, Richards [R] established invariance of the Gibbs measure for (1.17) when  $p = 5$ . In an attempt to study more dynamical properties of (1.17), one can construct Gibbs measure conditioned on mass by an argument similar to Theorem 1. In this case, an analogue of Theorem 1 holds for all (even)  $p$  when (1.17) is defocusing, and for  $p \leq 6$  when it is non-defocusing. However, an analogue of Theorem 2 holds only for  $p \leq 5$  due to lack of well-defined flow for gKdV (1.17) in the support of the Gibbs measure when  $p \geq 6$ . Note that KdV ( $p = 3$ ) and mKdV ( $p = 4$ ) are completely integrable. Hence, a question of ergodicity can be posed only for  $p \geq 5$ . See Remark 1.1.

REMARK 1.5. An interesting but straightforward comment is that the momentum  $P(u)$  is nothing but the Lévy stochastic area of the planar loop  $(\operatorname{Re} u(x), \operatorname{Im} u(x))$ ,  $0 \leq x < 2\pi$ ,

$$P(u) = i \int_{\mathbb{T}} u \bar{u}_x = \int_{\mathbb{T}} (\operatorname{Re} u) d(\operatorname{Im} u) - (\operatorname{Im} u) d(\operatorname{Re} u). \quad (1.19)$$

Note that this is not the actual area enclosed by the loop, but a signed version. A Brownian loop has infinitely many self-intersections. Regularizing the Brownian loop gives a loop with finitely many self-intersections. The ‘area’ is then computed through the path integral above, with each subregion bounded by non-intersecting part of the loop having area counted positive or negative depending on whether the boundary is traversed in the counterclockwise or clockwise direction, respectively. This includes the fact that the areas inside internal loops are multiply counted. Removing the regularization gives the Lévy stochastic area. Remarkably, unlike other stochastic integrals, the limit does not depend on the regularization procedure. For example, one can check directly that the Itô (left endpoint rule in the Riemann sum) and Stratonovich (midpoint rule) versions of (1.19) give the same result. The stochastic area has attracted a great deal of attention. Lévy [L] found the exact expression  $1/4(\cosh(x/2))^{-2}$  for its density under the standard Brownian motion measure. Our base Gaussian measure (1.8) is almost the same as the standard Brownian motion, and the analogous computation can be performed (see Section 2.1.) Our Gibbs measures  $\mu_0 = \mu_{a,b}$  are absolutely continuous with respect to the base Brownian motion, so most of the results about the stochastic area continue to hold, though, of course, there are no longer any exact formulas. The Lévy area is basically the only new element when one moves from the Wiener-Itô chaos of order one to order two. Therefore, it is a natural object to supplement the Brownian path itself, and this is the basis of the rough path theory [LQ]. It



seems a remarkable fact that the flow of NLS preserves the Lévy area.

ACKNOWLEDGMENTS. The authors would like to thank the anonymous referee for pointing out an error in the previous version of this paper as well as for helpful comments.

## 2. Proof of Theorem 1: Construction of the conditioned Gibbs measures.

### 2.1. Wiener measure conditioned on mass and momentum.

In this subsection, we construct the Wiener measure  $P_0$  conditioned on mass  $a$  and momentum  $b$  for any fixed  $a > 0$  and  $b \in \mathbb{R}$ . Given  $P_\varepsilon$  as in (1.11), we define  $P_0$  as a limit of  $P_\varepsilon$  by (1.12), where  $E$  is an arbitrary set in the  $\sigma$ -field  $\mathcal{F}$ . In the following, we show that (1.12) indeed defines a probability measure. For this purpose, we can simply take  $E$  to be in some generating family of  $\mathcal{F}$ . Let us choose the increasing family  $\mathcal{F}_N = \sigma(g_n; |n| \leq N)$  as such a generating family of  $\mathcal{F}$ .

Fix a nonnegative integer  $N$  and a Borel set  $F$  in  $\mathbb{C}^{2N+1}$ . Let  $E = \{\omega : (g_n; |n| \leq N) \in F\}$ . Then, by (1.10), we have

$$P_\varepsilon(E) = P\left((g_n; |n| \leq N) \in F \mid \int_{\mathbb{T}} |u|^2 \in A_\varepsilon(a), i \int_{\mathbb{T}} u \bar{u}_x \in B_\varepsilon(b)\right),$$

where  $A_\varepsilon(a)$  and  $B_\varepsilon(b)$  are neighborhoods shrinking nicely to  $a$  and  $b$  as  $\varepsilon \rightarrow 0$ . That is,

(a) For each  $\varepsilon > 0$ , we have

$$A_\varepsilon(a) \subset (a - \varepsilon, a + \varepsilon) \quad \text{and} \quad B_\varepsilon(b) \subset (b - \varepsilon, b + \varepsilon).$$

(b) There exists  $\alpha > 0$ , independent of  $\varepsilon$ , such that

$$|A_\varepsilon(a)| > \alpha\varepsilon \quad \text{and} \quad |B_\varepsilon(b)| > \alpha\varepsilon.$$

By (1.9), we have

$$\int_{\mathbb{T}} |u(x)|^2 dx = \sum_{n \in \mathbb{Z}} \langle \tilde{n} \rangle^{-2} |g_n|^2 \quad \text{and} \quad i \int_{\mathbb{T}} u \bar{u}_x dx = \sum_{n \in \mathbb{Z}} \langle \tilde{n} \rangle^{-2} \tilde{n} |g_n|^2, \quad (2.1)$$

where  $\tilde{n} = 2\pi n$  and  $\langle \tilde{n} \rangle = \sqrt{1 + \tilde{n}^2}$ . Therefore, by independence of  $\{g_n\}_{|n| \leq N}$  and  $\{g_n\}_{|n| \geq N+1}$ , we have

$$\begin{aligned}
P_\varepsilon(E) &= \int_F \frac{P\left(\sum_{|n|\geq N+1} \langle \tilde{n} \rangle^{-2} |g_n|^2 \in A_\varepsilon(\tilde{a}), \sum_{|n|\geq N+1} \langle \tilde{n} \rangle^{-2} \tilde{n} |g_n|^2 \in B_\varepsilon(\tilde{b})\right)}{P\left(\sum_{n \in \mathbb{Z}} \langle \tilde{n} \rangle^{-2} |g_n|^2 \in A_\varepsilon(a), \sum_{n \in \mathbb{Z}} \langle \tilde{n} \rangle^{-2} \tilde{n} |g_n|^2 \in B_\varepsilon(b)\right)} \\
&\quad \times \frac{e^{-1/2 \sum_{|n|\leq N} |\xi_n|^2}}{(2\pi)^{2N+1}} \prod_{|n|\leq N} d\xi_n, \tag{2.2}
\end{aligned}$$

where  $d\xi_n$  denotes the Lebesgue measure on  $\mathbb{C}$ , and  $A_\varepsilon(\tilde{a})$  and  $B_\varepsilon(\tilde{b})$  are the translates of  $A_\varepsilon(a)$  and  $B_\varepsilon(b)$  centered at

$$\tilde{a} = a - \sum_{|n|\leq N} \langle \tilde{n} \rangle^{-2} |\xi_n|^2, \quad \text{and} \quad \tilde{b} = b - \sum_{|n|\leq N} \langle \tilde{n} \rangle^{-2} \tilde{n} |\xi_n|^2, \tag{2.3}$$

respectively.

