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On the indecomposable modules in almost cyclic coherent Auslander-Reiten components

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Abstract. We establish an inequality between the dimensions of the endomorphism and extension spaces of the indecomposable modules in generalized standard almost cyclic coherent components of the Auslander-Reiten quivers of finite dimensional algebras over an arbitrary base field. As an application we provide a homological characterization, involving the Euler quadratic form, of the tame algebras with separating families of almost cyclic coherent Auslander-Reiten components.

1. Introduction and the main results.

Throughout the paper, K will denote a fixed field. By an algebra we mean a finite dimensional K-algebra with an identity, which we shall assume (without loss of generality) to be basic. For an algebra A, we denote by $\operatorname{mod} A$ the category of finite dimensional right A-modules, by $\operatorname{rad}(\operatorname{mod} A)$ the Jacobson radical of $\operatorname{mod} A$, and by $\operatorname{rad}^{\infty}(\operatorname{mod} A)$ the intersection of all powers $\operatorname{rad}^{i}(\operatorname{mod} A)$, $i \geq 1$, of $\operatorname{rad}(\operatorname{mod} A)$. We shall denote by Γ_A the *Auslander-Reiten quiver* of A, and by τ_A and τ_A^{-} the Auslander-Reiten translations $D\mathrm{Tr}$ and $\mathrm{Tr}D$, respectively. We will not distinguish between an indecomposable module in $\operatorname{mod} A$ and the vertex of Γ_A corresponding to it. Following [**30**], a component $\mathscr{C}$ of Γ_A is called *generalized standard* if $\operatorname{rad}^{\infty}(X, Y) = 0$ for all modules X and Y in $\mathscr{C}$. It has been proved in [**30**] that every generalized standard component $\mathscr{C}$ of Γ_A is *quasi-periodic*, that is, all but finitely many τ_A-orbits in $\mathscr{C}$ are periodic.

The Auslander-Reiten quiver is an important combinatorial and homological invariant of the module category $\operatorname{mod} A$ of an algebra A. Frequently, we may recover A and the category $\operatorname{mod} A$ from the behaviour of distinguished components of Γ_A in $\operatorname{mod} A$. For example, the important classes of tilted algebras, double tilted algebras, generalized double tilted algebras are the algebras whose Auslander-

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Reiten quiver admits a faithful generalized standard component with a section, double section, multisection, respectively (see [**15**], [**22**], [**23**], [**29**]).

In the representation theory of algebras a prominent role is played by the algebras whose Auslander-Reiten quiver admits a separating family of almost cyclic coherent components. Recall that a family $\mathscr{C} = (\mathscr{C}_i)_{i\in I}$ of components of Γ_A is called *separating* in $\operatorname{mod} A$ if the components in Γ_A split into three disjoint classes $\mathscr{P}_A$, $\mathscr{C}_A = \mathscr{C}$ and $\mathscr{Q}_A$ such that:

(S1) $\mathscr{C}_A$ is a sincere family of pairwise orthogonal generalized standard components;
(S2) $\operatorname{Hom}_A(\mathscr{Q}_A, \mathscr{P}_A) = 0$, $\operatorname{Hom}_A(\mathscr{Q}_A, \mathscr{C}_A) = 0$, $\operatorname{Hom}_A(\mathscr{C}_A, \mathscr{P}_A) = 0$;
(S3) any morphism from $\mathscr{P}_A$ to $\mathscr{Q}_A$ factors through $\operatorname{add}(\mathscr{C}_A)$.

We then say that $\mathscr{C}_A$ separates $\mathscr{P}_A$ from $\mathscr{Q}_A$ and write $\Gamma_A = \mathscr{P}_A \vee \mathscr{C}_A \vee \mathscr{Q}_A$. We also note that then $\mathscr{P}_A$ and $\mathscr{Q}_A$ are uniquely determined by $\mathscr{C}_A$ (see [**4**, (2.1)]). Further, a component Γ of Γ_A is called *almost cyclic* if all but finitely many modules of Γ lie on oriented cycles contained entirely in Γ. Further, a component Γ of Γ_A is called *coherent* if the following two conditions are satisfied:

(C1) For each projective module P in Γ there is an infinite sectional path
$P = X_1 \to X_2 \to \cdots \to X_i \to X_{i+1} \to X_{i+2} \to \cdots$;
(C2) For each injective module I in Γ there is an infinite sectional path
$\cdots \to Y_{j+2} \to Y_{j+1} \to Y_j \to \cdots \to Y_2 \to Y_1 = I$.

The authors proved in [**18**, Theorem A] that the Auslander-Reiten quiver Γ_A of an algebra A admits a separating family of almost cyclic coherent components if and only if A is a generalized multicoil enlargement of a finite family of concealed canonical algebras. Moreover, for such an algebra A, we have $\operatorname{gl\,dim} A \leq 3$, and $\operatorname{pd}_A X \leq 2$ or $\operatorname{id}_A X \leq 2$ for any indecomposable module X in $\operatorname{mod} A$ (see [**18**, Corollary B and Theorem E]). We note that an algebra C is concealed canonical [**11**] if and only if Γ_C admits a separating family of stable tubes (see [**12**]). More generally, it has been proved in [**13**] that the quasitilted algebras of canonical type are exactly the algebras for which the Auslander-Reiten quiver admits a separating family of semiregular tubes (ray and coray tubes). Further, by [**8**] the class of algebras A with $\operatorname{gl\,dim} A \leq 2$ and $\operatorname{pd}_A X \leq 1$ or $\operatorname{id}_A X \leq 1$ for any indecomposable module X in $\operatorname{mod} A$ is the class of quasitilted algebras, that is, the endomorphism algebras $\operatorname{End}_{\mathscr{H}}(T)$ of tilting objects T in hereditary abelian categories $\mathscr{H}$. It has been proved in [**7**] that the class of quasitilted algebras consists of the tilted algebras and the quasi-tilted algebras of canonical type.

The general structure of the module category $\operatorname{mod} A$ as well as the Auslander-Reiten quiver Γ_A of an algebra A with a separating family of almost cyclic coherent components have been described in [**18**, Theorem C and Corollary D]. In particular, the genus $g(A)$ of such an algebra A was defined in [**18**], and it was shown that A is not wild if and only if $g(A) \leq 1$. For K algebraically closed, this is equivalent to the tameness of A, or to the weak nonnegativity of the Tits quadratic form q_A of A (see [**18**, Theorem F]). Moreover, geometric and homological characterizations of tame algebras with separating families of almost cyclic coherent components over an algebraically closed field K have been established in [**19**, Theorem B], where algebraic geometry arguments were essentially applied.

One of the aims of the paper is to establish a homological characterization of the tame algebras with separating families of almost cyclic coherent components over an arbitrary field.

Recall that the *Euler form* of an algebra A of finite global dimension is the quadratic form $\chi_A : K_0(A) \to \boldsymbol{Z}$ on the Grothendieck group $K_0(A)$ of A such that

$$\chi_A([M]) = \sum_{i=0}^{\infty} (-1)^i \dim_K \operatorname{Ext}_A^i(M, M),$$

where $[M]$ is the class of a module M from $\operatorname{mod} A$ in $K_0(A)$ (see [**24**], [**25**]).

The following theorem is the first main result of the paper.

THEOREM 1.1. *Let A be a finite dimensional K-algebra over a field K with a separating family of almost cyclic coherent components in Γ_A. The following statements are equivalent*:

(i) $g(A) \leq 1$.
(ii) $\chi_A([M]) \geq 0$ *for any indecomposable module M in* $\operatorname{mod} A$.
(iii) $\dim_K \operatorname{Ext}_A^1(M, M) \leq \dim_K \operatorname{End}_A(M)$ *and* $\operatorname{Ext}_A^r(M, M) = 0$ *for any $r \geq 2$ and any indecomposable module M in* $\operatorname{mod} A$.

In the course of our proof of the above theorem, we establish also the following fact.

COROLLARY 1.2. *Let A be a finite dimensional K-algebra over a field K with a separating family of almost cyclic coherent components in Γ_A and $g(A) \leq 1$, and M be an indecomposable module in* $\operatorname{mod} A$. *The following statements are equivalent*:

(i) $\chi_A([M]) = 0$.
(ii) *There is a tame concealed canonical factor algebra C of A such that M lies in a stable tube $\mathscr{T}$ of Γ_C and the quasi-length of M in $\mathscr{T}$ is divisible by the rank of $\mathscr{T}$.*

We note that for a separating family $\mathscr{C}$ of almost cyclic coherent components in the Auslander-Reiten quiver Γ_A of an algebra A we may have indecomposable modules M in $\mathscr{C}$ with arbitrarily large $\chi_A([M])$ (see [**21**, (5.3)]). Moreover, the nonnegativity of the values of the Euler form on the classes of indecomposable modules is not the property of all tame algebras of finite global dimension. We refer to [**21**, (5.6)] for an example of a tame algebra A of global dimension 3 over an algebraically closed field which admits an infinite family X_n, $n \geq 1$, of finite dimensional indecomposable modules with $\chi_A([X_n]) = 1 - 3n$.

The proof of Theorem 1.1 is based on the following general result, which is the second main result of the paper.

Theorem 1.3. *Let A be a finite dimensional K-algebra over a field K, $\mathscr{C}$ be a generalized standard almost cyclic coherent component of Γ_A and M be an indecomposable module in $\mathscr{C}$. Then the following statements hold*:

(i) $\dim_K \operatorname{Ext}_A^1(M, M) \leq \dim_K \operatorname{End}_A(M)$.

(ii) $\dim_K \operatorname{Ext}_A^1(M, M) = \dim_K \operatorname{End}_A(M)$ *if and only if there is a factor algebra C and a generalized standard stable tube $\mathscr{T}$ of Γ_C such that M lies in $\mathscr{T}$ and the quasi-length of M in $\mathscr{T}$ is divisible by the rank of $\mathscr{T}$.*

We mention that by [**34**, Theorem 1] the additive category $\operatorname{add}(\mathscr{C})$ of an arbitrary generalized standard component $\mathscr{C}$ of an Auslander-Reiten quiver Γ_A is closed under extensions. We also note that the class of algebras whose Auslander-Reiten quiver admits generalized standard almost cyclic coherent components is wide and contains algebras of arbitrary nonzero, finite or infinite, global dimension. In particular, all multicoil enlargements (see Section 2) of concealed canonical algebras [**26**], generalized canonical algebras [**33**], and concealed generalized canonical algebras [**20**] have this property.

For basic background on the representation theory of algebras we refer to [**1**], [**5**], [**25**], [**27**], [**28**].

The main results of the paper were presented by the first-named author during the International Conference on Representations of Algebras, ICRA XIV held at Tokyo in August 2010.

2. Generalized standard stable tubes.

The aim of this section is to recall some facts on generalized standard stable tubes, applied in the proof of Theorem 1.3.

