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On the maximal L_p-L_q regularity of the Stokes problem with first order boundary condition; model problems

By Yoshihiro Shibata and Senjo Shimizu

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Abstract. In this paper, we proved the generalized resolvent estimate and the maximal L_p-L_q regularity of the Stokes equation with first order boundary condition in the half-space, which arises in the mathematical study of the motion of a viscous incompressible one phase fluid flow with free surface. The core of our approach is to prove the $\mathscr{R}$ boundedness of solution operators defined in a sector $\Sigma_{\epsilon,\gamma_0} = \{\lambda \in \boldsymbol{C} \setminus \{0\} \mid |\arg \lambda| \leq \pi - \epsilon, |\lambda| \geq \gamma_0\}$ with $0 < \epsilon < \pi/2$ and $\gamma_0 \geq 0$. This $\mathscr{R}$ boundedness implies the resolvent estimate of the Stokes operator and the combination of this $\mathscr{R}$ boundedness with the operator valued Fourier multiplier theorem of L. Weis implies the maximal L_p-L_q regularity of the non-stationary Stokes. For a densely defined closed operator A, we know that what A has maximal L_p regularity implies that the resolvent estimate of A in $\lambda \in \Sigma_{\epsilon,\gamma_0}$, but the opposite direction is not true in general (cf. Kalton and Lancien [**19**]). However, in this paper using the $\mathscr{R}$ boundedness of the operator family in the sector $\Sigma_{\epsilon,\lambda_0}$, we derive a systematic way to prove the resolvent estimate and the maximal L_p regularity at the same time.

1. Introduction.

This paper is concerned with the generalized resolvent estimate and the maximal L_p-L_q regularity of the Stokes problem with first order boundary condition in the half-space, which arises in the study of the free boundary problem of viscous incompressible one phase fluid flow. This problem is mathematically to find a time dependent domain Ω_t, t being time variable, in the n-dimensional Euclidean space $\boldsymbol{R}^n$, an n-vector of functions $v(x,t) = (v_1(x,t), \ldots, v_n(x,t))$ and a scalar function $p(x,t)$ satisfying the following Navier-Stokes equations:

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$$
\begin{aligned}
v_t + (v\cdot\nabla)v - \operatorname{Div} S(v,p) = f, \quad & \operatorname{div} v = 0 && \text{in } \Omega_t,\ t>0,\\
S(v,p)\boldsymbol{n}_t - c_\sigma H\boldsymbol{n}_t = 0, \quad & V_n = v\cdot\boldsymbol{n}_t && \text{on } \Gamma_t,\ t>0, \qquad (1.1)\\
v|_{t=0} = v_0, \quad & && \text{in } \Omega_0,
\end{aligned}
$$

where V_n is the velocity of the evolution of Γ_t in a normal direction, $\boldsymbol{n}_t = (\boldsymbol{n}_{t,1},\dots,\boldsymbol{n}_{t,n})$ is the unit outer normal to Γ_t, $S(v,p) = -pI + \mu D(v)$ is the Stokes stress tensor, $I = (\delta_{ij})$ is the $n\times n$ identity matrix, $D(v)$ is the Cauchy stress tensor with elements $D_{jk}(v) = D_j v_k + D_k v_j$ $(D_j = \partial/\partial x_j)$,

$$
\begin{aligned}
&i^{\text{th}} \text{ component of } \operatorname{Div} S(v,p)\\
&\quad = \sum_{j=1}^n D_j(\mu(D_i v_j + D_j v_i) - \delta_{ij}p) = \mu(\Delta v_i + D_i \operatorname{div} v) - D_i p,\\
&i^{\text{th}} \text{ component of } S(v,p)\boldsymbol{n}_t\\
&\quad = \sum_{j=1}^n \{\mu(D_i v_j + D_j v_i) - \delta_{ij}p\}\boldsymbol{n}_{t,j} = \sum_{j=1}^n \mu(D_i v_j + D_j v_i)\boldsymbol{n}_{t,j} - p\boldsymbol{n}_{t,i},
\end{aligned}
$$

$\operatorname{div} v = \sum_{j=1}^n D_j v_j$, $(v\cdot\nabla)v = \sum_{j=1}^n v_j D_j v$, H is the doubled mean curvature of Γ_t, μ is a positive constant describing viscosity, and c_σ is a positive constant describing the coefficient of surface tension.

The following two problems have been studied by many mathematicians: (1) the motion of an isolated liquid mass and (2) the motion of a viscous incompressible fluid contained in an ocean of infinite extent. In case (1), the initial domain Ω_0 is bounded. A local in time unique existence theorem was proved by Solonnikov [**39**], [**42**], [**44**], [**45**] in the L_2 Sobolev-Slobodetskii space, by Schweizer [**29**] in the semigroup setting, by Moglilevskiĭ and Solonnikov [**22**], [**45**] in the Hölder spaces when $c_\sigma > 0$; and by Solonnikov [**41**] and Mucha and Zajączkowski [**23**] in the L_p Sobolev-Slobodetskii space and by Shibata and Shimizu [**33**], [**34**] in the L_p in time and L_q in space setting when $c_\sigma = 0$. A global in time unique existence theorem for small initial velocity was proved by Solonnikov [**41**] in the L_p Sobolev-Slobodetskii space and by Shibata and Shimizu [**33**], [**34**] in the L_p in time and L_q in space setting when $c_\sigma = 0$; and by Solonnikov [**40**] in the L_2 Sobolev-Slobodetskii space and by Padula and Solonnikov [**25**] in the Hölder spaces under the additional assumption that the initial domain Ω_0 is sufficiently close to a ball when $c_\sigma > 0$. In case (2), the initial domain Ω_0 is a perturbed layer like: $\Omega_0 = \{x \in \boldsymbol{R}^3 \mid -b < x_3 < \eta(x'),\ x' = (x_1,x_2,x_3) \in \boldsymbol{R}^2\}$. A local in time unique existence theorem was proved by Beale [**7**], Allain [**3**] and Tani [**51**] in the L_2 Sobolev-Slobodetskii space when $c_\sigma > 0$ and by Abels [**1**] in the

L_p Sobolev-Slobodetskii space when $c_\sigma = 0$. A global in time unique existence theorem for small initial velocity was proved in the L_2 Sobolev-Slobodetskii space by Beale [**7**] and Tani and Tanaka [**52**] when $c_\sigma > 0$ and by Sylvester [**47**] when $c_\sigma = 0$. The decay rate was studied by Beale and Nishida [**9**], Sylvestre [**48**] and Hataya [**17**].

We remark that in the two phase fluid flow case, the free boundary problem also has been studied by Abels [**2**], Denisova and Solonnikov [**12**], [**13**], [**14**], [**15**], Giga and Takahashi [**18**], Takahashi [**49**], Nouri and Poupaud [**24**], Prüß and Simonett [**26**], [**27**], [**28**], Shibata and Shimizu [**32**], Shimizu [**36**], [**37**], [**38**], Tanaka [**50**] and references therein.

Our purpose is to prove a local in time unique existence theorem of (1.1) in the L_p in time and L_q in space setting and in the case where the initial domain Ω_0 satisfies more general assumptions including the above physical situation. In fact, the L_p in time and L_q in space approach relaxes the regularity assumption and compatibility condition on initial data and the general domain setting allows us to treat several different physical situations at the same time. The core of our approach is to prove the maximal L_p-L_q regularity of the Stokes problems with first order boundary condition in a general domain. Since to achieve our approach is a rather long journey, we decide to divide it into three parts. In this paper, we prove the maximal L_p-L_q regularity of Stokes equations with first order boundary condition in the half-space. And in the forthcoming papers, we shall discuss the same problem in a general domain and the local in time unique existence theorem of free boundary problems of the Navier-Stokes equations in a general domain in the L_p in time and L_q in space setting.

Another issue of this paper is to drive a systematic way to prove the resolvent estimate and the maximal L_p regularity at the same time in the model problem case. On the one hand, we know that the maximal regularity implies the resolvent estimate, but that the opposite direction is not true in general (cf. Kalton and Lancien [**19**]), but on the other hand, using the $\mathscr{R}$ boundedness of the operator family in the sector $\Sigma_{\epsilon,\lambda_0}$, we can prove the resolvent estimate and the maximal L_p regularity at the same time at least in the model problem case.

Now, we formulate our problem in this paper. Let $\boldsymbol{R}^n_+$ and $\boldsymbol{R}^n_0$ be a half-space and its boundary and let $\boldsymbol{Q}_+$ and $\boldsymbol{Q}_0$ be their cylindrical domains. Namely,

$$\boldsymbol{R}^n_+ = \{x = (x_1, \dots, x_n) \in \boldsymbol{R}^n \mid x_n > 0\}, \quad \boldsymbol{R}^n_0 = \{x = (x_1, \dots, x_n) \in \boldsymbol{R}^n \mid x_n = 0\},$$
$$\boldsymbol{Q}_+ = \{(x,t) \mid x \in \boldsymbol{R}^n_+,\ t > 0\} \qquad \boldsymbol{Q}_0 = \{(x,t) \mid x \in \boldsymbol{R}^n_0,\ t > 0\}.$$

Let $\boldsymbol{n} = (0, \dots, 0, -1)$ be the unit outer normal to $\boldsymbol{R}^n_0$. We consider the following four problems:

$$
\begin{aligned}
\lambda u - \operatorname{Div} S(u,\theta) = f, \quad \operatorname{div} u = g \quad & \text{in } \boldsymbol{R}^n_+,\\
S(u,\theta)\boldsymbol{n} = h \quad & \text{on } \boldsymbol{R}^n_0;
\end{aligned} \tag{1.2}
$$

$$
\begin{aligned}
U_t - \operatorname{Div} S(U,\Theta) = F, \quad \operatorname{div} U = G \quad & \text{in } \boldsymbol{Q}_+,\\
S(U,\Theta)\boldsymbol{n} = H \quad & \text{on } \boldsymbol{Q}_0,\\
U|_{t=0} = 0; \quad &
\end{aligned} \tag{1.3}
$$

$$
\begin{aligned}
\lambda u - \operatorname{Div} S(u,\theta) = f, \quad \operatorname{div} u = g \quad & \text{in } \boldsymbol{R}^n_+,\\
\lambda\eta + u_n = d \quad & \text{on } \boldsymbol{R}^n_0,\\
S(u,\theta)\boldsymbol{n} + (c_g - c_\sigma\Delta')\eta\boldsymbol{n} = h \quad & \text{on } \boldsymbol{R}^n_0;
\end{aligned} \tag{1.4}
$$

$$
\begin{aligned}
U_t - \operatorname{Div} S(U,\Theta) = F, \quad \operatorname{div} U = G \quad & \text{in } \boldsymbol{Q}_+,\\
Y_t + U_n = D \quad & \text{in } \boldsymbol{Q}_0,\\
S(U,\Theta)\boldsymbol{n} + (c_g - c_\sigma\Delta')Y\boldsymbol{n} = H \quad & \text{on } \boldsymbol{Q}_0,\\
U|_{t=0} = 0. \quad &
\end{aligned} \tag{1.5}
$$

Here, c_g and c_σ are positive constants; $U = (U_1,\ldots,U_n)$, $u = (u_1,\ldots,u_n)$, Θ, θ, Y and η are unknown functions while $F = (F_1,\ldots,F_n)$, $H = (H_1,\ldots,H_n)$, $f = (f_1,\ldots,f_n)$, $h = (h_1,\ldots,h_n)$, G, g, D and d are given functions; $\Delta'\eta = \sum_{j=1}^{n-1} D_j^2\eta$, and (1.2) and (1.4) are the corresponding generalized resolvent problems to the evolution equations (1.3) and (1.5), respectively.

To state our main results exactly, we introduce several symbols. Given ϵ $(0 < \epsilon < \pi/2)$ and $\gamma_0 \geq 0$, we set

$$\Sigma_{\epsilon,\gamma_0} = \{\lambda \in \boldsymbol{C} \setminus \{0\} \mid |\arg\lambda| \leq \pi - \epsilon,\ |\lambda| \geq \gamma_0\},$$

where $\boldsymbol{C}$ stands for the set of all complex numbers. Given domain G, $L_q(G)$ and $W^m_q(G)$ denote the usual Lebesgue space and Sobolev space while $\|\cdot\|_{L_q(G)}$ and $\|\cdot\|_{W^m_q(G)}$ denote their norms, respectively. For the differentiations of scalar θ and n-vector $u = (u_1,\ldots,u_n)$, we use the following symbol:

$$
\begin{aligned}
\nabla\theta &= (D_1\theta,\ldots,D_n\theta), & \nabla^2\theta &= (D_iD_j\theta \mid i,j = 1,\ldots,n),\\
\nabla u &= (D_iu_j \mid i,j = 1,\ldots,n), & \nabla^2 u &= (D_iD_ju_k \mid i,j,k = 1,\ldots,n).
\end{aligned}
$$

Given Banach space X with norm $\|\cdot\|_X$, X^n denotes the n-product space of X, that is $X^n = \{f = (f_1,\ldots,f_n) \mid f_i \in X\}$. The norm of X^n is also denoted by $\|\cdot\|_X$ for simplicity and

$$\|f\|_X = \sum_{j=1}^n \|f_j\|_X \quad \text{for } f = (f_1, \ldots, f_n) \in X^n.$$

Set

$$\hat{W}^1_q(G) = \bigl\{\theta \in L_{q,\mathrm{loc}}(G) \mid \nabla\theta \in L_q(G)^n\bigr\}, \quad \hat{W}^1_{q,0}(G) = \bigl\{\theta \in \hat{W}^1_q(G) \mid \theta|_{\partial G} = 0\bigr\},$$

where ∂G denotes the boundary of G. Let $\hat{W}^{-1}_q(G)$ denote the dual space of $\hat{W}^1_{q',0}(G)$, where $1/q + 1/q' = 1$. For $\theta \in \hat{W}^{-1}_q(G) \cap L_q(G)$, we have

$$\|\theta\|_{\hat{W}^{-1}_q(G)} = \sup\left\{\left|\int_G \theta\varphi\,dx\right| \mid \varphi \in \hat{W}^1_{q',0}(G),\ \|\nabla\varphi\|_{L_{q'}(G)} = 1\right\}.$$

For $1 \le p \le \infty$, $L_p(\boldsymbol{R}, X)$ and $W^m_p(\boldsymbol{R}, X)$ denote the usual Lebesgue space and Sobolev space of X-valued functions defined on the whole line $\boldsymbol{R}$, and $\|\cdot\|_{L_p(\boldsymbol{R},X)}$ and $\|\cdot\|_{W^m_p(\boldsymbol{R},X)}$ denote their norms, respectively. Set

$$\begin{aligned}
L_{p,0,\gamma_0}(\boldsymbol{R}, X) &= \{f : \boldsymbol{R} \to X \mid e^{-\gamma_0 t} f(t) \in L_p(\boldsymbol{R}, X),\ f(t) = 0 \text{ for } t < 0\},\\
W^m_{p,0,\gamma_0}(\boldsymbol{R}, X) &= \bigl\{f \in L_{p,0,\gamma_0}(\boldsymbol{R}, X) \mid e^{-\gamma_0 t} D^j_t f(t) \in L_p(\boldsymbol{R}, X),\ j = 1, \ldots, m\bigr\},\\
L_{p,0}(\boldsymbol{R}, X) &= L_{p,0,0}(\boldsymbol{R}, X), \quad W^m_{p,0}(\boldsymbol{R}, X) = W^m_{p,0,0}(\boldsymbol{R}, X).
\end{aligned}$$

Let $\mathscr{L}$ and $\mathscr{L}^{-1}_\lambda$ denote the Laplace transform and its inverse, that is

$$\mathscr{L}[f](\lambda) = \int_{-\infty}^{\infty} e^{-\lambda t} f(t)\,dt, \quad \mathscr{L}^{-1}_\lambda[g](t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\lambda t} g(\lambda)\,d\tau,$$

where $\lambda = \gamma + i\tau$. Given $s \in \boldsymbol{R}$ and X-valued function $f(t)$, we set

$$\Lambda^s_\gamma f(t) = \mathscr{L}^{-1}_\lambda[|\lambda|^s \mathscr{L}[f](\lambda)](t).$$

We introduce the Bessel potential space of X valued functions of order s as follows:

$$\begin{aligned}
H^s_{p,0,\gamma_0}(\boldsymbol{R}, X) &= \bigl\{f : \boldsymbol{R} \to X \mid e^{-\gamma t}\Lambda^s_\gamma f(t) \in L_p(\boldsymbol{R}, X)\\
&\qquad\qquad \text{for any } \gamma \ge \gamma_0,\ f(t) = 0 \text{ for } t < 0\bigr\},\\
H^s_{p,0}(\boldsymbol{R}, X) &= H^s_{p,0,0}(\boldsymbol{R}, X).
\end{aligned}$$

The following four theorems are our main results of the paper.

Theorem 1.1. *Let $0 < \epsilon < \pi/2$ and $1 < q < \infty$. Then, for any $\lambda \in \Sigma_{\epsilon,0}$,*

$$f \in L_q\big(\boldsymbol{R}^n_+\big)^n, \quad g \in \hat{W}^{-1}_q\big(\boldsymbol{R}^n_+\big) \cap W^1_q(\boldsymbol{R}^n_+), \quad h \in W^1_q\big(\boldsymbol{R}^n_+\big),$$

problem (1.2) *admits a unique solution $(u, \theta) \in W^2_q(\boldsymbol{R}^n_+)^n \times \hat{W}^1_q(\boldsymbol{R}^n_+)$ that satisfies the estimate*:

$$\begin{aligned}&\|(|\lambda| u, |\lambda|^{1/2}\nabla u, \nabla^2 u, \nabla\theta)\|_{L_q(\boldsymbol{R}^n_+)} \\ &\quad \le C\big\{\|(f, |\lambda|^{1/2} g, \nabla g, |\lambda|^{1/2} h, \nabla h)\|_{L_q(\boldsymbol{R}^n_+)} + |\lambda| \|g\|_{\hat{W}^{-1}_q(\boldsymbol{R}^n_+)}\big\}\end{aligned}$$

for some constant $C = C_{n,q,\epsilon,\mu}$ depending only on n, q, ϵ and μ.

Theorem 1.2. *Let $1 < p, q < \infty$ and $\gamma_0 \ge 0$. Then, for any*

$$\begin{gathered}F \in L_{p,0,\gamma_0}\big(\boldsymbol{R}, L_q(\boldsymbol{R}^n_+)^n\big), \quad G \in L_{p,0,\gamma_0}\big(\boldsymbol{R}, W^1_q(\boldsymbol{R}^n_+)\big) \cap W^1_{p,0,\gamma_0}\big(\boldsymbol{R}, \hat{W}^{-1}_q(\boldsymbol{R}^n_+)\big), \\ H \in L_{p,0,\gamma_0}\big(\boldsymbol{R}, W^1_q(\boldsymbol{R}^n_+)^n\big) \cap H^{1/2}_{p,0,\gamma_0}\big(\boldsymbol{R}, L_q(\boldsymbol{R}^n_+)^n\big),\end{gathered}$$

problem (1.3) *admits a unique solution (U, Θ) such that*

$$U \in \big(L_{p,0,\gamma_0}\big(\boldsymbol{R}, W^2_q(\boldsymbol{R}^n_+)^n\big) \cap W^1_{p,0,\gamma_0}\big(\boldsymbol{R}, L_q(\boldsymbol{R}^n_+)^n\big)\big), \quad \Theta \in L_{p,0,\gamma_0}\big(\boldsymbol{R}, \hat{W}^1_q(\boldsymbol{R}^n_+)\big)$$

that satisfy the estimate:

$$\begin{aligned}&\big\|e^{-\gamma t}(U_t, \gamma U, \Lambda^{1/2}_\gamma \nabla U, \nabla^2 U, \nabla\Theta)\big\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^n_+))} \\ &\quad \le C\big\{\big\|e^{-\gamma t}\big(F, \Lambda^{1/2}_\gamma G, \nabla G, \Lambda^{1/2}_\gamma H, \nabla H\big)\big\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^n_+))} \\ &\qquad + \|e^{-\gamma t}(G_t, \gamma G)\|_{L_p(\boldsymbol{R}, \hat{W}^{-1}_q(\boldsymbol{R}^n_+))}\big\}\end{aligned}$$

for any $\gamma \ge \gamma_0$ with some constant $C = C_{n,p,q,\mu}$ depending only on n, p, q and μ.

Theorem 1.3. *Let $0 < \epsilon < \pi/2$ and $1 < q < \infty$. Then, there exists a constant $\gamma_0 \ge 1$ depending on ϵ such that for any $\lambda \in \Sigma_{\epsilon,\gamma_0}$,*

$$f \in L_q(\boldsymbol{R}^n_+)^n, \quad g \in \hat{W}^{-1}_q(\boldsymbol{R}^n_+) \cap W^1_q(\boldsymbol{R}^n_+), \quad h \in W^1_q(\boldsymbol{R}^n_+), \quad d \in W^2_q(\boldsymbol{R}^n_+),$$

problem (1.4) *admits a unique solution $(u, \theta, \eta) \in W^2_q(\boldsymbol{R}^n_+)^n \times \hat{W}^1_q(\boldsymbol{R}^n_+) \times W^3_q(\boldsymbol{R}^n_+)$ that satisfies the estimate*:

$$\begin{aligned}
&\big\|(|\lambda|u, |\lambda|^{1/2}\nabla u, \nabla^2 u, \nabla\theta, |\lambda|^{1/2}\tilde\theta, \nabla\tilde\theta)\big\|_{L_q(\boldsymbol{R}^n_+)} + |\lambda|\|\eta\|_{W^2_q(\boldsymbol{R}^n_+)} + \|\eta\|_{W^3_q(\boldsymbol{R}^n_+)} \\
&\qquad \le C\big\{\big\|(f, |\lambda|^{1/2}g, \nabla g, |\lambda|^{1/2}h, \nabla h)\big\|_{L_q(\boldsymbol{R}^n_+)} + |\lambda|\|g\|_{\hat W^{-1}_q(\boldsymbol{R}^n_+)} + \|d\|_{W^2_q(\boldsymbol{R}^n_+)}\big\}, \\
&|\lambda|^{3/2}\|\eta\|_{W^1_q(\boldsymbol{R}^n_+)} \le C\big\{\big\|(f, |\lambda|^{1/2}g, \nabla g, |\lambda|^{1/2}h, \nabla h)\big\|_{L_q(\boldsymbol{R}^n_+)} \\
&\qquad\qquad + |\lambda|\|g\|_{\hat W^{-1}_q(\boldsymbol{R}^n_+)} + \|d\|_{W^2_q(\boldsymbol{R}^n_+)} + |\lambda|^{1/2}\|d\|_{W^1_q(\boldsymbol{R}^n_+)}\big\},
\end{aligned}$$

for some constant $C = C_{n,q,\epsilon,\mu}$ depending only on n, q, ϵ and μ.

THEOREM 1.4. *Let $1 < p, q < \infty$. Then, there exists a constant $\gamma_0 \ge 1$ such that for any*

$$\begin{aligned}
&F \in L_{p,0,\gamma_0}\big(\boldsymbol{R}, L_q(\boldsymbol{R}^n_+)^n\big), \quad G \in L_{p,0,\gamma_0}\big(\boldsymbol{R}, W^1_q(\boldsymbol{R}^n_+)\big) \cap W^1_{p,0,\gamma_0}\big(\boldsymbol{R}, \hat W^{-1}_q(\boldsymbol{R}^n_+)\big), \\
&H \in L_{p,0,\gamma_0}\big(\boldsymbol{R}, W^1_q(\boldsymbol{R}^n_+)^n\big) \cap H^{1/2}_{p,0,\gamma_0}\big(\boldsymbol{R}, L_q(\boldsymbol{R}^n_+)^n\big), \quad D \in L_{p,0,\gamma_0}\big(\boldsymbol{R}, W^2_q(\boldsymbol{R}^n_+)\big),
\end{aligned}$$

problem (1.5) *admits a unique solution (U, Θ, Y) such that*

$$\begin{gathered}
U \in L_{p,0,\gamma_0}\big(\boldsymbol{R}, W^2_q(\boldsymbol{R}^n_+)^n\big) \cap W^1_{p,0,\gamma_0}\big(\boldsymbol{R}, L_q(\boldsymbol{R}^n_+)^n\big), \quad \Theta \in L_{p,0,\gamma_0}\big(\boldsymbol{R}, \hat W^1_q(\boldsymbol{R}^n_+)\big), \\
Y \in L_{p,0,\gamma_0}\big(\boldsymbol{R}, W^3_q(\boldsymbol{R}^n_+)\big) \cap W^1_{p,0,\gamma_0}\big(\boldsymbol{R}, W^2_q(\boldsymbol{R}^n_+)\big)
\end{gathered}$$

that satisfy the estimate:

$$\begin{aligned}
&\big\|e^{-\gamma t}(U_t, \gamma U, \Lambda^{1/2}_\gamma \nabla U, \nabla^2 U, \nabla\Theta)\big\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^n_+))} \\
&\quad + \|e^{-\gamma t}(Y_t, \gamma Y)\|_{L_p(\boldsymbol{R}, W^2_q(\boldsymbol{R}^n_+))} + \|e^{-\gamma t} Y\|_{L_p(\boldsymbol{R}, W^3_q(\boldsymbol{R}^n_+))} \\
&\qquad \le C\big\{\big\|e^{-\gamma t}(F, \Lambda^{1/2}_\gamma G, \nabla G, \Lambda^{1/2}_\gamma H, \nabla H)\big\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^n_+))} \\
&\qquad\qquad + \|e^{-\gamma t}(G_t, \gamma G)\|_{L_p(\boldsymbol{R}, \hat W^{-1}_q(\boldsymbol{R}^n_+))} + \|e^{-\gamma t} D\|_{L_p(\boldsymbol{R}, W^2_q(\boldsymbol{R}^n_+))}\big\}
\end{aligned}$$

for any $\gamma \ge \gamma_0$ with some constant $C = C_{n,p,q,\mu}$ depending only on n, p, q and μ.

If we assume that $D \in H^{1/2}_{p,0,\gamma_0}(\boldsymbol{R}, W^1_q(\boldsymbol{R}^n_+))$ in addition, then $Y \in H^{3/2}_{p,0,\gamma_0}(\boldsymbol{R}, W^1_q(\boldsymbol{R}^n_+))$ and

$$
\begin{aligned}
&\|e^{-\gamma t}\Lambda_\gamma^{3/2}Y\|_{L_p(\boldsymbol{R},W^1_q(\boldsymbol{R}^n_+))} \\
&\quad \leq C\big\{\|e^{-\gamma t}(F,\Lambda_\gamma^{1/2}G,\nabla G,\Lambda_\gamma^{1/2}H,\nabla H)\|_{L_p(\boldsymbol{R},L_q(\boldsymbol{R}^n_+))} \\
&\qquad\quad + \|e^{-\gamma t}(G_t,\gamma G)\|_{L_p(\boldsymbol{R},\hat W^{-1}_q(\boldsymbol{R}^n_+))} + \|e^{-\gamma t}D\|_{L_p(\boldsymbol{R},W^2_q(\boldsymbol{R}^n_+)} \\
&\qquad\quad + \|e^{-\gamma t}\Lambda_\gamma^{1/2}D\|_{L_p(\boldsymbol{R},W^1_q(\boldsymbol{R}^n_+))}\big\}.
\end{aligned}
$$

Remark 1.5. The case of non-zero initial values in (1.3) and (1.5) can be treated by the semigroup, whose generation are guaranteed by Theorem 1.1 and Theorem 1.3 with $g=0$ and $h=0$, respectively.

Remark 1.6. We use the Fourier multiplier theorem with respect to time variable, so that it is natural to use Bessel potential spaces with respect to time variable. Especially, $\Lambda_\gamma^{1/2}$ plays an essential role to treat the original nonlinear problem.

Theorem 1.1 and Theorem 1.3 were proved by Shibata and Shimizu [**31**] and [**35**], respectively. Theorem 1.2 was essentially proved by Shibata and Shimizu [**34**]. Theorem 1.4 is only new. But it is the purpose of this paper that we investigate a systematic approach by means of the $\mathscr{R}$ boundedness of the operator family in the sector $\Sigma_{\epsilon,\gamma_0}$ to obtain the both of the generalized resolvent estimate and the maximal L_p-L_q regularity at the same time, and therefore we reprove the results obtained in [**31**], [**35**] and [**34**].

The paper is organized as follows. In Section 2, we introduce $\mathscr{R}$ boundedness, operator valued Fourier multiplier theorem and several results used in later sections. In Section 3, we prove the generalized resolvent estimate and the maximal L_p-L_q regularity in the whole space. In Section 4, we derive solution formulas to problems (1.2) and (1.3). In Section 5, we prepare several technical lemmas used to prove our main results. In Section 6, applying technical lemmas to the solution formulas, we prove Theorems 1.1 and 1.2 at the same time. In Section 7, we derive solution formulas of (1.4) and (1.5), and applying technical lemmas to these formulas, we prove Theorems 1.3 and 1.4 at the same time.

2. $\mathscr{R}$-boundedness and operator valued Fourier multiplier theorem.

Let X and Y be two Banach spaces, and $\|\cdot\|_X$ and $\|\cdot\|_Y$ denote their norms, respectively. Let $\mathscr{L}(X,Y)$ denote the set of all bounded linear operators from X into Y and $\mathscr{L}(X)=\mathscr{L}(X,X)$.