Now, define the density  $f_N(a, b)$  by

$$f_N(a, b) da db = P\left(\sum_{|n|\geq N} \langle \tilde{n} \rangle^{-2} |g_n|^2 \in da, \sum_{|n|\geq N} \langle \tilde{n} \rangle^{-2} \tilde{n} |g_n|^2 \in db\right). \tag{2.4}$$

Then, we have the following lemma on the regularity of  $f_N$ .

**LEMMA 2.1.** *Let  $\hat{f}_N$  be the characteristic function (Fourier transform) of  $f_N$ . Then, we have  $\hat{f}_N \in L^1(\mathbb{R}^2)$  with estimate:  $\|\hat{f}_N\|_{L^1(\mathbb{R}^2)} < C(N) < \infty$ , where  $C(N)$  is at most a power of  $N$ . In particular,  $f_N$  is bounded and uniformly continuous.*

**PROOF.** By computing the characteristic function of  $f_N$ , we have

$$\begin{aligned}
\hat{f}_N(s, t) &= \mathbb{E}\left[\exp\left(is \sum_{|n|\geq N} \langle \tilde{n} \rangle^{-2} |g_n|^2 + it \sum_{|n|\geq N} \langle \tilde{n} \rangle^{-2} \tilde{n} |g_n|^2\right)\right] \\
&= \prod_{|n|\geq N} \mathbb{E}\left[e^{i(s\langle \tilde{n} \rangle^{-2} + t\langle \tilde{n} \rangle^{-2} \tilde{n}) |g_n|^2}\right] \\
&= \prod_{n \geq N} \frac{1}{(1 - 2i\langle \tilde{n} \rangle^{-2}(s + t\tilde{n})) (1 - 2i\langle \tilde{n} \rangle^{-2}(s - t\tilde{n}))}. \tag{2.5}
\end{aligned}$$

For any  $n \geq N$ , we have  $\max(s + t\tilde{n}, s - t\tilde{n}) \geq \max(s, t\tilde{n})$ . Also, note that each factor in (2.5) is bounded by 1. Thus, considering the terms for  $n = N, \dots, N+3$  in (2.5), we have

$$|\hat{f}_N(s, t)| \leq C(N) \langle s \rangle^{-2} \langle t \rangle^{-2},$$

where  $C(N)$  is at most a power of  $N$ . Therefore, we have  $\|\hat{f}_N\|_{L^1_{s,t}} < C'(N) < \infty$ . Note that  $C'(N)$  is at most a power of  $N$ . We use this fact in Subsection 2.2.  $\square$

By Lemma 2.1, we have, for any  $N \geq 0$ ,

$$\begin{aligned} & \frac{P(\sum_{|n| \geq N} \langle n \rangle^{-2} |g_n|^2 \in A_\varepsilon(\tilde{a}), \sum_{|n| \geq N} \langle n \rangle^{-2} n |g_n|^2 \in B_\varepsilon(\tilde{b}))}{|A_\varepsilon(\tilde{a}) \times B_\varepsilon(\tilde{b})|} \\ &= \frac{1}{|A_\varepsilon(\tilde{a}) \times B_\varepsilon(\tilde{b})|} \int_{A_\varepsilon(\tilde{a}) \times B_\varepsilon(\tilde{b})} f_N(a', b') da' db' \longrightarrow f_N(\tilde{a}, \tilde{b}), \end{aligned} \quad (2.6)$$

as  $\varepsilon \rightarrow 0$ . By the uniform continuity of  $f_N$ , this convergence is uniform in  $\tilde{a}$  and  $\tilde{b}$ .

In taking the limit of (2.2) as  $\varepsilon \rightarrow 0$ , the expression  $f_0(a, b)$ , i.e. (2.4) with  $N = 0$ , appears in the denominator. Hence, we need to show that  $f_0(a, b) > 0$  for any  $a > 0$  and  $b \in \mathbb{R}$ . Indeed, we have

**PROPOSITION 2.2.** *Let  $a > 0$  and  $b \in \mathbb{R}$ . Then, we have  $f_0(a, b) > 0$ .*

Proposition 2.2 is intuitively obvious. However, since  $f_0$  involves an infinite number of random variables, we were not able to find any reference. The proof will be given at the end of this subsection.

Putting everything together, we have

$$\begin{aligned} & \frac{P(\sum_{|n| \geq N+1} \langle \tilde{n} \rangle^{-2} |g_n|^2 \in A_\varepsilon(\tilde{a}), \sum_{|n| \geq N+1} \langle \tilde{n} \rangle^{-2} \tilde{n} |g_n|^2 \in B_\varepsilon(\tilde{b}))}{P(\sum_n \langle \tilde{n} \rangle^{-2} |g_n|^2 \in A_\varepsilon(a), \sum_n \langle \tilde{n} \rangle^{-2} \tilde{n} |g_n|^2 \in B_\varepsilon(b))} \\ & \longrightarrow \frac{f_{N+1}(\tilde{a}, \tilde{b})}{f_0(a, b)}, \end{aligned} \quad (2.7)$$

where the convergence is uniform in  $\tilde{a}$  and  $\tilde{b}$ . Moreover, the left hand side of (2.7) is uniformly bounded for small  $\varepsilon > 0$  (for fixed  $a$  and  $b$ ), since  $\|f_{N+1}\|_{L^\infty} \leq \|\hat{f}_{N+1}\|_{L^1} < \infty$  and  $f_0(a, b) > 0$ . Hence, by (1.12), (2.2), and Lebesgue dominated convergence theorem, we have

$$P_0(E) = \lim_{\varepsilon \rightarrow 0} P_\varepsilon(E) = \int_F \frac{f_{N+1}(\tilde{a}, \tilde{b})}{f_0(a, b)} \frac{e^{-1/2 \sum_{|n| \leq N} |\xi_n|^2}}{(2\pi)^{2N+1}} \prod_{|n| \leq N} d\xi_n.$$

This shows that  $P_0$  is a well-defined probability measure. Lastly, note that it basically follows from the definition that  $P_\varepsilon$  converges weakly to  $P_0$ .