Recall that if $\boldsymbol{A}_\infty$ is the quiver $0 \to 1 \to 2 \to \cdots$ (with trivial valuations (1,1)), then $\boldsymbol{Z}\boldsymbol{A}_\infty$ is the translation quiver of the form:

$$\begin{tikzcd}[column sep=tiny, row sep=small]
 & (i-1,0) \arrow[dr] & & (i,0) \arrow[dr] & & (i+1,0) \arrow[dr] & & (i+2,0) \arrow[dr] & \\
\iddots \arrow[ur] & & (i-1,1) \arrow[ur] \arrow[dr] & & (i,1) \arrow[ur] \arrow[dr] & & (i+1,1) \arrow[ur] \arrow[dr] & & \ddots \\
 & \iddots \arrow[ur] & & (i-1,2) \arrow[ur] \arrow[dr] & & (i,2) \arrow[ur] \arrow[dr] & & \ddots & \\
 & & \iddots \arrow[ur] & & \ddots \quad \iddots \arrow[ur] & & \ddots & &
\end{tikzcd}$$

with $\tau(i,j) = (i-1,j)$ for $i \in \boldsymbol{Z}$, $j \in \boldsymbol{N}$. For $r \geq 1$, denote by $\boldsymbol{Z}\boldsymbol{A}_\infty/(\tau^r)$ the translation quiver Γ obtained from $\boldsymbol{Z}\boldsymbol{A}_\infty$ by identifying each vertex (i,j) of $\boldsymbol{Z}\boldsymbol{A}_\infty$ with the vertex $\tau^r(i,j)$ and each arrow $x \to y$ in $\boldsymbol{Z}\boldsymbol{A}_\infty$ with the arrow $\tau^r x \to \tau^r y$. The translation quiver of the form $\boldsymbol{Z}\boldsymbol{A}_\infty/(\tau^r)$ is called *stable tube of rank* r. A stable tube of rank 1 is said to be *homogeneous*. The τ-orbit of a stable tube Γ formed by all vertices having exactly one immediate predecessor (equivalently, successor) is said to be the *mouth* of Γ.

The following characterization of generalized standard stable tubes of an Auslander-Reiten quiver has been established in [**30**, Corollary 5.3] (see also [**31**, Lemma 3.1]).

Proposition 2.1. *Let A be an algebra and $\mathscr{T}$ a stable tube of Γ_A. The following statements are equivalent*:

(i) *$\mathscr{T}$ is generalized standard.*
(ii) *The mouth of $\mathscr{T}$ consists of pairwise orthogonal bricks.*
(iii) $\operatorname{rad}^\infty(X,X) = 0$ *for any module X in $\mathscr{T}$.*

An indecomposable module X is called a *brick* if its endomorphism algebra $\operatorname{End}_A(X)$ is a division algebra. We also note that the division algebras of all modules X lying on the mouth of a generalized standard stable tube of Γ_A are isomorphic.

Let A be an algebra and $\mathscr{T}$ be a stable tube of Γ_A. For every indecomposable module M in $\mathscr{T}$ there exists a unique sectional path $X_1 \to X_2 \to \cdots \to X_m = M$ (possibly $m = 1$) with X_1 lying on the mouth of $\mathscr{T}$, and m is called the *quasi-length* of M in $\mathscr{T}$ which we shall denote by $\operatorname{ql}(M)$. For an indecomposable module M in $\operatorname{mod} A$, we abbreviate $F_M = \operatorname{End}_A(M)/\operatorname{rad}\operatorname{End}_A(M)$. Since $\operatorname{End}_A(M)$ is a local algebra, then F_M is a division algebra (over the base field K of A).

The following facts have been established in [**31**, Proposition 3.5].

Proposition 2.2. *Let A be an algebra over a field K, $\mathscr{T}$ a generalized standard stable tube of rank r in Γ_A, and M be an indecomposable module in $\mathscr{T}$. The following statements hold*:

(i) $\dim_K \operatorname{End}_A(M) = (p+1)\dim_K F_M$, *where $p \geq 0$ is such that $pr < \operatorname{ql}(M) \leq (p+1)r$.*

(ii) $\dim_K \operatorname{Ext}_A^1(M,M) = p \dim_K F_M$, *where* $p \geq 0$ *is such that* $pr \leq \operatorname{ql}(M) < (p+1)r$.

As an immediate consequence we obtain the following facts (see [**31**, Corollary 3.6]).

COROLLARY 2.3. *Let* A *be an algebra over a field* K, $\mathscr{T}$ *a generalized standard stable tube of rank* r *in* Γ_A, *and* M *be an indecomposable module in* $\mathscr{T}$. *Then the following statements hold*:

(i) $\dim_K \operatorname{Ext}_A^1(M,M) \leq \dim_K \operatorname{End}_A(M)$.
(ii) $\dim_K \operatorname{Ext}_A^1(M,M) = \dim_K \operatorname{End}_A(M)$ *if and only if* r *divides* $\operatorname{ql}(M)$.

We end this section with the following result.

PROPOSITION 2.4. *Let* A *be an algebra,* $\mathscr{T}$ *be a faithful generalized standard stable tube in* Γ_A, *and* M *be an indecomposable module in* $\mathscr{T}$. *Then* $\operatorname{Ext}_A^n(M,M) = 0$ *for any* $n \geq 2$.

PROOF. It follows from [**30**, Lemma 5.9] that $\operatorname{pd}_A M \leq 1$ and $\operatorname{id}_A M \leq 1$, and consequently $\operatorname{Ext}_A^n(M,M) = 0$ for any $n \geq 2$. □

Recall that a component $\mathscr{C}$ of an Auslander-Reiten quiver Γ_A is called *faithful* if its annihilator $\operatorname{ann}_A(\mathscr{C})$ (the intersection of the annihilators $\operatorname{ann}_A(X)$ of all modules X in $\mathscr{C}$) is zero.

In the proofs of our results we need also facts on the compositions of irreducible morphisms. The following theorem has been proved in [**9**, Theorem 13.3].

THEOREM 2.5. *Let* A *be an algebra. If* $X_0 \xrightarrow{f_1} X_1 \to \cdots \to X_{n-1} \xrightarrow{f_n} X_n$ *is a sectional path of irreducible morphisms between indecomposable modules in* $\operatorname{mod} A$, *then the composed morphism* $f_1 f_2 \ldots f_n$ *lies in* $\operatorname{rad}^n(X_0, X_n)$ *but not in* $\operatorname{rad}^{n+1}(X_0, X_n)$. *In particular,* $f_1 f_2 \ldots f_n$ *is nonzero.*

Let A be an algebra and $f : X \to Y$ be an irreducible morphism in $\operatorname{mod} A$. Following [**14**], we say that the *right degree* of f is the smallest positive integer m such that there exists a morphism $g \in \operatorname{rad}^m(Y,Z) \setminus \operatorname{rad}^{m+1}(Y,Z)$, for some $Z \in \operatorname{mod} A$, such that $fg \in \operatorname{rad}^{m+2}(X,Z)$. If no such an integer m exists, then right degree of f is infinite. We define the *left degree* of f in a dual manner. The following result from [**14**, Proposition 1.14] will be applied.

PROPOSITION 2.6. *Let* A *be an algebra and let*

$$X_0 \to X_1 \to \cdots \to X_n \to \cdots$$

be an infinite sectional path in Γ_A. If all X_i are right stable, then all irreducible morphisms $\tau_A^j X_i \to \tau_A^j X_{i+1}$ and $\tau_A^j X_{i+1} \to \tau_A^{j-1} X_i$ with $j \leq 0$ and $i \geq 0$ have infinite right degree.

3. Generalized multicoil enlargements of algebras.

The aim of this section is to recall generalized multicoil enlargements of algebras from [**18**, Section 3], playing the fundamental role in our proof of Theorem 1.3. It has been proved in [**17**, Theorem A] that a component Γ of an Auslander-Reiten quiver is almost cyclic and coherent if and only if Γ is a generalized multicoil, that is, can be obtained, as a translation quiver, from a finite family of stable tubes by a sequence of admissible operations. We start with the concepts of one-point extensions and one-point coextensions of algebras. Let A be an algebra, let F be a division algebra over K, and let $M = {}_F M_A$ be an F-A-bimodule such that $M_A \in \operatorname{mod} A$ and K acts centrally on ${}_F M_A$. Then the *one-point extension* of A by M is the matrix K-algebra of the form

$$A[M] = \begin{bmatrix} A & 0 \\ {}_F M_A & F \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 \\ m & f \end{bmatrix};\ f \in F,\ a \in A,\ m \in M \right\}$$

with the usual addition and multiplication. Then the valued quiver $Q_{A[M]}$ of $A[M]$ contains the valued quiver Q_A of A as a convex subquiver, and there is an additional (extension) vertex which is a source. We may identify the category $\operatorname{mod} A[M]$ with the category whose objects are triples (V, X, φ), where $X \in \operatorname{mod} A$, $V \in \operatorname{mod} F$, and $\varphi : V_F \to \operatorname{Hom}_A(M, X)_F$ is an F-linear map. A morphism $h : (V, X, \varphi) \to (W, Y, \psi)$ is given by a pair (f, g), where $f : V \to W$ is F-linear, $g : X \to Y$ is a morphism in $\operatorname{mod} A$ and $\psi f = \operatorname{Hom}_A(M, g)\varphi$. Then the new indecomposable projective $A[M]$-module P is given by the triple $(F, M, \bullet)$, where $\bullet : F_F \to \operatorname{Hom}_A(M, M)_F$ assigns to the identity element of F the identity morphism of M. An important class of such one-point extensions occurs in the following situation. Let Λ be a basic K-algebra, P an indecomposable projective Λ-module, ${}_\Lambda\Lambda = P \oplus Q$, and assume that $\operatorname{Hom}_\Lambda(P, Q \oplus \operatorname{rad} P) = 0$. Since P is indecomposable projective, $S = P/\operatorname{rad} P$ is a simple Λ-module and hence $\operatorname{End}_\Lambda(S)$ is a division K-algebra. Moreover, the canonical homomorphism of algebras $\operatorname{End}_\Lambda(P) \to \operatorname{End}_\Lambda(S)$ is an isomorphism. Then we obtain isomorphisms of algebras

$$\Lambda \cong \operatorname{End}_\Lambda(\Lambda_\Lambda) \cong \begin{bmatrix} A & 0 \\ {}_F M_A & F \end{bmatrix} = A[M],$$

where $F = \operatorname{End}_\Lambda(P)$, $A = \operatorname{End}_\Lambda(Q)$, and $M = {}_F M_A = \operatorname{Hom}_\Lambda(Q, P) \cong \operatorname{rad} P$.

Clearly K acts centrally on ${}_F M_A$. We note that if the valued quiver of an algebra Λ has no oriented cycles then Λ can be obtained from a semisimple algebra by a sequence of one-point extensions of the above form. Dually, one defines also the *one-point coextension* of A by ${}_F M_A$ as the matrix algebra

$$[M]A = \begin{bmatrix} F & 0 \\ D({}_F M_A) & A \end{bmatrix}.$$

For each bimodule ${}_F M_A$ considered in the paper we assume that A is an algebra, $M_A \in \operatorname{mod} A$, F is a division algebra, and K acts centrally on ${}_F M_A$.