Definition 2.1. A family of operators $\mathscr{T}\subset\mathscr{L}(X,Y)$ is called $\mathscr{R}$-bounded, if there exist constants $C > 0$ and $p \in [1, \infty)$ such that for each $m \in \boldsymbol{N}$, $\boldsymbol{N}$ being the set of all natural numbers, $T_j \in \mathscr{T}$, $x_j \in X$ ($j = 1, \dots, N$) and for all sequences $\{r_j(u)\}_{j=1}^N$ of independent, symmetric, $\{-1, 1\}$-valued random variables on $[0, 1]$, there holds the inequality:

$$\int_0^1 \Big\| \sum_{j=1}^N r_j(u) T_j(x_j) \Big\|_Y^p \, du \le C \int_0^1 \Big\| \sum_{j=1}^N r_j(u) x_j \Big\|_X^p \, du. \tag{2.1}$$

The smallest such C is called $\mathscr{R}$-bound of $\mathscr{T}$, which is denoted by $\mathscr{R}(\mathscr{T})$.

Let $\mathscr{D}(\boldsymbol{R}, X)$ and $\mathscr{S}(\boldsymbol{R}, X)$ be the set of all X valued C^∞ functions having compact supports and the Schwartz space of rapidly decreasing X valued functions, while $\mathscr{D}'(\boldsymbol{R}, X) = \mathscr{L}(\mathscr{D}(\boldsymbol{R}), X)$ and $\mathscr{S}'(\boldsymbol{R}, X) = \mathscr{L}(\mathscr{S}(\boldsymbol{R}), X)$, respectively. Here, $\mathscr{D}(\boldsymbol{R}) = \mathscr{D}(\boldsymbol{R}, \boldsymbol{C})$ and $\mathscr{S}(\boldsymbol{R}) = \mathscr{S}(\boldsymbol{R}, \boldsymbol{C})$.

Definition 2.2. A Banach space X is said to be a UMD Banach space, if the Hilbert transform is bounded on $L_p(\boldsymbol{R}, X)$ for some (and then all) p with $1 < p < \infty$. Here, the Hilbert transform H of a function $f \in \mathscr{S}(\boldsymbol{R}, X)$ is defined by

$$Hf = \frac{1}{\pi} \lim_{\epsilon \to 0+} \int_{|t-s|>\epsilon} \frac{f(s)}{t-s} \, ds \quad (t \in \boldsymbol{R}).$$

Given $M \in L_{1,\mathrm{loc}}(\boldsymbol{R}, \mathscr{L}(X, Y))$, let us define the operator $T_M : \mathscr{F}^{-1}\mathscr{D}(\boldsymbol{R}, X) \to \mathscr{S}'(\boldsymbol{R}, Y)$ by the formula:

$$T_M \phi = \mathscr{F}^{-1}[M \mathscr{F}[\phi]], \quad (\mathscr{F}[\phi] \in \mathscr{D}(\boldsymbol{R}, X)). \tag{2.2}$$

Here and hereafter, $\mathscr{F}$ and $\mathscr{F}^{-1}$ denote the Fourier transform and its inversion formula, that is

$$\mathscr{F}[f](\xi) = \hat{f}(\xi) = \int_{\boldsymbol{R}^n} e^{-ix\cdot\xi} f(x) \, dx,$$
$$\mathscr{F}^{-1}[g](x) = \mathscr{F}_\xi^{-1}[g](x) = \frac{1}{(2\pi)^n} \int_{\boldsymbol{R}^n} e^{ix\cdot\xi} g(\xi) \, d\xi.$$

Theorem 2.3 (Weis [**55**]). *Let X and Y be two UMD Banach spaces and $1 < p < \infty$. Let M be a function in $C^1(\boldsymbol{R} \setminus \{0\}, \mathscr{L}(X, Y))$ such that*

$$\mathscr{R}(\{M(\rho) \mid \rho \in \boldsymbol{R} \setminus \{0\}\}) = \kappa_0 < \infty, \quad \mathscr{R}(\{\rho M'(\rho) \mid \rho \in \boldsymbol{R} \setminus \{0\}\}) = \kappa_1 < \infty.$$

Then, the operator T_M defined in (2.2) *is extended to a bounded linear operator from $L_p(\boldsymbol{R}, X)$ into $L_p(\boldsymbol{R}, Y)$. Moreover, denoting this extension by T_M, we have*

$$\|T_M\|_{\mathscr{L}(L_p(\boldsymbol{R},X),L_p(\boldsymbol{R},Y))} \leq C(\kappa_0 + \kappa_1)$$

for some positive constant C depending on p, X and Y.

THEOREM 2.4 (Bourgain [**10**]). *Let X be a UMD Banach space and $1 < p < \infty$. Let $m(\rho)$ be a scalar function in $C^1(\boldsymbol{R} \setminus \{0\})$ such that*

$$|m(\rho)| \leq M, \quad |\rho m'(\rho)| \leq M \quad (r \in \boldsymbol{R} \setminus \{0\})$$

for some positive constant M. Let T_m be a Fourier multiplier defined by the formula:

$$T_m f = \mathscr{F}^{-1}[m\mathscr{F}[f]] \quad (\mathscr{F}[f] \in \mathscr{D}(\boldsymbol{R}, X)).$$

Then, T_m is extended to a bounded linear operator on $L_p(\boldsymbol{R}, X)$. Moreover, denoting this extension by T_m, we have

$$\|T_m\|_{\mathscr{L}(L_p(\boldsymbol{R},X))} \leq CM$$

for some positive constant C depending on p and X.

REMARK 2.5. Theorem 2.4 was extended to the several variables case by Zimmermann [**56**].

In order to prove the $\mathscr{R}$-boundedness, we use the following two propositions whose proofs were given in Denk-Hieber-Prüß [**11**].

PROPOSITION 2.6. *Let $1 < q < \infty$ and Λ be an index set. Let $\{k_\lambda(x) \mid \lambda \in \Lambda\}$ be a family of functions in $L_{1,\mathrm{loc}}(\boldsymbol{R}^n)$ and let us define the operator K_λ of a function f by the formula*:

$$K_\lambda f(x) = \int_{\boldsymbol{R}^n} k_\lambda(x-y) f(y)\, dy \quad (\lambda \in \Lambda).$$

Assume that there exists a constant M independent of $\lambda \in \Lambda$ such that

$$\|K_\lambda f\|_{L_2(\boldsymbol{R}^n)} \leq M \|f\|_{L_2(\boldsymbol{R}^n)} \quad (f \in L_2(\boldsymbol{R}^n),\ \lambda \in \Lambda), \tag{2.3}$$

$$\sum_{|\alpha|=1} \left|\partial_x^\alpha k_\lambda(x)\right| \leq M|x|^{-(n+1)} \quad (x \in \boldsymbol{R}^n \setminus \{0\},\ \lambda \in \Lambda). \tag{2.4}$$

Then, the set $\{K_\lambda \mid \lambda \in \Lambda\}$ is an $\mathscr{R}$-bounded family in $\mathscr{L}(L_q(\boldsymbol{R}^n))$ and $\mathscr{R}(\{K_\lambda \mid \lambda \in \Lambda\}) \leq C_{n,q}M$ for some constant $C_{n,q}$ depending on n and q.

PROPOSITION 2.7. *Let G be a domain in $\boldsymbol{R}^n$, Λ an index set and $1 < q < \infty$. Let $\{k_\lambda(x,y) \mid \lambda \in \Lambda\}$ be a family of functions in $L_{1,\mathrm{loc}}(G \times G)$ and let us define the operator K_λ of a function f by the formula:*

$$T_\lambda f(x) = \int_G k(x,y)f(y)\,dy \quad (x \in G).$$

Assume that there exists a function $k_0(x,y) \in L_{1,\mathrm{loc}}(G \times G)$ such that

$$|k_\lambda(x,y)| \leq k_0(x,y) \quad ((x,y) \in G \times G,\ \lambda \in \Lambda). \tag{2.5}$$

Let us define the operator K_0 of a function f by the formula:

$$T_0 f(x) = \int_G k(x,y)f(y)\,dy \quad (x \in G,\ \lambda \in \Lambda).$$

If $T_0 \in \mathscr{L}(L_q(G))$, then the set $\{T_\lambda \mid \lambda \in \Lambda\}$ is an $\mathscr{R}$-bounded family in $\mathscr{L}(L_q(G))$ and $\mathscr{R}(\{T_\lambda \mid \lambda \in \Lambda\}) \leq C_{q,G}\|T_0\|_{\mathscr{L}(L_q(G))}$ for some positive constant C_q depending on q.

Finally, we give a theorem used in proving the resolvent estimate and the maximal L_p-L_q regularity.

THEOREM 2.8. *Let $1 < p,\ q < \infty$, $0 < \epsilon < \pi/2$ and $\gamma_0 \geq 0$. Let G be a domain in $\boldsymbol{R}^n$ and Φ_λ be a C^1 function of $\tau \in \boldsymbol{R}\setminus\{0\}$ when $\lambda = \gamma + i\tau \in \Sigma_{\epsilon,\gamma}$ with its value in $\mathscr{L}(L_q(G))$. Assume that the sets $\{\Phi_\lambda \mid \lambda \in \Sigma_{\epsilon,\gamma_0}\}$ and $\{\tau(d/d\tau)\Phi_\lambda \mid \lambda = \gamma + i\tau \in \Sigma_{\epsilon,\gamma_0}\}$ are $\mathscr{R}$-bounded families in $\mathscr{L}(L_q(G))$. In addition, we assume that there exists a constant M such that*

$$\mathscr{R}(\{\Phi_\lambda \mid \lambda \in \Sigma_{\epsilon,\gamma_0}\}) \leq M, \quad \mathscr{R}\left(\left\{\tau\frac{d}{d\tau}\Phi_\lambda \mid \lambda = \gamma + i\tau \in \Sigma_{\epsilon,\gamma_0}\right\}\right) \leq M.$$

Then, we have

$$\|\Phi_\lambda f\|_{L_q(G)} \leq M\|f\|_{L_q(G)} \quad (f \in L_q(G),\ \lambda \in \Sigma_{\epsilon,\gamma_0})$$

for some constant C_q depending on q.

Moreover, if we define the operator Ψ of a function $f \in L_p(\boldsymbol{R}, L_q(G))$ by the formula:

$$\Psi f(x,t) = \mathscr{L}_\lambda^{-1}[\Phi_\lambda \mathscr{L}[f](\lambda)](x,t) = e^{\gamma t}\mathscr{F}_\tau^{-1}[\Phi_\lambda \mathscr{F}[e^{-\gamma t}f](\tau)](t) \quad (\lambda = \gamma + i\tau)$$

where

$$\mathscr{F}[e^{-\gamma t}f](\tau) = \int_{-\infty}^{\infty} e^{-(\gamma+i\tau)t} f(x,t)\,dt,$$

then there exists a constant $C_{p,q}$ depending on p and q such that

$$\|e^{-\gamma t}\Psi f\|_{L_p(\boldsymbol{R},L_q(G))} \le C_{p,q} M \|e^{-\gamma t} f\|_{L_p(\boldsymbol{R},L_q(G))}$$

for any $\gamma \ge \gamma_0$.

Proof. Since the set $\{\Phi_\lambda \mid \lambda \in \Sigma_{\epsilon,\gamma_0}\}$ is an $\mathscr{R}$-bounded family in $\mathscr{L}(L_q(G))$, it is easy to see from the definition of the $\mathscr{R}$ boundedness that the set $\{\Phi_\lambda \mid \lambda \in \Sigma_{\epsilon,\gamma_0}\}$ is a bounded family in $\mathscr{L}(L_q(G))$. Moreover, we have

$$\|\Phi_\lambda f\|_{L_q(G)} \le \mathscr{R}(\{\Phi_\lambda \mid \lambda \in \Sigma_{\epsilon,\gamma_0}\})\|f\|_{L_q(G)} \le M\|f\|_{L_q(G)}.$$

On the other hand, by the assumption we have

$$\mathscr{R}\big(\big\{\Phi_{\gamma+i\tau} \mid \tau \in \boldsymbol{R}\setminus\{0\}\big\}\big) \le M, \quad \mathscr{R}\left(\left\{\tau\frac{d}{d\tau}\Phi_{\gamma+i\tau} \mid \tau \in \boldsymbol{R}\setminus\{0\}\right\}\right) \le M$$

for any $\gamma \ge \gamma_0$, and therefore applying Theorem 2.3 with $X = L_q(G)$ to the formula:

$$e^{-\gamma t}\Psi f(x,t) = \mathscr{F}_\tau^{-1}[\Phi_{\gamma+i\tau}\mathscr{F}[e^{-\gamma t}f](\tau)](t),$$

we have

$$\|e^{-\gamma t}\Psi f\|_{L_p(\boldsymbol{R},L_q(G))} \le C_{p,q} M \|e^{-\gamma t} f\|_{L_p(\boldsymbol{R},L_q(G))}. \qquad \square$$

3. Problems in the whole space.

In this section we consider the generalized resolvent problem and non-stationary Stokes equations in $\boldsymbol{R}^n$ as follows:

$$\lambda u - \mathrm{Div}\, S(u,\theta) = f, \quad \mathrm{div}\, u = g \quad \text{in } \boldsymbol{R}^n, \tag{3.1}$$

$$U_t - \mathrm{Div}\, S(U,\Theta) = F, \ \mathrm{div}\, U = G \quad \text{in } \boldsymbol{R}^n \times (0,\infty), \tag{3.2}$$

subject to the initial condition: $U(x,0) = 0$. We prove the following theorem.

THEOREM 3.1. *Let $1 < p, q < \infty$, $0 < \epsilon < \pi/2$ and $\gamma_0 \geq 0$.*

(1) *For any $\lambda \in \Sigma_{\epsilon,0}$, $f \in L_q(\boldsymbol{R}^n)^n$ and $g \in \hat{W}_q^{-1}(\boldsymbol{R}^n) \cap W_q^1(\boldsymbol{R}^n)$, problem* (3.1) *admits a unique solution $(u,\theta) \in W_q^2(\boldsymbol{R}^n)^n \cap \hat{W}_q^1(\boldsymbol{R}^n)$ that satisfies the following estimate*:

$$\begin{aligned}
&\big\|(|\lambda| u, |\lambda|^{1/2}\nabla u, \nabla^2 u, \nabla\theta)\big\|_{L_q(\boldsymbol{R}^n)} \\
&\quad \leq C_{n,q,\epsilon,\mu}\big\{\big\|(f, |\lambda|^{1/2} g, \nabla g)\big\|_{L_q(\boldsymbol{R}^n)} + |\lambda| \|g\|_{\hat{W}_q^{-1}(\boldsymbol{R}^n)}\big\}.
\end{aligned}$$

(2) *For any $F \in L_{p,0,\gamma_0}(\boldsymbol{R}, L_q(\boldsymbol{R}^n))$ and $G \in L_{p,0,\gamma_0}(\boldsymbol{R}, W_q^1(\boldsymbol{R}^n)) \cap W^1_{p,0,\gamma_0}(\boldsymbol{R}, \hat{W}_q^{-1}(\boldsymbol{R}^n))$, problem* (3.2) *admits a unique solution*

$$(U,\Theta) \in \big(L_{p,0,\gamma_0}(\boldsymbol{R}, W_q^2(\boldsymbol{R}^n)^n) \cap W^1_{p,0,\gamma_0}(\boldsymbol{R}, L_q(\boldsymbol{R}^n)^n)\big) \times L_{p,0,\gamma_0}\big(\boldsymbol{R}, \hat{W}_q^1(\boldsymbol{R}^n)\big)$$

that satisfies the estimate:

$$\begin{aligned}
&\big\|e^{-\gamma t}(U_t, \gamma U, \Lambda_\gamma^{1/2}\nabla U, \nabla^2 U, \nabla\Theta)\big\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^n))} \\
&\quad \leq C_{n,p,q,\mu}\big\{\big\|e^{-\gamma t}(F, \Lambda_\gamma^{1/2} G, \nabla G)\big\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^n))} + \|e^{-\gamma t}(G_t, \gamma G)\|_{L_p(\boldsymbol{R}, \hat{W}_q^{-1}(\boldsymbol{R}^n))}\big\}
\end{aligned}$$

for any $\gamma \geq \gamma_0$.

First, we reduce the problems (3.1) and (3.2) to the case where $g = 0$ and $G = 0$. To do this, we use the following lemma.

LEMMA 3.2. *Let $1 < p, q < \infty$.*

(1) *For any $g \in \hat{W}_q^{-1}(\boldsymbol{R}^n) \cap W_q^1(\boldsymbol{R}^n)$, there exists a $v \in W_q^2(\boldsymbol{R}^n)^n$ such that $\mathrm{div}\, v = g$ in $\boldsymbol{R}^n$ and there hold the estimates*:

$$\begin{aligned}\|v\|_{L_q(\boldsymbol{R}^n)} &\leq C_{n,q}\|g\|_{\hat{W}_q^{-1}(\boldsymbol{R}^n)},\\ \|\nabla v\|_{L_q(\boldsymbol{R}^n)} &\leq C_{n,q}\|g\|_{L_q(\boldsymbol{R}^n)},\\ \|\nabla^2 v\|_{L_q(\boldsymbol{R}^n)} &\leq C_{n,q}\|\nabla g\|_{L_q(\boldsymbol{R}^n)}.\end{aligned} \tag{3.3}$$

(2) *For any* $G \in W^1_{p,0}(\boldsymbol{R}, \hat{W}_q^{-1}(\boldsymbol{R}^n)) \cap L_{p,0}(\boldsymbol{R}, W_q^1(\boldsymbol{R}^n))$, *there exists a* $V \in L_{p,0}(\boldsymbol{R}, W_q^2(\boldsymbol{R}^n)^n)$ *such that* $\operatorname{div} V = G$ *in* $\boldsymbol{R}^n \times \boldsymbol{R}$ *and*

$$\begin{aligned}\|e^{-\gamma t}(V_t, \gamma V)\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^n))} &\leq C_{n,p,q}\|e^{-\gamma t}(G_t, \gamma G)\|_{L_p(\boldsymbol{R}, \hat{W}_q^{-1}(\boldsymbol{R}^n))},\\ \big\|e^{-\gamma t}\Lambda_\gamma^{1/2}\nabla V\big\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^n))} &\leq C_{n,p,q}\big\|e^{-\gamma t}\Lambda_\gamma^{1/2} G\big\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^n))},\\ \|e^{-\gamma t}\nabla^2 V\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^n))} &\leq C_{n,p,q}\|e^{-\gamma t}\nabla G\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^n))}\end{aligned} \tag{3.4}$$

for any $\gamma \geq 0$.

PROOF.

(1) Defining $v_j(x)$ by the formula:

$$v_j(x) = \mathscr{F}_\xi^{-1}\left[\frac{i\xi_j\hat{g}(\xi)}{|\xi|^2}\right](x),$$

and setting $v = (v_1, \ldots, v_n)$, obviously we have $\operatorname{div} v = g$ in $\boldsymbol{R}^n$. Moreover, by the Fourier multiplier theorem of S. G. Mihlin, we have

$$\|\nabla v\|_{L_q(\boldsymbol{R}^n)} \leq C_{n,q}\|g\|_{L_q(\boldsymbol{R}^n)}, \quad \|\nabla^2 v\|_{L_q(\boldsymbol{R}^n)} \leq C_{n,q}\|\nabla g\|_{L_q(\boldsymbol{R}^n)}.$$

In order to estimate $\|v\|_{L_q(\boldsymbol{R}^n)}$, we take any $\varphi \in C_0^\infty(\boldsymbol{R}^n)$ and we consider a vector of function $\Phi(x) = (\Phi_1(x), \ldots, \Phi_n(x))$ defined by the formula:

$$\Phi_j(x) = -\mathscr{F}\left[\frac{i\xi_j\mathscr{F}^{-1}[\varphi](\xi)}{|\xi|^2}\right](x).$$

Setting $(\eta, \zeta)_{\boldsymbol{R}^n} = \int_{\boldsymbol{R}^n}\eta(x)\zeta(x)\,dx$, by the definition we have $(v_j, \varphi)_{\boldsymbol{R}^n} = (g, \Phi_j)_{\boldsymbol{R}^n}$. Since

$$\|\nabla\Phi_j\|_{L_{q'}(\boldsymbol{R}^n)} \leq C_{n,q'}\|\varphi\|_{L_q'(\boldsymbol{R}^n)}$$

as follows from the Fourier multiplier theorem of S. G. Mihlin, we have

$$|(v_j, \varphi)_{\boldsymbol{R}^n}| \leq \|g\|_{\hat{W}_q^{-1}(\boldsymbol{R}^n)} \|\nabla \Phi_j\|_{L_{q'}(\boldsymbol{R}^n)} \leq C_{n,q'} \|g\|_{\hat{W}_q^{-1}(\boldsymbol{R}^n)} \|\varphi\|_{L_{q'}(\boldsymbol{R}^n)},$$

which implies that $\|v\|_{L_q(\boldsymbol{R}^n)} \leq C_{n,q'} \|g\|_{\hat{W}_q^{-1}(\boldsymbol{R}^n)}$. Therefore, we have proved (3.3).

(2) Regarding t as a parameter, defining $V_j(x,t)$ by the formula:

$$V_j(x,t) = \mathscr{F}_\xi^{-1}\left[\frac{i\xi_j \hat{G}(\xi,t)}{|\xi|^2}\right](x),$$

and setting $V = (V_1, \ldots, V_n)$, obviously we have $\operatorname{div} V = G$ in $\boldsymbol{R}^n$ for all $t > 0$. Moreover, $V_j(x,t) = 0$ for $t < 0$ because $G(x,t) = 0$ for $t < 0$ by the assumption. Since

$$\begin{aligned} &\big(D_t V_j(x,t), \gamma V_j(x,t), \Lambda_\gamma^{1/2} V_j(x,t)\big) \\ &\quad = \mathscr{F}_\xi^{-1}\left[\frac{i\xi_j(\hat{G}_t(\xi,t), \gamma\hat{G}(\xi,t), \mathscr{F}[\Lambda_\gamma^{1/2} G](\xi,t))}{|\xi|^2}\right](x), \end{aligned}$$

applying the same argument as in the proof of (1), we have (3.4). This completes the proof of the lemma. □

In view of Lemma 3.2, we set $u = v + w$ and $U = V + W$ in (3.1) and (3.2), respectively, and then setting $\tilde{f} = f - (\lambda v - \mu \operatorname{Div} D(v))$ and $\tilde{F} = F - (V_t - \mu \operatorname{Div} D(V))$, we see that (3.1) and (3.2) are converted to the following equations:

$$\lambda w - \operatorname{Div} S(w, \theta) = \tilde{f}, \quad \operatorname{div} w = 0 \quad \text{in } \boldsymbol{R}^n, \tag{3.5}$$

$$W_t - \operatorname{Div} S(W, \Theta) = \tilde{F}, \ \operatorname{div} W = 0 \quad \text{in } \boldsymbol{R}^n \times (0, \infty), \tag{3.6}$$

subject to the initial condition: $W(x,0) = 0$, respectively. By Lemma 3.2 we have

$$\|\tilde{f}\|_{L_q(\boldsymbol{R}^n)} \leq \|f\|_{L_q(\boldsymbol{R}^n)} + C_{n,q}\big\{|\lambda| \|g\|_{\hat{W}_q^{-1}(\boldsymbol{R}^n)} + \mu \|\nabla g\|_{L_q(\boldsymbol{R}^n)}\big\}, \tag{3.7}$$

$$\begin{aligned} &\|e^{-\gamma t}\tilde{F}\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^n))} \\ &\quad \leq \|e^{-\gamma t} F\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^n))} \\ &\qquad + C_{n,q}\big\{\|e^{-\gamma t} G_t\|_{L_p(\boldsymbol{R}, \hat{W}_q^{-1}(\boldsymbol{R}^n))} + \mu \|e^{-\gamma t} \nabla G\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^n))}\big\}. \end{aligned} \tag{3.8}$$

Since $\operatorname{Div} D(u) = \Delta u$ when $\operatorname{div} u = 0$, in what follows instead of (3.5) and (3.6) we consider the equations:

$$\lambda u - \mu\Delta u + \nabla\theta = f, \quad \operatorname{div} u = 0 \quad \text{in } \boldsymbol{R}^n, \tag{3.9}$$
$$U_t - \mu\Delta U + \nabla\Theta = F, \ \operatorname{div} U = 0 \quad \text{in } \boldsymbol{R}^n \times (0,\infty), \tag{3.10}$$

subject to the initial condition: $U(x,0) = 0$, respectively. By using the Fourier transform, we have the following solution formulas:

$$u(x) = \mathscr{F}_\xi^{-1}\left[\frac{P(\xi)\hat{f}(\xi)}{\lambda+\mu|\xi|^2}\right](x), \qquad \theta(x) = -\mathscr{F}_\xi^{-1}\left[\frac{i\xi\cdot\hat{f}(\xi)}{|\xi|^2}\right](x),$$
$$U(x,t) = \mathscr{L}_\lambda^{-1}\mathscr{F}_\xi^{-1}\left[\frac{P(\xi)\mathscr{L}\mathscr{F}[F](\xi,\lambda)}{\lambda+\mu|\xi|^2}\right](x,t), \quad \Theta(x,t) = -\mathscr{F}_\xi^{-1}\left[\frac{i\xi\cdot\hat{F}(\xi,t)}{|\xi|^2}\right](x),$$

where $P(\xi)$ is an $n\times n$ matrix whose (j,k) component $P_{jk}(\xi)$ is given by the formula: $P_{jk}(\xi) = \delta_{jk} - \xi_j\xi_k|\xi|^{-2}$ and δ_{jk} denote the Kronecker delta symbols defined by the formula: $\delta_{jk} = 1$ when $j = k$ and $\delta_{jk} = 0$ when $j \neq k$. Note that

$$\begin{aligned}
\mathscr{L}\mathscr{F}[F](\xi,\lambda) &= \int_{-\infty}^{\infty}\int_{\boldsymbol{R}^n} e^{-(\lambda t + i\xi\cdot x)}F(x,t)\,dxdt \\
&= \int_{-\infty}^{\infty}\int_{\boldsymbol{R}^n} e^{-i(\tau t+\xi\cdot x)}e^{-\gamma t}F(x,t)\,dxdt \\
&= \mathscr{F}[e^{-\gamma t}F](\xi,\tau) \quad (\lambda = \gamma + i\tau), \\
\mathscr{L}_\lambda^{-1}\mathscr{F}_\xi^{-1}[G(\xi,\lambda)](x,t) &= \frac{1}{(2\pi)^{n+1}}\int_{-\infty}^{\infty}\int_{\boldsymbol{R}^n} e^{\lambda t + i\xi\cdot x}G(\xi,\lambda)\,d\xi d\tau \\
&= \frac{e^{\gamma t}}{(2\pi)^{n+1}}\int_{-\infty}^{\infty}\int_{\boldsymbol{R}^n} e^{i(\tau t+\xi\cdot x)}G(\xi,\lambda)\,d\xi d\tau \\
&= e^{\gamma t}\mathscr{F}^{-1}[G](x,t).
\end{aligned}$$

By a technical reason, instead of u and U we consider

$$u_\epsilon(x) = \mathscr{F}_\xi^{-1}\left[\frac{e^{-\epsilon|\xi|^2}P(\xi)\hat{f}(\xi)}{\lambda+\mu|\xi|^2}\right](x), \quad \theta_\epsilon(x) = -\mathscr{F}_\xi^{-1}\left[\frac{ie^{-\epsilon|\xi|^2}\xi\cdot\hat{f}(\xi)}{|\xi|^2}\right](x),$$
$$U_\epsilon(x,t) = \mathscr{L}_\lambda^{-1}\mathscr{F}_\xi^{-1}\left[\frac{e^{-\epsilon|\xi|^2}P(\xi)\mathscr{L}\mathscr{F}[F](\xi,\lambda)}{\lambda+\mu|\xi|^2}\right](x,t),$$
$$\Theta_\epsilon(x,t) = -\mathscr{F}_\xi^{-1}\left[\frac{ie^{-\epsilon|\xi|^2}\xi\cdot\hat{F}(\xi,t)}{|\xi|^2}\right](x),$$

for $\epsilon > 0$. We see that

$$\begin{aligned} \lambda u_\epsilon - \mu\Delta u_\epsilon + \nabla\theta_\epsilon = f_\epsilon, \ \ \operatorname{div} u_\epsilon = 0 \quad \text{in } \boldsymbol{R}^n, \\ (U_\epsilon)_t - \mu\Delta U_\epsilon + \nabla\Theta_\epsilon = F_\epsilon, \ \operatorname{div} U_\epsilon = 0 \quad \text{in } \boldsymbol{R}^n \times (0,\infty), \end{aligned}$$

where $f_\epsilon(x) = \mathscr{F}_\xi^{-1}[e^{-\epsilon|\xi|^2}\hat f(\xi)](x)$ and $F_\epsilon(x,t) = \mathscr{F}_\xi^{-1}[e^{-\epsilon|\xi|^2}\hat F(\xi,t)](x)$. In order to estimate λu_ϵ and $(U_\epsilon)_t$, we consider a family of kernel functions $k_{\epsilon,\lambda}(x)$ and operators $K_{\epsilon,\lambda}$ defined by the formulas:

$$\begin{aligned} k_{\epsilon,\lambda}(x) &= \mathscr{F}_\xi^{-1}\left[\frac{\lambda e^{-\epsilon|\xi|^2}P(\xi)}{\lambda+\mu|\xi|^2}\right](x), \\ K_{\epsilon,\lambda}[f](x) &= \int_{\boldsymbol{R}^n} k_{\epsilon,\lambda}(x-y)f(y)\,dy = \mathscr{F}_\xi^{-1}\left[\frac{\lambda e^{-\epsilon|\xi|^2}P(\xi)\hat f(\xi)}{\lambda+\mu|\xi|^2}\right](x). \end{aligned}$$

We have

$$\begin{aligned} \lambda u_\epsilon(x) &= K_{\epsilon,\lambda}[f](x), \\ (U_\epsilon)_t(x,t) &= \mathscr{L}_\lambda^{-1}[K_{\epsilon,\lambda}[\mathscr{L}[F](\lambda)]](t) = e^{\gamma t}\mathscr{F}_\tau^{-1}[K_{\epsilon,\gamma+i\tau}[\mathscr{F}[e^{-\gamma t}F](\tau)]](t). \end{aligned}$$

Now, we prove that for any $0 < \sigma < \pi/2$ sets $\{K_{\epsilon,\lambda} \mid \lambda \in \Sigma_{\sigma,0}\}$ and $\{\tau(d/d\tau)K_{\epsilon,\lambda} \mid \lambda \in \Sigma_{\sigma,0}\}$ are $\mathscr{R}$ bounded families in $\mathscr{L}(L_q(\boldsymbol{R}^n))$, whose $\mathscr{R}$ bounds do not exceed some constant C which depends only on σ, μ and n. We start with the following well-known fact.