We will need the following lemma for the proof of Proposition 2.2.

LEMMA 2.3. *Assume that  $f_0(a^*, b^*) = 0$  for some  $a^* > 0$  and  $b^* \in \mathbb{R}$ . Then, there exists sufficiently large  $N_0 \in \mathbb{N}$  such that  $f_N(a, b) = 0$  on*

$$B := \left\{ (a, b) \in \mathbb{R}_+ \times \mathbb{R} : a \leq \frac{1}{2}a^*, |b| \leq |b^*| + 1 \right\} \quad (2.8)$$

for all  $N \geq N_0$ .

PROOF. First, note that, by symmetry, we have

$$f_N(a, b) = f_N(a, -b) \quad (2.9)$$

for any  $a, b \in \mathbb{R}$  and  $N \geq 0$ . Defining  $X_N$  and  $Y_N$  by

$$X_N = \sum_{|n| \geq N} \langle \tilde{n} \rangle^{-2} |g_n|^2 \quad \text{and} \quad Y_N = \sum_{|n| \geq N} \langle \tilde{n} \rangle^{-2} \tilde{n} |g_n|^2, \quad (2.10)$$

we have  $X_0 = X_1 + |g_0|^2$  and  $Y_0 = Y_1$ . Note that  $X_1$  and  $|g_0|^2$  are independent. Thus, we can write  $f_0$  as  $f_0 = f_1 *_a \chi_2^2$ , where  $\chi_2^2$  is the density for the (rescaled) chi square distribution with two degrees of freedom, corresponding to  $|g_0|^2 = (\operatorname{Re} g_0)^2 + (\operatorname{Im} g_0)^2$ , and  $*_a$  denotes the convolution only in the first variable of  $f_1$ . Recall that  $\chi_2^2(x) > 0$  for  $x > 0$  and  $= 0$  for  $x < 0$ .

Now, suppose that  $f_0(a^*, b^*) = 0$  for some  $a^* > 0$  and  $b^* \in \mathbb{R}$ . By (2.9), assume that  $b^* \geq 0$ . Then, from

$$0 = f_0(a^*, b^*) = \int_{x > 0} f_1(a^* - x, b^*) \chi_2^2(x) dx$$

and the positivity of  $\chi_2^2$  on  $\mathbb{R}_+$ , we have  $f_1(a, b^*) = 0$  for  $a \leq a^*$ . (Recall that  $f_1$  is continuous by Lemma 2.1.)

Let  $c_1(n)$  and  $c_2(n)$  be given by

$$c_1(n) = (1 + 4\pi^2 n^2)^{-1} \quad \text{and} \quad c_2(n) = 2\pi n c_1(n), \quad n \in \mathbb{N}. \quad (2.11)$$

Then, from (2.10), we have

$$X_1 = X_2 + c_1(1)(|g_1|^2 + |g_{-1}|^2) \quad \text{and} \quad Y_1 = Y_2 + c_2(1)(|g_1|^2 - |g_{-1}|^2).$$

Since  $f_1(a^*, b^*) = 0$ , we have

$$\begin{aligned} 0 &= f_1(a^*, b^*) \\ &= \int_0^\infty \int_0^\infty f_2(a^* - c_1(1)(x+y), b^* - c_2(1)(x-y)) \chi_2^2(x) \chi_2^2(y) dx dy. \end{aligned} \quad (2.12)$$

By change of variables  $p = x + y$  and  $q = x - y$ , we can write (2.12) as

$$0 = c \iint_{\substack{p>0 \\ |q|\leq p}} f_2(a^* - c_1(1)p, b^* - c_2(1)q) \chi_2^2\left(\frac{p+q}{2}\right) \chi_2^2\left(\frac{p-q}{2}\right) dp dq.$$

This implies that  $f_2(a, b) = 0$  on a triangular region

$$A_2 := \{(a, b) \in \mathbb{R}_+ \times \mathbb{R} : a \leq a^*, |b - b^*| \leq 2\pi(a^* - a)\}.$$

In particular,  $f_2(a^*, b^*) = 0$ . From (2.10), we have

$$X_2 = X_3 + c_1(2)(|g_2|^2 + |g_{-2}|^2) \quad \text{and} \quad Y_2 = Y_3 + c_2(2)(|g_2|^2 - |g_{-2}|^2),$$

where  $c_1(2)$  and  $c_2(2)$  are as in (2.11). Since  $f_2(a^*, b^*) = 0$ , we have

$$\begin{aligned} 0 &= f_2(a^*, b^*) \\ &= \int_0^\infty \int_0^\infty f_3(a^* - c_1(2)(x+y), b^* - c_2(2)(x-y)) \chi_2^2(x) \chi_2^2(y) dx dy. \end{aligned} \quad (2.13)$$

Once again, by change of variables  $p = x + y$  and  $q = x - y$ , we can write (2.13) as

$$0 = c \iint_{\substack{p>0 \\ |q|\leq p}} f_3(a^* - c_1(2)p, b^* - c_2(2)q) \chi_2^2\left(\frac{p+q}{2}\right) \chi_2^2\left(\frac{p-q}{2}\right) dp dq.$$

This implies that  $f_3(a, b) = 0$  on a triangular region

$$A_3 := \{(a, b) \in \mathbb{R}_+ \times \mathbb{R} : a \leq a^*, |b - b^*| \leq 4\pi(a^* - a)\}.$$

In particular, we have  $f_3(a^*, b^*) = 0$  and thus we can repeat the argument. In general, from  $f_N(a^*, b^*) = 0$ , we can show that  $f_{N+1}(a, b) = 0$  on a triangular

region

$$A_{N+1} := \{(a, b) \in \mathbb{R}_+ \times \mathbb{R} : a \leq a^*, |b - b^*| \leq 2\pi N(a^* - a)\}$$

by simply noting  $c_2(N)/c_1(N) = 2\pi N$ . By symmetry (2.9), we have  $f_{N+1}(a, b) = 0$  also on

$$\tilde{A}_{N+1} := \{(a, b) \in \mathbb{R}_+ \times \mathbb{R} : a \leq a^*, |b + b^*| \leq 2\pi N(a^* - a)\}.$$

Finally, by choosing  $N_0$  large such that  $\pi N_0 a^* \geq \max(1, |b^*|)$ , we see that  $B \subset A_N \cup \tilde{A}_N$  for  $N \geq N_0$  and hence  $f_N(a, b) = 0$  on  $B$  for  $N \geq N_0$ .  $\square$

Finally, we conclude this subsection by presenting the proof of Proposition 2.2.