For a division algebra F and $r \geq 1$, we denote by $T_r(F)$ the $r \times r$-lower triangular matrix algebra

$$\begin{bmatrix} F & 0 & 0 & \cdots & 0 & 0 \\ F & F & 0 & \cdots & 0 & 0 \\ F & F & F & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ F & F & F & \cdots & F & 0 \\ F & F & F & \cdots & F & F \end{bmatrix}.$$

Given a generalized standard component Γ of Γ_A, and an indecomposable module X in Γ, the *support* $\mathscr{S}(X)$ of the functor $\operatorname{Hom}_A(X,-)|_\Gamma$ is the K-linear category defined as follows. Let $\mathscr{H}_X$ denote the full subcategory of $\operatorname{mod} A$ consisting of the indecomposable modules M in Γ such that $\operatorname{Hom}_A(X, M) \neq 0$, and $\mathscr{I}_X$ denote the ideal of $\mathscr{H}_X$ consisting of the morphisms $f : M \to N$ (with M, N in $\mathscr{H}_X$) such that $\operatorname{Hom}_A(X, f) = 0$. We define $\mathscr{S}(X)$ to be the quotient category $\mathscr{H}_X / \mathscr{I}_X$. Following the above convention, we usually identify the K-linear category $\mathscr{S}(X)$ with its quiver.

From now on, let A be an algebra and Γ be a family of generalized standard infinite components of Γ_A. For an indecomposable brick X in Γ, called the *pivot*, one defines five admissible operations (ad 1)–(ad 5) and their dual (ad 1*)–(ad 5*) modifying the translation quiver $\Gamma = (\Gamma, \tau)$ to a new translation quiver (Γ', τ') and the algebra A to a new algebra A', depending on the shape of the support $\mathscr{S}(X)$ (see [**17**, Section 2] for the figures illustrating the modified translation quivers Γ'). Let $F = F_X = \operatorname{End}_A(X)$ be the division algebra associated to X.

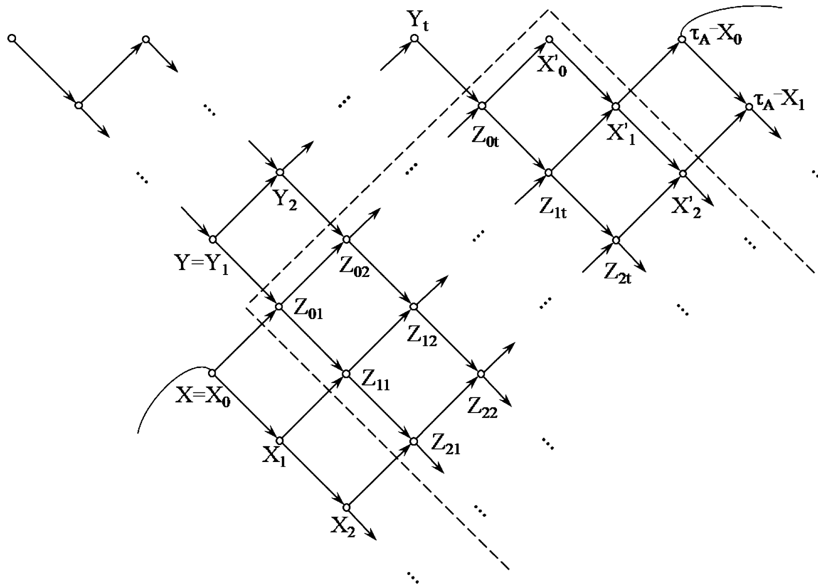
(ad 1) Assume $\mathscr{S}(X)$ consists of an infinite sectional path starting at X:

$$X = X_0 \to X_1 \to X_2 \to \cdots$$

In this case, we let $t \geq 1$ be a positive integer, $D = T_t(F)$ and $Y_1, Y_2, \ldots, Y_t$ denote the indecomposable injective D-modules with $Y = Y_1$ the unique indecomposable projective-injective D-module. We define the *modified algebra* A' of A to be the one-point extension

$$A' = (A \times D)[X \oplus Y]$$

and the *modified translation quiver* Γ' of Γ to be obtained by inserting in Γ the rectangle consisting of the modules $Z_{ij} = \left(F, X_i \oplus Y_j, \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right)$ for $i \geq 0$, $1 \leq j \leq t$, and $X'_i = (F, X_i, 1)$ for $i \geq 0$ as follows:



The translation τ' of Γ' is defined as follows: $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geq 1$, $j \geq 2$, $\tau' Z_{i1} = X_{i-1}$ if $i \geq 1$, $\tau' Z_{0j} = Y_{j-1}$ if $j \geq 2$, Z_{01} is projective, $\tau' X'_0 = Y_t$, $\tau' X'_i = Z_{i-1,t}$ if $i \geq 1$, $\tau'(\tau^{-1} X_i) = X'_i$ provided X_i is not an injective A-module, otherwise X'_i is injective in Γ'. For the remaining vertices of Γ', τ' coincides with the translation of Γ, or Γ_D, respectively.

If $t = 0$, we define the modified algebra A' to be the one-point extension $A' = A[X]$ and the modified translation quiver Γ' to be the translation quiver obtained from Γ by inserting only the sectional path consisting of the vertices X'_i, $i \geq 0$.

The nonnegative integer t is such that the number of infinite sectional paths parallel to $X_0 \to X_1 \to X_2 \to \cdots$ in the inserted rectangle equals $t + 1$. We call t

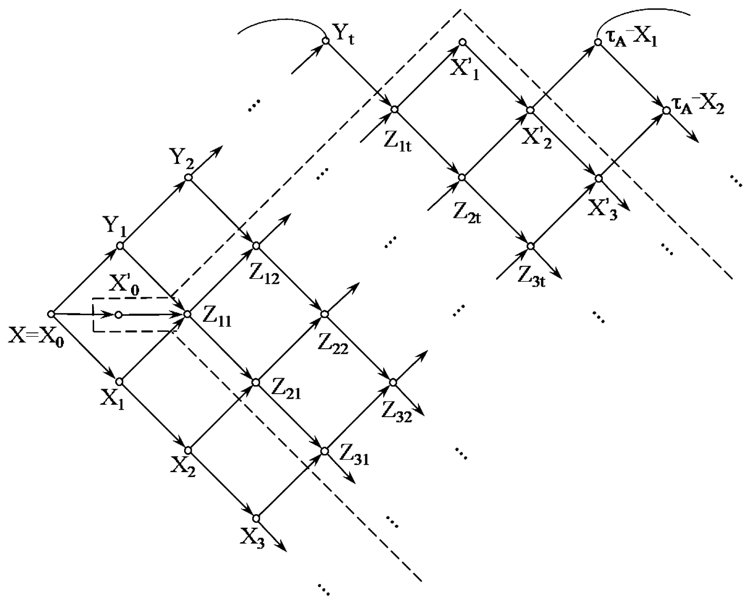
the *parameter* of the operation.

In case Γ is a stable tube, it is clear that any module on the mouth of Γ satisfies the condition for being a pivot for the above operation. Actually, the above operation is, in this case, the tube insertion as considered in [**6**].

(ad 2) Suppose that $\mathscr{S}(X)$ admits two sectional paths starting at X, one infinite and the other finite with at least one arrow:

$$Y_t \leftarrow \cdots \leftarrow Y_2 \leftarrow Y_1 \leftarrow X = X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots$$

where $t \geq 1$. In particular, X is necessarily injective. We define the *modified algebra* A' of A to be the one-point extension $A' = A[X]$ and the *modified translation quiver* Γ' of Γ to be obtained by inserting in Γ the rectangle consisting of the modules $Z_{ij} = \big(F, X_i \oplus Y_j, \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\big)$ for $i \geq 1$, $1 \leq j \leq t$, and $X_i' = (F, X_i, 1)$ for $i \geq 1$ as follows:



The translation τ' of Γ' is defined as follows: X_0' is projective-injective, $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geq 2$, $j \geq 2$, $\tau' Z_{i1} = X_{i-1}$ if $i \geq 1$, $\tau' Z_{1j} = Y_{j-1}$ if $j \geq 2$, $\tau' X_i' = Z_{i-1,t}$ if $i \geq 2$, $\tau' X_1' = Y_t$, $\tau'(\tau^{-1} X_i) = X_i'$ provided X_i is not an injective A-module, otherwise X_i' is injective in Γ'. For the remaining vertices of Γ', τ' coincides with the translation τ of Γ.

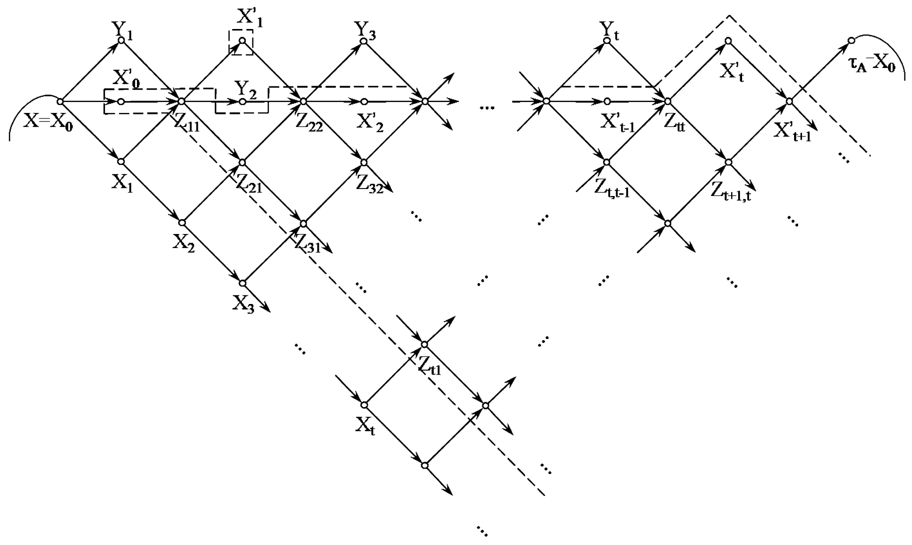
The integer $t \geq 1$ is such that the number of infinite sectional paths parallel to $X_0 \to X_1 \to X_2 \to \cdots$ in the inserted rectangle equals $t+1$. We call t the *parameter* of the operation.

(ad 3) Assume $\mathscr{S}(X)$ is the mesh-category of two parallel sectional paths:

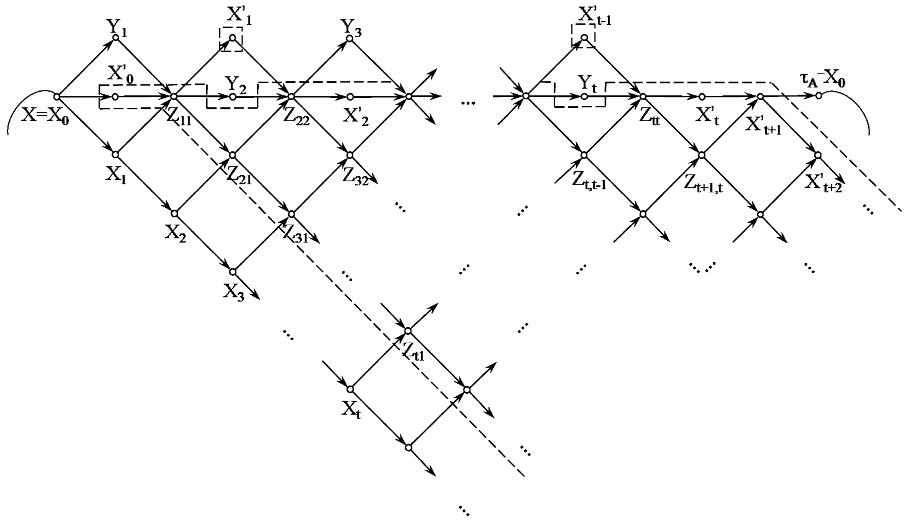
$$\begin{tikzcd}
 & Y_1 \arrow[r] & Y_2 \arrow[r] & \cdots \arrow[r] & Y_t & & \\
X = X_0 \arrow[ur, phantom] & & & & & & \\
\end{tikzcd}$$

$$\begin{tikzcd}
Y_1 \arrow[r] & Y_2 \arrow[r] & \cdots \arrow[r] & Y_t & & \\
X = X_0 \arrow[r] \arrow[u] & X_1 \arrow[r] \arrow[u] & \cdots \arrow[r] & X_{t-1} \arrow[r] \arrow[u] & X_t \arrow[r] & \cdots
\end{tikzcd}$$

where $t \geq 2$. In particular, X_{t-1} is necessarily injective. Moreover, we consider the translation subquiver $\overline{\Gamma}$ of Γ obtained by deleting the arrows $Y_i \to \tau_A^{-1} Y_{i-1}$. We assume that the union $\widehat{\Gamma}$ of connected components of $\overline{\Gamma}$ containing the vertices $\tau_A^{-1} Y_{i-1}$, $2 \leq i \leq t$, is a finite translation quiver. Then $\overline{\Gamma}$ is a disjoint union of $\widehat{\Gamma}$ and a cofinite full translation subquiver Γ^*, containing the pivot X. We define the *modified algebra* A' of A to be the one-point extension $A' = A[X]$ and the *modified translation quiver* Γ' of Γ to be obtained from Γ^* by inserting the rectangle consisting of the modules $Z_{ij} = \left(F, X_i \oplus Y_j, \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right)$ for $i \geq 1$, $1 \leq j \leq t$, and $X'_i = (F, X_i, 1)$ for $i \geq 1$ as follows:



if t is odd, while



if t is even. The translation τ' of Γ' is defined as follows: X_0' is projective, $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geq 2$, $2 \leq j \leq t$, $\tau' Z_{i1} = X_{i-1}$ if $i \geq 1$, $\tau' X_i' = Y_i$ if $1 \leq i \leq t$, $\tau' X_i' = Z_{i-1,t}$ if $i \geq t+1$, $\tau' Y_j = X_{j-2}'$ if $2 \leq j \leq t$, $\tau'(\tau^{-1} X_i) = X_i'$, if $i \geq t$ provided X_i is not injective in Γ, otherwise X_i' is injective in Γ'. For the remaining vertices of Γ', τ' coincides with the translation τ of Γ^*. We note that X_{t-1}' is injective.

The integer $t \geq 2$ is such that the number of infinite sectional paths parallel to $X_0 \to X_1 \to X_2 \to \cdots$ in the inserted rectangle equals $t+1$. We call t the *parameter* of the operation.

(ad 4) Suppose that $\mathscr{S}(X)$ consists of an infinite sectional path, starting at X

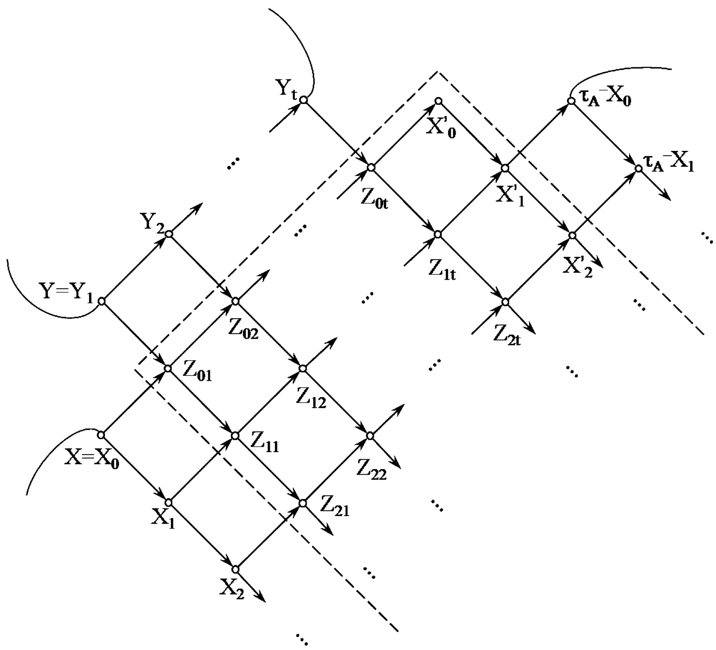
$$X = X_0 \to X_1 \to X_2 \to \cdots$$

and

$$Y = Y_1 \to Y_2 \to \cdots \to Y_t$$

with $t \geq 1$, is a finite sectional path in Γ such that $F_Y = F = F_X$. Let r be a positive integer. Moreover, we consider the translation subquiver $\overline{\Gamma}$ of Γ obtained by deleting the arrows $Y_i \to \tau_A^{-1} Y_{i-1}$. We assume that the union $\widehat{\Gamma}$ of connected components of $\overline{\Gamma}$ containing the vertices $\tau_A^{-1} Y_{i-1}$, $2 \leq i \leq t$, is a finite translation quiver. Then $\overline{\Gamma}$ is a disjoint union of $\widehat{\Gamma}$ and a cofinite full translation subquiver

Γ^*, containing the pivot X. For $r = 0$ we define the *modified algebra* A' of A to be the one-point extension $A' = A[X \oplus Y]$ and the *modified translation quiver* Γ' of Γ to be obtained from Γ^* by inserting the rectangle consisting of the modules $Z_{ij} = \left(F, X_i \oplus Y_j, \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right)$ for $i \geq 0$, $1 \leq j \leq t$, and $X'_i = (F, X_i, 1)$ for $i \geq 1$ as follows:

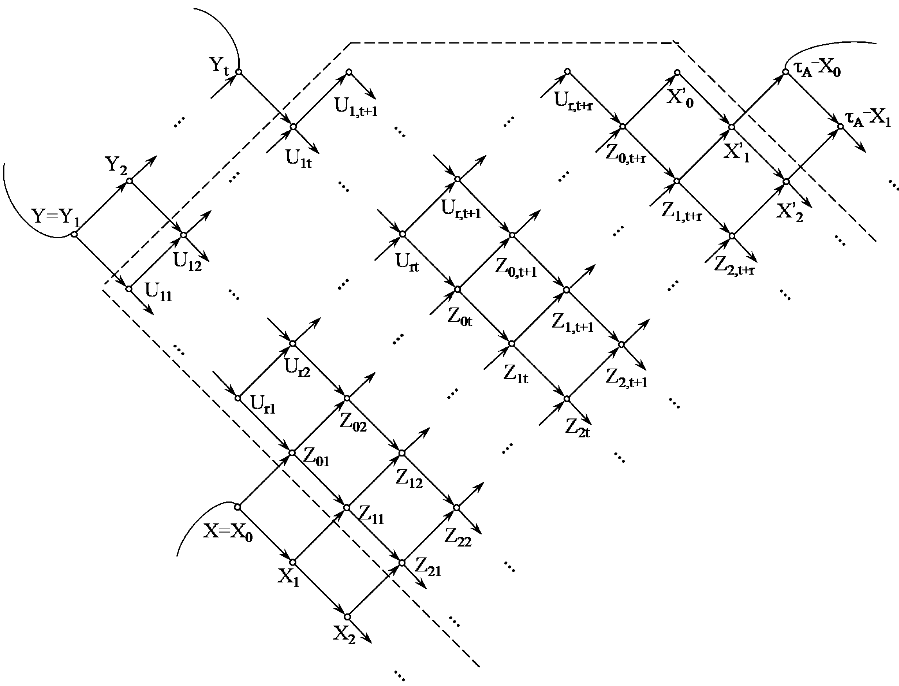


The translation τ' of Γ' is defined as follows: $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geq 1$, $j \geq 2$, $\tau' Z_{i1} = X_{i-1}$ if $i \geq 1$, $\tau' Z_{0j} = Y_{j-1}$ if $j \geq 2$, Z_{01} is projective, $\tau' X'_0 = Y_t$, $\tau' X'_i = Z_{i-1,t}$ if $i \geq 1$, $\tau'(\tau^{-1} X_i) = X'_i$ provided X_i is not injective in Γ, otherwise X'_i is injective in Γ'. For the remaining vertices of Γ', τ' coincides with the translation of Γ^*.

For $r \geq 1$, let $G = T_r(F)$, and let $U_{1,t+1}, U_{2,t+1}, \ldots, U_{r,t+1}$ denote the indecomposable projective G-modules, $U_{r,t+1}, U_{r,t+2}, \ldots, U_{r,t+r}$ denote the indecomposable injective G-modules, with $U_{r,t+1}$ the unique indecomposable projective-injective G-module. We define the *modified algebra* A' of A to be the triangular matrix algebra of the form:

$$A' = \begin{bmatrix} A & 0 & 0 & \cdots & 0 & 0 \\ Y & F & 0 & \cdots & 0 & 0 \\ Y & F & F & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ Y & F & F & \cdots & F & 0 \\ X \oplus Y & F & F & \cdots & F & F \end{bmatrix}$$

with $r+2$ columns and rows and the *modified translation quiver* Γ' of Γ to be obtained from Γ^* by inserting the rectangles consisting of the modules $U_{kl} = Y_l \oplus U_{k,t+1}$ for $1 \le k \le r$, $1 \le l \le t$, and $Z_{ij} = \left(F, X_i \oplus U_{rj}, \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right)$ for $i \ge 0$, $1 \le j \le t+r$, and $X'_i = (F, X_i, 1)$ for $i \ge 0$ as follows:



The translation τ' of Γ' is defined as follows: $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \ge 1$, $j \ge 2$, $\tau' Z_{i1} = X_{i-1}$ if $i \ge 1$, $\tau' Z_{0j} = U_{r,j-1}$ if $2 \le j \le t+r$, Z_{01}, U_{k1}, $1 \le k \le r$ are projective, $\tau' U_{kl} = U_{k-1,l-1}$ if $2 \le k \le r$, $2 \le l \le t+r$, $\tau' U_{1l} = Y_{l-1}$ if $2 \le l \le t+1$, $\tau' X'_0 = U_{r,t+r}$, $\tau' X'_i = Z_{i-1,t+r}$ if $i \ge 1$, $\tau'(\tau^{-1} X_i) = X'_i$ provided X_i is not injective in Γ, otherwise X'_i is injective in Γ'. For the remaining vertices of Γ', τ' coincides with the translation of Γ^*, or Γ_G, respectively.

We note that the quiver $Q_{A'}$ of A' is obtained from the quiver of the double one-point extension $A[X][Y]$ by adding a path of length $r+1$ with source at the extension vertex of $A[X]$ and sink at the extension vertex of $A[Y]$.

The integers $t \geq 1$ and $r \geq 0$ are such that the number of infinite sectional paths parallel to $X_0 \to X_1 \to X_2 \to \cdots$ in the inserted rectangles equals $t+r+1$. We call $t+r$ the *parameter* of the operation.

To the definition of the next admissible operation we need also the finite versions of the admissible operations (ad 1), (ad 2), (ad 3), (ad 4), which we denote by (fad 1), (fad 2), (fad 3) and (fad 4), respectively. In order to obtain these operations we replace all infinite sectional paths of the form $X_0 \to X_1 \to X_2 \to \cdots$ (in the definitions of (ad 1), (ad 2), (ad 3), (ad 4)) by the finite sectional paths of the form $X_0 \to X_1 \to X_2 \to \cdots \to X_s$. For the operation (fad 1) $s \geq 0$, for (fad 2) and (fad 4) $s \geq 1$, and for (fad 3) $s \geq t-1$. In all above operations X_s is injective (see [**17**] or [**18**] for the details).