Lemma 3.3. *Let $0 < \sigma < \pi/2$ and $\lambda \in \Sigma_{\sigma,0}$. Then, $|\lambda + \mu|\xi|^2| \geq \sin(\sigma/2)(|\lambda| + \mu|\xi|^2)$.*

Lemma 3.4. *Let $0 < \sigma < \pi/2$ and $s \in \boldsymbol{R}$. Set $\boldsymbol{N}_0 = \boldsymbol{N} \cup \{0\}$. Then, for any $\lambda \in \Sigma_{\sigma,0}$ and multi-index $\alpha = (\alpha_1,\ldots,\alpha_n) \in \boldsymbol{N}_0^n$, we have*

$$\left|D_\xi^\alpha(\lambda+\mu|\xi|^2)^s\right| \leq C_{\alpha,s,\mu,\sigma}(|\lambda|^{1/2}+|\xi|)^{2s-|\alpha|}.$$

Proof. Setting $f(t) = t^s$, by the Bell formula and Lemma 3.3 we have

$$\begin{aligned} &\left|D_\xi^\alpha(\lambda+\mu|\xi|^2)^s\right| \\ &\quad \leq C_\alpha \sum_{\ell=1}^{|\alpha|}\left|f^{(\ell)}(\lambda+\mu|\xi|^2)\right| \sum_{\substack{\alpha_1+\cdots+\alpha_\ell=\alpha \\ |\alpha_i|\geq 1}} \left|D_\xi^{\alpha_1}(\lambda+\mu|\xi|^2)\right|\cdots\left|D_\xi^{\alpha_\ell}(\lambda+\mu|\xi|^2)\right| \end{aligned}$$

$$\leq C_{\alpha,s,\mu,\sigma}\sum_{\ell=1}^{|\alpha|}(|\lambda|^{1/2}+|\xi|)^{2(s-\ell)}\sum_{k+2(\ell-k)=|\alpha|}|\xi|^k$$
$$\leq C_{\alpha,s,\mu,\sigma}(|\lambda|^{1/2}+|\xi|)^{2s-|\alpha|}$$

where we have used the facts that $2\ell-k=|\alpha|$ and $|\xi|^k \leq (|\lambda|^{1/2}+|\xi|)^k = (|\lambda|^{1/2}+|\xi|)^{2\ell-|\alpha|}$. $\square$

Now, we check the conditions stated in Proposition 2.6. By the Parseval formula, we have

$$\|K_{\epsilon,\lambda}[f]\|_{L_2(\boldsymbol{R}^n)} = \left(\int_{\boldsymbol{R}^n}\left|\frac{\lambda e^{-\epsilon|\xi|^2}P(\xi)\hat{f}(\xi)}{\lambda+\mu|\xi|^2}\right|^2 d\xi\right)^{1/2}$$
$$\leq C_\sigma\|f\|_{L_2(\boldsymbol{R}^n)} \qquad (\lambda\in\Sigma_{\sigma,0}). \tag{3.11}$$

Noting that

$$\tau\frac{\partial}{\partial\tau}\frac{\lambda}{\lambda+\mu|\xi|^2} = \frac{i\mu\tau|\xi|^2}{(\lambda+\mu|\xi|^2)^2},$$

we have also

$$\left\|\tau\frac{\partial}{\partial\tau}K_{\epsilon,\lambda}[f]\right\|_{L_2(\boldsymbol{R}^n)} = \left(\int_{\boldsymbol{R}^n}\left|\frac{i\mu\tau|\xi|^2e^{-\epsilon|\xi|^2}P(\xi)\hat{f}(\xi)}{(\lambda+\mu|\xi|^2)^2}\right|^2 d\xi\right)^{1/2}$$
$$\leq C_\sigma\|f\|_{L_2(\boldsymbol{R}^n)} \qquad (\lambda\in\Sigma_{\sigma,0}), \tag{3.12}$$

where we have used the fact that

$$\left|\frac{i\mu\tau|\xi|^2}{(\lambda+\mu|\xi|^2)^2}\right| \leq C_\sigma\frac{\mu|\lambda||\xi|^2}{(|\lambda|+\mu|\xi|^2)^2} \leq C_\sigma.$$

To continue the estimate, we use the following lemma.

Lemma 3.5. *For any $\epsilon>0$ and multi-index $\alpha\in\boldsymbol{N}_0^n$, there exists a constant C_α independent of ϵ and ξ such that*

$$\left|D_\xi^\alpha e^{-\epsilon|\xi|^2}\right| \leq C_\alpha e^{-(\epsilon/2)|\xi|^2}|\xi|^{-|\alpha|}.$$

PROOF. Setting $f(t) = e^{-\epsilon t}$, by the Bell formula we have

$$\begin{aligned}\bigl|D_\xi^\alpha e^{-\epsilon|\xi|^2}\bigr| &\leq C_\alpha \sum_{\ell=1}^{|\alpha|} |f^{(\ell)}(|\xi|^2)| \sum_{\substack{\alpha_1+\cdots+\alpha_\ell=\alpha \\ |\alpha_i|\geq 1}} \bigl|D_\xi^{\alpha_1}|\xi|^2\bigr| \cdots \bigl|D_\xi^{\alpha_\ell}|\xi|^2\bigr| \\ &\leq C_\alpha \sum_{\ell=1}^{|\alpha|} e^{-\epsilon|\xi|^2}\epsilon^\ell |\xi|^{2\ell-|\alpha|} \leq C_\alpha e^{-(\epsilon/2)|\xi|^2}|\xi|^{-|\alpha|},\end{aligned}$$

where we have used the facts that $|D_\xi^\alpha|\xi|^2| \leq 2|\xi|^{2-|\alpha|}$ and $e^{-\epsilon|\xi|^2}(\epsilon|\xi|^2)^\ell \leq C_\ell e^{-(\epsilon/2)|\xi|^2}$ for some constant C_ℓ independent of ϵ and ξ. This completes the proof of the lemma. □

By Lemma 3.5 and the Leibniz formula, we have

$$\left|D_\xi^\alpha\left(i\xi_j \frac{\lambda e^{-\epsilon|\xi|^2}P(\xi)}{\lambda+\mu|\xi|^2}\right)\right| \leq C_{\alpha,\sigma,\mu}|\xi|^{1-|\alpha|},$$

$$\left|D_\xi^\alpha\left\{\tau\frac{\partial}{\partial\tau}\left(i\xi_j \frac{\lambda e^{-\epsilon|\xi|^2}P(\xi)}{\lambda+\mu|\xi|^2}\right)\right\}\right| \leq C_{\alpha,\sigma,\mu}|\xi|^{1-|\alpha|}$$

for $j = 1,\dots,n$ and $(\lambda,\xi) \in \Sigma_{\sigma,0} \times (\boldsymbol{R}^n \setminus \{0\})$, which combined with Lemma 3.6 below due to Shibata and Shimizu [**30**, Theorem 2.3] implies that

$$|D_j k_{\epsilon,\lambda}(x)| \leq C_{\sigma,\mu}|x|^{-(n+1)}, \quad \left|D_j\left\{\tau\frac{\partial}{\partial\tau}(k_{\epsilon,\lambda}(x)\right\}\right| \leq C_{\sigma,\mu}|x|^{-(n+1)} \tag{3.13}$$

for any $\lambda \in \Sigma_{\sigma,0}$ and $x \in \boldsymbol{R}^n \setminus \{0\}$.

LEMMA 3.6. *Let N and n be a non-negative integer and positive integer, respectively. Let $0 < \sigma \leq 1$ and set $s = N + \sigma - n$. Let $\ell(\sigma)$ be a number defined in such a way that $\ell(\sigma) = 0$ when $0 < \sigma < 1$ and $\ell(\sigma) = 1$ when $\sigma = 1$. Let $f(\xi)$ be a function in $C^{N+\ell(\sigma)+1}(\boldsymbol{R}^n \setminus \{0\})$ which satisfies the following two conditions*:

(1) *$D_\xi^\gamma f \in L_1(\boldsymbol{R}^n)$ for any multi-index $\gamma \in \boldsymbol{N}_0^n$ with $|\gamma| \leq N$.*
(2) *For any multi-index $\gamma \in \boldsymbol{N}_0^n$ with $|\gamma| \leq N + 1 + \ell(\sigma)$ there exists a number C_σ such that*

$$\bigl|D_\xi^\gamma f(\xi)\bigr| \leq C_\sigma|\xi|^{s-|\gamma|} \quad (\xi \in \boldsymbol{R}^n \setminus \{0\}).$$

Then, there exists a constant $C_{n,s}$ depending essentially only on n and α such that

$$\left|\mathscr{F}_\xi^{-1}[f](x)\right| \leq C_{n,\alpha}\Big(\max_{|\gamma|\leq N+1+\ell(\sigma)} C_\gamma\Big)|x|^{-(n+s)} \quad (x \in \boldsymbol{R}^n \setminus \{0\}).$$

In view of (3.12), (3.11) and (3.13) we can apply Proposition 2.6 and therefore, there exists a constant $M_{n,q,\sigma,\mu} > 0$ depending essentially only on n, q, σ and μ such that

$$\begin{aligned} &\mathscr{R}(\{K_{\epsilon,\lambda} \mid \lambda \in \Sigma_{\sigma,0}\}) \leq M_{n,q,\sigma,\mu}, \\ &\mathscr{R}\left(\left\{\tau\frac{\partial}{\partial\tau}K_{\epsilon,\lambda} \mid \lambda \in \Sigma_{\sigma,0}\right\}\right) \leq M_{n,q,\sigma,\mu}, \end{aligned} \tag{3.14}$$

which combined with Theorem 2.8 implies that

$$\|\lambda u_\epsilon\|_{L_q(\boldsymbol{R}^n)} \leq C_{n,q,\sigma,\mu}\|f\|_{L_q(\boldsymbol{R}^n)}, \tag{3.15}$$

$$\|e^{-\gamma t}(U_\epsilon)_t\|_{L_p(\boldsymbol{R},L_q(\boldsymbol{R}^n))} \leq C_{n,p,q,\mu}\|e^{-\gamma t}F\|_{L_p(\boldsymbol{R},L_q(\boldsymbol{R}^n))} \tag{3.16}$$

for any $\gamma \geq 0$.

Now, we discuss the limit process. We have

$$u_\epsilon(x) - u_{\epsilon'}(x) = \mathscr{F}_\xi^{-1}\left[\frac{(e^{-\epsilon|\xi|^2} - e^{-\epsilon'|\xi|^2})P(\xi)}{\lambda+\mu|\xi|^2}\hat{f}(\xi)\right](x).$$

Since

$$e^{-\epsilon|\xi|^2} - e^{-\epsilon'|\xi|^2} = -(\epsilon-\epsilon')|\xi|^2\int_0^1 e^{-(\theta\epsilon+(1-\theta)\epsilon')|\xi|^2}\,d\theta,$$

by Lemma 3.5 and the Leibniz formula we have

$$\left|D_\xi^\alpha(e^{-\epsilon|\xi|^2} - e^{-\epsilon'|\xi|^2})\right| \leq C_\alpha|\epsilon-\epsilon'||\xi|^{2-|\alpha|}e^{-(\min(\epsilon,\epsilon')/2)|\xi|^2},$$

which implies that

$$\left|D_\xi^\alpha\left\{(\tau\partial_\tau)^\ell\left(\frac{(e^{-\epsilon|\xi|^2} - e^{-\epsilon'|\xi|^2})P(\xi)}{\lambda+\mu|\xi|^2}\right)\right\}\right| \leq C_{\alpha,\sigma,\mu}|\epsilon-\epsilon'||\xi|^{-|\alpha|}e^{-(\min(\epsilon,\epsilon')/2)|\xi|^2}$$

for $\ell = 0, 1$. Setting

$$k_{\epsilon,\epsilon',\lambda}(x) = \mathscr{F}_\xi^{-1}\left[\frac{(e^{-\epsilon|\xi|^2} - e^{-\epsilon'|\xi|^2})P(\xi)}{\lambda + \mu|\xi|^2}\right](x),$$
$$K_{\epsilon,\epsilon',\lambda}[f](x) = \int_{\boldsymbol{R}^n} k_{\epsilon,\epsilon',\lambda}(x-y)f(y)\,dy$$

for $\lambda \in \Sigma_{\sigma,0}$, by the same argument as in the proof of (3.14) we have

$$\begin{aligned}
&\mathscr{R}(\{K_{\epsilon,\epsilon',\lambda} \mid \lambda \in \Sigma_{\sigma,0}\}) \leq M_{n,q,\sigma,\mu}|\epsilon - \epsilon'|,\\
&\mathscr{R}\left(\left\{\tau\frac{\partial}{\partial\tau}K_{\epsilon,\epsilon',\lambda} \mid \lambda \in \Sigma_{\sigma,0}\right\}\right) \leq M_{n,q,\sigma,\mu}|\epsilon - \epsilon'|.
\end{aligned} \tag{3.17}$$

Since

$$u_\epsilon - u_{\epsilon'} = K_{\epsilon,\epsilon',\lambda}[f], \quad U_\epsilon - U_{\epsilon'} = e^{\gamma t}\mathscr{F}_\tau^{-1}[K_{\epsilon,\epsilon',\lambda}[\mathscr{F}[e^{-\gamma t}F](\tau)]],$$

combining (3.14) and Theorem 2.8 implies that

$$\begin{aligned}
\|u_\epsilon - u_{\epsilon'}\|_{L_q(\boldsymbol{R}^n)} &\leq C_{n,q,\sigma,\mu}|\epsilon - \epsilon'|\|f\|_{L_q(\boldsymbol{R}^n)},\\
\|e^{-\gamma t}(U_\epsilon - U_{\epsilon'})\|_{L_p(\boldsymbol{R},L_q(\boldsymbol{R}^n))} &\leq C_{n,p,q,\mu}|\epsilon - \epsilon'|\|e^{-\gamma t}F\|_{L_p(\boldsymbol{R},L_q(\boldsymbol{R}^n))}.
\end{aligned}$$

Therefore, $\{u_\epsilon\}_{\epsilon>0}$ and $\{U_\epsilon(x,t)\}_{\epsilon>0}$ are Cauchy sequences in $L_q(\boldsymbol{R}^n)$ and $L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^n))$, respectively, which implies that there exist $u \in L_q(\boldsymbol{R}^n)$ and $U \in L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^n))$ such that

$$\lim_{\epsilon\to 0+}\|u_\epsilon - u\|_{L_q(\boldsymbol{R}^n)} = 0, \quad \lim_{\epsilon\to 0+}\|e^{-\gamma t}(U_\epsilon - U)\|_{L_p(\boldsymbol{R},L_q(\boldsymbol{R}^n))} = 0 \tag{3.18}$$

for any $\gamma \geq 0$, respectively. Combining (3.15) and (3.18) implies that

$$\|\lambda u\|_{L_q(\boldsymbol{R}^n)} \leq C_{n,q,\sigma,\mu}\|f\|_{L_q(\boldsymbol{R}^n)}.$$

On the other hand, to estimate U_t we use the following fact.

THEOREM 3.7 (cf. [**16**]). *Let $1 < p < \infty$, let Ω be a domain in $\boldsymbol{R}^n$ and let X be a reflexive Banach space. Let p' be a conjugate exponent of p, that is $1/p + 1/p' = 1$. Then,*

$$L_p(\Omega, X)^* = L_{p'}(\Omega, X^*), \quad L_p(\Omega, X)^{**} = L_p(\Omega, X),$$

where Y^* stands for the dual space of Y.

Note that $C_0^\infty(\boldsymbol{R}^{n+1})$ is dense in $L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^n))$ when $1 \leq p < \infty$ and $1 \leq q < \infty$. Given any $\varphi \in C_0^\infty(\boldsymbol{R}^{n+1})$, we have

$$(e^{-\gamma t}U_t, \varphi)_{\boldsymbol{R}^{n+1}} = -(U, (e^{-\gamma t}\varphi)_t)_{\boldsymbol{R}^{n+1}} = (U - U_\epsilon + U_\epsilon, (e^{-\gamma t}\varphi)_t)_{\boldsymbol{R}^{n+1}},$$

and therefore

$$\begin{aligned}
&\big|(e^{-\gamma t}U_t, \varphi)_{\boldsymbol{R}^{n+1}}\big| \\
&\quad \leq \|e^{-\gamma t}(U - U_\epsilon)\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^n))}\big(\gamma\|\varphi\|_{L_{p'}(\boldsymbol{R}, L_{q'}(\Omega))} + \|\varphi_t\|_{L_{p'}(\boldsymbol{R}, L_{q'}(\boldsymbol{R}^n))}\big) \\
&\qquad + \|e^{-\gamma t}(U_\epsilon)_t\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^n))}\|\varphi\|_{L_{p'}(\boldsymbol{R}, L_{q'}(\boldsymbol{R}^n))}.
\end{aligned}$$

Letting $\epsilon \to 0+$ and using (3.16) and (3.18), we have

$$|(e^{-\gamma t}U_t, \varphi)_{\boldsymbol{R}^{n+1}}| \leq C_{n,p,q,\mu}\|e^{-\gamma t}F\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^n))}\|\varphi\|_{L_{p'}(\boldsymbol{R}, L_{q'}(\boldsymbol{R}^n))},$$

which combined with Theorem 3.7 implies that

$$\|e^{-\gamma t}U_t\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^n))} \leq C_{n,p,q,\mu}\|e^{-\gamma t}F\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^n))}$$

for any $\gamma \geq 0$.

Analogously, considering that

$$\begin{aligned}
k^1_{\epsilon,\lambda}(x) &= \mathscr{F}_\xi^{-1}\left[\frac{\gamma e^{-\epsilon|\xi|^2}P(\xi)}{\lambda + \mu|\xi|^2}\right](x); \\
k^2_{\epsilon,\lambda}(x) &= \mathscr{F}_\xi^{-1}\left[\frac{|\lambda|^{1/2}e^{-\epsilon|\xi|^2}\xi_j P(\xi)}{\lambda + \mu|\xi|^2}\right](x); \\
k^3_{\epsilon,\lambda}(x) &= \mathscr{F}_\xi^{-1}\left[\frac{\xi_j\xi_k e^{-\epsilon|\xi|^2}P(\xi)}{\lambda + \mu|\xi|^2}\right](x),
\end{aligned}$$

we can show the following estimates:

$$\begin{aligned}
\gamma\|e^{-\gamma t}U\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^n))} &\leq C_{n,p,q,\mu}\|e^{-\gamma t}F\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^n))}; \\
|\lambda|^{1/2}\|D_j u\|_{L_q(\boldsymbol{R}^n)} &\leq C_{n,q,\sigma,\mu}\|f\|_{L_q(\boldsymbol{R}^n)}, \\
\big\|e^{-\gamma t}\Lambda_\gamma^{1/2}D_j U\big\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^n))} &\leq C_{n,p,q,\mu}\|e^{-\gamma t}F\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^n))};
\end{aligned}$$

$$\|D_jD_ku\|_{L_q(\boldsymbol{R}^n)} \leq C_{n,q,\sigma,\mu}\|f\|_{L_q(\boldsymbol{R}^n)},$$
$$\|e^{-\gamma t}D_jD_kU\|_{L_p(\boldsymbol{R},L_q(\boldsymbol{R}^n))} \leq C_{n,p,q,\mu}\|e^{-\gamma t}F\|_{L_p(\boldsymbol{R},L_q(\boldsymbol{R}^n))}$$

for any $\lambda \in \Sigma_{\sigma,0}$ and $\gamma \geq 0$, respectively. Therefore, we have proved the existence of solutions u and U to problems (3.9) and (3.10), which satisfy the estimates:

$$\begin{aligned}
&\big\|(|\lambda|u, |\lambda|^{1/2}\nabla u, \nabla^2 u)\big\|_{L_q(\boldsymbol{R}^n)} \leq C_{n,q,\sigma,\mu}\|f\|_{L_q(\boldsymbol{R}^n)},\\
&\big\|e^{-\gamma t}(U_t, \gamma U, \Lambda_\gamma^{1/2}\nabla U, \nabla^2 U)\big\|_{L_p(\boldsymbol{R},L_q(\boldsymbol{R}^n))}\\
&\qquad\qquad \leq C_{n,p,q,\mu}\|e^{-\gamma t}F\|_{L_p(\boldsymbol{R},L_q(\boldsymbol{R}^n))},
\end{aligned} \tag{3.19}$$

respectively. About the estimates of the pressure terms θ and Θ, we use the estimates (3.19) and the equations (3.9) and (3.10). The uniqueness of solutions follows from the existence of solutions to the dual problem. What $F = 0$ when $t < 0$ implies that U also vanishes when $t < 0$. In fact, we know the estimate

$$\gamma\|e^{-\gamma t}U\|_{L_p(\boldsymbol{R},L_q(\boldsymbol{R}^n))} \leq C_{n,p,q,\mu}\|e^{-\gamma t}F\|_{L_q(\boldsymbol{R},L_q(\boldsymbol{R}^n))} \tag{3.20}$$

for $\gamma \geq \gamma_0$ with some $\gamma_0 > 0$. Since $F = 0$ when $t < 0$, we have

$$\begin{aligned}
\gamma\|U\|_{L_p((-\infty,0),L_q(\boldsymbol{R}^n))} &\leq \gamma\|e^{-\gamma t}U\|_{L_p((-\infty,0),L_q(\boldsymbol{R}^n))} \leq \gamma\|e^{-\gamma t}U\|_{L_p(\boldsymbol{R},L_q(\boldsymbol{R}^n))}\\
&\leq C\|e^{-\gamma t}F\|_{L_p(\boldsymbol{R},L_q(\boldsymbol{R}^n))} = C\|e^{-\gamma t}F\|_{L_p((0,\infty),L_q(\boldsymbol{R}^n))}\\
&\leq C\|e^{-\gamma_0 t}F\|_{L_p((0,\infty),L_q(\boldsymbol{R}^n))}.
\end{aligned}$$

Letting $\gamma \to \infty$, we have $\|U\|_{L_p((-\infty,0),L_q(\boldsymbol{R}^n))} = 0$, which implies that $U = 0$ when $t < 0$. This completes the proof of Theorem 3.1.

4. Solution formula of the model problem without surface tension in the half-space.

In this section we consider the following generalized resolvent problem and non-stationary Stokes equations in $\boldsymbol{R}^n_+$:

$$\lambda u - \mu\Delta u + \nabla\theta = f, \quad \operatorname{div} u = g \quad \text{in } \boldsymbol{R}^n_+, \quad S(u,\theta)\boldsymbol{n} = h \quad \text{on } \boldsymbol{R}^n_0, \tag{4.1}$$
$$U_t - \mu\Delta U + \nabla\Theta = F, \ \operatorname{div} U = G \quad \text{in } \boldsymbol{Q}_+, \quad S(U,\Theta)\boldsymbol{n} = H \quad \text{on } \boldsymbol{Q}_0, \tag{4.2}$$

subject to the initial condition: $U|_{t=0} = 0$, respectively. In order to reduce (4.1) and (4.2) to the case where $g = 0$ and $G = 0$, respectively, we use the following lemma.

Lemma 4.1. *Let $1 < p, q < \infty$ and $\gamma_0 \geq 0$.*

(1) *For any $g \in \hat{W}_q^{-1}(\boldsymbol{R}_+^n) \cap W_q^1(\boldsymbol{R}_+^n)$, there exists a $v \in W_q^2(\boldsymbol{R}_+^n)^n$ such that* $\operatorname{div} v = g$ *in Ω and there hold the estimates*:

$$\begin{aligned} \|v\|_{L_q(\boldsymbol{R}_+^n)} &\leq C_{n,q}\|g\|_{\hat{W}_q^{-1}(\boldsymbol{R}_+^n)}, \\ \|\nabla v\|_{L_q(\boldsymbol{R}_+^n)} &\leq C_{n,q}\|g\|_{L_q(\boldsymbol{R}_+^n)}, \\ \|\nabla^2 v\|_{L_q(\boldsymbol{R}_+^n)} &\leq C_{n,q}\|\nabla g\|_{L_q(\boldsymbol{R}_+^n)}. \end{aligned}$$

(2) *For any $G \in L_{p,0,\gamma_0}(\boldsymbol{R}, W_q^1(\boldsymbol{R}_+^n)) \cap W_{p,0}^1(\boldsymbol{R}, \hat{W}_q^{-1}(\boldsymbol{R}_+^n))$, there exists a V such that*

$$V \in L_{p,0,\gamma_0}\big(\boldsymbol{R}, W_q^2(\boldsymbol{R}_+^n)^n\big) \cap W_{p,0,\gamma_0}^1\big(\boldsymbol{R}, L_q(\boldsymbol{R}_+^n)^n\big)$$

and $\operatorname{div} V = G$ *in $\boldsymbol{Q}_+$. Moreover, for any $\gamma \geq \gamma_0$ there hold the estimates*:

$$\begin{aligned} \|e^{-\gamma t}(V_t, \gamma V)\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}_+^n))} &\leq C_{n,q}\|e^{-\gamma t}(G_t, \gamma G)\|_{L_p(\boldsymbol{R}, \hat{W}_q^{-1}(\boldsymbol{R}_+^n))}, \\ \big\|e^{-\gamma t}\Lambda_\gamma^{1/2}\nabla V\big\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}_+^n))} &\leq C_{n,q}\big\|e^{-\gamma t}\Lambda_\gamma^{1/2} G\big\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}_+^n))}, \\ \|e^{-\gamma t}\nabla^2 V\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}_+^n))} &\leq C_{n,q}\|e^{-\gamma t}\nabla G\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}_+^n))}. \end{aligned}$$

Proof.