**PROOF OF PROPOSITION 2.2.** Suppose that  $f_0(a^*, b^*) = 0$  for some  $a^* > 0$  and  $b^* \in \mathbb{R}$ . By Lemma 2.3, there exists  $N_0 \in \mathbb{N}$  such that  $f_N = 0$  on  $B$  for all  $N \geq N_0$ , where  $B$  is defined in (2.8). Recall that  $f_N$  is nonnegative and  $f_N(a, b) = 0$  for  $a < 0$ . Then, by  $(a, b) \in \mathbb{R}_+ \times \mathbb{R} \subset B \cup \{a > (1/2)a^*\} \cup \{|b| \geq |b^*| + 1\}$ , we have

$$\begin{aligned} 1 &= \int_{\mathbb{R}} \int_0^\infty f_N(a, b) da db \\ &\leq \iint_B f_N(a, b) da db + \iint_{a > (1/2)a^*} f_N(a, b) da db + \iint_{|b| > |b^*| + 1} f_N(a, b) da db \\ &= 0 + P\left(X_N > \frac{1}{2}a^*\right) + P(|Y_N| > |b^*| + 1), \end{aligned} \tag{2.14}$$

for all  $N \geq N_0$ , where  $X_N$  and  $Y_N$  are as in (2.10). Once we prove

$$P\left(X_N > \frac{1}{2}a^*\right) < \frac{1}{2}, \tag{2.15}$$

$$P(|Y_N| > |b^*| + 1) < \frac{1}{2}, \tag{2.16}$$

for some  $N$ , (2.14) together with (2.15) and (2.16) leads to a contradiction, and hence  $f_0(a, b) > 0$  for all  $a > 0$  and  $b \in \mathbb{R}$ .

Therefore, it remains to prove (2.15) and (2.16) for large  $N$ . First, we prove (2.16). Write  $Y_N$  as

$$Y_N = \sum_{n \geq N} \frac{2\pi n}{1 + 4\pi^2 n^2} (|g_n|^2 - |g_{-n}|^2).$$

Since  $\mathbb{E}[|g_n|^2 - |g_{-n}|^2] = 0$ , we have  $\mathbb{E}[|Y_N|^2] \leq CN^{-1}$ . Then, by Chebyshev's inequality, we conclude that

$$P(|Y_N| > |b^*| + 1) \lesssim \mathbb{E}[|Y_N|^2] \leq CN^{-1}.$$

Hence, there exists  $N_1$  such that (2.16) holds for all  $N \geq N_1$ .

Next, we prove (2.15). Fix large dyadic  $N_2 = 2^k$  (to be chosen later). Let  $\sigma_j = C2^{-(1/2)j}$  such that  $\sum_{j=1}^{\infty} \sigma_j = 1$ . Then, for  $N \geq N_2$ , we have

$$\begin{aligned} P\left(X_N > \frac{1}{2}a^*\right) &\leq \sum_{j=k}^{\infty} P\left(\left(\sum_{2^j \leq |n| < 2^{j+1}} (1 + 4\pi^2 n^2)^{-2} |g_n|^2\right)^{1/2} > \frac{1}{2}\sigma_j a^*\right) \\ &\leq \sum_{j=k}^{\infty} P\left(\left(\sum_{2^j \leq |n| < 2^{j+1}} |g_n|^2\right)^{1/2} > c_{a^*} \sigma_j 2^j\right), \end{aligned}$$

where  $c_{a^*} > 0$  is a constant depending only on  $a^*$ . By the large deviation estimate (e.g. see Lemma 4.2 in [OQV]), we obtain

$$P\left(X_N > \frac{1}{2}a^*\right) \leq \sum_{j=k}^{\infty} e^{-c'_{a^*} \sigma_j^2 2^{2j}} \leq e^{-\tilde{c}_{a^*} 2^k} < \frac{1}{2}$$

for sufficiently large  $k \in \mathbb{N}$ . By choosing  $N \geq \max(N_0, N_1, N_2)$ , (2.14) together with (2.15) and (2.16) leads to a contradiction. This completes the proof of Proposition 2.2.  $\square$

## 2.2. Gibbs measure conditioned on mass and momentum.

In the previous subsection, we constructed the Wiener measure  $P_0$  conditioned on mass and momentum as a limit of conditioned Wiener measures  $P_\varepsilon$ . In this subsection, we define the conditioned Gibbs measure  $\mu_0 = \mu_{a,b}$  by (1.13). In the defocusing case, (1.13) defines a probability measure. In the focusing case, however, we need to show (1.14); the weight  $e^{1/p \int_{\mathbb{T}} |u|^p}$  is integrable with respect to  $P_0$  for  $p \leq 6$  (with sufficiently small mass when  $p = 6$ ).

Bourgain [B2] proved a similar integrability result of the weight  $e^{1/p \int_{\mathbb{T}} |u|^p}$  with respect to the (unconditioned) Wiener measure  $P$  in (1.8) via dyadic pigeonhole principle and a large deviation estimate. In the following, we also use dyadic pigeonhole principle and a large deviation estimate (for the conditioned Wiener

measure  $P_0$ ) to show that the conditioned Gibbs measure  $\mu_0$  is a well-defined probability measure. Indeed, Lemma 2.4 below establishes a uniform large deviation estimate for  $P_\varepsilon$ ,  $\varepsilon > 0$ , and we prove the  $L^1$ -boundedness of the weight  $e^{1/p \int_{\mathbb{T}} |u|^p}$  with respect to  $P_\varepsilon$ , uniformly in sufficiently small  $\varepsilon > 0$ . See (2.23).

First, we present a *uniform* large deviation lemma for the conditioned Wiener measure  $P_\varepsilon$ ,  $\varepsilon > 0$ .