(ad 5) We define the *modified algebra* A' of A (respectively, *modified translation quiver* Γ' of Γ) in the following three steps: first we are doing on A (respectively, Γ) one of the operations (fad 1), (fad 2) or (fad 3), next a finite number (possibly empty) of the operation (fad 4) and finally the operation (ad 4), and in such a way that the sectional paths starting from all the new projective vertices have a common cofinite (infinite) sectional subpath.

Finally, together with each of the admissible operations (ad 1)–(ad 5), we consider its dual, denoted by (ad 1*)–(ad 5*). These ten operations are called the *admissible operations*. Following [**17**] a connected translation quiver Γ is said to be a *generalized multicoil* if Γ can be obtained from a finite family $\mathscr{T}_1, \mathscr{T}_2, \ldots, \mathscr{T}_s$ of stable tubes by an iterated application of admissible operations (ad 1), (ad 1*), (ad 2), (ad 2*), (ad 3), (ad 3*), (ad 4), (ad 4*), (ad 5) or (ad 5*). If $s=1$, such a translation quiver Γ is said to be a *generalized coil*. The admissible operations of types (ad 1), (ad 2), (ad 3), (ad 1*), (ad 2*) and (ad 3*) have been introduced in [**2**], [**3**], [**4**], and the admissible operations (ad 4) and (ad 4*) for $r=0$ in [**16**].

Finally, let C be a (not necessarily connected) algebra and $\mathscr{T}_C$ a family of pairwise orthogonal generalized standard stable tubes of Γ_C. We say that an algebra A is a *generalized multicoil enlargement* of C using modules from $\mathscr{T}_C$ if A is obtained from C by an iteration of admissible operations of types (ad 1)–(ad 5) and (ad 1*)–(ad 5*) performed either on stable tubes of $\mathscr{T}_C$, or on generalized multicoils obtained from stable tubes of $\mathscr{T}_C$ by means of operations done so far.

The following theorem follows from Proposition 2.1 and the proof of Theorem A in [**18**].

THEOREM 3.1. *Let A be an algebra, $\mathscr{C}$ be a component of Γ_A, and $\Lambda = A/\operatorname{ann}_A \mathscr{C}$. Then the following statements are equivalent*:

(i) *$\mathscr{C}$ is generalized standard and a generalized multicoil.*

(ii) *Λ is a generalized multicoil enlargement of an algebra C using modules from a generalized standard family $\mathscr{T}_C$ of stable tubes of Γ_C and $\mathscr{C}$ is the generalized standard multicoil obtained from $\mathscr{T}_C$ by the admissible operations leading from C to Λ.*

We need also results from [**18**, Theorems A, C, E] on the algebras with separating families of almost cyclic coherent components.

Theorem 3.2. *Let A be an algebra. The following statements are equivalent*:

(i) *Γ_A admits a separating family of almost cyclic coherent components.*

(ii) *A is a generalized multicoil enlargement of a concealed canonical algebra C using modules of a separating family $\mathscr{T}_C$ of stable tubes of Γ_C.*

Theorem 3.3. *Let A be an algebra with a separating family $\mathscr{C}_A$ of almost cyclic coherent components in Γ_A, and $\Gamma_A = \mathscr{P}_A \vee \mathscr{C}_A \vee \mathscr{Q}_A$ the induced decomposition of Γ_A. Then the following statements hold*:

(i) *There is a unique factor algebra A_l of A which is a quasitilted algebra of canonical type having a separating family $\mathscr{T}_{A_l}$ of coray tubes such that $\Gamma_{A_l} = \mathscr{P}_{A_l} \vee \mathscr{T}_{A_l} \vee \mathscr{Q}_{A_l}$, $\mathscr{P}_{A_l} = \mathscr{P}_A$, and A is obtained from A_l by a sequence of admissible operations of types* (ad 1)–(ad 5) *using modules from $\mathscr{T}_{A_l}$.*

(ii) *There is a unique factor algebra A_r of A which is a quasitilted algebra of canonical type having a separating family $\mathscr{T}_{A_r}$ of ray tubes such that $\Gamma_{A_r} = \mathscr{P}_{A_r} \vee \mathscr{T}_{A_r} \vee \mathscr{Q}_{A_r}$, $\mathscr{Q}_{A_r} = \mathscr{Q}_A$, and A is obtained from A_r by a sequence of admissible operations of types* (ad 1*)–(ad 5*) *using modules from $\mathscr{T}_{A_r}$.*

(iii) $\mathrm{pd}_A X \leq 1$ *for any module X in $\mathscr{P}_A$.*

(iv) $\mathrm{id}_A X \leq 1$ *for any module X in $\mathscr{Q}_A$.*

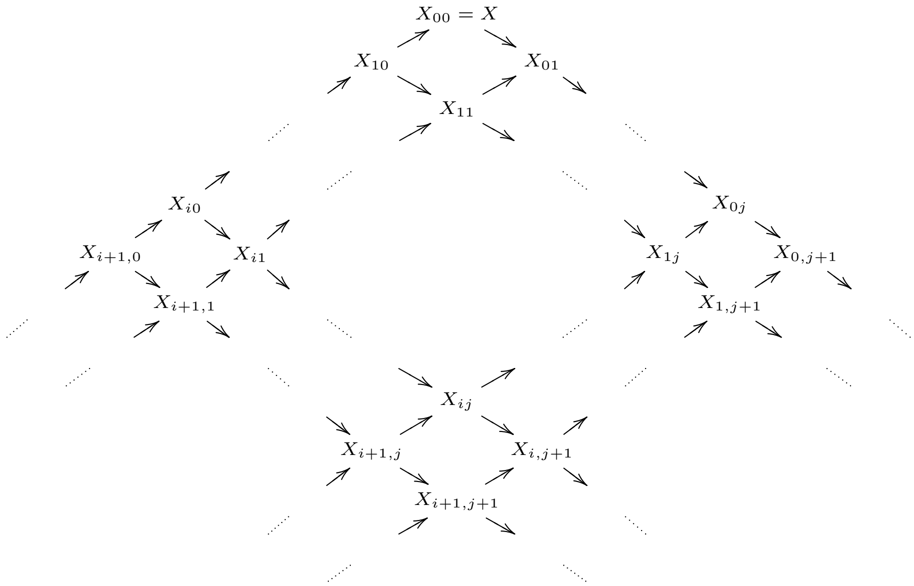
(v) $\mathrm{pd}_A X \leq 2$ *and* $\mathrm{id}_A X \leq 2$ *for any module X in $\mathscr{C}_A$.*

(vi) $\mathrm{gl\,dim}\, A \leq 3$.

The algebra A_l (respectively, A_r) in Theorem 3.3 is called the *left* (respectively, *right*) *quasitilted* algebra of A.

4. Proof of Theorem 1.3.

In the proof we need the following notion. A proper subtube of an Auslander-Reiten quiver Γ_A is a full translation subquiver $\mathscr{T}(X, a, b)$, $a, b \geq 1$, obtained from the translation quiver $\mathscr{T}(X)$ of the form



with the set of vertices X_{rs}, the set of arrows $X_{r+1,s} \to X_{rs}$, $X_{rs} \to X_{r,s+1}$, $r, s \geq 0$, and the translation τ defined on X_{rs}, $r \geq 0$, $s \geq 1$, by $\tau(X_{rs}) = X_{r+1,s-1}$, by identifying the vertices $X_{i+a,j}$ with $X_{i,j+b}$ for all pairs $i, j \geq 0$. Observe that then

$$\{X_{ij}; i \geq 0, 0 \leq j < b\} = \{X_{ij}; 0 \leq i < a, j \geq 0\}$$

is a complete set of pairwise different vertices of $\mathscr{T}(X, a, b)$.

(i) Let $\mathscr{C}$ be a generalized standard almost cyclic coherent component of Γ_A. Consider the quotient algebra $\Lambda = A/\operatorname{ann}_A(\mathscr{C})$. Then $\mathscr{C}$ is a generalized standard component of Γ_Λ. Further, it follows from [**34**, Theorem 1] that the additive category $\operatorname{add}(\mathscr{C})$ of $\mathscr{C}$ is closed under extensions in $\operatorname{mod} A$, and hence also in $\operatorname{mod} \Lambda$. Then for every indecomposable module M in $\mathscr{C}$ we have an isomorphism of K-vector spaces $\operatorname{Ext}^1_A(M, M) \cong \operatorname{Ext}^1_\Lambda(M, M)$, and clearly the equality $\operatorname{End}_A(M) = \operatorname{End}_\Lambda(M)$, because M is a Λ-module. Therefore, we may assume that $\operatorname{ann}_A(\mathscr{C}) = 0$, that is, $\mathscr{C}$ is a faithful component of Γ_A. Then it follows from Theorem 3.1 that there is a quotient algebra C of A (not necessarily connected) and a family $\mathscr{T}_1, \mathscr{T}_2, \ldots, \mathscr{T}_s$ of pairwise orthogonal generalized standard stable tubes in Γ_C such that A is a generalized multicoil enlargement of C using modules from $\mathscr{T}_1, \mathscr{T}_2, \ldots, \mathscr{T}_s$, and $\mathscr{C}$ is the generalized multicoil obtained from the stable tubes $\mathscr{T}_1, \mathscr{T}_2, \ldots, \mathscr{T}_s$ by an iterated application of the translation quiver admissible operations corresponding to the algebra admissible operations of types (ad 1)–(ad 5) and (ad 1*)–(ad 5*) leading from C to A.

For each arrow $X \xrightarrow{\alpha} Y$ in $\mathscr{C}$ we choose an irreducible morphism $f_\alpha : X \to Y$. We may assume that $f_\xi f_\eta$ belongs to $\operatorname{rad}^3(\operatorname{mod} A)$ for any mesh

$$\begin{tikzcd}[column sep=small] \tau_A Z \arrow[dr, "\xi"'] & & Z \\ & W \arrow[ur, "\eta"'] & \end{tikzcd}$$

with Z lying on the mouth of $\mathscr{C}$ and $f_\alpha f_\beta + f_\gamma f_\delta \in \operatorname{rad}^3(\operatorname{mod} A)$ for any mesh in $\mathscr{C}$ of the form

$$\begin{tikzcd}[column sep=small] & U \arrow[dr, "\beta"] & \\ \tau_A Z \arrow[ur, "\alpha"] \arrow[dr, "\gamma"'] & & Z \\ & V \arrow[ur, "\delta"'] & \end{tikzcd}$$

and $f_\alpha f_\beta + f_\gamma f_\delta + f_\epsilon f_\sigma \in \operatorname{rad}^3(\operatorname{mod} A)$ for any mesh in $\mathscr{C}$ of the form

$$\begin{tikzcd}[column sep=small] & U \arrow[dr, "\beta"] & \\ \tau_A Z \arrow[ur, "\alpha"] \arrow[r, "\gamma"] \arrow[dr, "\epsilon"'] & W \arrow[r, "\delta"] & Z \\ & V \arrow[ur, "\sigma"'] & \end{tikzcd} .$$

Observe that for any irreducible morphism $f : X \to Y$ with X and Y from $\mathscr{C}$, there are automorphisms $b : X \to X$ and $c : Y \to Y$ such that

$$bf_\alpha + \operatorname{rad}^2(X, Y) = f + \operatorname{rad}^2(X, Y) = f_\alpha c + \operatorname{rad}^2(X, Y),$$

where $X \xrightarrow{\alpha} Y$ is the corresponding arrow in $\mathscr{C}$. This follows from the fact that

$$\dim_{F_X}\left(\frac{\operatorname{rad}(X, Y)}{\operatorname{rad}^2(X, Y)}\right) = 1 \quad \text{and} \quad \dim\left(\frac{\operatorname{rad}(X, Y)}{\operatorname{rad}^2(X, Y)}\right)_{F_Y} = 1,$$

where $F_X = \operatorname{End}_A(X)/\operatorname{rad}\operatorname{End}_A(X)$, $F_Y = \operatorname{End}_A(Y)/\operatorname{rad}\operatorname{End}_A(Y)$. We shall prove the required inequality $\dim_K \operatorname{Ext}^1_A(M, M) \leq \dim_K \operatorname{End}_A(M)$, for all indecomposable modules M in $\mathscr{C}$, by induction on the number of admissible operations leading from C to A, equivalently from $\mathscr{T}_1, \mathscr{T}_2, \ldots, \mathscr{T}_s$ to $\mathscr{C}$.