(1) Throughout the paper, given function $f(x)$ defined on $\boldsymbol{R}^n$ and $F(x,t)$ defined on $\boldsymbol{R}_+^n \times \boldsymbol{R}$, f^e, F^e and f^o, F^o denote their even and odd extensions, respectively, that is

$$f^e(x) = \begin{cases} f(x) & \text{for } x_n > 0 \\ f(x', -x_n) & \text{for } x_n < 0, \end{cases} \qquad f^o(x) = \begin{cases} f(x) & \text{for } x_n > 0 \\ -f(x', -x_n) & \text{for } x_n < 0, \end{cases}$$

$$F^e(x,t) = \begin{cases} F(x,t) & \text{for } x_n > 0 \\ F(x', -x_n, t) & \text{for } x_n < 0, \end{cases} \qquad F^o(x,t) = \begin{cases} F(x,t) & \text{for } x_n > 0 \\ -F(x', -x_n, t) & \text{for } x_n < 0, \end{cases}$$

where $x' = (x_1, \ldots, x_{n-1})$. Setting

$$v_j(x) = -\mathscr{F}_\xi^{-1}\left[\frac{i\xi_j \mathscr{F}[g^o](\xi)}{|\xi|^2}\right](x), \quad v(x) = (v_1(x), \ldots, v_n(x)),$$

we have $\operatorname{div} v = g^o$ in $\boldsymbol{R}^n$. First, we prove that $\|v\|_{L_q(\boldsymbol{R}^n_+)} \le C_{n,q}\|g\|_{\hat{W}^{-1}_q(\boldsymbol{R}^n_+)}$. From the proof of Lemma 3.2 we see that

$$\|v_j\|_{L_q(\boldsymbol{R}^n)} \le C_{n,q}\|g^o\|_{\hat{W}^{-1}_q(\boldsymbol{R}^n)}, \tag{4.3}$$

while we have

$$\|g^o\|_{\hat{W}^{-1}_q(\boldsymbol{R}^n)} \le 2\|g\|_{\hat{W}^{-1}_q(\boldsymbol{R}^n_+)}. \tag{4.4}$$

In fact, we choose $\varphi \in C_0^\infty(\boldsymbol{R}^n)$ arbitrarily and we observe that

$$(g^o, \varphi)_{\boldsymbol{R}^n} = \int_{\boldsymbol{R}^n_+} g(x)(\varphi(x) - \varphi(x', -x_n))\, dx.$$

Since $\varphi(x) - \varphi(x', -x_n) \in \hat{W}^1_{q',0}(\boldsymbol{R}^n_+)$, we have

$$|(g^o, \varphi)_{\boldsymbol{R}^n}| \le 2\|g\|_{\hat{W}^{-1}_q(\boldsymbol{R}^n_+)}\|\nabla\varphi\|_{L_{q'}(\boldsymbol{R}^n_+)} \le 2\|g\|_{\hat{W}^{-1}_q(\boldsymbol{R}^n_+)}\|\nabla\varphi\|_{L_{q'}(\boldsymbol{R}^n)},$$

which implies (4.4). Combining (4.3) and (4.4) yields that

$$\|v_j\|_{L_q(\boldsymbol{R}^n_+)} \le \|v_j\|_{L_q(\boldsymbol{R}^n)} \le C_{n,q}\|g^o\|_{W^{-1}_q(\boldsymbol{R}^n)} \le 2C_{n,q}\|g\|_{\hat{W}^{-1}_q(\boldsymbol{R}^n_+)}.$$

By the Fourier multiplier theorem of S. G. Mihlin, we have

$$\|\nabla v_j\|_{L_q(\boldsymbol{R}^n)} \le C_{n,q}\|g^o\|_{L_q(\boldsymbol{R}^n)} \le 2C_{n,q}\|g\|_{L_q(\boldsymbol{R}^n_+)}.$$

Since $D_k g^o = (D_k g)^o$ for $k = 1, \ldots, n-1$, we have

$$\|\nabla D_k v_j\|_{L_q(\boldsymbol{R}^n)} \le C_{n,q}\|(D_k g)^o\|_{L_q(\boldsymbol{R}^n)} \le 2C_{n,q}\|D_k g\|_{L_q(\boldsymbol{R}^n_+)} \quad (j = 1, \ldots, n).$$

Moreover, if we write

$$D_n^2 v_k(x) = \mathscr{F}_\xi^{-1}\left[\frac{\xi_n^2 \mathscr{F}[(D_k g)^o](\xi)}{|\xi|^2}\right](x),$$

we have

$$\left\|D_n^2 v_k\right\|_{L_q(\boldsymbol{R}^n)} \le C_{n,q}\|(D_k g)^o\|_{L_q(\boldsymbol{R}^n)} \le 2C_{n,q}\|D_k g\|_{L_q(\boldsymbol{R}^n_+)}. \tag{4.5}$$

Since $\operatorname{div} v = g^0$ in $\boldsymbol{R}^n$, we have $D_n^2 v_n = D_n g - \sum_{k=1}^{n-1} D_n^2 v_k$ in $\boldsymbol{R}_+^n$, which combined with (4.5) yields that

$$\left\| D_n^2 v_n \right\|_{L_q(\boldsymbol{R}_+^n)} \leq C_{n,q} \|\nabla g\|_{L_q(\boldsymbol{R}_+^n)}.$$

Summing up, we have proved the assertion (1).

(2) Defining V_j and V by the formulas:

$$V_j(x,t) = -\mathscr{F}_\xi^{-1}\left[\frac{i\xi_j \mathscr{F}[G^o](\xi,t)]}{|\xi|^2}\right](x), \quad V(x,t) = (V_1(x,t), \dots, V_n(x,t)),$$

regarding t as a parameter and using the same argument as in the proof of the assertion (1), we have the assertion (2). This completes the proof of Lemma 4.1. $\square$

Setting $u = v + w$, $\tilde{f} = f - (\lambda v - \mu \operatorname{Div} D(v))$, $\tilde{h} = h - \mu D(v)\boldsymbol{n}$ in (4.1) and $U = V + W$, $\tilde{F} = F - (V_t - \mu \operatorname{Div} D(V))$, $\tilde{H} = F - \mu D(V)\boldsymbol{n}$ in (4.2), respectively, we have

$$\lambda w - \mu \Delta w + \nabla \theta = \tilde{f}, \quad \operatorname{div} w = 0 \quad \text{in } \boldsymbol{R}_+^n, \quad S(w,\theta)\boldsymbol{n} = \tilde{h} \quad \text{on } \boldsymbol{R}_0^n, \tag{4.6}$$

$$W_t - \mu \Delta W + \nabla \Theta = \tilde{F}, \ \operatorname{div} W = 0 \quad \text{in } \boldsymbol{Q}_+, \quad S(W,\Theta)\boldsymbol{n} = \tilde{H} \quad \text{on } \boldsymbol{Q}_0, \tag{4.7}$$

subject to $W|_{t=0} = 0$. By Lemma 4.1 we have

$$\|\tilde{f}\|_{L_q(\boldsymbol{R}_+^n)} \leq \|f\|_{L_q(\boldsymbol{R}_+^n)} + C_{n,q}\left\{|\lambda| \|g\|_{\hat{W}_q^{-1}(\boldsymbol{R}_+^n)} + \mu \|\nabla g\|_{L_q(\boldsymbol{R}_+^n)}\right\},$$

$$\begin{aligned} &|\lambda|^{1/2} \|\tilde{h}\|_{L_q(\boldsymbol{R}_+^n)} + \|\nabla \tilde{h}\|_{L_q(\boldsymbol{R}_+^n)} \\ &\quad \leq |\lambda|^{1/2} \|h\|_{L_q(\boldsymbol{R}_+^n)} + \|\nabla h\|_{L_q(\boldsymbol{R}_+^n)} + C_{n,q}\left\{|\lambda|^{1/2} \|g\|_{L_q(\boldsymbol{R}_+^n)} + \mu \|\nabla g\|_{L_q(\boldsymbol{R}_+^n)}\right\}, \end{aligned}$$

$$\begin{aligned} &\|e^{-\gamma t} \tilde{F}\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}_+^n))} \\ &\quad \leq \|e^{-\gamma t} F\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}_+^n))} \\ &\qquad + C_{n,p,q}\left\{\|e^{-\gamma t} G_t\|_{L_p(\boldsymbol{R}, \hat{W}_q^{-1}(\boldsymbol{R}_+^n))} + \mu \|e^{-\gamma t} \nabla G\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}_+^n))}\right\}, \end{aligned}$$

$$\begin{aligned} &\left\| e^{-\gamma t}(\Lambda_\gamma^{1/2} \tilde{H}, \nabla \tilde{H}) \right\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}_+^n))} \\ &\quad \leq \left\| e^{-\gamma t}(\Lambda_\gamma^{1/2} H, \nabla H) \right\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}_+^n))} + C_{n,p,q}\mu \left\| e^{-\gamma t}(\Lambda_\gamma^{1/2} G, \nabla G) \right\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}_+^n))}. \end{aligned}$$

Below, we consider (4.1) with $g = 0$ and (4.2) with $G = 0$, respectively. First of all, we reduce these problems to the case where $f = 0$ and $F = 0$. For this purpose, setting $\iota f = (f_1^o, \ldots, f_{n-1}^o, f_n^e)$ and $\iota F = (F_1^o, \ldots, F_{n-1}^o, F_n^e)$, let us define $(v(x), \tau(x))$ and $(V(x,t), \Upsilon(x,t))$ by the formulas:

$$v(x) = \mathscr{F}_\xi^{-1}\left[\frac{P(\xi)\mathscr{F}[\iota f](\xi)}{\lambda + \mu|\xi|^2}\right](x), \quad \tau(x) = -\mathscr{F}_\xi^{-1}\left[\frac{i\xi \cdot \mathscr{F}[\iota f](\xi)}{|\xi|^2}\right](x),$$
$$V(x,t) = \mathscr{L}_\lambda^{-1}\mathscr{F}_\xi^{-1}\left[\frac{P(\xi)\mathscr{L}\mathscr{F}[\iota F](\xi,\lambda)}{\lambda + \mu|\xi|^2}\right](x,t),$$
$$\Upsilon(x,t) = -\mathscr{F}_\xi^{-1}\left[\frac{i\xi \cdot \mathscr{F}[\iota F](\xi,t)}{|\xi|^2}\right](x).$$

Employing the same argument as in Section 3, we have

$$\begin{gathered}(v, \tau) \in W_q^2(\boldsymbol{R}^n)^n \times \hat{W}_q^1(\boldsymbol{R}^n),\\ \lambda v - \mu\Delta v + \nabla\tau = \iota f, \quad \operatorname{div} v = 0 \quad \text{in } \boldsymbol{R}^n,\\ \big\|(|\lambda| v, |\lambda|^{1/2}\nabla v, \nabla^2 v, \nabla\tau)\big\|_{L_q(\boldsymbol{R}^n)}\\ \le C_{n,q,\sigma,\mu}\|\iota f\|_{L_q(\boldsymbol{R}^n)} \le 2C_{n,q,\sigma,\mu}\|f\|_{L_q(\boldsymbol{R}_+^n)} \end{gathered}\tag{4.8}$$

for any $\lambda \in \Sigma_{\sigma,0}$ and $0 < \sigma < \pi/2$. And also, when $F \in L_{p,0}(\boldsymbol{R}, L_q(\boldsymbol{R}_+^n)^n)$ we have

$$\begin{gathered}V \in L_{p,0}\big(\boldsymbol{R}, W_q^2(\boldsymbol{R}^n)^n\big) \cap W_{p,0}^1\big(\boldsymbol{R}, L_q(\boldsymbol{R}^n)\big), \quad \Upsilon \in L_{p,0}\big(\boldsymbol{R}, \hat{W}_q^1(\boldsymbol{R}^n)\big),\\ V_t - \mu\Delta V + \nabla\Upsilon = \iota F, \quad \operatorname{div} V = 0 \quad \text{in } \boldsymbol{R}^n \times (0,\infty), \quad V|_{t=0} = 0,\\ \big\|e^{-\gamma t}(V_t, \gamma V, \Lambda_\gamma^{1/2}\nabla V, \nabla^2 V, \nabla\Upsilon)\big\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^n))}\\ \le C_{n,p,q,\mu}\|e^{-\gamma t}F\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}_+^n))} \end{gathered}\tag{4.9}$$

for any $\gamma \ge 0$, where we have used $\|e^{-\gamma t}\iota F\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^n))} \le 2\|e^{-\gamma t}F\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}_+^n))}$. Moreover, from the definition of ιf and ιF it follows that

$$D_n v_n \,|_{\boldsymbol{R}_0^n} = 0, \;\; \tau \,|_{\boldsymbol{R}_0^n} = 0, \;\; D_n V_n \,|_{\boldsymbol{R}_0^n} = 0, \;\; \Upsilon \,|_{\boldsymbol{R}_0^n} = 0$$

(cf. Shibata and Shimizu [**31**], [**34**]).

Now, setting $u = v + w$, $\theta = \tau + \kappa$, $\tilde{h} = h - \mu D(v)\boldsymbol{n}$ in (4.1) with $g = 0$ and $U = V + W$, $\Theta = \Upsilon + \Xi$, $\tilde{H} = H - \mu D(V)\boldsymbol{n}$ in (4.2) with $G = 0$, respectively, we have

$$\lambda w - \operatorname{Div} S(w,\kappa) = 0, \quad \operatorname{div} w = 0 \quad \text{in } \boldsymbol{R}^n_+, \quad S(w,\kappa)\boldsymbol{n} = \tilde{h} \quad \text{on } \boldsymbol{R}^n_0, \tag{4.10}$$

$$W_t - \operatorname{Div} D(W,\Xi) = 0, \ \operatorname{div} W = 0 \quad \text{in } \boldsymbol{Q}_+, \quad S(W,\Xi)\boldsymbol{n} = \tilde{H} \quad \text{on } \boldsymbol{Q}_0, \tag{4.11}$$

subject to $W|_{t=0} = 0$. By (4.8) and (4.9) we have

$$\begin{aligned}
&\big\|(|\lambda|^{1/2}\tilde{h}, \nabla\tilde{h})\big\|_{L_q(\boldsymbol{R}^n_+)} \leq \big\|(|\lambda|^{1/2}h, \nabla h)\big\|_{L_q(\boldsymbol{R}^n_+)} + C_{n,q,\sigma,\mu}\|f\|_{L_q(\boldsymbol{R}^n_+)}, \\
&\big\|e^{-\gamma t}\big(\Lambda_\gamma^{1/2}\tilde{H}, \nabla\tilde{H}\big)\big\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^n_+))} \\
&\quad \leq \big\|e^{-\gamma t}\big(\Lambda_\gamma^{1/2}H, \nabla H\big)\big\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^n_+))} + C_{p,q,n,\mu}\|e^{-\gamma t}F\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^n_+))}
\end{aligned}$$

for any $\lambda \in \Sigma_{\sigma,0}$ and $\gamma \geq 0$, and $\tilde{H} = 0$ for $t < 0$. Therefore, in what follows we consider (4.1) and (4.2) under the conditions that $f = 0$, $g = 0$ and $F = 0$, $G = 0$, respectively. Since $\operatorname{Div} S(u,\theta) = \mu\Delta u - \nabla\theta$ when $\operatorname{div} u = 0$, in what follows we consider the problems:

$$\begin{aligned}
&\lambda u - \mu\Delta u + \nabla\theta = 0, \quad \operatorname{div} u = 0 && \text{in } \boldsymbol{R}^n_+, \\
&\mu(D_n u_j + D_j u_n) = -h_j \ (j = 1, \ldots, n-1) && \text{on } \boldsymbol{R}^n_0, \\
&2\mu D_n u_n - \theta = -h_n && \text{on } \boldsymbol{R}^n_0,
\end{aligned} \tag{4.12}$$

$$\begin{aligned}
&U_t - \mu\Delta U + \nabla\Theta = 0, \quad \operatorname{div} U = 0 && \text{in } \boldsymbol{Q}_+, \\
&\mu(D_n U_j + D_j U_n) = -H_j \ (j = 1, \ldots, n-1) && \text{on } \boldsymbol{Q}_0, \\
&2\mu D_n U_n - \Theta = -H_n && \text{on } \boldsymbol{Q}_0,
\end{aligned} \tag{4.13}$$

subject to $U|_{t=0} = 0$ under the conditions that $h \in W^1_q(\boldsymbol{R}^n_+)$ and $H \in L_{p,0}(\boldsymbol{R}, W^1_q(\boldsymbol{R}^n_+)) \cap H^{1/2}_{p,0}(\boldsymbol{R}, L_q(\boldsymbol{R}^n_+))$. To get the solution formula to (4.12), we apply the partial Fourier transform with respect to $x' = (x_1, \ldots, x_{n-1})$ that is defined by the formula:

$$\hat{v}(\xi', x_n) = \int_{\boldsymbol{R}^{n-1}} e^{-ix'\cdot\xi'} v(x', x_n)\, dx', \quad \xi' = (\xi_1, \ldots, \xi_{n-1}) \tag{4.14}$$

to (4.12) and therefore, setting

$$A = |\xi'|, \ B = \sqrt{\lambda\mu^{-1} + |\xi'|^2}, \tag{4.15}$$

we have

$$
\begin{aligned}
\lambda\hat u_j(\xi',x_n)+\mu A^2\hat u_j(\xi',x_n)-\mu D_n^2\hat u_j(\xi',x_n)+i\xi_j\hat\theta(\xi',x_n)&=0 \qquad (x_n>0),\\
\lambda\hat u_n(\xi',x_n)+\mu A^2\hat u_n(\xi',x_n)-\mu D_n^2\hat u_n(\xi',x_n)+D_n\hat\theta(\xi',x_n)&=0 \qquad (x_n>0),\\
\sum_{j=1}^{n-1} i\xi_j\hat u_j(\xi',x_n)+D_n\hat u_n(\xi',x_n)&=0 \qquad (x_n>0),\\
\mu\big(D_n\hat u_j(\xi',0)+i\xi_j\hat u_n(\xi',0)\big)&=-\hat h_j(\xi',0),\\
2\mu D_n\hat h_n(\xi',0)-\hat\theta(\xi',0)&=-\hat h_n(\xi',0),
\end{aligned}
\tag{4.16}
$$

where j runs through 1 to $n-1$. Setting

$$
\hat u_j(\xi',x_n)=\alpha_j e^{-Ax_n}+\beta_j e^{-Bx_n},\quad \hat\theta(\xi',x_n)=\gamma e^{-Ax_n}
$$

and inserting these formulas into (4.16), we have

$$
\begin{gathered}
\mu\alpha_j(B^2-A^2)+i\xi_j\gamma=0,\qquad \mu\alpha_n(B^2-A^2)-A\gamma=0,\\
\sum_{k=1}^{n-1} i\xi_k\alpha_k-A\alpha_n=0,\qquad \sum_{k=1}^{n-1} i\xi_k\beta_k-B\beta_n=0,\\
\mu(A\alpha_j+B\beta_j+i\xi_j(\alpha_n+\beta_n))=\hat h_j(\xi',0),\quad 2\mu(A\alpha_n+B\beta_n)+\gamma=\hat h_n(\xi',0),
\end{gathered}
\tag{4.17}
$$

where j runs through 1 to $n-1$. Solving (4.17) and setting

$$
D(A,B)=B^3+AB^2+3A^2B-A^3, \tag{4.18}
$$

we have

$$
\begin{aligned}
\alpha_j&=\frac{i\xi_j}{\mu(B-A)D(A,B)}\Big\{2iB\sum_{k=1}^{n-1}\xi_k\hat h_k(\xi',0)-(A^2+B^2)\hat h_n(\xi',0)\Big\},\\
\beta_j&=\frac{i\xi_j}{\mu B(B-A)D(A,B)}\Big\{(A^2+B^2-4AB)\sum_{k=1}^{n-1} i\xi_k\hat h_k(\xi',0)+2AB^2\hat h_n(\xi',0)\Big\}\\
&\quad+\frac{1}{\mu B}\hat h_j(\xi',0),\\
\alpha_n&=-\frac{1}{\mu(B-A)D(A,B)}\Big\{2AB\sum_{k=1}^{n-1} i\xi_k\hat h_k(\xi',0)-(A^2+B^2)A\hat h_n(\xi',0)\Big\},
\end{aligned}
$$

$$\beta_n = \frac{1}{\mu(B-A)D(A,B)}\Big\{(A^2+B^2)\sum_{k=1}^{n-1} i\xi_k\hat{h}_k(\xi',0) - 2A^3\hat{h}_n(\xi',0)\Big\},$$

$$\gamma = -\frac{A+B}{D(A,B)}\Big\{2B\sum_{k=1}^{n-1} i\xi_k h_k(\xi',0) - (A^2+B^2)\hat{h}_n(\xi',0)\Big\},$$

where j runs through 1 to $n-1$. Therefore, setting

$$\mathscr{M}(A,B,x_n) = \frac{e^{-Bx_n}-e^{-Ax_n}}{B-A}, \tag{4.19}$$

we have

$$\begin{aligned}
u_j(x) = &-\mathscr{F}_{\xi'}^{-1}\Big[\frac{2i\xi_j A}{\mu D(A,B)}\mathscr{M}(A,B,x_n)\big(i\xi'\cdot\hat{h}'(\xi',0) - B\hat{h}_n(\xi',0)\big)\Big](x') \\
&-\mathscr{F}_{\xi'}^{-1}\Big[\frac{e^{-Bx_n}(B-A)\xi_j}{\mu BD(A,B)}\xi'\cdot\hat{h}'(\xi',0)\Big](x') \\
&+\mathscr{F}_{\xi'}^{-1}\Big[\frac{2i\xi_j e^{-Ax_n}}{\mu D(A,B)}\big(i\xi'\cdot\hat{h}'(\xi',0) + (B-A)\hat{h}_n(\xi',0)\big)\Big](x') \\
&+\mathscr{F}_{\xi'}^{-1}\Big[\frac{e^{-Bx_n}}{\mu B}\hat{h}_j(\xi',0)\Big](x') \quad (j=1,\ldots,n-1),
\end{aligned} \tag{4.20}$$

$$\begin{aligned}
u_n(x) = &\mathscr{F}_{\xi'}^{-1}\Big[\frac{A}{\mu D(A,B)}\mathscr{M}(A,B,x_n)\big(2Bi\xi'\cdot\hat{h}'(\xi',0) - (A^2+B^2)\hat{h}_n(\xi',0)\big)\Big](x') \\
&+\mathscr{F}_{\xi'}^{-1}\Big[\frac{e^{-Bx_n}}{\mu D(A,B)}\big((B-A)i\xi'\cdot\hat{h}'(\xi',0) + A(A+B)\hat{h}_n(\xi',0)\big)\Big](x'),
\end{aligned} \tag{4.21}$$

$$\theta(x) = -\mathscr{F}_{\xi'}^{-1}\Big[\frac{(A+B)e^{-Ax_n}}{D(A,B)}\big(2Bi\xi'\cdot\hat{h}'(\xi',0) - (A^2+B^2)\hat{h}_n(\xi',0)\big)\Big](x'), \tag{4.22}$$

where we have set $\xi'\cdot\hat{h}' = \sum_{k=1}^{n-1}\xi_k\hat{h}_k$ for the notational simplicity and $\mathscr{F}_{\xi'}^{-1}$ denotes the Fourier inverse transform with respect to $\xi' = (\xi_1,\ldots,\xi_{n-1})$, that is

$$\mathscr{F}_{\xi'}^{-1}[g(\xi')](x') = \frac{1}{(2\pi)^{n-1}}\int_{\boldsymbol{R}^{n-1}} e^{ix'\cdot\xi'} g(\xi')\,d\xi'.$$

Using the partial Fourier transform with respect to x' and the Laplace transform with respect to t, we have the following solution formula to (4.13) as follows:

$$\begin{aligned}
U_j(x,t) = &-\mathscr{L}_\lambda^{-1}\mathscr{F}_{\xi'}^{-1}\Big[\frac{2i\xi_j A\mathscr{M}(A,B,x_n)}{\mu D(A,B)}\big(i\xi'\cdot\mathscr{L}\mathscr{F}[H'](\xi',0,\lambda) \\
&\qquad\qquad - B\mathscr{L}\mathscr{F}[H_n](\xi',0,\lambda)\big)\Big](x',t) \\
&-\mathscr{L}_\lambda^{-1}\mathscr{F}_{\xi'}^{-1}\Big[\frac{e^{-Bx_n}(B-A)\xi_j}{\mu BD(A,B)}\xi'\cdot\mathscr{L}\mathscr{F}[H'](\xi',0,\lambda)\Big](x',t) \\
&+\mathscr{L}_\lambda^{-1}\mathscr{F}_{\xi'}^{-1}\Big[\frac{2i\xi_j e^{-Ax_n}}{\mu D(A,B)}\big(i\xi'\cdot\mathscr{L}\mathscr{F}[H'](\xi',0,\lambda) \\
&\qquad\qquad + (B-A)\mathscr{L}\mathscr{F}[H_n](\xi',0,\lambda)\big)\Big](x',t) \\
&+\mathscr{L}_\lambda^{-1}\mathscr{F}_{\xi'}^{-1}\Big[\frac{e^{-Bx_n}}{\mu B}\mathscr{L}\mathscr{F}[H_j](\xi',0,\lambda)\Big](x',t) \quad (j=1,\ldots,n-1),
\end{aligned} \tag{4.23}$$

$$\begin{aligned}
U_n(x,t) = &\mathscr{L}_\lambda^{-1}\mathscr{F}_{\xi'}^{-1}\Big[\frac{A\mathscr{M}(A,B,x_n)}{\mu D(A,B)}\big(2Bi\xi'\cdot\mathscr{L}\mathscr{F}[H'](\xi',0,\lambda) \\
&\qquad\qquad - (A^2+B^2)\mathscr{L}\mathscr{F}[H_n](\xi',0,\lambda)\big)\Big](x',t) \\
&+\mathscr{L}_\lambda^{-1}\mathscr{F}_{\xi'}^{-1}\Big[\frac{e^{-Bx_n}}{\mu D(A,B)}\big((B-A)i\xi'\cdot\mathscr{L}\mathscr{F}[H'](\xi',0,\lambda) \\
&\qquad\qquad + A(A+B)\mathscr{L}\mathscr{F}[H_n](\xi',0,\lambda)\big)\Big](x',t),
\end{aligned} \tag{4.24}$$

$$\begin{aligned}
\Theta(x,t) = &-\mathscr{L}_\lambda^{-1}\mathscr{F}_{\xi'}^{-1}\Big[\frac{(A+B)e^{-Ax_n}}{D(A,B)}\big(2Bi\xi'\cdot\mathscr{L}\mathscr{F}[H'](\xi',0,\lambda) \\
&\qquad\qquad - (A^2+B^2)\mathscr{L}\mathscr{F}[H_n](\xi',0,\lambda)\big)\Big](x',t),
\end{aligned} \tag{4.25}$$

where we have set

$$\begin{aligned}
\mathscr{L}\mathscr{F}[H](\xi',x_n,\lambda) &= \int_{\boldsymbol{R}^n} e^{-\lambda t+i\xi'\cdot x'}H(x',x_n,t)\,dtdx' \\
&= \mathscr{F}_t\mathscr{F}_{x'}[e^{-\gamma t}H(\cdot,x_n,\cdot)](\xi',\tau) \quad (\lambda=\gamma+i\tau), \\
\mathscr{L}_\lambda^{-1}\mathscr{F}_{\xi'}^{-1}[G(\xi,x_n,\lambda)](x',t) &= \frac{1}{(2\pi)^n}\int_{\boldsymbol{R}^n} e^{\lambda t+i\xi'\cdot x'}G(\xi',x_n,\lambda)\,d\tau d\xi' \\
&= e^{\gamma t}\mathscr{F}_\tau^{-1}\mathscr{F}_{\xi'}^{-1}[G(\xi',x_n,\gamma+i\tau)](x',t).
\end{aligned}$$

5. Technical lemmas.

In this section, we show several estimates of Fourier multipliers, which will be used to estimate solution formulas obtained in Section 4. First of all, we introduce two classes of multipliers. Let $0 < \epsilon < \pi/2$ and $\gamma_0 \geq 0$. Let $m(\lambda, \xi')$ be a function defined on $\Sigma_{\epsilon,\gamma_0}$ which is infinitely many times differentiable with respect to τ and ξ' when $\lambda = \gamma + i\tau \in \Sigma_{\epsilon,\gamma_0}$ and $\xi' \in \boldsymbol{R}^{n-1} \setminus \{0\}$. If there exists a real number s such that for any multi-index $\alpha' = (\alpha_1, \ldots, \alpha_{n-1}) \in \boldsymbol{N}_0^{n-1}$ and $(\lambda, \xi') \in \Sigma_{\epsilon,\gamma_0} \times (\boldsymbol{R}^{n-1} \setminus \{0\})$ there hold the estimates:

$$\begin{aligned}\left|D_{\xi'}^{\alpha'} m(\lambda, \xi')\right| &\leq C_{\alpha',\epsilon,\gamma_0,\mu}(|\lambda|^{1/2} + A)^{s-|\alpha|},\\ \left|D_{\xi'}^{\alpha'}\left(\tau\frac{\partial m}{\partial \tau}(\lambda, \xi')\right)\right| &\leq C_{\alpha',\epsilon,\gamma_0,\mu}(|\lambda|^{1/2} + A)^{s-|\alpha|}\end{aligned} \tag{5.1}$$

for some constant $C_{\alpha',\epsilon,\gamma_0,\mu}$ depending on α', ϵ, γ_0 and μ only, then $m(\lambda, \xi')$ is called a multiplier of order s with type 1. If there exists a real number s such that for any multi-index $\alpha' = (\alpha_1, \ldots, \alpha_{n-1}) \in \boldsymbol{N}_0^{n-1}$ and $(\lambda, \xi') \in \Sigma_{\epsilon,\gamma_0} \times (\boldsymbol{R}^{n-1} \setminus \{0\})$ there hold the estimates:

$$\begin{aligned}\left|D_{\xi'}^{\alpha'} m(\lambda, \xi')\right| &\leq C_{\alpha',\epsilon,\gamma_0,\mu}(|\lambda|^{1/2} + A)^{s} A^{-|\alpha|},\\ \left|D_{\xi'}^{\alpha'}\left(\tau\frac{\partial m}{\partial \tau}(\lambda, \xi')\right)\right| &\leq C_{\alpha',\epsilon,\gamma_0,\mu}(|\lambda|^{1/2} + A)^{s} A^{-|\alpha|}\end{aligned} \tag{5.2}$$

for some constant $C_{\alpha',\epsilon,\gamma_0,\mu}$ depending on α', ϵ, γ_0 and μ only, then $m(\lambda, \xi')$ is called a multiplier of order s with type 2. In what follows, we denote the set of all multipliers defined on $\Sigma_{\epsilon,\gamma_0} \times (\boldsymbol{R}^{n-1} \setminus \{0\})$ of order s with type ℓ ($\ell = 1, 2$) by $\boldsymbol{M}_{s,\ell,\epsilon,\gamma_0}$. For example, the Riesz kernel $\xi_j/|\xi'|$ belongs to $\boldsymbol{M}_{0,2,\epsilon,0}$ ($j = 1, \ldots, n-1$). A function $|\lambda|^s = (\gamma^2 + \tau^2)^{s/2}$ belongs to $\boldsymbol{M}_{2s,1,\epsilon,0}$ when $s \geq 0$. A function $\lambda|\lambda|^{-1/2}$ belongs to $\boldsymbol{M}_{1,1,\epsilon,\gamma_0}$. The following lemma follows from the definition of $\boldsymbol{M}_{s,\ell,\epsilon,\gamma_0}$ and the Leibniz rule.