LEMMA 2.4. *Let  $R \geq 5N^{1/2}$  and  $M \sim N$ . Then, we have*

$$P_\varepsilon \left( \sum_{|n-M| \leq N} |g_n|^2 \geq R^2 \right) \leq C e^{-(1/8)R^2} \quad (2.17)$$

*uniformly for sufficiently small  $\varepsilon \geq 0$ .*

PROOF. By Chebyshev's inequality, we have

$$P_\varepsilon \left( \sum_{|n-M| \leq N} |g_n|^2 \geq R^2 \right) \leq e^{-tR^2} \mathbb{E}_{P_\varepsilon} [e^{t \sum_{|n-M| \leq N} |g_n|^2}]. \quad (2.18)$$

Set  $t = 1/4$ . We estimate  $\mathbb{E}_{P_\varepsilon} [e^{1/4 \sum_{|n-M| \leq N} |g_n|^2}]$  in the following. As in (2.2), we can write it as

$$\begin{aligned} & \mathbb{E}_{P_\varepsilon} [e^{1/4 \sum_{|n-M| \leq N} |g_n|^2}] \\ &= \int_{\mathbb{C}^{2N+1}} \frac{P(\sum_{|n-M| \geq N+1} \langle \tilde{n} \rangle^{-2} |g_n|^2 \in A_\varepsilon(\tilde{a}), \sum_{|n-M| \geq N+1} \langle \tilde{n} \rangle^{-2} \tilde{n} |g_n|^2 \in B_\varepsilon(\tilde{b}))}{P(\sum_n \langle \tilde{n} \rangle^{-2} |g_n|^2 \in A_\varepsilon(a), \sum_n \langle \tilde{n} \rangle^{-2} \tilde{n} |g_n|^2 \in B_\varepsilon(b))} \\ & \quad \times \frac{e^{-1/4 \sum_{|n-M| \leq N} |\xi_n|^2}}{(2\pi)^{2N+1}} \prod_{|n-M| \leq N} d\xi_n, \end{aligned} \quad (2.19)$$

where  $\tilde{a}$  and  $\tilde{b}$  are given by

$$\tilde{a} = a - \sum_{|n-M| \leq N} \langle \tilde{n} \rangle^{-2} |\xi_n|^2, \quad \text{and} \quad \tilde{b} = b - \sum_{|n-M| \leq N} \langle \tilde{n} \rangle^{-2} \tilde{n} |\xi_n|^2. \quad (2.20)$$

By repeating the argument in Subsection 2.1, we can show that the right hand side of (2.19) is uniformly bounded for small  $\varepsilon > 0$ .

More precisely, define the density  $\tilde{f}_N(a, b)$  by



$$\tilde{f}_N(a, b) da db = P\left(\sum_{|n-M|\geq N} \langle \tilde{n} \rangle^{-2} |g_n|^2 \in da, \sum_{|n-M|\geq N} \langle \tilde{n} \rangle^{-2} \tilde{n} |g_n|^2 \in db\right).$$

Then, as in Subsection 2.1, one can prove

$$\begin{aligned} & \frac{P\left(\sum_{|n-M|\geq N+1} \langle \tilde{n} \rangle^{-2} |g_n|^2 \in A_\varepsilon(\tilde{a}), \sum_{|n-M|\geq N+1} \langle \tilde{n} \rangle^{-2} \tilde{n} |g_n|^2 \in B_\varepsilon(\tilde{b})\right)}{P\left(\sum_n \langle \tilde{n} \rangle^{-2} |g_n|^2 \in A_\varepsilon(a), \sum_n \langle \tilde{n} \rangle^{-2} \tilde{n} |g_n|^2 \in B_\varepsilon(b)\right)} \\ & \longrightarrow \frac{\tilde{f}_{N+1}(\tilde{a}, \tilde{b})}{f_0(a, b)}, \end{aligned} \quad (2.21)$$

where the convergence is uniform in  $\tilde{a}$  and  $\tilde{b}$ . Moreover, by showing  $\|\tilde{f}_N\|_{L^\infty} < \infty$  as before, we see that the left hand side of (2.21) is uniformly bounded for small  $\varepsilon > 0$ . (Recall that  $a$  and  $b$  are fixed.) By (2.19), (2.21), and Lebesgue dominated convergence theorem, we have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \mathbb{E}_{P_\varepsilon} \left[ e^{1/4 \sum_{|n-M|\leq N} |g_n|^2} \right] &= \int_{\mathbb{C}^{2N+1}} \frac{\tilde{f}_{N+1}(\tilde{a}, \tilde{b})}{f_0(a, b)} \frac{e^{-1/4 \sum_{|n-M|\leq N} |\xi_n|^2}}{(2\pi)^{2N+1}} \prod_{|n-M|\leq N} d\xi_n \\ &\leq \frac{\|\tilde{f}_{N+1}\|_{L^\infty}}{f_0(a, b)} \int_{\mathbb{C}^{2N+1}} \frac{e^{-1/4 \sum_{|n-M|\leq N} |\xi_n|^2}}{(2\pi)^{2N+1}} \prod_{|n-M|\leq N} d\xi_n \\ &\leq \frac{\|\tilde{f}_{N+1}\|_{L^\infty}}{f_0(a, b)} 2^{2N+1}, \end{aligned}$$

where the last inequality follows from change of variables. Also, by an analogous argument to the proof of Lemma 2.1, we see that  $\|\tilde{f}_{N+1}\|_{L^\infty} \leq \|(\tilde{f}_{N+1})^\wedge\|_{L^1}$  is bounded at most by a power of  $N$ . Hence, we have

$$\mathbb{E}_{P_\varepsilon} \left[ e^{1/4 \sum_{|n-M|\leq K} |g_n|^2} \right] \lesssim 2^{3N} \quad (2.22)$$

for all sufficiently small  $\varepsilon > 0$ . Therefore, (2.17) follows from (2.18) and (2.22) as long as  $R^2 \geq (24 \ln 2)N$ .  $\square$

In the following, we show the  $L^1$ -boundedness of the weight  $e^{1/p \int_{\mathbb{T}} |u|^p}$  with respect to  $P_\varepsilon$ , uniformly for sufficiently small  $\varepsilon \geq 0$ , for  $p \leq 6$  (with sufficiently small mass when  $p = 6$ .) This, in particular, shows that  $\mu_\varepsilon$  in (1.15) is a well-defined probability measure.