In the case when $\mathscr{C}$ is a generalized standard stable tube, so $s = 1$ and $\mathscr{C} = \mathscr{T}_1$, the required inequality follows from Corollary 2.3. Therefore, we may assume that

$\mathscr{C}$ is not a stable tube, and hence $A \neq C$.

Let n be the number of admissible operations of types (ad 1)–(ad 5) and (ad 1*)–(ad 5*) leading from C to A.

Assume $n = 1$. Then we can only apply an admissible operation of type (ad 1) or (ad 1*), and $s = 1$. By duality we may assume that the admissible operation is of type (ad 1). Assume that $\mathscr{C}$ is obtained from $\mathscr{T}_1$ by applying an operation of type (ad 1). Then $A = C[X]$ or $A = (C \times D)[X \oplus Y]$, where X is the pivot of the operation (ad 1) in the stable tube $\mathscr{T}_1$, $D = T_t(F)$ is the lower $t \times t$ triangular matrix algebra over a division K-algebra F for some $t \geq 1$, and Y is the unique indecomposable projective-injective D-module (see definition of (ad 1)).

Let M be an indecomposable A-module in $\mathscr{C}$. If M is a directing module in $\operatorname{mod} A$, then by [**25**, (2.4)(8)] we get $\operatorname{Ext}^1_A(M, M) = 0$, $\operatorname{End}_A(M) = F_M$, and the required inequality holds. Assume that M is nondirecting. If M is a C-module, then M lies in the stable tube $\mathscr{T}_1 = \boldsymbol{Z}\boldsymbol{A}_\infty/(\tau^r)$ of Γ_C. Then, applying Corollary 2.3, we have $\dim_K \operatorname{Ext}^1_A(M, M) \leq \dim_K \operatorname{End}_A(M)$, and we receive the equality if and only if r divides $\operatorname{ql}(M)$. If M is not a C-module, then M lies in the infinite rectangle $\mathscr{S}(Z_{01})$ of $\mathscr{C}$ consisting of the A-modules Z_{pq}, for $p \geq 0$, $1 \leq q \leq t+1$, where Z_{01} is the projective A-module and $Z_{p,t+1} = X'_p$ (see definition of (ad 1)). Let $M = Z_{pq}$ and k be a nonnegative integer with $rk \leq p < r(k+1)$. Let W be the target of the unique maximal sectional path from infinity to the mouth of $\mathscr{C}$ passing through M. Moreover, let $W \xrightarrow{\sigma} R$ be the arrow with source W and $R \xrightarrow{\varrho} \tau_A^- W$ the arrow with target $\tau_A^- W$. Put $v = f_\varrho$ and $u = f_\sigma$. Then uv belongs to $\operatorname{rad}^3(\operatorname{mod} A)$. Observe that any path in $\mathscr{C}$ from M to M has length $(2r+t+1)i$ for some $i \geq 0$. This implies that $\operatorname{rad}^{(2r+t+1)i+1}(M, M) = \operatorname{rad}^{(2r+t+1)(i+1)}(M, M)$ for all $i \geq 0$. We claim that $\operatorname{rad}^m(M, M) = 0$ for all $m \geq (2r+t+1)(k+1)$. It is enough to show that $\operatorname{rad}^m(M, M) \subset \operatorname{rad}^{m+1}(M, M)$ for any $m \geq (2r+t+1)(k+1)$. Indeed, then $\operatorname{rad}^{(2r+t+1)(k+1)}(M, M) = \operatorname{rad}^\infty(M, M) = 0$ because $\mathscr{C}$ is generalized standard. Let $m \geq (2r+t+1)(k+1)$ and $\Phi \in \operatorname{rad}^m(M, M)$. Then we have the equality $\Phi + \operatorname{rad}^{m+1}(M, M) = (\sum \psi_i a_i uv b_i) + \operatorname{rad}^{m+1}(M, M)$, where $a_i uv b_i$ are the composites of m irreducible morphisms including u and v, and ψ_i are invertible elements of $\operatorname{End}_A(M)$. Since uv lies in $\operatorname{rad}^3(\operatorname{mod} A)$, we get $\Phi + \operatorname{rad}^{m+1}(M, M) = 0 + \operatorname{rad}^{m+1}(M, M)$ and hence Φ belongs to $\operatorname{rad}^{m+1}(M, M)$. This proves our claim. In particular, if $p \leq r-1$, then $\operatorname{rad}(M, M) = \operatorname{rad}^{2r+t+1}(M, M) = 0$, and hence $\operatorname{End}_A(M) \cong \operatorname{End}_A(M)/\operatorname{rad}(M, M)$. Assume that $p > r-1$. Let

$$V_s \xrightarrow{\alpha_{s-1}} V_{s-1} \to \cdots \to V_1 \xrightarrow{\alpha_0} V_0 = M$$

be the unique maximal sectional path in $\mathscr{C}$ passing through M, formed by arrows pointing to infinity, and

$$M = W_0 \xrightarrow{\beta_0} W_1 \to \cdots \to W_{l-1} \xrightarrow{\beta_{l-1}} W_l = W$$

be the sectional path in $\mathscr{C}$ formed by arrows pointing to the mouth. Note that we have $s = p + t + 1 - q$ and $l = p + (k+1)(t+1) - q$. Clearly, $W = W_l$ and V_s lie on the mouth of $\mathscr{C}$. Put $g_i = f_{\alpha_i}$, for $0 \leq i \leq s-1$, and $h_i = f_{\beta_i}$, for $0 \leq i \leq l-1$. Since $p > r-1$, the above two sectional paths intersect. Let $f : M \to M$ be the composed morphism $h_0 h_1 \dots h_{r+t} g_{r-1} \dots g_1 g_0$. We shall prove that $f^j \in \operatorname{rad}^{(2r+t+1)j}(M,M) \setminus \operatorname{rad}^{(2r+t+1)j+1}(M,M)$ for all $1 \leq j \leq k$. First observe that

$$f^j + \operatorname{rad}^{(2r+t+1)j+1}(M,M) = f^{(j)} + \operatorname{rad}^{(2r+t+1)j+1}(M,M),$$

where $f^{(j)} = h_0 h_1 \dots h_{(r+t+1)j-1} g_{rj-1} \dots g_1 g_0$ for $1 \leq j \leq k$. Since the morphism $g_{rj-1} \dots g_1 g_0$ is not in $\operatorname{rad}^{rj+1}(\operatorname{mod} A)$ by Theorem 2.5 and the h_i have infinite right degree by Proposition 2.6, it follows that $f^{(j)}$ is not in $\operatorname{rad}^{(2r+t+1)j+1}(M,M)$. Hence, for any $1 \leq j \leq k$, f^j does not belong to $\operatorname{rad}^{(2r+t+1)j+1}(M,M)$. Therefore, the K-vector space $\operatorname{End}_A(M)$ admits the following chain of subspaces

$$\begin{aligned} 0 = \operatorname{rad}^{(2r+t+1)(k+1)}(M,M) &\subset \operatorname{rad}^{(2r+t+1)k}(M,M) \subset \cdots \\ &\subset \operatorname{rad}^{2r+t+1}(M,M) \subset \operatorname{End}_A(M) \end{aligned}$$

such that $\operatorname{rad}^{(2r+t+1)i}(M,M)/\operatorname{rad}^{(2r+t+1)(i+1)}(M,M) = \operatorname{rad}^{(2r+t+1)i}(M,M)/\operatorname{rad}^{(2r+t+1)i+1}(M,M)$ is a right F_M-module generated by $f^i + \operatorname{rad}^{(2r+t+1)i+1}(M,M)$, for each $0 \leq i \leq k$. Hence, we get $\dim_K \operatorname{End}_A(M) = (k+1) \dim_K F_M$.

We shall now calculate $\dim_K \operatorname{Ext}^1_A(M,M)$. Note that from the Auslander-Reiten formula we have an isomorphism $\operatorname{Ext}^1_A(M,M) \cong D\underline{\operatorname{Hom}}_A(\tau_A^- M, M)$ of K-vector spaces. Moreover, $\operatorname{Ext}^1_A(Z_{01}, Z_{01}) = 0$, because Z_{01} is the projective A-module, and hence, we may assume that $M \neq Z_{01}$. First observe that any path in $\mathscr{C}$ from $\tau_A^- M$ to M has length $(2r+t+1)i - 2$ for some $i \geq 1$. This implies that $\operatorname{rad}(\tau_A^- M, M) = \operatorname{rad}^{2r+t-1}(\tau_A^- M, M)$ and $\operatorname{rad}^{(2r+t+1)i-1}(\tau_A^- M, M) = \operatorname{rad}^{(2r+t+1)(i+1)-2}(\tau_A^- M, M)$ for all $i \geq 1$. Similarly, as above, we prove that $\operatorname{rad}^m(\tau_A^- M, M) = 0$ for all $m \geq 2r+t$. In particular, if $p \leq r-1$, then $\operatorname{Hom}_A(\tau_A^- M, M) = \operatorname{rad}^{2r+t-1}(\tau_A^- M, M) = 0$. Suppose that $p > r-1$. Let

$$\tau_A^- M = U_0 \xrightarrow{\gamma_0} U_1 \to \cdots \to U_{r+t-1} \xrightarrow{\gamma_{r+t-1}} U_{r+t} = V_{r-1}$$

be the sectional path in $\mathscr{C}$ of length $r+t$ starting at $\tau_A^- M$ and formed by arrows pointing to the mouth. Put $h = f_{\gamma_0} \dots f_{\gamma_{r+t-1}} f_{\alpha_{r-2}} \dots f_{\alpha_0} : \tau_A^- M \to M$. Then, as above, we show that

$$hf^{j-1} \in \operatorname{rad}^{(2r+t+1)j-2}(\tau_A^- M, M) \setminus \operatorname{rad}^{(2r+t+1)j-1}(\tau_A^- M, M)$$

for all $1 \le j \le k$ such that $rk \le p < r(k+1)$. Therefore, the K-vector space $\operatorname{Hom}_A(\tau_A^- M, M)$ admits the following chain of subspaces

$$\begin{aligned} \operatorname{rad}^{(2r+t+1)(k+1)-2}(\tau_A^- M, M) &\subset \operatorname{rad}^{(2r+t+1)k-2}(\tau_A^- M, M) \subset \cdots \\ &\subset \operatorname{rad}^{2r+t-1}(\tau_A^- M, M) = \operatorname{Hom}_A(\tau_A^- M, M) \end{aligned}$$

such that

$$\frac{\operatorname{rad}^{(2r+t+1)i-2}(\tau_A^- M, M)}{\operatorname{rad}^{(2r+t+1)(i+1)-2}(\tau_A^- M, M)} = \frac{\operatorname{rad}^{(2r+t+1)i-2}(\tau_A^- M, M)}{\operatorname{rad}^{(2r+t+1)i-1}(\tau_A^- M, M)}$$

is a right F_M-module generated by $hf^{i-1} + \operatorname{rad}^{(2r+t+1)i-1}(\tau_A^- M, M)$, for each $1 \le i \le k$. Hence, we get

$$\dim_K \operatorname{Ext}_A^1(M, M) = \dim_K D\underline{\operatorname{Hom}}_A(\tau_A^- M, M) = k \dim_K F_M,$$

so the required inequality holds.