Lemma 5.1. *Let s_1, $s_2 \in \boldsymbol{R}$.*

(1) *Given $m_i \in \boldsymbol{M}_{s_i,1,\epsilon,\gamma_0}$ $(i = 1, 2)$, we have $m_1 m_2 \in \boldsymbol{M}_{s_1+s_2,1,\epsilon,\gamma_0}$.*
(2) *Given $\ell_i \in \boldsymbol{M}_{s_i,i,\epsilon,\gamma_0}$ $(i = 1, 2)$, we have $\ell_1 \ell_2 \in \boldsymbol{M}_{s_1+s_2,2,\epsilon,\gamma_0}$.*
(3) *Given $n_i \in \boldsymbol{M}_{s_i,2,\epsilon,\gamma_0}$ $(i = 1, 2)$, we have $n_1 n_2 \in \boldsymbol{M}_{s_1+s_2,2,\epsilon,\gamma_0}$.*

From now on, we show several lemmas which will be used to estimate solution formulas given in Section 4.

Lemma 5.2. *Let $s \in \boldsymbol{R}$ and $0 < \epsilon < \pi/2$. Let A, B and $D(A, B)$ be symbols defined in* (4.15) *and* (4.18)*, respectively. Then, there exists a positive constant c depending on ϵ and μ such that*

$$c(|\lambda|^{1/2} + A) \leq \operatorname{Re} B \leq |B| \leq (\mu^{-1}|\lambda|)^{1/2} + A, \tag{5.3}$$

$$c(|\lambda|^{1/2} + A)^3 \leq |D(A, B)| \leq 6((\mu^{-1}|\lambda|)^{1/2} + A)^3. \tag{5.4}$$

Moreover, we have $B^s \in \boldsymbol{M}_{s,1,\epsilon,0}$, $(A + B)^s \in \boldsymbol{M}_{s,2,\epsilon,0}$ and $D(A, B)^s \in \boldsymbol{M}_{3s,2,\epsilon,0}$ for any $s \in \boldsymbol{R}$. If $s \geq 0$, then $A^s \in \boldsymbol{M}_{s,2,\epsilon,0}$.

Proof. The inequalities (5.3) and (5.4) were proved in Shibata and Shimizu [**31**, Lemma 4.4]. Now, we prove that $B^s \in \boldsymbol{M}_{s,1,\epsilon,0}$. Employing the same argument as in the proof of Lemma 3.4, we have

$$\left| D_{\xi'}^{\alpha'} B^s \right| \leq C_{\alpha',\epsilon,\mu,s} (|\lambda|^{1/2} + A)^{s - |\alpha'|}.$$

Using the formula: $\tau \partial_\tau B^s = i(2\mu)^{-1} s \tau (\lambda \mu^{-1} + A^2)^{s/2-1}$, we have

$$\begin{aligned} \left| D_{\xi'}^{\alpha'} \left(\tau \frac{\partial B^s}{\partial \tau} \right) \right| &\leq C_{\alpha',\epsilon,\mu,s} |\tau| (|\lambda|^{1/2} + A)^{s-2-|\alpha'|} \\ &\leq C_{\alpha',\epsilon,\mu,s} \frac{|\lambda|}{(|\lambda|^{1/2} + A)^2} (|\lambda|^{1/2} + A)^{s-|\alpha'|} \\ &\leq C_{\alpha',\epsilon,\mu,s} (|\lambda|^{1/2} + A)^{s-|\alpha'|}. \end{aligned}$$

Combining these estimates implies that $(|\lambda|^{1/2} + A)^s \in \boldsymbol{M}_{s,1,\epsilon,0}$.

By the Bell formula, we have also

$$\left| D_{\xi'}^{\alpha'} A^s \right| \leq C_{\alpha',s} A^{s-|\alpha'|}. \tag{5.5}$$

Since $A^s \leq (|\lambda|^{1/2} + A)^s$ when $s \geq 0$, it follows from (5.5) that $A^s \in \boldsymbol{M}_{s,2,\epsilon,0}$ when $s \geq 0$.

Setting $f(t) = t^s$ for $t > 0$, by the Bell formula we have

$$\left| D_{\xi'}^{\alpha'} (A + B)^s \right| \leq C_{\alpha'} \sum_{\ell=1}^{|\alpha'|} |f^{(\ell)}(A + B)| \sum_{\substack{\alpha'_1 + \cdots + \alpha'_\ell = \alpha' \\ |\alpha'_i| \geq 1}} \left| D_{\xi'}^{\alpha'_1} (A + B) \right| \cdots \left| D_{\xi'}^{\alpha'_\ell} (A + B) \right|$$

$$\leq C_{\alpha',\epsilon,\mu,s}\sum_{\ell=1}^{|\alpha'|}(|\lambda|^{1/2}+A)^{s-\ell}(|\lambda|^{1/2}+A)^{\ell}A^{-|\alpha'|}$$
$$\leq C_{\alpha',\epsilon,\mu,s}(|\lambda|^{1/2}+A)^{s}A^{-|\alpha'|},$$

where we have used (5.5) and

$$\left|D_{\xi'}^{\alpha'}B\right| \leq C_{\alpha',\epsilon,\mu}(|\lambda|^{1/2}+A)^{1-|\alpha'|} \leq C_{\alpha',\epsilon,\mu,s}(|\lambda|^{1/2}+A)A^{-|\alpha'|},$$
$$c(|\lambda|^{1/2}+A) \leq \operatorname{Re} B \leq \operatorname{Re}(A+B) \leq A+|B| \leq 2\big((|\lambda|\mu^{-1})^{1/2}+A\big).$$

Since $\tau\partial_\tau(A+B)^s = 2^{-1}s(A+B)^{s-1}\tau\mu^{-1}(\lambda\mu^{-1}+A-2)^{-1/2}$, by the Leibniz rule we have

$$\left|D_{\xi'}^{\alpha'}\left(\tau\frac{\partial(A+B)^s}{\partial\tau}\right)\right|$$
$$\leq \frac{|s|}{2}|\tau|\sum_{\beta'+\gamma'=\alpha'}\frac{\alpha'!}{\beta'!\gamma'!}\left|D_{\xi'}^{\beta'}(A+B)^{s-1}\right|\left|D_{\xi'}^{\gamma'}B^{-1}\right|$$
$$\leq C_{\alpha',\epsilon,\mu,s}|\lambda|\sum_{\beta'+\gamma'=\alpha'}(|\lambda|^{1/2}+A)^{s-1}A^{-|\beta'|}(|\lambda|^{1/2}+A)^{-1}A^{-|\gamma'|}$$
$$\leq C_{\alpha',\epsilon,\mu,s}|\lambda|(|\lambda|^{1/2}+A)^{-2}(|\lambda|^{1/2}+A)^{s}A^{-|\alpha'|} \leq C_{\alpha',\epsilon,\mu,s}(|\lambda|^{1/2}+A)^{s}A^{-|\alpha'|}.$$

Combining these estimates implies that $(A+B)^s \in \boldsymbol{M}_{s,2,\epsilon,0}$.

Since $A,\ B \in \boldsymbol{M}_{1,2,\epsilon,0}$ and $D(A,B)$ is a cubic polynomial with respect to A and B, by Lemma 5.1 we have $D(A,B) \in \boldsymbol{M}_{3,2,\epsilon,0}$. Setting $f(t)=t^s$ for $t>0$, by the Bell formula and (5.4) we have

$$\left|D_{\xi'}^{\alpha'}D(A,B)^s\right|$$
$$\leq C_{\alpha'}\sum_{\ell=1}^{|\alpha'|}|f^{(\ell)}(D(A,B))|\sum_{\substack{\alpha'_1+\cdots+\alpha'_\ell=\alpha'\\ |\alpha'_i|\geq 1}}\left|D_{\xi'}^{\alpha'_1}D(A,B)\right|\cdots\left|D_{\xi'}^{\alpha'_\ell}D(A,B)\right|$$
$$\leq C_{\alpha',\epsilon,\mu,s}\sum_{\ell=1}^{|\alpha'|}(|\lambda|^{1/2}+A)^{3(s-\ell)}\sum_{\substack{\alpha'_1+\cdots+\alpha'_\ell=\alpha'\\ |\alpha'_i|\geq 1}}(|\lambda|^{1/2}+A)^{3}A^{-|\alpha'_1|}\cdots(|\lambda|^{1/2}+A)^{3}A^{-|\alpha'_\ell|}$$
$$\leq C_{\alpha',\epsilon,\mu,s}(|\lambda|^{1/2}+A)^{3s}A^{-|\alpha'|}.$$

We have $\tau\partial_\tau D(A,B)^s = s\tau D(A,B)^{s-1}E(A,B)$ with $E(A,B) = i(3/2)\mu^{-1}B + i\mu^{-1}A + i(3/2)\mu^{-1}B^{-1}A^2$. Since $E(A,B) \in \boldsymbol{M}_{1,2,\epsilon,0}$ as follows from Lemma 5.1, by the Leibniz formula we have

$$\begin{aligned}
&\left|D_{\xi'}^{\alpha'}\left(\tau\frac{\partial D(A,B)^s}{\partial\tau}\right)\right| \\
&\quad\le C_{\alpha'}|s||\tau|\sum_{\beta'+\gamma'=\alpha'}\left|D_{\xi'}^{\beta'}D(A,B)^{s-1}\right|\left|D_{\xi'}^{\gamma'}E(A,B)\right| \\
&\quad\le C_{\alpha',\epsilon,\mu,s}|\lambda|\sum_{\beta'+\gamma'=\alpha'}(|\lambda|^{1/2}+A)^{3(s-1)}A^{-|\beta'|}(|\lambda|^{1/2}+A)A^{-|\gamma'|} \\
&\quad\le C_{\alpha',\epsilon,\mu,s}(|\lambda|^{1/2}+A)^{3s}A^{-|\alpha'|}.
\end{aligned}$$

Combining these estimates implies that $D(A,B)^s \in \boldsymbol{M}_{3s,1,\epsilon,0}$. This completes the proof of the lemma. $\square$

Lemma 5.3. *Let $\ell = 0,1$ and $0 < \epsilon < \pi/2$. We use the symbols defined in* (4.15) *and* (4.19)*. Then, for any multi-index $\alpha' \in \boldsymbol{N}_0^{n-1}$ and $(\lambda,\xi',x_n) \in \Sigma_{\epsilon,\gamma_0}\times(\boldsymbol{R}^{n-1}\setminus\{0\})\times(0,\infty)$, we have*

$$\begin{aligned}
\left|D_{\xi'}^{\alpha'}\{(\tau\partial_\tau)^\ell e^{-Bx_n}\}\right| &\le C_{\alpha',\epsilon,\mu}(|\lambda|^{1/2}+A)^{-|\alpha'|}e^{-d(|\lambda|^{1/2}+A)x_n}, \\
\left|D_{\xi'}^{\alpha'}e^{-Ax_n}\right| &\le C_{\alpha'}A^{-|\alpha'|}e^{-(1/2)Ax_n}, \\
\left|D_{\xi'}^{\alpha'}\{(\tau\partial_\tau)^\ell\mathscr{M}(A,B,x_n)\}\right| &\le C_{\alpha',\epsilon,\mu}(x_n \text{ or } |\lambda|^{-1/2})e^{-dAx_n}A^{-|\alpha'|},
\end{aligned}$$

where d is a positive constant which depends on ϵ and μ but is independent of α'.

Proof. We write

$$\mathscr{M}(A,B,x_n) = -x_n\int_0^1 e^{-((1-\theta)A+\theta B)x_n}\,d\theta.$$

Setting $f(t) = e^{-tx_n}$, by the Bell formula we have

$$
\begin{aligned}
&\left|D_{\xi'}^{\alpha'} e^{-((1-\theta)A+\theta B)x_n}\right| = \left|D_{\xi'}^{\alpha'} f((1-\theta)A+\theta B)\right| \\
&\quad \le C_{\alpha'} \sum_{\ell=1}^{|\alpha'|} \left|f^{(\ell)}((a-\theta)A+\theta B)\right| \sum_{\substack{\alpha'_1+\cdots+\alpha'_\ell=\alpha' \\ |\alpha'_i|\ge 1}} \left|D_{\xi'}^{\alpha'_1}((1-\theta)A+\theta B)\right| \\
&\qquad\qquad \cdots \left|D_{\xi'}^{\alpha'_\ell}((1-\theta)A+\theta B)\right| \\
&\quad \le C_{\alpha'} \sum_{\ell=1}^{|\alpha'|} x_n^\ell e^{-((1-\theta)A+c\theta(|\lambda|^{1/2}+A))x_n} \big((1-\theta)A^{1-|\alpha'_1|}+\theta(|\lambda|^{1/2}+A)^{1-|\alpha'_1|}\big) \\
&\qquad\qquad \cdots \big((1-\theta)A^{1-|\alpha'_\ell|}+\theta(|\lambda|^{1/2}+A)^{1-|\alpha'_\ell|}\big),
\end{aligned}
$$

where we have used $|e^{-((1-\theta)A+\theta B)x_n}| = e^{-((1-\theta)A+\theta \operatorname{Re} B)x_n}$ and (5.3). When $\theta=0$, we have

$$
\left|D_{\xi'}^{\alpha'} e^{-Ax_n}\right| \le C_{\alpha'} \sum_{\ell=1}^{|\alpha'|} x_n^\ell e^{-Ax_n} A^{\ell-|\alpha'|} \le C_{\alpha'} e^{-(1/2)Ax_n} A^{-|\alpha'|}. \tag{5.6}
$$

When $\theta=1$, we have

$$
\begin{aligned}
\left|D_{\xi'}^{\alpha'} e^{-Bx_n}\right| &\le C_{\alpha'} \sum_{\ell=1}^{|\alpha'|} x_n^\ell e^{-c(|\lambda|^{1/2}+A)x_n} (|\lambda|^{1/2}+A)^{\ell-|\alpha'|} \\
&\le C_{\alpha'} e^{-(c/2)(|\lambda|^{1/2}+A)x_n} (|\lambda|^{1/2}+A)^{-|\alpha'|}.
\end{aligned} \tag{5.7}
$$

For general $0<\theta<1$, since we may assume that $0<c\le 1$ without loss of generality, we have

$$
\begin{aligned}
&\left|D_{\xi'}^{\alpha'} e^{-((1-\theta)A+\theta B)x_n}\right| \\
&\quad \le C_{\alpha'} \sum_{\ell=1}^{|\alpha'|} x_n^\ell e^{-((1-\theta)A+\theta c(|\lambda|^{1/2}+A))x_n} \big((1-\theta)A+\theta(|\lambda|^{1/2}+A)\big)^\ell A^{-|\alpha'|} \\
&\quad \le C_{\alpha'} \sum_{\ell=1}^{|\alpha'|} x_n^\ell e^{-c((1-\theta)A+\theta(|\lambda|^{1/2}+A))x_n} \big((1-\theta)A+\theta(|\lambda|^{1/2}+A)\big)^\ell A^{-|\alpha'|} \\
&\quad \le C_{\alpha'} e^{-(c/2)\big((1-\theta)A+\theta(|\lambda|^{1/2}+A)\big)x_n} A^{-|\alpha'|},
\end{aligned} \tag{5.8}
$$

which implies that

$$
\begin{aligned}
\left|D_{\xi'}^{\alpha'} \mathscr{M}(A,B,x_n)\right| &\le C_{\alpha'} \int_0^1 e^{-(c/2)((1-\theta)A+\theta(|\lambda|^{1/2}+A))x_n}\, d\theta x_n A^{-|\alpha'|} \\
&= C_{\alpha'} \int_0^1 e^{-(c/2)Ax_n} e^{-\theta(c/2)|\lambda|^{1/2}x_n}\, d\theta x_n A^{-|\alpha'|}.
\end{aligned}
$$

On the one hand, integrating the last formula with respect to θ, we have

$$
\left|D_{\xi'}^{\alpha'} \mathscr{M}(A,B,x_n)\right| \le C_{\alpha'}(c/2)^{-1}|\lambda|^{-1/2} e^{-(c/2)Ax_n} A^{-|\alpha'|}, \tag{5.9}
$$

but on the other hand, using the estimate: $e^{-\theta(c/2)|\lambda|^{1/2}x_n} \le 1$, we have

$$
\left|D_{\xi'}^{\alpha'} \mathscr{M}(A,B,x_n)\right| \le C_{\alpha'} x_n e^{-(c/2)Ax_n} A^{-|\alpha'|}. \tag{5.10}
$$

Since $\partial_\tau e^{-Bx_n} = -i(2\mu)^{-1} x_n B^{-1} e^{-Bx_n}$, by the Leibniz formula, Lemma 5.2 and (5.7)

$$
\begin{aligned}
\left|D_{\xi'}^{\alpha'}\left(\tau\partial_\tau e^{-Bx_n}\right)\right| &\le C_{\alpha'} x_n \sum_{\beta'+\gamma'=\alpha'} \left|D_{\xi'}^{\beta'}(\tau\partial_\tau B)\right| \left|D_{\xi'}^{\gamma'} e^{-Bx_n}\right| \\
&\le C_{\alpha'} x_n \sum_{\beta'+\gamma'=\alpha'} (|\lambda|^{1/2}+A)^{1-|\beta'|} e^{-c(|\lambda|^{1/2}+A)x_n} (|\lambda|^{1/2}+A)^{-|\gamma'|} \\
&\le C_{\alpha'} e^{-(c/2)(|\lambda|^{1/2}+A)x_n} (|\lambda|^{1/2}+A)^{-|\alpha'|}.
\end{aligned}
$$

Since

$$
\tau\frac{\partial}{\partial\tau}\mathscr{M}(A,B,x_n) = i\frac{\mu^{-1}\tau x_n^2}{2}\int_0^1 \theta B^{-1} e^{-((1-\theta)A+\theta B)x_n}\, d\theta,
$$

by the Leibniz rule, Lemma 5.2 and (5.8) we have

$$
\begin{aligned}
&\left|D_{\xi'}^{\alpha'}(\tau\partial_\tau \mathscr{M}(A,B,x_n))\right| \\
&\quad\le C_{\alpha'}|\lambda| \sum_{\beta'+\gamma'=\alpha'} \int_0^1 \theta(|\lambda|^{1/2}+A)^{-1-|\beta'|} e^{-(c/2)((1-\theta)A+\theta(|\lambda|^{1/2}+A))x_n} A^{-|\gamma'|}\, d\theta x_n^2 \\
&\quad\le C_{\alpha'} x_n \frac{|\lambda|}{(|\lambda|^{1/2}+A)^2} \int_0^1 \theta(|\lambda|^{1/2}+A) x_n e^{-(c/2)((1-\theta)A+\theta(|\lambda|^{1/2}+A))x_n} A^{-|\alpha'|}\, d\theta
\end{aligned}
$$

$$\begin{aligned}
&\le C_{\alpha'} x_n \int_0^1 e^{-(c/4)((1-\theta)A+\theta(|\lambda|^{1/2}+A))x_n}\,d\theta A^{-|\alpha'|}\\
&= C_{\alpha'} x_n \int_0^1 e^{-(c/4)(A+\theta|\lambda|^{1/2})x_n}\,d\theta A^{-|\alpha'|}.
\end{aligned}$$

Therefore, by the same argument as in obtaining (5.9) and (5.10) we have

$$\left|D_{\xi'}^{\alpha'}(\tau\partial_\tau \mathscr{M}(A,B,x_n))\right| \le C_{\alpha'}(x_n \text{ or } |\lambda|^{-1/2})e^{-(c/4)Ax_n}A^{-|\alpha'|}.$$

This completes the proof of the lemma. $\square$

Lemma 5.4. *Let $0<\epsilon<\pi/2$, $1<q<\infty$ and $\gamma_0\ge 0$ and we use the symbols defined in* (4.14), (4.15) *and* (4.19). *Let $m_i \in \boldsymbol{M}_{0,i,\epsilon,\gamma_0}$ $(i=1,2)$, and we define the operators $K_j(\lambda)$ $(j=1,2,3,4,5)$ for $\lambda\in\Sigma_{\epsilon,\gamma_0}$ by the formulas*:

$$\begin{aligned}
[K_1(\lambda)g](x) &= \int_0^\infty \mathscr{F}_{\xi'}^{-1}\big[m_1(\lambda,\xi')|\lambda|^{1/2}e^{-B(x_n+y_n)}\hat g(\xi',y_n)\big](x')\,dy_n,\\
[K_2(\lambda)g](x) &= \int_0^\infty \mathscr{F}_{\xi'}^{-1}\big[m_2(\lambda,\xi')Ae^{-B(x_n+y_n)}\hat g(\xi',y_n)\big](x')\,dy_n,\\
[K_3(\lambda)g](x) &= \int_0^\infty \mathscr{F}_{\xi'}^{-1}\big[m_2(\lambda,\xi')Ae^{-A(x_n+y_n)}\hat g(\xi',y_n)\big](x')\,dy_n,\\
[K_4(\lambda)g](x) &= \int_0^\infty \mathscr{F}_{\xi'}^{-1}\big[m_2(\lambda,\xi')A^2\mathscr{M}(A,B,x_n+y_n)\hat g(\xi',y_n)\big](x')\,dy_n,\\
[K_5(\lambda)g](x) &= \int_0^\infty \mathscr{F}_{\xi'}^{-1}\big[m_2(\lambda,\xi')|\lambda|^{1/2}A\mathscr{M}(A,B,x_n+y_n)\hat g(\xi',y_n)\big](x')\,dy_n.
\end{aligned}$$

Then, for $\ell=1,2$ and $j=1,2,3,4,5$, the sets $\{(\tau\partial_\tau)^\ell K_j(\lambda)\mid\lambda\in\Sigma_{\epsilon,\gamma_0}\}$ are $\mathscr{R}$-bounded families in $\mathscr{L}(L_q(\boldsymbol{R}^n_+))$, whose $\mathscr{R}$ bounds do not exceed some constant $C_{n,q,\epsilon,\gamma_0,\mu}$ depending essentially only on n, q, ϵ, γ_0 and μ.

Proof. In what follows, we say that the family of operator $\{A(\lambda)\mid\lambda\in\Sigma_{\epsilon,\gamma_0}\}$ has the required properties if $\{(\tau\partial_\tau)^\ell A(\lambda)\mid\lambda\in\Sigma_{\epsilon,\gamma_0}\}$ are $\mathscr{R}$-bounded families in $\mathscr{L}(L_q(\boldsymbol{R}^n))$, whose $\mathscr{R}$ bounds do not exceed some constant $C_{n,q,\epsilon,\gamma_0,\mu}$ which depends essentially only on n, q, ϵ, γ_0 and μ. First, we consider $K_1(\lambda)$. Setting $k_{1,\lambda}(x)=\mathscr{F}_{\xi'}^{-1}[m_1(\lambda,\xi')|\lambda|^{1/2}e^{-Bx_n}](x')$, we have

$$[K_1(\lambda)g](x) = \int_{\boldsymbol{R}^n_+} k_{1,\lambda}(x'-y',x_n+y_n)g(y)\,dy.$$

We prove that there exists a constant $C_{n,\epsilon,\gamma_0,\mu}$ depending essentially only on n, ϵ, γ_0 and μ such that

$$|k_{1,\lambda}(x)| \leq C_{n,\epsilon,\gamma_0,\mu}|x|^{-n} \quad (\lambda \in \Sigma_{\epsilon,\gamma_0},\ x \in \boldsymbol{R}^n \setminus \{0\}), \tag{5.11}$$

$$\left|\tau\frac{\partial}{\partial\tau}k_{1,\lambda}(x)\right| \leq C_{n,\epsilon,\gamma_0,\mu}|x|^{-n} \quad (\lambda \in \Sigma_{\epsilon,\gamma_0},\ x \in \boldsymbol{R}^n \setminus \{0\}). \tag{5.12}$$

By the assumption, the Leibniz rule and Lemma 5.3 we have

$$\begin{aligned}&\left|D_{\xi'}^{\alpha'}(m_1(\lambda,\xi')|\lambda|^{1/2}e^{-Bx_n})\right| \\ &\quad \leq C_{\alpha',\epsilon,\gamma_0,\mu}|\lambda|^{1/2}(|\lambda|^{1/2}+A)^{-|\alpha'|}e^{-d(|\lambda|^{1/2}+A)x_n}.\end{aligned} \tag{5.13}$$

Using the identity:

$$e^{ix'\cdot\xi'} = \sum_{j=1}^{n-1}\frac{x_j}{i|x'|^2}\frac{\partial}{\partial\xi_j}e^{ix'\cdot\xi'},$$

$k_{1,\lambda}(x)$ can be written in the form:

$$k_{1,\lambda}(x) = \sum_{|\alpha'|=n}\left(\frac{ix'}{|x'|^2}\right)^{\alpha'}\left(\frac{1}{2\pi}\right)^{n-1}\int_{\boldsymbol{R}^{n-1}} e^{ix'\cdot\xi'}D_{\xi'}^{\alpha'}\left(m_1(\lambda,\xi')|\lambda|^{1/2}e^{-Bx_n}\right)d\xi'.$$

Applying (5.13) to the above formula and using the change of variables: $\xi' = |\lambda|^{1/2}\eta'$ imply that

$$\begin{aligned}|k_{1,\lambda}(x)| &\leq C_{n,\epsilon,\gamma_0,\mu}|x'|^{-n}\int_{\boldsymbol{R}^{n-1}}|\lambda|^{1/2}(|\lambda|^{1/2}+|\xi'|)^{-n}\,d\xi' \\ &= C_{n,\epsilon,\gamma_0,\mu}|x'|^{-n}\int_{\boldsymbol{R}^{n-1}}(1+|\eta'|)^{-n}\,d\eta'.\end{aligned}$$

Moreover, by (5.13) we have

$$\begin{aligned}|k_{1,\lambda}(x)| &\leq \left(\frac{1}{2\pi}\right)^{n-1}\int_{\boldsymbol{R}^{n-1}}C_{\epsilon,\gamma_0,\mu}|\lambda|^{1/2}e^{-d(|\lambda|^{1/2}+|\xi'|)x_n}\,d\xi' \\ &\leq \left(\frac{1}{2\pi}\right)^{n-1}\frac{C_{\epsilon,\gamma_0,\mu}n!}{(dx_n)^n}\int_{\boldsymbol{R}^{n-1}}|\lambda|^{1/2}(|\lambda|^{1/2}+|\xi'|)^{-n}\,d\xi'\end{aligned}$$

$$= \left(\frac{1}{2\pi}\right)^{n-1} \frac{C_{\epsilon,\gamma_0,\mu} n!}{(dx_n)^n} \int_{\boldsymbol{R}^{n-1}} (1+|\eta'|)^{-n}\, d\eta'.$$

Combining above two estimations implies (5.11).

Recall that $\tau\partial_\tau k_{1,\lambda}(x) = \mathscr{F}_{\xi'}^{-1}[\tau\partial_\tau(m_1(\lambda,\xi')|\lambda|^{1/2}e^{-Bx_n})](x')$. Noting that

$$\begin{aligned}
&\tau\partial_\tau\big(m_1(\lambda,\xi')|\lambda|^{1/2}e^{-Bx_n}\big) \\
&\quad = \tau\frac{\partial m_1(\lambda,\xi')}{\partial\tau}|\lambda|^{1/2}e^{-Bx_n} + \frac{\tau^2}{2}|\lambda|^{-\frac{3}{2}}e^{-Bx_n} + m_1(\lambda,\xi')|\lambda|^{1/2}\left(\tau\frac{\partial e^{-Bx_n}}{\partial\tau}\right),
\end{aligned}$$

by the Leibniz rule, the assumption and Lemma 5.3 we have

$$\begin{aligned}
&\left|D_{\xi'}^{\alpha'}\left\{\tau\frac{\partial}{\partial\tau}\big(|\lambda|^{1/2}e^{-Bx_n}\big)\right\}\right| \\
&\quad \le C_{\alpha',\epsilon,\gamma_0,\mu}\Bigg\{\sum_{\beta'+\gamma'=\alpha'}(|\lambda|^{1/2}+A)^{-|\beta'|}|\lambda|^{1/2}(|\lambda|^{1/2}+A)^{-|\gamma'|}e^{-d(|\lambda|^{1/2}+A)x_n} \\
&\qquad\qquad + \frac{\tau^2}{|\lambda|^{3/2}}(|\lambda|^{1/2}+A)^{-|\alpha'|}e^{-d(|\lambda|^{1/2}+A)x_n}\Bigg\} \\
&\quad \le C_{\alpha',\epsilon,\gamma_0,\mu}|\lambda|^{1/2}(|\lambda|^{1/2}+A)^{-|\alpha'|}e^{-d(|\lambda|^{1/2}+A)x_n}. \qquad (5.14)
\end{aligned}$$

Employing the same argument as in proving (5.11) by (5.13), we have (5.12) by using (5.14).

Now, using Proposition 2.7, we prove that $K_1(\lambda)$ has the required properties. For this purpose, in view of (5.11) and (5.12) we set $k_0(x) = C_{n,\epsilon,\gamma_0,\mu}|x|^{-n}$ and we define the operator K_0 by the formula:

$$[K_0 g](x) = \int_{\boldsymbol{R}^n_+} k_0(x'-y', x_n+y_n)g(y)\, dy.$$

We prove that K_0 is a bounded linear operator on $L_q(\boldsymbol{R}^n_+)$, whose bound does not exceed a constant $C_{n,\epsilon,\gamma_0,\mu}$. By the Young inequality we have

$$\begin{aligned}
\|K_0[g](\cdot,x_n)\|_{L_q(\boldsymbol{R}^{n-1})} &\le \int_0^\infty \|k_0(\cdot,x_n+y_n)\|_{L_1(\boldsymbol{R}^{n-1})}\|g(\cdot,y_n)\|_{L_q(\boldsymbol{R}^{n-1})}\, dy_n \\
&\le C_{n,\epsilon,\gamma_0,\mu}\int_0^\infty \frac{\|g(\cdot,y_n)\|_{L_q(\boldsymbol{R}^{n-1})}}{x_n+y_n}\, dy_n. \qquad (5.15)
\end{aligned}$$

To continue the estimate (5.15) we use the following lemma.