Note that it suffices to prove that

$$\begin{aligned}
& \int_0^\infty e^\lambda P_\varepsilon \left( \int_{\mathbb{T}} |u|^p \geq p\lambda \right) d\lambda \\
&= \int_0^\infty e^\lambda P \left( \int_{\mathbb{T}} |u|^p \geq p\lambda \mid \int_{\mathbb{T}} |u|^2 \in A_\varepsilon(a), i \int_{\mathbb{T}} u \bar{u}_x \in B_\varepsilon(b) \right) d\lambda \leq C_p < \infty
\end{aligned} \tag{2.23}$$

for all sufficiently small  $\varepsilon > 0$ . The estimate (2.23) follows once we prove

$$P_\varepsilon \left( \int_{\mathbb{T}} |u|^p \geq p\lambda \right) \leq \begin{cases} C e^{-c\lambda^{1+\delta}} & \text{when } p < 6. \\ C e^{-(1+\delta)\lambda} & \text{when } p = 6. \end{cases} \tag{2.24}$$

for  $\lambda > 1$  (with some  $\delta > 0$ ), uniformly in small  $\varepsilon > 0$ .

Before proving (2.24), let us introduce some notations. Given  $M_0 \in \mathbb{N}$ , let  $\mathbb{P}_{>M_0}$  denote the Dirichlet projection onto the frequencies  $\{|n| > M_0\}$ . i.e.  $\mathbb{P}_{>M_0} u = \sum_{|n| > M_0} \hat{u}_n e^{2\pi i n x}$ .  $\mathbb{P}_{\leq M_0}$  is defined in a similar manner. Given  $j \in \mathbb{N}$ , let  $M_j = 2^j M_0$ . We use the notation  $|n| \sim M_j$  to denote the set of integers  $|n| \in (M_{j-1}, M_j]$ , and denote by  $\mathbb{P}_{M_j}$  the Dirichlet projection onto the dyadic block  $(M_{j-1}, M_j]$ , i.e.  $\mathbb{P}_{M_j} u = \sum_{|n| \sim M_j} \hat{u}_n e^{2\pi i n x}$ .

Without loss of generality, assume  $\varepsilon \leq a$ . Then, we have  $\int |u|^2 \leq 2a =: K$ . By Sobolev inequality (or equivalently, by Hausdorff-Young inequality followed by Hölder inequality on the Fourier side in this particular case),

$$\|\mathbb{P}_{\leq M_0} u\|_{L^p(\mathbb{T})} \leq c M_0^{1/2-1/p} \|\mathbb{P}_{\leq M_0} u\|_{L^2(\mathbb{T})}. \tag{2.25}$$

Hence, we have

$$\int_{\mathbb{T}} |\mathbb{P}_{\leq M_0} u|^p \leq \frac{p}{2} \lambda \quad \text{on } \int_{\mathbb{T}} |u|^2 \leq K, \tag{2.26}$$

by choosing

$$M_0 = c_0 \lambda^{2/(p-2)} K^{-p/(p-2)} \sim c_0 \lambda^{2/(p-2)} a^{-p/(p-2)} \tag{2.27}$$

for some  $c_0 > 0$ . Let  $\sigma_j = C 2^{-\delta j}$ ,  $j = 1, 2, \dots$  for some small  $\delta > 0$  where  $C = C(\delta)$  is chosen such that  $\sum_{j=1}^\infty \sigma_j = 1$ . Then, we have

$$P_\varepsilon \left( \int_{\mathbb{T}} |\mathbb{P}_{>M_0} u|^p > \frac{p}{2} \lambda \right) \leq \sum_{j=0}^\infty P_\varepsilon \left( \|\mathbb{P}_{M_j} u\|_{L^p(\mathbb{T})} > \sigma_j \left( \frac{p}{2} \lambda \right)^{1/p} \right). \tag{2.28}$$

By Sobolev inequality as in (2.25), we have

$$\|\mathbb{P}_{M_j} u\|_{L^p(\mathbb{T})} \leq cM_j^{1/2-1/p} \|\mathbb{P}_{M_j} u\|_{L^2(\mathbb{T})}. \quad (2.29)$$

From (1.9), we have

$$\|\mathbb{P}_{M_j} u\|_{L^2(\mathbb{T})}^2 = \sum_{|n| \sim M_j} |\hat{u}_n|^2 = \sum_{|n| \sim M_j} (1 + (2\pi n)^2)^{-1} |g_n|^2. \quad (2.30)$$

From (2.29) and (2.30), the right hand side of (2.28) is bounded by

$$\sum_{j=0}^{\infty} P_\varepsilon \left( \sum_{|n| \sim M_j} |g_n|^2 \geq R_j^2 \right), \quad \text{where } R_j := c' \sigma_j \lambda^{1/p} M_j^{1/p-1/2} (1 + M_j^2)^{1/2}. \quad (2.31)$$

Note that  $R_j \gtrsim M_j^{1/2+1/p} \gg M_j^{1/2}$ . By applying Lemma 2.4 to (2.31), we obtain

$$\begin{aligned} P_\varepsilon \left( \int_{\mathbb{T}} |\mathbb{P}_{>M_0} u|^p > \frac{p}{2} \lambda \right) &\lesssim \sum_{j=0}^{\infty} e^{-(1/8)R_j^2} \lesssim \sum_{j=0}^{\infty} e^{-c'' \sigma_j^2 \lambda^{2/p} M_j^{(p+2)/p}} \\ &\lesssim \sum_{j=0}^{\infty} e^{-\tilde{c}(2^j)^{(p+2)/p-2\delta} \lambda^{2/p} M_0^{(p+2)/p}} \lesssim e^{-c\lambda^{2/p} M_0^{(p+2)/p}}. \end{aligned} \quad (2.32)$$

Hence, from (2.32) and (2.27), we have

$$P_\varepsilon \left( \int_{\mathbb{T}} |u|^p > p\lambda \right) \leq C \exp \left\{ -c \lambda^{1+(6-p)/(p-2)} a^{-(p+2)/(p-2)} \right\} \quad (2.33)$$

and (2.24) follows. Note that when  $p = 6$ , we need to take  $a$  sufficiently small such that the coefficient of  $\lambda$  in (2.33) is less than  $-1$ .

### 2.3. Weak convergence.

Finally, we prove weak convergence of  $\mu_\varepsilon$  defined in (1.15) to  $\mu_0$ . Let  $f$  be a bounded continuous function on  $H^{1/2-\gamma}(\mathbb{T})$  for some small  $\gamma > 0$ .