Assume $n \ge 2$. Then there is a generalized multicoil enlargement B of C using modules from the stable tubes $\mathscr{T}_1, \mathscr{T}_2, \dots, \mathscr{T}_s$, a finite family $\Omega_1, \Omega_2, \dots, \Omega_l$ of generalized standard generalized multicoils in Γ_B such that B is obtained from C by iterated application of $n-1$ admissible operations of types (ad 1)–(ad 5) and (ad 1*)–(ad 5*), $\Omega_1, \Omega_2, \dots, \Omega_l$ are obtained from the stable tubes $\mathscr{T}_1, \mathscr{T}_2, \dots, \mathscr{T}_s$ by the corresponding translation quiver admissible operations, A is obtained from B by one of the admissible operations of types (ad 1)–(ad 5) and (ad 1*)–(ad 5*), and $\mathscr{C}$ is obtained from $\Omega_1, \Omega_2, \dots, \Omega_l$ by the corresponding translation quiver admissible operation. If the admissible operation leading from B to A is of type (ad 1), (ad 1*), (ad 2), (ad 2*), (ad 3) or (ad 3*), then $l = 1$, and hence $\mathscr{C}$ is obtained from Ω_1 by the corresponding translation quiver admissible operation.

If the n-th admissible operation is of type (ad 1), then $A = B[X]$ or $A = (B \times D)[X \oplus Y]$, where X is the pivot of the operation (ad 1) in the generalized multicoil Ω_1, $D = T_t(F)$ is the lower $t \times t$ triangular matrix algebra over a division K-algebra F for some $t \ge 1$, and Y is the unique indecomposable projective-injective D-module (see definition of (ad 1)). Let M be an indecomposable

A-module in $\mathscr{C}$. Again, if M is a directing module in $\operatorname{mod} A$, then, by [**25**, (2.4)(8)], we get $\operatorname{Ext}_A^1(M, M) = 0$, $\operatorname{End}_A(M) = F_M$ and the required inequality holds. Assume that M is nondirecting. If M is a B-module, then M lies in the generalized multicoil Ω_1 of Γ_B. Then, by our inductive assumption, we have $\dim_K \operatorname{Ext}_A^1(M, M) \leq \dim_K \operatorname{End}_A(M)$. If M is not a B-module, then M lies in the infinite rectangle $\mathscr{S}(Z_{01})$ of $\mathscr{C}$ consisting of the A-modules Z_{pq}, for $p \geq 0$, $1 \leq q \leq t+1$, where Z_{01} is the projective A-module and $Z_{p,t+1} = X'_p$ (see definition of (ad 1)). Note that from the Auslander-Reiten formula we have an isomorphism $\operatorname{Ext}_A^1(M, M) \cong D\underline{\operatorname{Hom}}_A(\tau_A^- M, M)$ of K-vector spaces. Let $M = Z_{pq}$. From the definition of (ad 1) we know that there are at most two immediately successors of M in $\mathscr{C}$. We have three cases to consider. If M is an injective A-module, then $q = t+1$, $\operatorname{Ext}_A^1(M, M) = 0$ and the required inequality holds. Assume that M is a noninjective A-module. If M is the starting vertex of a mesh with exactly one middle term, then $q = t+1$ and we get

$$\begin{array}{ccccc} M & & & & \tau_A^- M \\ & \overset{\sigma}{\searrow} & & \nearrow^{\varrho} & \\ & & N & & \end{array}$$

where $N = Z_{p+1,t+1}$. Let

$$\tau_A^- M = \tau_A^- Z_{p,t+1} = N_{p+1} \to N_{p+2} \to \cdots \to N_{p+l} \to \cdots$$

where $l \geq 1$, be the sectional path (finite or infinite) in $\mathscr{C}$ formed by arrows pointing to infinity. Put $v = f_\varrho$ and $u = f_\sigma$. By our assumption, we have uv belongs to $\operatorname{rad}^3(M, \tau_A^- M)$. Let s be the length of shortest nontrivial path in $\mathscr{C}$ from M to N_j, for $j \geq p+1$. Then $\operatorname{Hom}_A(M, N_j) = \operatorname{rad}^s(M, N_j)$, $j \geq p+1$. We shall show that $\operatorname{rad}^m(M, N_j) = \operatorname{rad}^{m+1}(M, N_j)$ for any $m \geq s$. This will imply that $\operatorname{Hom}_A(M, N_j) = \operatorname{rad}^\infty(M, N_j) = 0$ for all $j \geq p+1$, because $\mathscr{C}$ is generalized standard. Let $m \geq s$ and $\Phi \in \operatorname{rad}^m(M, N_j)$, with $j \geq p+1$. Then $\Phi + \operatorname{rad}^{m+1}(M, N_j) = (\sum \psi_i a_i uv b_i) + \operatorname{rad}^{m+1}(M, N_j)$, where $a_i uv b_i$ are the composites of m irreducible morphisms including u and v, and ψ_i are invertible elements of $\operatorname{Hom}_A(M, N_j)$. Since uv lies in $\operatorname{rad}^3(\operatorname{mod} A)$, we get $\Phi + \operatorname{rad}^{m+1}(M, N_j) = 0 + \operatorname{rad}^{m+1}(M, N_j)$, and hence Φ belongs to $\operatorname{rad}^{m+1}(M, N_j)$, $j \geq p+1$. This proves our claim. Hence, using additionally the definition of (ad 1), we infer that $\operatorname{Hom}_A(M, \tau_A M) = 0$. Note that from the Auslander-Reiten formula we have an isomorphism $\operatorname{Ext}_A^1(M, M) \cong D\overline{\operatorname{Hom}}_A(M, \tau_A M)$ of K-vector spaces. Therefore,

$$\dim_K \operatorname{Ext}_A^1(M, M) = \dim_K D\overline{\operatorname{Hom}}_A(M, \tau_A M) \leq \dim_K \operatorname{Hom}_A(M, \tau_A M) = 0,$$

and the required inequality holds. If M is the starting vertex of a mesh with exactly two middle terms, then we have the following mesh

$$\begin{tikzcd} & L \arrow[dr, "\psi"] & \\ M \arrow[ur, "\varphi"] \arrow[dr, "\xi"'] & & \tau_A^- M \\ & N \arrow[ur, "\eta"'] & \end{tikzcd}$$

where $L = Z_{p,q+1}$, $N = Z_{p+1,q}$, $\tau_A^- M = Z_{p+1,q+1}$, $p \geq 0$, $1 \leq q \leq t$ or $L = \tau_A^- Z_{p-1,t+1}$, $N = Z_{p+1,t+1}$, $\tau_A^- M = \tau_A^- Z_{p,t+1}$, $p \geq 0$. Let

$$\Sigma : \ V_s \xrightarrow{\alpha_{s-1}} V_{s-1} \to \cdots \to V_1 \xrightarrow{\alpha_0} V_0 = M$$

be the unique maximal sectional path in $\mathscr{C}$ passing through M, formed by arrows pointing to infinity, and

$$\Theta : \ \tau_A^- M = U_0 \xrightarrow{\gamma_0} U_1 \to \cdots \to U_{l-1} \xrightarrow{\gamma_{l-1}} U_l$$

be the maximal sectional path in $\mathscr{C}$ starting at $\tau_A^- M$ and formed by arrows pointing to the mouth. Note that, in the case of a Möbius strip configuration created by an operation of type (ad 4) or (ad 4*), it could happen that we have an infinite sectional path in $\mathscr{C}$ starting at $\tau_A^- M$ and formed by finite number of arrows pointing to the mouth followed by arrows pointing to the infinity, but then $\mathrm{Hom}_A(\tau_A^- M, M) = 0$. Clearly, V_s lies on the mouth of $\mathscr{C}$. From the definition of a generalized multicoil we know that U_l is an injective A-module which does not lie on the mouth of $\mathscr{C}$ or U_l lies on the mouth of $\mathscr{C}$. In the first case, any sectional path in $\mathscr{C}$ from U_l to V_j, where $0 \leq j \leq s$, factors through a projective A-module. Therefore, $\mathrm{Ext}_A^1(M, M) = D\underline{\mathrm{Hom}}_A(\tau_A^- M, M) = 0$. So, the required inequality holds. In the second case, we consider two subcases. In the first subcase, the intersection of Θ and Σ is empty. The A-module U_l is the starting vertex of a mesh with exactly one middle term or is the middle term of a mesh with exactly three middle terms, then, similarly as above, we prove that $\mathrm{Hom}_A(\tau_A^- M, M) = 0$. So, the required inequality holds. In the second subcase, the intersection of Θ and Σ contains an A-module $U_i = V_j$, for some $0 \leq i \leq l$ and $0 \leq j \leq s$. Moreover, we know that the above two sectional paths intersect only finitely many times. Let $U_{l_1}, U_{l_2}, \ldots, U_{l_k}$ be the set of all A-modules in $\mathscr{C}$ such that $U_{l_i} = V_{j_i}$, with $1 \leq i \leq k$, $0 \leq j_i \leq s$. Without loss of generality we can assume that $l_1 < l_2 < \cdots < l_k$. Then $j_1 < j_2 < \cdots < j_k$. Since the morphism $f_{\alpha_{j_i-1}} \ldots f_{\alpha_1} f_{\alpha_0}$ is in $\mathrm{rad}^{j_i}(\mathrm{mod}\, A) \setminus \mathrm{rad}^{j_i+1}(\mathrm{mod}\, A)$ by Theorem 2.5 and the f_{γ_j}, $0 \le j < l_i$, have infinite right degree by Proposition 2.6, it follows that $f_i = f_{\gamma_0} \dots f_{\gamma_{l_i-1}} f_{\alpha_{j_i-1}} \dots f_{\alpha_0} : \tau_A^- M \to M$ belongs to $\mathrm{rad}^{l_i+j_i}(\tau_A^- M, M) \setminus \mathrm{rad}^{l_i+j_i+1}(\tau_A^- M, M)$ for all $1 \le i \le k$. Note that we have $l_i + j_i = l_1 + j_1 + (i-1)(a+b)$, where $1 \le i \le k$, a is the number of pairwise disjoint rays and b is the number of pairwise disjoint corays in a maximal proper subtube $\mathscr{T}(Z_{p1}, a, b)$ of $\mathscr{C}$, for some $Z_{p1} \in \mathscr{S}(Z_{01})$. If there is a nonzero path from $\tau_A^- M$ to M passing through a projective A-module which is the starting vertex of a mesh with exactly two middle terms, then $D\underline{\mathrm{Hom}}_A(\tau_A^- M, M) = 0$ and the required inequality holds. Therefore, although there may exist nonzero path from $\tau_A^- M$ to M passing through a projective-injective A-module which is in a mesh with exactly three middle terms, any generator of $\underline{\mathrm{Hom}}_A(\tau_A^- M, M)$ is of the form $\underline{f}_i$, for some $1 \le i \le k$, where $\underline{f}_i$ is the class of f_i in $\underline{\mathrm{Hom}}_A(\tau_A^- M, M)$. Note that any nonzero path in $\mathscr{C}$ from $\tau_A^- M$ to M we can lengthen to a nonzero path in $\mathscr{C}$ from M to M. Indeed, we have $M \to N \to \tau_A^- M \to U_1 \to \dots \to U_{l_1}$. Moreover, the path