LEMMA 5.5 (cf. [**46**]). *Let $k(t,s)$ be a function defined on $(0,\infty)\times(0,\infty)$ which satisfies the condition: $k(\lambda t,\lambda s)=\lambda^{-1}k(t,s)$ for any $\lambda>0$ and $(t,s)\in(0,\infty)\times(0,\infty)$. In addition, we assume that for some $1\le q<\infty$*

$$\int_0^\infty |k(1,s)|s^{-1/q}\,ds = A_q < \infty.$$

If we define the integral operator T by the formula:

$$[Tf](t) = \int_0^\infty k(t,s)f(s)\,ds,$$

then T is a bounded linear operator on $L_q((0,\infty))$ and

$$\|Tf\|_{L_q((0,\infty))} \le A_q\|f\|_{L_q((0,\infty))}.$$

If we set $k(x_n,y_n)=C_{n,\epsilon,\gamma_0,\mu}/(x_n+y_n)$, then $k(\lambda x_n,\lambda y_n)=\lambda^{-1}k(x_n,y_n)$ and for $1<q<\infty$ we have

$$\int_0^\infty k(1,y_n)y_n^{-1/q}\,dy_n = C_{n,\epsilon,\gamma_0,\mu}\int_0^\infty \frac{dy_n}{(1+y_n)y_n^{1/q}} = A_{n,\epsilon,\gamma_0,\mu}<\infty.$$

Applying Lemma 5.5 to (5.15), we have

$$\|K_0[g]\|_{L_q(\boldsymbol{R}^n_+)} \le A_{n,\epsilon,\gamma_0,\mu}\|g\|_{L_q(\boldsymbol{R}^n_+)},$$

which combined with Proposition 2.7 implies that $K_1(\lambda)$ has the required properties.

Now, we consider $K_2(\lambda)$. If we set $k_{2,\lambda}(x)=\mathscr{F}^{-1}_{\xi'}[m_2(\lambda,\xi')Ae^{-Bx_n}](x')$, then the operator $K_2(\lambda)$ is given by the formula:

$$[K_2(\lambda)g](x) = \int_{\boldsymbol{R}^n_+} k_{2,\lambda}(x'-y',x_n+y_n)g(y)\,dy.$$

Therefore, as we proved that $K_1(\lambda)$ has the required properties by using (5.11) and (5.12), to prove that $K_2(\lambda)$ has the required properties it is sufficient to prove that for any $\lambda\in\Sigma_{\epsilon,\lambda_0}$ and $x\in\boldsymbol{R}^n\setminus\{0\}$ there hold the estimates:

$$|k_{2,\lambda}(x)| \leq C_{n,\epsilon,\gamma_0,\mu}|x|^{-n}, \quad |\tau\partial_\tau k_{2,\lambda}(x)| \leq C_{n,\epsilon,\gamma_0,\mu}|x|^{-n}. \tag{5.16}$$

By the assumption, the Leibniz rule, and Lemma 5.3 we have

$$\begin{aligned}
&\left|D_{\xi'}^{\alpha'}(m_2(\lambda,\xi')Ae^{-Bx_n})\right| \\
&\quad \leq C_{\alpha'} \sum_{\beta'+\gamma'+\delta'=\alpha'} \left|D_{\xi'}^{\beta'} m_2(\lambda,\xi')\right|\left|D_{\xi'}^{\gamma'} A\right|\left|D_{\xi'}^{\delta'} e^{-Bx_n}\right| \\
&\quad \leq C_{\alpha',\epsilon,\gamma_0,\mu} \sum_{\beta'+\gamma'+\delta'=\alpha'} A^{-\beta'} A^{1-|\gamma'|}(|\lambda|^{1/2}+A)^{-|\delta'|}e^{-d(|\lambda|^{1/2}+A)x_n} \\
&\quad \leq C_{\alpha',\epsilon,\gamma_0,\mu}|\xi'|^{1-|\alpha'|}e^{-d(|\lambda|^{1/2}+A)x_n}.
\end{aligned} \tag{5.17}$$

Since $\tau\partial(m_2(\lambda,\xi')Ae^{-Bx_n}) = \partial_\tau m_2(\lambda,\xi')Ae^{-Bx_n} + m_2(\lambda)A\partial_\tau e^{-Bx_n}$, employing the same argument as in (5.17), by the assumption, the Leibniz rule and Lemma 5.3 we have also

$$\left|D_{\xi'}^{\alpha'}\left\{\tau\frac{\partial}{\partial\tau}\big(m_2(\lambda,\xi')Ae^{-Bx_n}\big)\right\}\right| \leq C_{\alpha',\epsilon,\gamma_0,\mu}|\xi'|^{1-|\alpha'|}e^{-d(|\lambda|^{1/2}+A)x_n}. \tag{5.18}$$

In view of (5.17) and (5.18), we apply Lemma 3.6, replacing n by $n-1$ to obtain

$$|k_{2,\lambda}(x)| \leq C_{n,\epsilon,\gamma_0,\mu}|x'|^{-n}, \quad |\tau\partial_\tau k_{2,\lambda}(x)| \leq C_{n,\epsilon,\gamma_0,\mu}|x'|^{-n}. \tag{5.19}$$

On the other hand, using (5.17) with $\alpha'=0$ and the change of variables: $x_n\xi'=\eta'$, we have

$$\begin{aligned}
\left|(\tau\partial_\tau)^\ell k_{2,\lambda}(x)\right| &\leq C_{0,\epsilon,\gamma_0,\mu}\left(\frac{1}{2\pi}\right)^{n-1}\int_{\boldsymbol{R}^{n-1}} |\xi'|e^{-d|\xi'|x_n}\,d\xi' \\
&= (x_n)^{-n}C_{0,\epsilon,\gamma_0,\mu}\left(\frac{1}{2\pi}\right)^{n-1}\int_{\boldsymbol{R}^{n-1}} |\eta'|e^{-d|\eta'|}\,d\eta',
\end{aligned}$$

for $\ell=0$ and 1, which combined with (5.17) and (5.18) implies (5.16), and therefore $K_2(\lambda)$ has the required properties.

Now, we consider $K_3(\lambda)$. Setting $k_{3,\lambda}(x) = \mathscr{F}_{\xi'}^{-1}[m_2(\lambda,\xi')Ae^{-Ax_n}](x')$, we have

$$[K_3(\lambda)g](x) = \int_{\boldsymbol{R}_+^n} k_{3,\lambda}(x'-y',x_n+y_n)g(y)\,dy,$$

so that to prove that $K_3(\lambda)$ has the required properties it is sufficient to prove that

$$|k_{3,\lambda}(x)| \leq C_{n,\epsilon,\gamma_0,\mu}|x|^{-n}, \quad |\tau\partial_\tau k_{3,\lambda}(x)| \leq C_{n,\epsilon,\gamma_0,\mu}|x|^{-n}. \tag{5.20}$$

By the Leibniz rule, the assumption and Lemma 5.3 we have

$$\left|D_{\xi'}^{\alpha'}\left\{(\tau\partial_\tau)^\ell(m_2(\lambda,\xi')Ae^{-Ax_n})\right\}\right| \leq C_{\alpha',\epsilon,\gamma_0,\mu}A^{1-|\alpha'|}e^{-(1/2)Ax_n} \quad (\ell = 0, 1),$$

which combined with Lemma 3.6 implies (5.20).

Now, we consider $K_4(\lambda)$. Setting $k_{4,\lambda}(x) = \mathscr{F}_{\xi'}^{-1}[m_2(\lambda,\xi')A^2\mathscr{M}(A,B,x_n)]$ $\times(x')$, we have

$$[K_4(\lambda)g](x) = \int_{\boldsymbol{R}_+^n} k_{4,\lambda}(x'-y', x_n+y_n)g(y)\,dy,$$

so that to prove that $K_4(\lambda)$ has the required properties it is sufficient to prove that

$$|k_{4,\lambda}(x)| \leq C_{n,\epsilon,\gamma_0,\mu}|x|^{-n}, \quad |\tau\partial_\tau k_{4,\lambda}(x)| \leq C_{n,\epsilon,\gamma_0,\mu}|x|^{-n}. \tag{5.21}$$

By the Leibniz rule, the assumption and Lemma 5.3, we have

$$\begin{aligned}&\left|D_{\xi'}^{\alpha'}\left\{(\tau\partial_\tau)^\ell(m_2(\lambda,\xi')A^2\mathscr{M}(A,B,x_n))\right\}\right| \\ &\quad \leq C_{\alpha',\epsilon,\gamma_0,\mu}A^{2-|\alpha'|}x_ne^{-dAx_n} \leq C_{\alpha',\epsilon,\gamma_0,\mu}\left(\frac{d}{2}\right)^{-1}A^{1-|\alpha'|}e^{-(d/2)Ax_n} \quad (\ell=0,1),\end{aligned}$$

which combined with Lemma 3.6 implies (5.21).

Finally, we consider $K_5(\lambda)$. Setting $k_{5,\lambda}(x) = \mathscr{F}_{\xi'}^{-1}[m_2(\lambda,\xi')|\lambda|^{1/2}A$ $\cdot\mathscr{M}(A,B,x_n)](x')$, we have

$$[K_5(\lambda)g](x) = \int_{\boldsymbol{R}_+^n} k_{5,\lambda}(x'-y', x_n+y_n)g(y)\,dy,$$

so that to prove that $K_5(\lambda)$ has the required properties it is sufficient to prove that

$$|k_{5,\lambda}(x)| \leq C_{n,\epsilon,\gamma_0,\mu}|x|^{-n}, \quad |\tau\partial_\tau k_{5,\lambda}(x)| \leq C_{n,\epsilon,\gamma_0,\mu}|x|^{-n}. \tag{5.22}$$

By the Leibniz rule, the assumption and Lemma 5.3, we have

$$\left|D_{\xi'}^{\alpha'}\{(\tau\partial_\tau)^\ell(m_2(\lambda,\xi')|\lambda|^{1/2}A\mathscr{M}(A,B,x_n))\}\right| \\ \leq C_{\alpha',\epsilon,\gamma_0,\mu}A^{1-|\alpha'|}x_ne^{-dAx_n} \quad (\ell=0,1),$$

which combined with Lemma 3.6 implies (5.22). This completes the proof of the lemma. □

Now, we show two lemmas which will be used to estimate the solutions (u,θ) and (U,Θ), respectively.

Lemma 5.6. *Let $0<\epsilon<\pi/2$, $\gamma_0\geq 0$ and $1<q<\infty$ and we use the symbols defined in* (4.14), (4.15) *and* (4.19). *Given $k_0\in \boldsymbol{M}_{-1,1,\epsilon,\gamma_0}$ and $k_1\in \boldsymbol{M}_{-2,2,\epsilon,\gamma_0}$, we define the operators $L_j(\lambda)$ $(j=1,2,3,4,5)$ by the following formulas*:

$$[L_1(\lambda)g](x)=\int_0^\infty \mathscr{F}_{\xi'}^{-1}\big[k_0(\lambda,\xi')e^{-B(x_n+y_n)}\hat g(\xi',y_n)\big](x')\,dy_n,$$

$$[L_2(\lambda)g](x)=\int_0^\infty \mathscr{F}_{\xi'}^{-1}\big[k_1(\lambda,\xi')Ae^{-B(x_n+y_n)}\hat g(\xi',y_n)\big](x')\,dy_n,$$

$$[L_3(\lambda)g](x)=\int_0^\infty \mathscr{F}_{\xi'}^{-1}\big[k_1(\lambda,\xi')Ae^{-A(x_n+y_n)}\hat g(\xi',y_n)\big](x')\,dy_n,$$

$$[L_4(\lambda)g](x)=\int_0^\infty \mathscr{F}_{\xi'}^{-1}\big[k_1(\lambda,\xi')A^2\mathscr{M}(A,B,x_n+y_n)\hat g(\xi',y_n)\big](x')\,dy_n,$$

$$[L_5(\lambda)g](x)=\int_0^\infty \mathscr{F}_{\xi'}^{-1}\big[k_1(\lambda,\xi')|\lambda|^{1/2}A\mathscr{M}(A,B,x_n+y_n)\hat g(\xi',y_n)\big](x')\,dy_n.$$

Then, for $\ell=0,1$, $i=1,2,3,4,5$ and $j,k=1,\ldots,n$, the following sets:

$$\big\{(\tau\partial_\tau)^\ell(\lambda L_i(\lambda))\mid\lambda\in\Sigma_{\epsilon,\gamma_0}\big\},\qquad \big\{(\tau\partial_\tau)^\ell(\gamma L_i(\lambda))\mid\lambda\in\Sigma_{\epsilon,\gamma_0}\big\},$$
$$\big\{(\tau\partial_\tau)^\ell(|\lambda|^{1/2}D_jL_i(\lambda))\mid\lambda\in\Sigma_{\epsilon,\gamma_0}\big\},\quad \big\{(\tau\partial_\tau)^\ell(D_jD_kL_i(\lambda))\mid\lambda\in\Sigma_{\epsilon,\gamma_0}\big\}$$

are $\mathscr{R}$-bounded families in $\mathscr{L}(L_q(\boldsymbol{R}^n_+))$, whose $\mathscr{R}$ bounds do not exceed some constant $C_{n,q,\epsilon,\gamma_0,\mu}$ depending only on n, q, ϵ, γ_0 and μ.

Proof. In what follows, we say that the set $\{A(\lambda)\mid\lambda\in\Sigma_{\epsilon,\gamma_0}\}$ has the required property if $\{(\tau\partial_\tau)^\ell A(\lambda)\mid\lambda\in\Sigma_{\epsilon,\gamma_0}\}$ $(\ell=0,1)$ are $\mathscr{R}$ bounded families in $\mathscr{L}(L_q(\boldsymbol{R}^n_+))$ whose $\mathscr{R}$ bounds do not exceed some constant $C_{n,q,\epsilon,\gamma_0,\mu}$ depending only on n, q, ϵ, γ_0 and μ. First, we consider $L_1(\lambda)$. We write

$$\lambda[L_1(\lambda)g](x) = \int_0^\infty \mathscr{F}_{\xi'}^{-1}\big[\lambda|\lambda|^{-1/2}k_0(\lambda,\xi')|\lambda|^{1/2}e^{-B(x_n+y_n)}\hat{g}(\xi',y_n)\big](x')\,dy_n,$$

$$\gamma[L_1(\lambda)g](x) = \int_0^\infty \mathscr{F}_{\xi'}^{-1}\big[\gamma|\lambda|^{-1/2}k_0(\lambda,\xi')|\lambda|^{1/2}e^{-B(x_n+y_n)}\hat{g}(\xi',y_n)\big](x')\,dy_n,$$

$$|\lambda|^{1/2}D_n[L_1(\lambda)g](x) = -\int_0^\infty \mathscr{F}_{\xi'}^{-1}\big[Bk_0(\lambda,\xi')|\lambda|^{1/2}e^{-B(x_n+y_n)}\hat{g}(\xi',y_n)\big](x')\,dy_n.$$

Since $\lambda|\lambda|^{1/2}$, $\gamma|\lambda|^{-1/2}$ and $B \in \boldsymbol{M}_{1,1,\epsilon,\gamma_0}$, by Lemma 5.1 $\lambda|\lambda|^{-1/2}k_0$, $\gamma|\lambda|^{-1/2}k_0$ and $Bk_0 \in \boldsymbol{M}_{0,1,\epsilon,\gamma_0}$, so that Lemma 5.4 implies that the sets: $\{\lambda L_1(\lambda) \mid \lambda \in \Sigma_{\epsilon,\gamma_0}\}$, $\{\gamma L_1(\lambda) \mid \lambda \in \Sigma_{\epsilon,\gamma_0}\}$ and $\{|\lambda|^{1/2}D_nL_1(\lambda) \mid \lambda \in \Sigma_{\epsilon,\gamma_0}\}$ have the required properties.

For $j,k = 1,\ldots,n-1$, we write

$$\begin{aligned}
&|\lambda|^{1/2}D_j[L_1(\lambda)g](x)\\
&\quad= \int_0^\infty \mathscr{F}_{\xi'}^{-1}\big[|\lambda|^{1/2}(i\xi_jA^{-1})k_0(\lambda,\xi')Ae^{-B(x_n+y_n)}\hat{g}(\xi',y_n)\big](x')\,dy_n,
\end{aligned}$$

$$D_jD_k[L_1(\lambda)g](x) = \int_0^\infty \mathscr{F}_{\xi'}^{-1}\big[(i\xi_j)(i\xi_kA^{-1})k_0(\lambda,\xi')Ae^{-B(x_n+y_n)}\hat{g}(\xi',y_n)\big](x')\,dy_n,$$

$$D_jD_n[L_1(\lambda)g](x) = -\int_0^\infty \mathscr{F}_{\xi'}^{-1}\big[(i\xi_jA^{-1})Bk_0(\lambda,\xi')Ae^{-B(x_n+y_n)}\hat{g}(\xi',y_n)\big](x')\,dy_n.$$

Since $|\lambda|^{1/2} \in \boldsymbol{M}_{1,1,\epsilon,\gamma_0}$, $i\xi_jA^{-1} \in \boldsymbol{M}_{0,2,\epsilon,\gamma_0}$, $i\xi_j \in \boldsymbol{M}_{1,2,\epsilon,\gamma_0}$ and $B \in \boldsymbol{M}_{1,1,\epsilon,\gamma_0}$, by Lemma 5.1 $|\lambda|^{1/2}(i\xi_jA^{-1})k_0$, $(i\xi_j)(i\xi_kA^{-1})k_0$ and $(i\xi_jA^{-1})Bk_0$ belong to $\boldsymbol{M}_{0,2,\epsilon,\gamma_0}$, respectively, so that Lemma 5.4 implies that the sets $\{|\lambda|^{1/2}D_jL_1(\lambda) \mid \lambda \in \Sigma_{\epsilon,\gamma_0}\}$, $\{D_jD_kL_1(\lambda) \mid \lambda \in \Sigma_{\epsilon,\gamma_0}\}$ and $\{D_jD_nL_1(\lambda) \mid \lambda \in \Sigma_{\epsilon,\gamma_0}\}$ $(j,k = 1,\ldots,n-1)$ have the required properties. Since $D_n^2L_1(\lambda) = \lambda\mu^{-1}L_1(\lambda) + \sum_{j=1}^{n-1}D_j^2L_1(\lambda)$, we see easily that the set $\{D_n^2L_1(\lambda) \mid \lambda \in \Sigma_{\epsilon,\gamma_0}\}$ has the required properties.

Now, we consider the operator $L_2(\lambda)$. For $j,k = 1,\ldots,n-1$, we write

$$\begin{aligned}
&\big(\lambda[L_2(\lambda)g](x), \gamma[L_2(\lambda)g](x), |\lambda|^{1/2}D_j[L_2(\lambda)g](x), D_jD_k[L_2(\lambda)g](x),\\
&|\lambda|^{1/2}D_n[L_2(\lambda)g](x), D_nD_j[L_2(\lambda)g](x), D_n^2[L_2(\lambda)g](x)\big)\\
&\quad= \int_0^\infty \mathscr{F}_{\xi'}^{-1}\big[\big(\lambda,\gamma,|\lambda|^{1/2}(i\xi_j),(i\xi_j)(i\xi_k),-|\lambda|^{1/2}B,\\
&\qquad\qquad (-i\xi_j)B, B^2\big)k_1(\lambda,\xi')Ae^{-B(x_n+y_n)}\hat{g}(\xi',y_n)\big](x')\,dy_n.
\end{aligned}$$

By Lemma 5.1, λk_1, γk_1, $|\lambda|^{1/2}(i\xi_j)k_1$, $(i\xi_j)(i\xi_k)k_1$, $|\lambda|^{1/2}Bk_1$, $(i\xi_j)Bk_1$ and B^2k_1 belong to $\boldsymbol{M}_{0,2,\epsilon,\gamma_0}$, so that Lemma 5.4 implies that the sets: $\{\lambda L_2(\lambda) \mid \lambda \in \Sigma_{\epsilon,\gamma_0}\}$, $\{\gamma L_2(\lambda) \mid \lambda \in \Sigma_{\epsilon,\gamma_0}\}$, $\{|\lambda|^{1/2} D_j L_2(\lambda) \mid \lambda \in \Sigma_{\epsilon,\gamma_0}\}$, $\{D_j D_k L_2(\lambda) \mid \lambda \in \Sigma_{\epsilon,\gamma_0}\}$ $(j, k = 1, \ldots, n)$ have the required properties, respectively. Analogously, we see that the same assertions hold true for the operator $L_3(\lambda)$.

Now, we consider $L_4(\lambda)$. For $j, k = 1, \ldots, n-1$, we write

$$\begin{aligned}
&\big(\lambda[L_4(\lambda)g](x), \gamma[L_4(\lambda)g](x), |\lambda|^{1/2} D_j[L_4(\lambda)g](x), D_jD_k[L_4(\lambda)g](x)\big) \\
&= \int_0^\infty \mathscr{F}^{-1}_{\xi'}\big[\big(\lambda, \gamma, |\lambda|^{1/2}(i\xi_j), (i\xi_j)(i\xi_k)\big) \\
&\qquad\qquad \cdot k_1(\lambda, \xi')A^2 \mathscr{M}(A, B, x_n + y_n)\hat{g}(\xi', y_n)\big](x')\, dy_n.
\end{aligned}$$

By Lemma 5.1, λk_1, γk_1, $|\lambda|^{1/2}(i\xi_j)k_1$ and $(i\xi_j)(i\xi_k)k_1$ belong to $\boldsymbol{M}_{0,2,\epsilon,\gamma_0}$, so that Lemma 5.4 implies that the sets: $\{\lambda L_4(\lambda) \mid \lambda \in \Sigma_{\epsilon,\gamma_0}\}$, $\{\gamma L_4(\lambda) \mid \lambda \in \Sigma_{\epsilon,\gamma_0}\}$ and $\{D_j D_k L_4(\lambda) \mid \lambda \in \Sigma_{\epsilon,\gamma_0}\}$ have the required properties, respectively. Since

$$D_n \mathscr{M}(A, B, x_n) = -e^{-Bx_n} - A\mathscr{M}(A, B, x_n), \tag{5.23}$$

we have

$$\begin{aligned}
&\big(|\lambda|^{1/2} D_n[L_4(\lambda)g](x), D_j D_n[L_4(\lambda)g](x)\big) \\
&\quad = -\int_0^\infty \mathscr{F}^{-1}_{\xi'}\big[(|\lambda|^{1/2}, i\xi_j)Ak_1(\lambda, \xi')Ae^{-B(x_n+y_n)}\hat{g}(\xi', y_n)\big](x')\, dy_n \\
&\qquad - \int_0^\infty \mathscr{F}^{-1}_{\xi'}\big[(|\lambda|^{1/2}, i\xi_j)Ak_1(\lambda, \xi')A^2\mathscr{M}(A, B, x_n + y_n)\hat{g}(\xi', y_n)\big](x')\, dy_n.
\end{aligned}$$

By Lemma 5.1, $|\lambda|^{1/2}Ak_1$ and $(i\xi_j)Ak_1$ belong to $\boldsymbol{M}_{0,2,\epsilon,\gamma_0}$, so that Lemma 5.4 implies that the sets: $\{|\lambda|^{1/2} D_n L_4(\lambda) \mid \lambda \in \Sigma_{\epsilon,\gamma_0}\}$ and $\{D_j D_n L_4(\lambda) \mid \lambda \in \Sigma_{\epsilon,\gamma_0}\}$ have the required properties, respectively. Since $D_n^2 \mathscr{M}(A, B, x_n) = (A + B)e^{-Bx_n} + A^2\mathscr{M}(A, B, x_n)$, we have

$$\begin{aligned}
D_n^2[L_4(\lambda)g](x) &= \int_0^\infty \mathscr{F}^{-1}_{\xi'}\big[(A + B)Ak_1(\lambda, \xi')Ae^{-B(x_n+y_n)}\hat{g}(\xi', \lambda)\big](x')\, dy_n \\
&\quad + \int_0^\infty \mathscr{F}^{-1}_{\xi'}\big[A^2k_1(\lambda, \xi')A^2\mathscr{M}(A, B, x_n + y_n)\hat{g}(\xi', \lambda)\big](x')\, dy_n.
\end{aligned}$$

By Lemma 5.1 $(A+B)Ak_1$ and A^2k_1 belong to $\boldsymbol{M}_{0,2,\epsilon,\gamma_0}$, so that Lemma 5.4 implies that the set: $\{D_n^2 L_4(\lambda) \mid \lambda \in \Sigma_{\epsilon,\gamma_0}\}$ has the required properties. Analogously, we see that the same assertions hold true for the operator $L_5(\lambda)$, which completes the proof of the lemma. $\square$

LEMMA 5.7. *Let $0 < \epsilon < \pi/2$, $\gamma_0 \geq 0$ and $1 < q < \infty$ and we use the symbols defined in* (4.14), (4.15) *and* (4.19). *Given $k_2 \in \boldsymbol{M}_{0,1,\epsilon,\gamma_0}$ and $k_3 \in \boldsymbol{M}_{-1,2,\epsilon,\gamma_0}$, we define the operators $L_j(\lambda)$ $(j = 6,7,8,9,10)$ by the formulas*:

$$[L_6(\lambda)g](x) = \int_0^\infty \mathscr{F}_{\xi'}^{-1}\bigl[k_2(\lambda,\xi')e^{-B(x_n+y_n)}\hat{g}(\xi',y_n)\bigr](x')\,dy_n,$$

$$[L_7(\lambda)g](x) = \int_0^\infty \mathscr{F}_{\xi'}^{-1}\bigl[k_3(\lambda,\xi')Ae^{-B(x_n+y_n)}\hat{g}(\xi',y_n)\bigr](x')\,dy_n,$$

$$[L_8(\lambda)g](x) = \int_0^\infty \mathscr{F}_{\xi'}^{-1}\bigl[k_3(\lambda,\xi')Ae^{-A(x_n+y_n)}\hat{g}(\xi',y_n)\bigr](x')\,dy_n,$$

$$[L_9(\lambda)g](x) = \int_0^\infty \mathscr{F}_{\xi'}^{-1}\bigl[k_3(\lambda,\xi')A^2\mathscr{M}(A,B,x_n+y_n)\hat{g}(\xi',y_n)\bigr](x')\,dy_n,$$

$$[L_{10}(\lambda)g](x) = \int_0^\infty \mathscr{F}_{\xi'}^{-1}\bigl[k_3(\lambda,\xi')|\lambda|^{1/2}A\mathscr{M}(A,B,x_n+y_n)\hat{g}(\xi',y_n)\bigr](x')\,dy_n.$$

Then, for $\ell = 0,1$, $i = 6,7,8,9,10$ and $j = 1,\ldots,n$, the sets:

$$\bigl\{(\tau\partial_\tau)^\ell|\lambda|^{1/2}L_i(\lambda) \mid \lambda \in \Sigma_{\epsilon,\gamma_0}\bigr\}, \quad \bigl\{(\tau\partial_\tau)^\ell D_j L_i(\lambda) \mid \lambda \in \Sigma_{\epsilon,\gamma_0}\bigr\}$$

are $\mathscr{R}$-bounded families in $\mathscr{L}(L_q(\boldsymbol{R}_+^n))$ whose $\mathscr{R}$ bounds do not exceed some constant $C_{n,q,\epsilon,\gamma_0,\mu}$ depending only on n, q, ϵ, γ_0 and μ.

PROOF. As in the proof of Lemma 5.6, we say that the set $\{A(\lambda) \mid \lambda \in \Sigma_{\epsilon,\gamma_0}\}$ has the required property if $\{(\tau\partial_\tau)^\ell A(\lambda) \mid \lambda \in \Sigma_{\epsilon,\gamma_0}\}$ $(\ell = 0,1)$ are $\mathscr{R}$ bounded families in $\mathscr{L}(L_q(\boldsymbol{R}_+^n))$ whose $\mathscr{R}$ bounds do not exceed some constant $C_{n,q,\epsilon,\gamma_0,\mu}$ depending only on n, q, ϵ, γ_0 and μ. We consider $L_6(\lambda)$ and we write

$$|\lambda|^{1/2}[L_6(\lambda)g](x) = \int_0^\infty \mathscr{F}_{\xi'}^{-1}\bigl[k_2(\lambda,\xi')|\lambda|^{1/2}e^{-B(x_n+y_n)}\hat{g}(\xi',y_n)\bigr](x')\,dy_n.$$

Since $k_2 \in \boldsymbol{M}_{0,1,\epsilon,\gamma_0}$, Lemma 5.4 implies that the set $\{|\lambda|^{1/2}L_6(\lambda) \mid \lambda \in \Sigma_{\epsilon,\gamma_0}\}$ has the required properties. For $j = 1,\ldots,n-1$ we write

$$D_j[L_6(\lambda)g](x) = \int_0^\infty \mathscr{F}_{\xi'}^{-1}\bigl[(i\xi_j A^{-1})k_2(\lambda,\xi')Ae^{-B(x_n+y_n)}\hat{g}(\xi',y_n)\bigr](x')\,dy_n.$$

Since $i\xi_j A^{-1} \in \boldsymbol{M}_{0,2,\epsilon,\gamma_0}$, by Lemma 5.4 $(i\xi_j A^{-1})k_2 \in \boldsymbol{M}_{0,1,\epsilon,\gamma_0}$, so that Lemma 5.4 implies that the set $\{D_j L_6(\lambda) \mid \lambda \in \Sigma_{\epsilon,\gamma_0}\}$ has the required properties. Finally, using the identity

$$B = \mu^{-1}(\lambda|\lambda|^{-1/2}B^{-1})|\lambda|^{1/2} + (AB^{-1})A,$$

we write

$$\begin{aligned} &D_n[L_6(\lambda)g](x) \\ &\quad = \mu^{-1}\int_0^\infty \mathscr{F}^{-1}_{\xi'}\big[(\lambda|\lambda|^{-1/2}B^{-1})k_2(\lambda,\xi')|\lambda|^{1/2}e^{-B(x_n+y_n)}\hat g(\xi',y_n)\big](x')\,dy_n \\ &\qquad + \int_0^\infty \mathscr{F}^{-1}_{\xi'}\big[(AB^{-1})k_2(\lambda,\xi')Ae^{-B(x_n+y_n)}\hat g(\xi',y_n)\big](x')\,dy_n. \end{aligned}$$

By Lemma 5.1, $\lambda|\lambda|^{-1/2}B^{-1}k_2 \in \boldsymbol{M}_{0,1,\epsilon,\gamma_0}$ and $AB^{-1}k_2 \in \boldsymbol{M}_{0,2,\epsilon,\gamma_0}$, so that Lemma 5.4 implies that the set $\{D_n L_6(\lambda) \mid \lambda \in \Sigma_{\epsilon,\gamma_0}\}$ has the required properties.