We first consider the defocusing case. If a sequence of functions  $u_n$  converges to  $u$  in  $H^{1/2-\gamma}(\mathbb{T})$  with  $\gamma < p^{-1}$ , then we have  $u_n \rightarrow u$  in  $L^p(\mathbb{T})$  by Sobolev inequality. Thus,  $e^{-\int_{\mathbb{T}} |u|^p}$  is bounded and continuous on  $H^{1/2-\gamma}(\mathbb{T})$ . Then, by weak convergence of  $P_\varepsilon$  to  $P_0$ , we have

$$Z_\varepsilon = \int e^{-1/p \int_{\mathbb{T}} |u|^p} dP_\varepsilon \longrightarrow \int e^{-1/p \int_{\mathbb{T}} |u|^p} dP_0 = Z_0 \quad \text{as } \varepsilon \rightarrow 0.$$

Since  $f(u)e^{-1/p \int_{\mathbb{T}} |u|^p}$  is also bounded and continuous on  $H^{1/2-\gamma}(\mathbb{T})$ , we have

$$\begin{aligned} \int f d\mu_\varepsilon &= Z_\varepsilon^{-1} \int f(u) e^{-1/p \int_{\mathbb{T}} |u|^p} dP_\varepsilon \\ &\longrightarrow Z_0^{-1} \int f(u) e^{-1/p \int_{\mathbb{T}} |u|^p} dP_0 = \int f d\mu_0 \quad \text{as } \varepsilon \rightarrow 0. \end{aligned}$$

This shows that  $\mu_\varepsilon$  converges weakly to  $\mu_0$  in the defocusing case.

Next, we consider the focusing case. First, we prove

$$Z_\varepsilon = \int e^{1/p \int_{\mathbb{T}} |u|^p} dP_\varepsilon \longrightarrow \int e^{1/p \int_{\mathbb{T}} |u|^p} dP_0 = Z_0 \quad \text{as } \varepsilon \rightarrow 0. \quad (2.34)$$

Let  $g(u) = e^{1/p \int_{\mathbb{T}} |u|^p}$ . By Chebyshev's inequality with the uniform integrability (2.24), we have

$$\int_{g>B} g(u) dP_\varepsilon \leq CB^{-\delta} \quad (2.35)$$

for all small  $\varepsilon \geq 0$ . Then, (2.34) follows once we note that

$$|Z_\varepsilon - Z_0| \leq \left| \int_{g>B} g(u) dP_\varepsilon \right| + \left| \int_{g \leq B} g(u) (dP_\varepsilon - dP_0) \right| + \left| \int_{g>B} g(u) dP_0 \right|,$$

where the second term goes to 0 by the weak convergence of  $P_\varepsilon$  to  $P_0$ .

Let  $f$  be a bounded continuous function  $f$  on  $H^{1/2-\gamma}(\mathbb{T})$ . Then, by writing

$$\begin{aligned} &\int f d\mu_\varepsilon - \int f d\mu_0 \\ &= Z_\varepsilon^{-1} \int f(u) g(u) dP_\varepsilon - Z_0^{-1} \int f(u) g(u) dP_0 \\ &= Z_0^{-1} \left( \int f(u) g(u) dP_\varepsilon - \int f(u) g(u) dP_0 \right) + (Z_\varepsilon^{-1} - Z_0^{-1}) \int f(u) g(u) dP_\varepsilon, \end{aligned}$$

it follows from (2.34) that the second term on the right hand side goes to zero. The first term goes to zero by the uniform integrability (2.24) with Chebyshev's

inequality as before. Hence,  $\mu_\varepsilon$  converges weakly to  $\mu_0$ . This completes the proof of Theorem 1.

### 3. Proof of Theorem 2: Invariance of the conditioned Gibbs measures.

In this section, we show that the conditioned Gibbs measure  $\mu_0$  is invariant under the flow of NLS (1.1). In fact, one can directly establish the invariance of the conditioned Gibbs measure  $\mu_0$  by following the argument developed by Bourgain [B2], [B3]. This argument is based on approximating the PDE flow by finite dimensional Hamiltonian systems with invariant finite dimensional Gibbs measures. For such an argument, one needs the following large deviation estimate (with  $\varepsilon = 0$ .)

LEMMA 3.1. *Let  $s < 1/2$ . Then, we have*

$$P_\varepsilon(\|u\|_{H^s} > \Lambda) \leq C_s e^{-c\Lambda^2}, \quad (3.1)$$

uniformly in small  $\varepsilon \geq 0$ .

PROOF. This basically follows from the proof of (2.33) in Subsection 2.2. Given  $s < 1/2$ , choose  $p > 2$  such that  $s = 1/2 - 1/p$ . Then, we have

$$\|\mathbb{P}_{\leq M_0} u\|_{H^s(\mathbb{T})} \leq cM_0^{1/2-1/p} \|\mathbb{P}_{\leq M_0} u\|_{L^2(\mathbb{T})}. \quad (3.2)$$

(Compare this with (2.25).) By repeating the computation in Subsection 2.2 (with  $\Lambda = \lambda^{1/p}$ ), we obtain

$$P_\varepsilon(\|u\|_{H^s} > \Lambda) \leq C_s \exp\{-c\Lambda^{p(1+(6-p)/(p-2))} a^{-(p+2)/(p-2)}\}. \quad (3.3)$$

Then, (3.1) follows since  $p(1 + (6 - p)/(p - 2)) > 2$  for  $p > 2$ .  $\square$

Bourgain's argument [B2], [B3] requires a combination of PDE and probabilistic techniques. In the following, however, we simply show how the invariance of the conditioned Gibbs measure  $\mu_0$  follows, as a corollary, from a priori invariance of Gibbs measures  $\mu_\varepsilon$ ,  $\varepsilon > 0$ .

CASE 1:  $p \leq 6$ . In this case, the flow of (1.1) is globally defined in  $H^{1/2-\delta}(\mathbb{T})$  for small  $\delta = \delta(p) > 0$ , thanks to [B1], [B5]. Let  $\mathcal{S}_t$  be the flow map of (1.1):  $u_0 \mapsto u(t) = \mathcal{S}_t u_0$ . Then,  $\mathcal{S}_t$  is well-defined and continuous on  $H^{1/2-\delta}(\mathbb{T})$ .

Given a bounded continuous function  $\phi$  on  $H^{1/2-\delta}(\mathbb{T})$ ,  $\phi \circ \mathcal{S}_t$  is bounded and

continuous on  $H^{1/2-\delta}(\mathbb{T})$ . By weak convergence of  $\mu_\varepsilon$  to  $\mu_0$  and invariance of  $\mu_\varepsilon$  under the flow of (1.1), we have

$$\int \phi d\mu_0 = \lim_{\varepsilon \rightarrow 0} \int \phi d\mu_\varepsilon = \lim_{\varepsilon \rightarrow 0} \int \phi \circ \mathcal{S}_t d\mu_\varepsilon = \int \phi \circ \mathcal{S}_t d\mu_0.$$

This proves invariance of  $\mu_0$  for  $p \leq 6$ .