By Lemmas 5.1 and 5.4, we see easily that the assertions for $L_7(\lambda)$, $L_8(\lambda)$, $L_9(\lambda)$ and $L_{10}(\lambda)$ hold true. This completes the proof of the lemma. □

6. Proofs of Theorem 1.1 and Theorem 1.2.

In this section, applying Lemma 5.6, Lemma 5.7 and Theorem 2.8 to the solution formulas given in (4.20), (4.21), (4.22), (4.23), (4.24) and (4.25), we prove Theorems 1.1 and 1.2. Using a trick due to Volevich [**54**] and (5.23), and writing $\nabla'\cdot h' = \sum_{k=1}^{n-1} D_k h_k$, from (4.20) we have

$$\begin{aligned} &u_j(x) = \\ &\quad 2\mu^{-1}\sum_{k=1}^{n-1}\int_0^\infty \mathscr{F}^{-1}_{\xi'}\Big[\frac{\xi_j(\xi_k A^{-1})}{D(A,B)}A^2\mathscr{M}(A,B,x_n+y_n)\mathscr{F}[D_n h_k](\xi',y_n)\Big](x')\,dy_n \\ &\quad + 2\mu^{-1}\int_0^\infty \mathscr{F}^{-1}_{\xi'}\Big[\frac{i\xi_j}{D(A,B)}Ae^{-B(x_n+y_n)}\mathscr{F}[\nabla'\cdot h'](\xi',y_n)\Big](x')\,dy_n \\ &\quad + 2\mu^{-1}\int_0^\infty \mathscr{F}^{-1}_{\xi'}\Big[\frac{i\xi_j}{D(A,B)}A^2\mathscr{M}(A,B,x_n+y_n)\mathscr{F}[\nabla'\cdot h'](\xi',y_n)\Big](x')\,dy_n \\ &\quad + 2\mu^{-1}\int_0^\infty \mathscr{F}^{-1}_{\xi'}\Big[\frac{2i(\xi_j A^{-1})B}{D(A,B)}A^2\mathscr{M}(A,B,x_n+y_n)\mathscr{F}[D_n h_n](\xi',y_n)\Big](x')\,dy_n \end{aligned}$$

$$
\begin{aligned}
&- 2\mu^{-1} \int_0^\infty \mathscr{F}_{\xi'}^{-1}\Big[\frac{B}{D(A,B)} A e^{-B(x_n+y_n)} \mathscr{F}[D_j h_n](\xi', y_n)\Big](x')\, dy_n \\
&- 2\mu^{-1} \int_0^\infty \mathscr{F}_{\xi'}^{-1}\Big[\frac{B}{D(A,B)} A^2 \mathscr{M}(A,B,x_n+y_n) \mathscr{F}[D_j h_n](\xi', y_n)\Big](x')\, dy_n \\
&- \sum_{k=1}^{n-1} \mu^{-1} \int_0^\infty \mathscr{F}_{\xi'}^{-1}\Big[\frac{(B-A)\xi_j(\xi_k A^{-1})}{BD(A,B)} A e^{-B(x_n+y_n)} \mathscr{F}[D_n h_k](\xi', y_n)\Big](x')\, dy_n \\
&- \mu^{-1} \int_0^\infty \mathscr{F}_{\xi'}^{-1}\Big[\frac{(B-A)(\xi_j A^{-1})}{D(A,B)} A e^{-B(x_n+y_n)} \mathscr{F}[\nabla'\cdot h'](\xi', y_n)\Big](x')\, dy_n \\
&- \sum_{k=1}^{n-1} 2\mu^{-1} \int_0^\infty \mathscr{F}_{\xi'}^{-1}\Big[\frac{\xi_j(\xi_k A^{-1})}{D(A,B)} A e^{-A(x_n+y_n)} \mathscr{F}[D_n h_k](\xi', y_n)\Big](x')\, dy_n \\
&- 2\mu^{-1} \int_0^\infty \mathscr{F}_{\xi'}^{-1}\Big[\frac{i\xi_j}{D(A,B)} A e^{-A(x_n+y_n)} \mathscr{F}[\nabla'\cdot h'](\xi', y_n)\Big](x')\, dy_n \\
&- 2\mu^{-1} \int_0^\infty \mathscr{F}_{\xi'}^{-1}\Big[\frac{2i(B-A)(\xi_j A^{-1})}{D(A,B)} A e^{-A(x_n+y_n)} \mathscr{F}[D_n h_n](\xi', y_n)\Big](x')\, dy_n \\
&- 2\mu^{-1} \int_0^\infty \mathscr{F}_{\xi'}^{-1}\Big[\frac{B-A}{D(A,B)} A e^{-A(x_n+y_n)} \mathscr{F}[D_j h_n](\xi', y_n)\Big](x')\, dy_n \\
&+ \mu^{-1} \int_0^\infty \mathscr{F}_{\xi'}^{-1}\big[B^{-1} e^{-B(x_n+y_n)} \mathscr{F}[D_n h_j](\xi', y_n)\big](x')\, dy_n \\
&- \mu^{-2} \int_0^\infty \mathscr{F}_{\xi'}^{-1}\Big[\frac{\lambda|\lambda|^{-1/2}}{B^2} e^{-B(x_n+y_n)} \mathscr{F}[|\lambda|^{1/2} h_j](\xi', y_n)\Big](x')\, dy_n \\
&+ \mu^{-1} \sum_{k=1}^{n-1} \int_0^\infty \mathscr{F}_{\xi'}^{-1}\Big[\frac{i\xi_k A^{-1}}{B^2} A e^{-B(x_n+y_n)} \mathscr{F}[D_k h_j](\xi', y_n)\Big](x')\, dy_n.
\end{aligned}
$$

Lemma 5.1 and Lemma 5.2 imply that the symbols:

$$
\begin{aligned}
&\frac{\xi_j(\xi_k A^{-1})}{D(A,B)}, \quad \frac{\xi_j}{D(A,B)}, \quad \frac{(\xi_j A^{-1})B}{D(A,B)}, \quad \frac{B}{D(A,B)}, \\
&\frac{(B-A)\xi_j(\xi_k A^{-1})}{BD(A,B)}, \quad \frac{(B-A)(\xi_j A^{-1})}{D(A,B)}, \quad \frac{B-A}{D(A,B)}, \quad \frac{\xi_k A^{-1}}{B^2}
\end{aligned}
\tag{6.1}
$$

belong to $\boldsymbol{M}_{-2,2,\epsilon,0}$ and that the symbols:

$$B^{-1},\ \frac{\lambda|\lambda|^{-1/2}}{B^2} \tag{6.2}$$

belong to $\boldsymbol{M}_{-1,1,\epsilon,0}$. Therefore, by Lemma 5.6 and Theorem 2.8 we have

$$\|(|\lambda|u_j, |\lambda|^{1/2}\nabla u_j, \nabla^2 u_j)\|_{L_q(\boldsymbol{R}^n_+)} \leq C_{n,q,\epsilon,\mu}\|(\nabla h, |\lambda|^{1/2}h)\|_{L_q(\boldsymbol{R}^n_+)}$$

for any $\lambda \in \Sigma_{\epsilon,0}$ and $j = 1, \ldots, n-1$, and

$$\begin{aligned}
&\|e^{-\gamma t}((U_j)_t, \gamma U_j, \Lambda_\gamma^{1/2}\nabla U_j, \nabla^2 U_j)\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^n_+))} \\
&\quad \leq C_{n,p,q,\mu}\|e^{-\gamma t}(\nabla H, \Lambda_\gamma^{1/2} H)\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^n_+))}
\end{aligned}$$

for any $\gamma \geq 0$ and $j = 1, \ldots, n-1$.

Using a trick due to Volevich [**54**] and the relations: $B - A = \mu^{-1}\lambda(A+B)^{-1}$ and $B = (\mu^{-1}\lambda + A^2)B^{-1}$, from (4.21) we have

$$\begin{aligned}
&u_n(x) = \\
&\quad \mu^{-1}\sum_{k=1}^{n-1}\int_0^\infty \mathscr{F}_{\xi'}^{-1}\left[\frac{2iB(\xi_k A^{-1})}{D(A,B)}A^2\mathscr{M}(A,B,x_n+y_n)\mathscr{F}[D_n h_k](\xi',y_n)\right](x')\,dy_n \\
&\quad - \mu^{-1}\int_0^\infty \mathscr{F}_{\xi'}^{-1}\left[\frac{2iB}{D(A,B)}Ae^{-B(x_n+y_n)}\mathscr{F}[\nabla'\cdot h'](\xi',y_n)\right](x')\,dy_n \\
&\quad - \mu^{-1}\int_0^\infty \mathscr{F}_{\xi'}^{-1}\left[\frac{2iB}{D(A,B)}A^2\mathscr{M}(A,B,x_n+y_n)\mathscr{F}[\nabla'\cdot h'](\xi',y_n)\right](x')\,dy_n \\
&\quad - \mu^{-1}\int_0^\infty \mathscr{F}_{\xi'}^{-1}\left[\frac{2A}{D(A,B)}A^2\mathscr{M}(A,B,x_n+y_n)\mathscr{F}[D_n h_n](\xi',y_n)\right](x')\,dy_n \\
&\quad - i\mu^{-1}\sum_{k=1}^{n-1}\int_0^\infty \mathscr{F}_{\xi'}^{-1}\left[\frac{2i\xi_k}{D(A,B)}Ae^{-B(x_n+y_n)}\mathscr{F}[D_k h_n](\xi',y_n)\right](x')\,dy_n \\
&\quad - i\mu^{-1}\sum_{k=1}^{n-1}\int_0^\infty \mathscr{F}_{\xi'}^{-1}\left[\frac{2i\xi_k}{D(A,B)}A^2\mathscr{M}(A,B,x_n+y_n)\mathscr{F}[D_k h_n](\xi',y_n)\right](x')\,dy_n \\
&\quad - \mu^{-2}\int_0^\infty \mathscr{F}_{\xi'}^{-1}\left[\frac{\lambda|\lambda|^{-1/2}}{D(A,B)}A|\lambda|^{1/2}\mathscr{M}(A,B,x_n+y_n)\mathscr{F}[D_n h_n](\xi',y_n)\right](x')\,dy_n \\
&\quad - \mu^{-2}\int_0^\infty \mathscr{F}_{\xi'}^{-1}\left[\frac{\lambda|\lambda|^{-1/2}}{D(A,B)}Ae^{-B(x_n+y_n)}\mathscr{F}[|\lambda|^{1/2}h_n](\xi',y_n)\right](x')\,dy_n
\end{aligned}$$

$$
\begin{aligned}
&- \mu^{-2} \int_0^\infty \mathscr{F}^{-1}_{\xi'}\Big[\frac{\lambda|\lambda|^{-1/2}}{D(A,B)} A^2 \mathscr{M}(A,B,x_n+y_n)\mathscr{F}[|\lambda|^{1/2}h_n](\xi',y_n)\Big](x')\,dy_n \\
&+ \mu^{-1}\sum_{k=1}^{n-1}\int_0^\infty \mathscr{F}^{-1}_{\xi'}\Big[\frac{i(\xi_k A^{-1})(B-A)}{D(A,B)} A e^{-B(x_n+y_n)}\mathscr{F}[D_n h_k](\xi',y_n)\Big](x')\,dy_n \\
&- \mu^{-2}\sum_{k=1}^{n-1}\int_0^\infty \mathscr{F}^{-1}_{\xi'}\Big[\frac{iB(\xi_k A^{-1})\lambda|\lambda|^{-1/2}}{D(A,B)(A+B)} A e^{-B(x_n+y_n)}\mathscr{F}[|\lambda|^{1/2}h_k](\xi',y_n)\Big](x')\,dy_n \\
&+ \mu^{-1}\int_0^\infty \mathscr{F}^{-1}_{\xi'}\Big[\frac{A+B}{D(A,B)} A e^{-B(x_n+y_n)}\mathscr{F}[D_n h_n](\xi',y_n)\Big](x')\,dy_n \\
&- \mu^{-2}\int_0^\infty \mathscr{F}^{-1}_{\xi'}\Big[\frac{(A+B)\lambda|\lambda|^{-1/2}}{D(A,B)B} A e^{-B(x_n+y_n)}\mathscr{F}[|\lambda|^{1/2}h_n](\xi',y_n)\Big](x')\,dy_n \\
&+ \mu^{-1}\sum_{k=1}^{n-1}\int_0^\infty \mathscr{F}^{-1}_{\xi'}\Big[\frac{(A+B)(i\xi_k)}{D(A,B)B} A e^{-B(x_n+y_n)}\mathscr{F}[D_k h_n](\xi',y_n)\Big](x')\,dy_n.
\end{aligned}
$$

Lemma 5.1 and Lemma 5.2 imply that the symbols:

$$
\begin{aligned}
&\frac{A}{D(A,B)},\quad \frac{\lambda|\lambda|^{-1/2}}{D(A,B)},\quad \frac{(B(\xi_k A^{-1})\lambda|\lambda|^{-1/2}}{D(A,B)B}, \\
&\frac{A+B}{D(A,B)},\quad \frac{(A+B)\lambda|\lambda|^{-1/2}}{D(A,B)B},\quad \frac{(A+B)\xi_k}{D(A,B)B}
\end{aligned}
\tag{6.3}
$$

and the symbols appearing in (6.1) belong to $\boldsymbol{M}_{-2,2,\epsilon,0}$, and therefore by Lemma 5.6 and Theorem 2.8 we have

$$
\big\|(|\lambda|u_n, |\lambda|^{1/2}\nabla u_n, \nabla^2 u_n)\big\|_{L_q(\boldsymbol{R}^n_+)} \le C_{n,q,\epsilon,\mu}\|(\nabla h, |\lambda|^{1/2}h)\|_{L_q(\boldsymbol{R}^n_+)}
$$

for any $\lambda \in \Sigma_{\epsilon,0}$, and

$$
\begin{aligned}
&\big\|e^{-\gamma t}\big((U_n)_t, \gamma U_n, \Lambda^{1/2}_\gamma \nabla U_n, \nabla^2 U_n\big)\big\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^n_+))} \\
&\qquad \le C_{n,p,q,\mu}\big\|e^{-\gamma t}(\nabla H, \Lambda^{1/2}_\gamma H)\big\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^n_+))}
\end{aligned}
$$

for any $\gamma \ge 0$.

Concerning the pressure terms θ and Θ, we know that $\nabla\theta = -\lambda u + \mu\Delta u$ and $\nabla\Theta = -U_t + \mu\Delta U$, and therefore

$$\|\nabla\theta\|_{L_q(\boldsymbol{R}^n_+)} \leq C_{n,q,\epsilon,\mu}\big\|(\nabla h, |\lambda|^{1/2}h)\big\|_{L_q(\boldsymbol{R}^n_+)} \quad (\lambda \in \Sigma_{\epsilon,0}),$$

$$\|e^{-\gamma t}\nabla\Theta\|_{L_p(\boldsymbol{R},L_q(\boldsymbol{R}^n_+))} \leq C_{n,p,q,\mu}\big\|e^{-\gamma t}(\nabla H, \Lambda_\gamma^{1/2}H)\big\|_{L_p(\boldsymbol{R},L_q(\boldsymbol{R}^n_+))} \quad (\gamma \geq 0).$$

This completes the proofs of Theorem 1.1 and Theorem 1.2.

7. About the surface tension problem.

In this section, we consider problems (1.5) and (1.4) and we prove Theorem 1.3 and Theorem 1.4. Let (v,τ) and (V,Υ) be solutions to problems:

$$\begin{aligned} \lambda v - \operatorname{Div} S(v,\tau) = f, \quad \operatorname{div} v = g \quad &\text{in } \boldsymbol{R}^n_+,\\ S(v,\tau)\boldsymbol{n} = h \quad &\text{on } \boldsymbol{R}^n_0; \end{aligned} \tag{7.1}$$

$$\begin{aligned} V_t - \operatorname{Div} S(V,\Upsilon) = F, \quad \operatorname{div} V = G \quad &\text{in } \boldsymbol{Q}_+,\\ S(V,\Upsilon)\boldsymbol{n} = H \quad &\text{on } \boldsymbol{Q}_0,\\ V|_{t=0} = 0, & \end{aligned} \tag{7.2}$$

and we set $u = v + w$, $\theta = \tau + \kappa$ in (1.4) and $U = V + W$, $\Theta = \Upsilon + \Xi$ in (1.5), respectively. Then, the problems (1.4) and (1.5) convert to the following problems, respectively:

$$\begin{aligned} \lambda w - \operatorname{Div} S(w,\kappa) = 0, \quad \operatorname{div} w = 0 \quad &\text{in } \boldsymbol{R}^n_+,\\ \lambda\eta + w_n = d - v_n \quad &\text{on } \boldsymbol{R}^n_0,\\ S(w,\kappa)\boldsymbol{n} + (c_g - c_\sigma\Delta')\eta\boldsymbol{n} = 0 \quad &\text{on } \boldsymbol{R}^n_0; \end{aligned} \tag{7.3}$$

$$\begin{aligned} W_t - \operatorname{Div} S(W,\Xi) = 0, \quad \operatorname{div} W = 0, \quad &\text{in } \boldsymbol{Q}_+,\\ Y_t + W_n = D - V_n \quad &\text{on } \boldsymbol{Q}_0,\\ S(W,\Xi)\boldsymbol{n} + (c_g - c_\sigma\Delta')Y\boldsymbol{n} = 0 \quad &\text{on } \boldsymbol{Q}_0,\\ W|_{t=0} = 0, \quad Y|_{t=0} = 0. & \end{aligned} \tag{7.4}$$

Therefore, instead of (7.3) and (7.4) we consider problems (1.4) with $f = 0$, $g = 0$ and $h = 0$, and (1.5) with $F = 0$, $G = 0$ and $H = 0$ in what follows. Namely, we consider the following problems:

$$\begin{aligned} \lambda u - \operatorname{Div} S(u,\theta) = 0, \quad \operatorname{div} u = 0 \quad &\text{in } \boldsymbol{R}^n_+,\\ \lambda\eta + u_n = d \quad &\text{on } \boldsymbol{R}^n_0, \end{aligned}$$

$$S(u,\theta)\boldsymbol{n} + (c_g - c_\sigma\Delta')\eta\boldsymbol{n} = 0 \quad \text{on } \boldsymbol{R}^n_0; \tag{7.5}$$

$$\begin{aligned} U_t - \operatorname{Div} S(U,\Theta) = 0, \quad \operatorname{div} U = 0, &\quad \text{in } \boldsymbol{Q}_+, \\ Y_t + U_n = D &\quad \text{on } \boldsymbol{Q}_0, \\ S(U,\Theta)\boldsymbol{n} + (c_g - c_\sigma\Delta')Y\boldsymbol{n} = 0 &\quad \text{on } \boldsymbol{Q}_0, \\ U|_{t=0} = 0, \quad Y|_{t=0} = 0. & \end{aligned} \tag{7.6}$$

First, we derive the solution formula of the problem (7.5). In Shibata and Shimizu [**35**], the solution formula of the problem (7.5) was obtained, but for the completeness we discuss it again. As was done in the Section 4, applying the partial Fourier transform with respect to $x' = (x_1, \ldots, x_{n-1})$ to (7.5), we have

$$\begin{aligned} (\lambda + \mu A^2)\hat{u}_j(\xi', x_n) - \mu D_n^2 \hat{u}_j(\xi', x_n) + i\xi_j \hat{\theta}(\xi', x_n) = 0 &\qquad (x_n > 0), \\ (\lambda + \mu A^2)\hat{u}_n(\xi', x_n) - \mu D_n^2 \hat{u}_n(\xi', x_n) + D_n \hat{\theta}(\xi', x_n) = 0 &\qquad (x_n > 0), \\ \sum_{j=1}^{n-1} i\xi_j \hat{u}_j(\xi', x_n) + D_n \hat{u}_n(\xi', x_n) = 0 &\qquad (x_n > 0), \\ \lambda\hat{\eta}(\xi', 0) + \hat{u}_n(\xi', 0) = \hat{d}(\xi', 0), & \\ \mu(D_n \hat{u}_j(\xi', 0) + i\xi_j \hat{u}_n(\xi', 0)) = 0, & \\ 2\mu D_n \hat{u}_n(\xi', 0) - \hat{\theta}(\xi', 0) + (c_g + c_\sigma A^2)\hat{\eta} = 0, & \end{aligned} \tag{7.7}$$

where j runs through 1 to $n-1$ in the 1st equation and 5th equation. Setting $\hat{h}_j(\xi', 0) = 0$ $(j = 1, \ldots, n-1)$ and $\hat{h}_n(\xi', 0) = (c_g + c_\sigma A^2)\hat{\eta}(\xi', 0)$ in (4.16), by (4.20), (4.21) and (4.22) we have

$$\begin{aligned} \hat{u}_j(\xi', x_n) = &-\frac{(A^2+B^2)e^{-Ax_n} i\xi_j}{\mu(B-A)D(A,B)}(c_g + c_\sigma A^2)\hat{\eta}(\xi', 0) \\ &+ \frac{2ABe^{-Bx_n} i\xi_j}{\mu(B-A)D(A,B)}(c_g + c_\sigma A^2)\hat{\eta}(\xi', 0), \end{aligned} \tag{7.8}$$

$$\begin{aligned} \hat{u}_n(\xi', x_n) = &\frac{A(A^2+B^2)e^{-Ax_n}}{\mu(B-A)D(A,B)}(c_g + c_\sigma A^2)\hat{\eta}(\xi', 0) \\ &- \frac{2A^3 e^{-Bx_n}}{\mu(B-A)D(A,B)}(c_g + c_\sigma A^2)\hat{\eta}(\xi', 0), \end{aligned} \tag{7.9}$$

$$\hat{\theta}(\xi', x_n) = \frac{(A+B)(A^2+B^2)e^{-Ax_n}}{D(A,B)}(c_g + c_\sigma A^2)\hat{\eta}(\xi', 0). \tag{7.10}$$

Therefore, combining the 4th equation in (7.7) and (7.9) we have

$$\hat{\eta}(\xi', 0) = \frac{\mu^{-1} D(A, B)}{(A + B)L(A, B)} \hat{d}(\xi', 0), \tag{7.11}$$

where we have set

$$\begin{aligned} L(A, B) &= (B - A)D(A, B) + \mu^{-2} A(c_g + c_\sigma A^2) \\ &= (A^2 + B^2)^2 - 4A^3 B + \mu^{-2} A(c_g + c_\sigma A^2). \end{aligned} \tag{7.12}$$

Let $\varphi(x_n)$ be a function in $C^\infty(\boldsymbol{R})$ such that $\varphi(x_n) = 1$ when $x_n < 1$ and $\varphi(x_n) = 0$ when $x_n > 2$ and in view of (7.11) we set

$$\eta(x) = \varphi(x_n) \mathscr{F}_{\xi'}^{-1} \left[\frac{\mu^{-1} D(A, B) e^{-A x_n}}{(A + B)L(A, B)} \hat{d}(\xi', 0) \right] (x'). \tag{7.13}$$

Moreover, we define $Y(x, t)$ by the formula:

$$Y(x, t) = \varphi(x_n) \mathscr{L}_\lambda^{-1} \mathscr{F}_{\xi'}^{-1} \left[\frac{\mu^{-1} D(A, B) e^{-A x_n}}{(A + B)L(A, B)} \mathscr{L}\mathscr{F}[D](\xi', 0, \lambda) \right] (x', t), \tag{7.14}$$

where D is a given function in (7.6). We show the following lemma.

Lemma 7.1. *Let $1 < p, q < \infty$ and $0 < \epsilon < \pi/2$. Then, there exists a $\gamma_0 \geq 1$ depending on ϵ such that the following assertions hold true*:

(1) *If $\lambda \in \Sigma_{\epsilon, \gamma_0}$ and $d \in W_q^2(\boldsymbol{R}_+^n)$, then $\eta \in W_q^3(\boldsymbol{R}_+^n)$ and*

$$\begin{aligned} |\lambda| \|\eta\|_{W_q^2(\boldsymbol{R}_+^n)} + \|\eta\|_{W_q^3(\boldsymbol{R}_+^n)} &\leq C_{n,q,\epsilon,\gamma_0} \|d\|_{W_q^2(\boldsymbol{R}_+^n)}, \\ |\lambda|^{3/2} \|\eta\|_{W_q^1(\boldsymbol{R}_+^n)} &\leq C_{n,q,\epsilon,\gamma_0} |\lambda|^{1/2} \|d\|_{W_q^1(\boldsymbol{R}_+^n)}. \end{aligned}$$

(2) *If $D \in L_{p,0,\gamma_0}(\boldsymbol{R}, W_q^2(\boldsymbol{R}_+^n))$, then $Y \in L_{p,0,\gamma_0}(\boldsymbol{R}, W_q^2(\boldsymbol{R}_+^n)) \cap W_{p,0,\gamma_0}^1(\boldsymbol{R}, W_q^2(\boldsymbol{R}_+^n))$ and*

$$\begin{aligned} &\|e^{-\gamma t}(Y_t, \gamma Y)\|_{L_p(\boldsymbol{R}, W_q^2(\boldsymbol{R}_+^n))} + \|e^{-\gamma t} Y\|_{L_p(\boldsymbol{R}, W_q^3(\boldsymbol{R}_+^n))} \\ &\quad \leq C_{n,p,q,\gamma_0} \|e^{-\gamma t} D\|_{L_p(\boldsymbol{R}, W_q^2(\boldsymbol{R}_+^n))} \end{aligned}$$

for any $\gamma \leq \gamma_0$. If $D \in H_{p,0,\gamma_0}^{1/2}(\boldsymbol{R}, W_q^1(\boldsymbol{R}_+^n))$, then $Y \in H_{p,0,\gamma_0}^{3/2}(\boldsymbol{R}, W_q^1(\boldsymbol{R}_+^n))$

and

$$\|e^{-\gamma t}\Lambda_\gamma^{3/2}Y\|_{L_p(\boldsymbol{R},W_q^1(\boldsymbol{R}_+^n))} \leq C_{n,p,q,\gamma_0}\|e^{-\gamma t}\Lambda_\gamma^{1/2}D\|_{L_p(\boldsymbol{R},W_q^1(\boldsymbol{R}_+^n))}$$

for any $\gamma \geq \gamma_0$.

For a while we assume that Lemma 7.1 holds true. In view of (7.8), (7.9) and (7.10), setting $z(x) = (c_g - c_\sigma\Delta')\eta(x)$, we define $u(x)$, $\theta(x)$ and $\tilde\theta(x)$ by the formulas:

$$\begin{aligned}
u_j(x) &= -\mathscr{F}_{\xi'}^{-1}\left[\frac{(A^2+B^2)e^{-Ax_n}i\xi_j}{\mu(B-A)D(A,B)}\hat z(\xi',0)\right](x') \\
&\quad + \mathscr{F}_{\xi'}^{-1}\left[\frac{2ABe^{-Bx_n}i\xi_j}{\mu(B-A)D(A,B)}\hat z(\xi',0)\right](x'), \\
u_n(x) &= \mathscr{F}_{\xi'}^{-1}\left[\frac{A(A^2+B^2)e^{-Ax_n}}{\mu(B-A)D(A,B)}\hat z(\xi',0)\right](x') \\
&\quad - \mathscr{F}_{\xi'}^{-1}\left[\frac{2A^3e^{-Bx_n}}{\mu(B-A)D(A,B)}\hat z(\xi',0)\right](x'), \\
\theta(x) &= \mathscr{F}_{\xi'}^{-1}\left[\frac{(A+B)(A^2+B^2)e^{-Ax_n}}{D(A,B)}\hat z(\xi',0)\right](x').
\end{aligned}$$

By the observations in Section 6 and Lemma 7.1 we have

$$\begin{aligned}
&u(x) \in W_q^2(\boldsymbol{R}_+^n)^n, \quad \theta \in \hat W_q^1(\boldsymbol{R}_+^n), \\
&\|(\lambda u, |\lambda|^{1/2}\nabla u, \nabla^2 u, \nabla\theta)\|_{L_q(\boldsymbol{R}_+^n)} \\
&\quad \leq C_{n,q,\epsilon,\gamma_0}\|(\nabla z, |\lambda|^{1/2}z)\|_{L_q(\boldsymbol{R}_+^n)} \leq C_{n,q,\epsilon,\gamma_0}\|d\|_{W_q^2(\boldsymbol{R}_+^n)}
\end{aligned}$$

for any $\lambda \in \Sigma_{\epsilon,\gamma_0}$, where we have used the fact that $|\lambda|^{1/2} \leq |\lambda|$ when $\lambda \in \Sigma_{\epsilon,\gamma_0}$. Moreover, (u,η,θ) solves the problem (7.5).

And also, setting $Z(x,t) = (c_g - c_\sigma\Delta')Y(x,t)$ we define $U(x,t)$ and $\Theta(x,t)$ by the formulas:

$$\begin{aligned}
U_j(x,t) &= -\mathscr{L}_\lambda^{-1}\mathscr{F}_{\xi'}^{-1}\left[\frac{(A^2+B^2)e^{-Ax_n}i\xi_j}{\mu(B-A)D(A,B)}\mathscr{L}\mathscr{F}[Z](\xi',0,\lambda)\right](x',t) \\
&\quad + \mathscr{L}_\lambda^{-1}\mathscr{F}_{\xi'}^{-1}\left[\frac{2ABe^{-Bx_n}i\xi_j}{\mu(B-A)D(A,B)}\mathscr{L}\mathscr{F}[Z](\xi',0,\lambda)\right](x',t),
\end{aligned}$$

$$
\begin{aligned}
U_n(x,t) &= \mathscr{L}_\lambda^{-1}\mathscr{F}_{\xi'}^{-1}\left[\frac{A(A^2+B^2)e^{-Ax_n}}{\mu(B-A)D(A,B)}\mathscr{L}\mathscr{F}[Z](\xi',0,\lambda)\right](x',t)\\
&\qquad -\mathscr{L}_\lambda^{-1}\mathscr{F}_{\xi'}^{-1}\left[\frac{2A^3e^{-Bx_n}}{\mu(B-A)D(A,B)}\mathscr{L}\mathscr{F}[Z](\xi',0,\lambda)\right](x',t),\\
\Theta(x,t) &= \mathscr{L}_\lambda^{-1}\mathscr{F}_{\xi'}^{-1}\left[\frac{(A+B)(A^2+B^2)e^{-Ax_n}}{D(A,B)}\mathscr{L}\mathscr{F}[Z](\xi',0,\lambda)\right](x',t).
\end{aligned}
$$

By the observations in Section 6 and Lemma 7.1 we have

$$
\begin{aligned}
&U \in L_{p,0,\gamma_0}\big(\boldsymbol{R}, W_q^2(\boldsymbol{R}_+^n)^n\big)\cap W^1_{p,0,\gamma_0}\big(\boldsymbol{R}, L_q(\boldsymbol{R}_+^n)^n\big), \quad \Theta\in L_{p,0,\gamma_0}\big(\boldsymbol{R}, \hat{W}_q^1(\boldsymbol{R}_+^n)\big),\\
&\big\|e^{-\gamma t}\big(U_t, \gamma U, \Lambda_\gamma^{1/2}\nabla U, \nabla^2 U, \nabla\Theta\big)\big\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}_+^n))}\\
&\quad \le C_{n,p,q,\gamma_0}\big\|e^{-\gamma t}(\nabla Z, \Lambda_\gamma^{1/2}Z)\big\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}_+^n))} \le C_{p,q,\gamma_0}\|e^{-\gamma t}D\|_{L_p(\boldsymbol{R}, W_q^2(\boldsymbol{R}_+^n))}
\end{aligned}
$$

for any $\gamma\ge\gamma_0$, where we have used the fact that

$$
\big\|e^{-\gamma t}\Lambda_\gamma^{1/2}Z\big\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}_+^n))} \le C_{n,p,q}\gamma^{-1/2}\|e^{-\gamma t}Z_t\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}_+^n))} \quad (\gamma\ge\gamma_0\ge 1)
$$

which follows from Theorem 2.4 and the inequalities:

$$
\big|(\tau\partial_\tau)^\ell(|\lambda|^{1/2}\lambda^{-1})\big| \le C\gamma^{-1/2} \quad (\lambda=\gamma+i\tau\in\Sigma_{\epsilon,\gamma_0},\ \ell=0,1).
$$

Moreover, (U,Θ,Y) solves the problem (7.6). Therefore, to complete the proofs of Theorem 1.3 and Theorem 1.4, it is sufficient to prove Lemma 7.1.

To prove Lemma 7.1, we start with the following lemma.

Lemma 7.2. *Let $0<\epsilon<\pi/2$. Then, there exists a $\gamma_0\ge 1$ depending only on ϵ, c_g, c_σ and μ such that for any $\lambda\in\Sigma_{\epsilon,\gamma_0}$, any $\xi'\in\boldsymbol{R}^{n-1}\setminus\{0\}$, any multi-index $\alpha'=(\alpha_1,\dots,\alpha_{n-1})\in\boldsymbol{N}_0^{n-1}$ and $\ell=0,1$ we have*

$$
\begin{aligned}
&\big|D_{\xi'}^{\alpha'}\{(\tau\partial_\tau)^\ell L(A,B)^{-1}\}\big|\\
&\quad \le C_{\alpha',\epsilon,\gamma_0,\mu}\big(|\lambda|(|\lambda|^{1/2}+A)^2 + A(c_g+c_\sigma A^2)\big)^{-1}A^{-|\alpha'|}. \qquad (7.15)
\end{aligned}
$$

Proof. When $\ell=0$, the assertion was proved by Shibata and Shimizu [**35**]. Since

$$
\partial_\tau L(A,B)^{-1} = -L(A,B)^{-2}i\mu(2(A^2+B^2)-A^3B^{-1}),
$$

by the Leibniz rule, Lemma 5.2 and (7.15) with $\ell = 0$ we have

$$\begin{aligned}
&\left|D^{\alpha'}_{\xi'}(\tau\partial_\tau L(A,B)^{-1})\right| \\
&\leq C_{\alpha'}\mu^{-1}|\tau| \sum_{\beta'+\gamma'+\delta'=\alpha'} \left|D^{\beta'}_{\xi'} L(A,B)^{-1}\right|\left|D^{\gamma'}_{\xi'} L(A,B)^{-1}\right| \\
&\qquad\qquad\qquad\qquad \cdot \left|D^{\delta'}_{\xi'}(2(A^2+B^2) - A^3B^{-1})\right| \\
&\leq C_{\alpha',\epsilon,\gamma_0,\mu}|\lambda|\bigl\{|\lambda|((|\lambda|^{1/2}+A)^2 + A(c_g + c_\sigma A^2)\bigr\}^{-2} A^{-|\alpha'|} \\
&\leq C_{\alpha',\epsilon,\gamma_0,\mu}\bigl\{|\lambda|((|\lambda|^{1/2}+A)^2 + A(c_g + c_\sigma A^2)\bigr\}^{-1} A^{-|\alpha'|},
\end{aligned}$$

which completes the proof of the lemma. □

A proof of Lemma 7.1. Let η and Y be functions defined in (7.13) and (7.14). To estimate $\lambda\eta$ and $(Y_t, \gamma Y)$, using the identity:

$$(A+B)L(A,B) = \lambda\mu^{-1}D(A,B) + A(A+B)\mu^{-2}(c_g + c_\sigma A^2),$$

we write

$$\begin{aligned}
\eta(x) &= \varphi(x_n)\biggl\{\frac{1}{\lambda}\mathscr{F}^{-1}_{\xi'}\bigl[e^{-Ax_n}\hat{d}(\xi',0)\bigr](x') \\
&\qquad - \mathscr{F}^{-1}_{\xi'}\left[\frac{\mu^{-2}(c_g + c_\sigma A^2)}{\lambda L(A,B)} A e^{-Ax_n}\hat{d}(\xi',0)\right](x')\biggr\}, \\
Y(x,t) &= \varphi(x_n)\biggl\{\mathscr{L}^{-1}_\lambda\mathscr{F}^{-1}_{\xi'}\left[\frac{1}{\lambda}e^{-Ax_n}\mathscr{L}\mathscr{F}[D](\xi',0,\lambda)\right](x',t) \\
&\qquad - \mathscr{L}^{-1}_\lambda\mathscr{F}^{-1}_{\xi'}\left[\frac{\mu^{-2}(c_g + c_\sigma A^2)}{\lambda L(A,B)} A e^{-Ax_n}\mathscr{L}\mathscr{F}[D](\xi',0,\lambda)\right](x',t)\biggr\}.
\end{aligned} \tag{7.16}$$

We set

$$\begin{aligned}
\eta_1(x) &= \frac{1}{\lambda}K[d](x), \quad K[d](x) = \mathscr{F}^{-1}_{\xi'}\bigl[e^{-Ax_n}\hat{d}(\xi',0)\bigr](x'), \\
Y_1(x,t) &= \mathscr{L}^{-1}_\lambda\mathscr{F}^{-1}_{\xi'}\left[\frac{1}{\lambda}e^{-Ax_n}\mathscr{L}\mathscr{F}[D](\xi',0,\lambda)\right](x',t) \\
&= \mathscr{L}^{-1}\left[\frac{1}{\lambda}\mathscr{L}[K[D(\cdot,0,t)](x')](\lambda)\right](t).
\end{aligned} \tag{7.17}$$

By Lemma 5.3 and Fourier multiplier theorem of S. G. Mihlin, we have

$$\sup_{x_n \geq 0} \|K[d](\cdot, x_n)\|_{L_q(\boldsymbol{R}^{n-1})} \leq C_{n,q}\|d(\cdot, 0)\|_{L_q(\boldsymbol{R}^{n-1})}, \tag{7.18}$$

where $L_q(\boldsymbol{R}^{n-1})$ is the usual L_q space of functions with respect to $x' = (x_1, \dots, x_{n-1})$ variables. By Theorem 2.4 we have

$$\begin{aligned}
&\big\|e^{-\gamma t}(D_t Y_1, \gamma Y_1)(\cdot, x_n, t)\big\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^{n-1}))} \\
&\qquad \leq C_p \big\|e^{-\gamma t} K[D(\cdot, t)](\cdot, x_n)\big\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^{n-1}))}, \\
&\big\|e^{-\gamma t}(\Lambda_\gamma^{3/2} Y_1)(\cdot, x_n, t)\big\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^{n-1}))} \\
&\qquad \leq C_p \big\|e^{-\gamma t} K[(\Lambda_\gamma^{1/2} D)(\cdot, t)](\cdot, x_n)\big\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^{n-1}))}
\end{aligned} \tag{7.19}$$

for any $\gamma \geq 0$. Combining (7.19) and (7.18), we have

$$\begin{aligned}
&\sup_{x_n \geq 0} \big\|e^{-\gamma t}(D_t Y_1, \gamma Y_1)(\cdot, x_n, t)\big\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^{n-1}))} \\
&\qquad \leq C_{p,q} \big\|e^{-\gamma t} D(\cdot, 0, \cdot)\big\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^{n-1}))}, \\
&\sup_{x_n \geq 0} \big\|e^{-\gamma t}(\Lambda_\gamma^{3/2} Y_1)(\cdot, x_n, t)\big\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^{n-1}))} \\
&\qquad \leq C_{p,q} \big\|e^{-\gamma t}(\Lambda_\gamma^{1/2} D)(\cdot, 0, \cdot)\big\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^{n-1}))}.
\end{aligned} \tag{7.20}$$

To estimate the derivatives of $K[d]$, by a trick due to Volevich we write

$$\begin{aligned}
K[d](x) = -\int_0^\infty \mathscr{F}_{\xi'}^{-1}\bigg[e^{-A(x_n+y_n)}\bigg(&\sum_{\ell=1}^{n-1}(i\xi_\ell A^{-1})\mathscr{F}[D_\ell d](\xi', y_n) \\
&+ \mathscr{F}[D_n d](\xi', y_n)\bigg)\bigg](x')\, dy_n.
\end{aligned}$$

Therefore, for $j, k = 1, \dots, n-1$ we have

$$\begin{aligned}
D_j K[d](x) = -\int_0^\infty \mathscr{F}_{\xi'}^{-1}\bigg[A e^{-A(x_n+y_n)}(i\xi_j A^{-1})\bigg(&\sum_{\ell=1}^{n-1}(i\xi_\ell A^{-1})\mathscr{F}[D_\ell d](\xi', y_n) \\
&+ \mathscr{F}[D_n d](\xi', y_n)\bigg)\bigg](x')\, dy_n,
\end{aligned}$$

$$D_nK[d](x) = \int_0^\infty \mathscr{F}^{-1}_{\xi'}\Big[Ae^{-A(x_n+y_n)}\Big(\sum_{\ell=1}^{n-1}(i\xi_\ell A^{-1})\mathscr{F}[D_\ell d](\xi', y_n) + \mathscr{F}[D_n d](\xi', y_n)\Big)\Big](x')\,dy_n,$$

$$D_jD_kK[d](x) = -\int_0^\infty \mathscr{F}^{-1}_{\xi'}\Big[Ae^{-A(x_n+y_n)}(i\xi_j A^{-1}) \cdot \Big(\sum_{\ell=1}^{n-1}(i\xi_\ell A^{-1})\mathscr{F}[D_kD_\ell d](\xi', y_n) + \mathscr{F}[D_kD_n d](\xi', y_n)\Big)\Big](x')\,dy_n,$$

$$D_jD_nK[d](x) = \int_0^\infty \mathscr{F}^{-1}_{\xi'}\Big[Ae^{-A(x_n+y_n)}\Big(\sum_{\ell=1}^{n-1}(i\xi_\ell A^{-1})\mathscr{F}[D_jD_\ell d](\xi', y_n) + \mathscr{F}[D_jD_n d](\xi', y_n)\Big)\Big](x')\,dy_n,$$

$$D_n^2K[d](x) = -\int_0^\infty \mathscr{F}^{-1}_{\xi'}\Big[Ae^{-A(x_n+y_n)}\sum_{m=1}^{n-1}(i\xi_m A^{-1}) \cdot \Big(\sum_{\ell=1}^{n-1}(i\xi_\ell A^{-1})\mathscr{F}[D_mD_\ell d](\xi', x_n) + \mathscr{F}[D_mD_n d](\xi', y_n)\Big)\Big](x')\,dy_n.$$

Since $\xi_j A^{-1} \in \boldsymbol{M}_{0,2,\epsilon,\gamma_0}$, applying Lemma 5.4, we have

$$\|\nabla^\ell K[d]\|_{L_q(\boldsymbol{R}^n_+)} \leq C_{n,q}\|\nabla^\ell d\|_{L_q(\boldsymbol{R}^n_+)} \tag{7.21}$$

for $\ell = 1, 2$ where $\nabla^1 d = \nabla d$. Aplying (7.21) to the formulas of $\eta_1(x)$ and $Y_1(x,t)$ in (7.17) and using Theorem 2.4 for the estimate of Y_1, we have

$$\begin{aligned} |\lambda|^{1+s}\|\nabla\eta_1\|_{L_q(\boldsymbol{R}^n_+)} &\leq C_{n,q}|\lambda|^s\|\nabla d\|_{L_q(\boldsymbol{R}^n_+)} \quad (s = 0, 1/2), \\ |\lambda|\|\nabla^2\eta_1\|_{L_q(\boldsymbol{R}^n_+)} &\leq C_{n,q}\|\nabla^2 d\|_{L_q(\boldsymbol{R}^n_+)}, \end{aligned} \tag{7.22}$$

$$\begin{aligned} \|e^{-\gamma t}\nabla(D_tY_1, \gamma Y_1)\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^n_+))} &\leq C_{p,q}\|e^{-\gamma t}\nabla D\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^n_+))}, \\ \big\|e^{-\gamma t}\nabla\Lambda^{3/2}_\gamma Y_1\big\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^n_+))} &\leq C_{p,q}\big\|e^{-\gamma t}\nabla\Lambda^{1/2}_\gamma D\big\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^n_+))}, \\ \|e^{-\gamma t}\nabla^2(D_tY_1, \gamma Y_1)\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^n_+))} &\leq C_{p,q}\|e^{-\gamma t}\nabla^2 D\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^n_+))}. \end{aligned} \tag{7.23}$$

In view of (7.16), we set

$$\eta_2(x) = -\mathscr{F}^{-1}_{\xi'}\left[\frac{\mu^{-2}(c_g + c_\sigma A^2)}{\lambda L(A,B)} A e^{-Ax_n} \hat{d}(\xi', 0)\right](x'),$$

$$Y_2(x,t) = -\mathscr{L}^{-1}_\lambda \mathscr{F}^{-1}_{\xi'}\left[\frac{\mu^{-2}(c_g + c_\sigma A^2)}{\lambda L(A,B)} A e^{-Ax_n} \mathscr{L}\mathscr{F}[D](\xi', 0, \lambda)\right](x', t),$$

and then $\eta(x) = \varphi(x_n)(\eta_1(x) + \eta_2(x))$ and $Y(x,t) = \varphi(x_n)(Y_1(x,t) + Y_2(x,t))$. By using a trick due to Volevich, we write

$$\begin{aligned}\lambda\eta_2(x) &= \int_0^\infty \mathscr{F}^{-1}_{\xi'}\left[\frac{\mu^{-2}(c_g + c_\sigma A^2)}{L(A,B)} A e^{-A(x_n+y_n)} \mathscr{F}[D_n d](\xi', y_n)\right](x')\,dy_n \\ &\quad + \sum_{\ell=1}^{n-1} \int_0^\infty \mathscr{F}^{-1}_{\xi'}\left[\frac{\mu^{-2}(c_g + c_\sigma A^2)(i\xi_\ell A^{-1})}{L(A,B)}\right. \\ &\qquad\qquad \left. \cdot\, A e^{-A(x_n+y_n)} \mathscr{F}[D_\ell d](\xi', y_n)\right](x')\,dy_n.\end{aligned}$$

Lemma 7.2 and Lemma 5.1 imply that the symbols: $(c_g + c_\sigma A^2)/(L(A,B))$ and $((c_g + c_\sigma A^2)(i\xi_\ell A^{-1}))/(L(A,B))$ belong to $\boldsymbol{M}_{-1,2,\epsilon,\gamma_0}$. Applying Lemma 5.7, we have

$$\begin{aligned}|\lambda|^{1+s}\|\eta_2\|_{W^1_q(\boldsymbol{R}^n_+)} &\le C_{n,q,\epsilon,\gamma_0}|\lambda|^s\|\nabla d\|_{L_q(\boldsymbol{R}^n_+)} \quad \left(\lambda \in \Sigma_{\epsilon,\gamma_0},\ s = 0, \frac{1}{2}\right), \\ \|e^{-\gamma t}(D_t Y_2, \gamma Y_2)\|_{L_p(\boldsymbol{R}, W^1_q(\boldsymbol{R}^n_+))} &\le C_{n,p,q,\gamma_0}\|e^{-\gamma t}\nabla D\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^n_+))} \ (\gamma \ge \gamma_0), \\ \left\|e^{-\gamma t}\Lambda^{3/2}_\gamma Y_2\right\|_{L_p(\boldsymbol{R}, W^1_q(\boldsymbol{R}^n_+))} &\le C_{n,p,q,\gamma_0}\left\|e^{-\gamma t}\nabla\Lambda^{1/2} D\right\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^n_+))} \ (\gamma \ge \gamma_0).\end{aligned} \tag{7.24}$$

Moreover, for $j, \dots, n-1$ we have

$$\begin{aligned}\lambda D_j\eta_2(x) &= \int_0^\infty \mathscr{F}^{-1}_{\xi'}\left[\frac{\mu^{-2}(c_g + c_\sigma A^2)}{L(A,B)} A e^{-A(x_n+y_n)} \mathscr{F}[D_j D_n d](\xi', y_n)\right](x')\,dy_n \\ &\quad + \sum_{\ell=1}^{n-1} \int_0^\infty \mathscr{F}^{-1}_{\xi'}\left[\frac{\mu^{-2}(c_g + c_\sigma A^2)}{L(A,B)}\right. \\ &\qquad\qquad \left. \cdot\, A e^{-A(x_n+y_n)} \mathscr{F}[D_j D_\ell d](\xi', y_n)\right](x')\,dy_n,\end{aligned}$$

$$
\begin{aligned}
\lambda D_n \eta_2(x) = &- \sum_{m=1}^{n-1} \int_0^\infty \mathscr{F}_{\xi'}^{-1}\Bigg[\frac{\mu^{-2}(c_g + c_\sigma A^2)(i\xi_m A^{-1})}{L(A,B)} \\
&\qquad\qquad \cdot A e^{-A(x_n+y_n)} \mathscr{F}[D_m D_n d](\xi', y_n)\Bigg](x')\,dy_n \\
&- \sum_{\ell,m=1}^{n-1} \int_0^\infty \mathscr{F}_{\xi'}^{-1}\Bigg[\frac{\mu^{-2}(c_g + c_\sigma A^2)(i\xi_m A^{-1})}{L(A,B)} \\
&\qquad\qquad \cdot A e^{-A(x_n+y_n)} \mathscr{F}[D_\ell D_m d](\xi', y_n)\Bigg](x')\,dy_n.
\end{aligned}
$$

Lemma 7.2 and Lemma 5.1 imply that the symbols: $((c_g + c_\sigma A^2)(i\xi_m A^{-1}))/(L(A,B))$ belong to $\boldsymbol{M}_{-1,2,\epsilon,\gamma_0}$. Applying Lemma 5.7, we have

$$
\begin{aligned}
|\lambda|\|\nabla^2\eta_2\|_{L_q(\boldsymbol{R}^n_+)} &\le C_{n,q,\epsilon,\gamma_0}\|\nabla^2 d\|_{L_q(\boldsymbol{R}^n_+)} \quad (\lambda \in \Sigma_{\epsilon,\gamma_0}), \\
\|e^{-\gamma t}\nabla^2(D_t Y_2, \gamma Y_2)\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^n_+))} &\le C_{n,p,q,\gamma_0}\|e^{-\gamma t}\nabla^2 D\|_{L_p(\boldsymbol{R}, L_q(\boldsymbol{R}^n_+))} \quad (\gamma \ge \gamma_0).
\end{aligned}
\tag{7.25}
$$

Noting that $\|d(\cdot,0)\|_{L_q(\boldsymbol{R}^{n-1})} \le C_{n,q}\|d\|_{W^1_q(\boldsymbol{R}^n_+)}$, by (7.16), (7.18), (7.20), (7.22), (7.23), (7.24) and (7.25) we have

$$
\begin{aligned}
|\lambda|\|\eta\|_{W^2_q(\boldsymbol{R}^n_+)} &\le C_{n,q,\epsilon,\gamma_0}\|d\|_{W^2_q(\boldsymbol{R}^n_+)} && (\lambda \in \Sigma_{\epsilon,\gamma_0}), \\
|\lambda|^{3/2}\|\eta\|_{W^1_q(\boldsymbol{R}^n_+)} &\le C_{n,q,\epsilon,\gamma_0}|\lambda|^{1/2}\|d\|_{W^1_q(\boldsymbol{R}^n_+)} && (\lambda \in \Sigma_{\epsilon,\gamma_0}), \\
\|e^{-\gamma t}(Y_t, \gamma Y)\|_{L_p(\boldsymbol{R}, W^2_q(\boldsymbol{R}^n_+))} &\le C_{n,p,q,\gamma_0}\|e^{-\gamma t} D\|_{L_p(\boldsymbol{R}, W^2_q(\boldsymbol{R}^n_+))} && (\gamma \ge \gamma_0), \\
\big\|e^{-\gamma t}\Lambda_\gamma^{3/2} Y\big\|_{L_p(\boldsymbol{R}, W^1_q(\boldsymbol{R}^n_+))} &\le C_{n,p,q,\gamma_0}\|e^{-\gamma t}\Lambda_\gamma^{1/2} D\|_{L_p(\boldsymbol{R}, W^1_q(\boldsymbol{R}^n_+))} && (\gamma \ge \gamma_0).
\end{aligned}
$$

To estimate the 3rd spatial derivatives of η and Y, in view of (7.13) and (7.14), we set

$$
\begin{aligned}
\eta_3(x) &= \mathscr{F}_{\xi'}^{-1}\left[\frac{\mu^{-1} D(A,B) e^{-Ax_n}}{(A+B)L(A,B)}\hat{d}(\xi', 0)\right](x'), \\
Y_3(x,t) &= \mathscr{L}_\lambda^{-1}\mathscr{F}_{\xi'}^{-1}\left[\frac{\mu^{-1} D(A,B) e^{-Ax_n}}{(A+B)L(A,B)}\mathscr{L}\mathscr{F}[D](\xi', 0, \lambda)\right](x', t).
\end{aligned}
$$

Note that $\eta(x) = \varphi(x_n)\eta_3(x)$ and $Y(x,t) = \varphi(x_n)Y_3(x,t)$. Applying a trick due to Volevich, we write

$$\eta_3(x) = \int_0^\infty \mathscr{F}^{-1}_{\xi'}\left[\frac{\mu^{-1}D(A,B)}{(A+B)L(A,B)}Ae^{-A(x_n+y_n)}\mathscr{F}[d](\xi',y_n)\right](x')\,dy_n$$
$$-\int_0^\infty \mathscr{F}^{-1}_{\xi'}\left[\frac{\mu^{-1}D(A,B)}{(A+B)L(A,B)}e^{-A(x_n+y_n)}\mathscr{F}[D_nd](\xi',y_n)\right](x')\,dy_n.$$

And then, for $j,k=1,\dots,n-1$ we write

$$D_jD_k\eta_3(x) = \int_0^\infty \mathscr{F}^{-1}_{\xi'}\left[\frac{\mu^{-1}D(A,B)}{(A+B)L(A,B)}Ae^{-A(x_n+y_n)}\mathscr{F}[D_jD_kd](\xi',y_n)\right](x')\,dy_n$$
$$-\int_0^\infty \mathscr{F}^{-1}_{\xi'}\left[\frac{\mu^{-1}D(A,B)(i\xi_jA^{-1})}{(A+B)L(A,B)}\cdot Ae^{-A(x_n+y_n)}\mathscr{F}[D_kD_nd](\xi',y_n)\right](x')\,dy_n,$$

$$D_jD_n\eta_3(x) = \sum_{\ell=1}^{n-1}\int_0^\infty \mathscr{F}^{-1}_{\xi'}\left[\frac{\mu^{-1}D(A,B)(i\xi_\ell A^{-1})}{(A+B)L(A,B)}\cdot Ae^{-A(x_n+y_n)}\mathscr{F}[D_\ell D_kd](\xi',y_n)\right](x')\,dy_n$$
$$+\int_0^\infty \mathscr{F}^{-1}_{\xi'}\left[\frac{\mu^{-1}D(A,B)}{(A+B)L(A,B)}Ae^{-A(x_n+y_n)}\mathscr{F}[D_jD_nd](\xi',y_n)\right](x')\,dy_n,$$
$$D_n^2\eta_3(x) = -\int_0^\infty \mathscr{F}^{-1}_{\xi'}\left[\frac{\mu^{-1}D(A,B)}{(A+B)L(A,B)}Ae^{-A(x_n+y_n)}\mathscr{F}[\Delta'd](\xi',y_n)\right](x')\,dy_n$$
$$+\sum_{\ell=1}^{n-1}\int_0^\infty \mathscr{F}^{-1}_{\xi'}\left[\frac{\mu^{-1}D(A,B)(i\xi_\ell A^{-1})}{(A+B)L(A,B)}\cdot Ae^{-A(x_n+y_n)}\mathscr{F}[D_\ell D_nd](\xi',y_n)\right](x')\,dy_n.$$

Lemma 7.2, Lemma 5.2 and Lemma 5.1 imply that the symbols:

$$\frac{D(A,B)}{(A+B)L(A,B)},\quad \frac{D(A,B)(i\xi_jA^{-1})}{(A+B)L(A,B)},\quad \frac{D(A,B)(i\xi_\ell A^{-1})}{(A+B)L(A,B)}$$

belong to $\boldsymbol{M}_{-1,2,\epsilon,\gamma_0}$, so that applying Lemma 5.7 we have

$$\|\nabla^2\eta_3\|_{W^1_q(\boldsymbol{R}^n_+)} \le C_{n,q,\epsilon,\gamma_0}\|\nabla^2 d\|_{L_q(\boldsymbol{R}^n_+)}\quad (\lambda\in\Sigma_{\epsilon,\gamma_0}),$$
$$\|e^{-\gamma t}\nabla^2 Y_3\|_{L_p(\boldsymbol{R},W^1_q(\boldsymbol{R}^n_+))} \le C_{n,q,\epsilon,\gamma_0}\|e^{-\gamma t}\nabla^2 D\|_{L_p(\boldsymbol{R},L_q(\boldsymbol{R}^n_+))}\quad (\gamma\ge\gamma_0),$$

which completes the proof of Lemma 7.1. □

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Yoshihiro Shibata
Department of Mathematics and
Research Institute of Science and Engineering
JST CREST
Waseda University
Ohkubo 3-4-1, Shinjuku-ku
Tokyo 169-8555, Japan
E-mail: yshibata@waseda.jp

Senjo Shimizu
Department of Mathematics
Faculty of Science
Shizuoka University
Shizuoka 422-8529, Japan
E-mail: ssshimi@ipc.shizuoka.ac.jp