CASE 2:  $p > 6$ . (This is relevant only in the defocusing case.)

In this case, there is no a priori global-in-time flow of (1.1) on  $H^{1/2-\delta}(\mathbb{T})$ . However, by Bourgain's argument [B2], [B3],  $\mu_\varepsilon$  is invariant under the flow of NLS (1.1) for each  $\varepsilon > 0$ , and we show invariance of  $\mu_0$  as a corollary to the invariance of  $\mu_\varepsilon$ ,  $\varepsilon > 0$ .

Let  $K$  be a compact set in  $H^s(\mathbb{T})$  with  $s = 1/2-$ . Then, there exists  $\Lambda = \Lambda(K) > 0$  such that  $\|u\|_{H^s} \leq \Lambda$  for  $u \in K$ . By the (deterministic) local well-posedness [B2], there exists  $t_0 > 0$  such that NLS (1.1) is well-posed on  $[0, t_0]$  for initial data  $u_0$  with  $\|u_0\|_{H^s} \leq \Lambda + 1$ . Moreover, for each small  $\theta > 0$ , there exists  $\delta > 0$  such that

$$\mathcal{S}_{t_0}(K + B_\delta) \subset \mathcal{S}_{t_0}K + B_\theta. \quad (3.4)$$

Then, by weak convergence of  $\mu_\varepsilon$  to  $\mu_0$ , we have

$$\mu_0(K) \leq \mu_0(K + B_\delta) \leq \liminf_{\varepsilon \rightarrow 0} \mu_\varepsilon(K + B_\delta).$$

By invariance of  $\mu_\varepsilon$  and (3.4),

$$\begin{aligned} &= \liminf_{\varepsilon \rightarrow 0} \mu_\varepsilon(\mathcal{S}_{t_0}(K + B_\delta)) \leq \liminf_{\varepsilon \rightarrow 0} \mu_\varepsilon(\mathcal{S}_{t_0}K + B_\theta) \\ &\leq \limsup_{\varepsilon \rightarrow 0} \mu_\varepsilon(\mathcal{S}_{t_0}K + B_\theta) \leq \limsup_{\varepsilon \rightarrow 0} \mu_\varepsilon(\mathcal{S}_{t_0}K + \overline{B_\theta}) \\ &\leq \mu_0(\mathcal{S}_{t_0}K + \overline{B_\theta}), \end{aligned}$$

where the last inequality follows once again from the weak convergence of  $\mu_\varepsilon$  to  $\mu_0$ . By letting  $\theta \rightarrow 0$ , we have  $\mu_0(K) \leq \mu_0(\mathcal{S}_{t_0}K)$ . Given arbitrary  $t > 0$ , we can iterate the above argument and obtain  $\mu_0(K) \leq \mu_0(\mathcal{S}_tK)$ . By the time-reversibility of the NLS flow, we obtain

$$\mu_0(K) = \mu_0(\mathcal{S}_tK).$$

This proves invariance of  $\mu_0$  for  $p > 6$ .

## References

- [B1] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I, Schrödinger equations, *Geom. Funct. Anal.*, **3** (1993), 107–156.
- [B2] J. Bourgain, Periodic nonlinear Schrödinger equation and invariant measures, *Comm. Math. Phys.*, **166** (1994), 1–26.
- [B3] J. Bourgain, Nonlinear Schrödinger equations, In: Hyperbolic Equations and Frequency Interactions (Park City, UT, 1995), IAS/Park City Math. Ser., **5**, Amer. Math. Soc., Providence, RI, 1999, pp. 3–157.
- [B4] J. Bourgain, Global Solutions of Nonlinear Schrödinger Equations, Amer. Math. Soc. Colloq. Publ., **46**, Amer. Math. Soc., Providence, RI, 1999.
- [B5] J. Bourgain, A remark on normal forms and the “ $I$ -method” for periodic NLS, *J. Anal. Math.*, **94** (2004), 125–157.
- [FT] L. D. Faddeev and L. A. Takhtajan, Hamiltonian Methods in the Theory of Solitons, Translated from the Russian by A. G. Reyman, Springer Ser. Soviet Math., Springer-Verlag, Berlin, 1987.
- [F] H. Furstenberg, Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions, *J. Analyse Math.*, **31** (1977), 204–256.
- [GKP] B. Grébert, T. Kappeler and J. Pöschel, Normal form theory for the NLS equation, arXiv:0907.3938v1 [math.AP].
- [LRS] J. Lebowitz, H. Rose and E. Speer, Statistical mechanics of the nonlinear Schrödinger equation, *J. Statist. Phys.*, **50** (1988), 657–687.
- [L] P. Lévy, Le mouvement brownien plan (French), *Amer. J. Math.*, **62** (1940), 487–550.
- [LQ] T. Lyons and Z. Qian, System Control and Rough Paths, Oxford Math. Monogr., Oxford Science Publications, Oxford University Press, Oxford, 2002.
- [OQV] T. Oh, J. Quastel, and B. Valkó, Interpolation of Gibbs measures with white noise for Hamiltonian PDE, *J. Math. Pures Appl.*, **97** (2012), 391–410.
- [R] G. Richards, Invariance of the Gibbs measure for the periodic quartic gKdV, arXiv: 1209.4337 [math.AP].
- [ZM] V. E. Zaharov and S. V. Manakov, The complete integrability of the nonlinear Schrödinger equation (Russian), *Teoret. Mat. Fiz.*, **19** (1974), 332–343.
- [ZS] V. E. Zakharov and A. B. Shabat, Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media, *Soviet Physics JETP*, **34** (1972), 62–69.
- [Z] P. E. Zhidkov, Korteweg-de Vries and Nonlinear Schrödinger Equations: Qualitative Theory, Lecture Notes in Math., **1756**, Springer-Verlag, 2001.

Tadahiro OH

Department of Mathematics  
 Princeton University  
 Fine Hall, Washington Rd., Princeton  
 NJ 08544-1000, USA  
 E-mail: hirooh@math.princeton.edu

Jeremy QUASTEL

Departments of Mathematics and Statistics  
 University of Toronto  
 40 St. George St, Toronto  
 ON M5S 2E4, Canada  
 E-mail: quastel@math.toronto.edu