$$N \xrightarrow{\eta} \tau_A^- M = U_0 \xrightarrow{\gamma_0} U_1 \to \dots \to U_{l_1-1} \xrightarrow{\gamma_{l_1-1}} U_{l_1}$$

is sectional and f_ξ has infinite right degree by Proposition 2.6. Then, as above, we show that the morphism $f_\xi f_\eta f_{\gamma_0} \dots f_{\gamma_{l_i-1}} f_{\alpha_{j_i-1}} \dots f_{\alpha_0} : M \to M$ belongs to $\mathrm{rad}^{l_i+j_i+2}(M, M) \setminus \mathrm{rad}^{l_i+j_i+3}(M, M)$, for all $1 \le i \le k$. Hence

$$\begin{aligned} \dim_K \mathrm{End}_A(M) &\ge \dim_K \mathrm{Hom}_A(\tau_A^- M, M) \ge \dim_K D\underline{\mathrm{Hom}}_A(\tau_A^- M, M) \\ &= \dim_K \mathrm{Ext}_A^1(M, M). \end{aligned}$$

If the n-th operation is of type (ad 1*), then the proof is dual.

If the n-th admissible operation is of type (ad 2), then $A = B[X]$, where X is the pivot of the operation (ad 2) in the generalized multicoil Ω_1. Let M be an indecomposable A-module in $\mathscr{C}$. If M is a B-module, then M lies in the generalized multicoil Ω_1 of Γ_B. Then, by our inductive assumption, we have $\dim_K \mathrm{Ext}_A^1(M, M) \le \dim_K \mathrm{End}_A(M)$. If M is not a B-module, then M is nondirecting and lies in the infinite rectangle $\mathscr{S}(X_0')$ of $\mathscr{C}$ consisting of the A-modules Z_{pq}, for $p \ge 1$, $1 \le q \le t+1$, and X_0', where X_0' is the projective-injective A-module and $Z_{p,t+1} = X_p'$ (see definition of (ad 2)). Again, from the Auslander-Reiten formula we have an isomorphism $\mathrm{Ext}_A^1(M, M) \cong D\underline{\mathrm{Hom}}_A(\tau_A^- M, M)$ of K-vector spaces. Let $M = Z_{pq}$. From the definition of (ad 2) we know that there are at most two immediate successors of M in $\mathscr{C}$. We have three cases to consider. If M is an injective A-module, then $q = t+1$, $\mathrm{Ext}_A^1(M, M) = 0$ and the required inequality holds. Assume that M is a noninjective A-module. If M is the starting vertex of a mesh with exactly one middle term, then $q = t+1$ and we get

$$\begin{tikzcd} M \arrow[dr, "\sigma"'] & & \tau_A^- M \\ & N \arrow[ur, "\varrho"'] & \end{tikzcd}$$

where $N = Z_{p+1,t+1}$. Let

$$\tau_A^- M = \tau_A^- Z_{p,t+1} = N_{p+1} \to N_{p+2} \to \cdots \to N_{p+l} \to \cdots$$

where $l \geq 1$, be the sectional path (finite or infinite) in $\mathscr{C}$ formed by arrows pointing to infinity. Put $v = f_\varrho$ and $u = f_\sigma$. By our assumption, we have uv belongs to $\mathrm{rad}^3(M, \tau_A^- M)$. Similarly, as above, we prove that $\mathrm{Hom}_A(M, N_j) = 0$ for all $j \geq p+1$. Hence, using additionally the definition of (ad 2), we infer that $\mathrm{Hom}_A(M, \tau_A M) = 0$. Note that from the Auslander-Reiten formula we have an isomorphism $\mathrm{Ext}_A^1(M, M) \cong D\overline{\mathrm{Hom}}_A(M, \tau_A M)$ of K-vector spaces. Therefore,

$$\dim_K \mathrm{Ext}_A^1(M, M) = \dim_K D\overline{\mathrm{Hom}}_A(M, \tau_A M) \leq \dim_K \mathrm{Hom}_A(M, \tau_A M) = 0,$$

and the required inequality holds. If M is the starting vertex of a mesh with exactly two middle terms, then we have the following mesh

$$\begin{tikzcd} & L \arrow[dr] & \\ M \arrow[ur] \arrow[dr] & & \tau_A^- M \\ & N \arrow[ur] & \end{tikzcd}$$

where $L = Z_{p,q+1}$, $N = Z_{p+1,q}$, $\tau_A^- M = Z_{p+1,q+1}$, $p \geq 1$, $1 \leq q \leq t$ or $L = \tau_A^- Z_{p-1,t+1}$, $N = Z_{p+1,t+1}$, $\tau_A^- M = \tau_A^- Z_{p,t+1}$, $p \geq 1$. Since we can lengthen any nonzero path in $\mathscr{C}$ from $\tau_A^- M$ to M to a nonzero path in $\mathscr{C}$ from M to M (by a path $M \to N \to \tau_A^- M$ of length two), the required inequality follows from the previous considerations. Moreover, $\mathrm{Ext}_A^1(X_0', X_0') = 0$, because X_0' is the projective-injective A-module. So, the required inequality holds for $M = X_0'$. If the n-th operation is of type (ad 2*), then the proof is dual.

If the n-th admissible operation is of type (ad 3), then $A = B[X]$, where X is the pivot of the operation (ad 3) in the generalized multicoil Ω_1. Let M be an indecomposable A-module in $\mathscr{C}$. If M is a B-module, then M lies in the generalized multicoil Ω_1 of Γ_B. Then by our inductive assumption we have

$\dim_K \operatorname{Ext}_A^1(M,M) \leq \dim_K \operatorname{End}_A(M)$. If M is not a B-module, then M is nondirecting and lies in the infinite rectangle $\mathscr{S}(X_0')$ of $\mathscr{C}$ consisting of the A-modules Z_{pq}, for $p \geq 1$, $1 \leq q \leq t+1$, and X_0', where X_0' is the projective A-module and $Z_{p,t+1} = X_p'$ (see definition of (ad 3)). First observe that, for $M = Z_{pq}$, $p > q$, $1 \leq q \leq t$ and for $M = Z_{tt}$, we have the following mesh

$$\begin{tikzcd} & L \arrow[dr] & \\ M \arrow[ur] \arrow[dr] & & \tau_A^- M \\ & N \arrow[ur] & \end{tikzcd}$$

where $L = Z_{p,q+1}$, $N = Z_{p+1,q}$, $\tau_A^- M = Z_{p+1,q+1}$. Moreover, for $M = Z_{qq}$, $1 \leq q \leq t-1$, we have the following mesh

$$\begin{tikzcd} & L \arrow[dr] & \\ M \arrow[ur] \arrow[r] \arrow[dr] & R \arrow[r] & \tau_A^- M \\ & N \arrow[ur] & \end{tikzcd}$$

where $L = Z_{q,t+1}$, $R = Y_{q+1}$, $N = Z_{q+1,q}$, $\tau_A^- M = Z_{q+1,q+1}$ or $L = Y_{q+1}$, $R = Z_{q,t+1}$, $N = Z_{q+1,q}$, $\tau_A^- M = Z_{q+1,q+1}$. Since we can lengthen any nonzero path in $\mathscr{C}$ from $\tau_A^- M$ to M to a nonzero path in $\mathscr{C}$ from M to M (by a path $M \to N \to \tau_A^- M$ of length two), the required inequality follows from the previous considerations. Since X_0' is a projective A-module and $Z_{t-1,t+1}$ is an injective A-module, we get $\operatorname{Ext}_A^1(X_0', X_0') = 0$ and $\operatorname{Ext}_A^1(Z_{t-1,t+1}, Z_{t-1,t+1}) = 0$. So, the required inequality holds also for $M = X_0'$ and $M = Z_{t-1,t+1}$.

We shall now prove the required inequality for all indecomposable A-modules $M = Z_{p,t+1}$, with $p \geq 1$, $p \neq t-1$. From the definition of (ad 3) we know that there are at most two immediate successors of $Z_{p,t+1}$, $p \geq t$, in $\mathscr{C}$ and there is at least one mesh in $\mathscr{C}$ of the form

$$\begin{tikzcd} Z_{p,t+1} \arrow[dr, "\eta"'] & & \tau_A^- Z_{p,t+1} \\ & Z_{p+1,t+1} \arrow[ur, "\xi"'] & \end{tikzcd}$$

starting at $Z_{p,t+1}$, with $p \geq t$. Put $w = f_\eta$ and $h = f_\xi$. By our assumption, we have that wh belongs to $\operatorname{rad}^3(Z_{p,t+1}, \tau_A^- Z_{p,t+1})$. Let us first examine M for $1 \leq p \leq t-2$. Let s be the length of the shortest nontrivial path in $\mathscr{C}$ from M to $\tau_A^- Z_{j,t+1}$, for

$j \geq t$. Then $\mathrm{Hom}_A(M, \tau_A^- Z_{j,t+1}) = \mathrm{rad}^s(M, \tau_A^- Z_{j,t+1})$, for $j \geq t$. We shall show that $\mathrm{rad}^m(M, \tau_A^- Z_{j,t+1}) = \mathrm{rad}^{m+1}(M, \tau_A^- Z_{j,t+1})$ for any $m \geq s$. This will imply that $\mathrm{Hom}_A(M, \tau_A^- Z_{j,t+1}) = \mathrm{rad}^\infty(M, \tau_A^- Z_{j,t+1}) = 0$, for all $j \geq t$, because $\mathscr{C}$ is generalized standard. Let $m \geq s$ and $\Phi \in \mathrm{rad}^m(M, \tau_A^- Z_{j,t+1})$, with $j \geq t$. Then $\Phi + \mathrm{rad}^{m+1}(M, \tau_A^- Z_{j,t+1}) = (\sum \psi_i a_i w h b_i) + \mathrm{rad}^{m+1}(M, \tau_A^- Z_{j,t+1})$, where $a_i w h b_i$ are the composites of m irreducible morphisms including w and h, and ψ_i are invertible elements of $\mathrm{Hom}_A(M, \tau_A^- Z_{j,t+1})$. Since wh lies in $\mathrm{rad}^3(\mathrm{mod}\, A)$, we get $\Phi + \mathrm{rad}^{m+1}(M, \tau_A^- Z_{j,t+1}) = 0 + \mathrm{rad}^{m+1}(M, \tau_A^- Z_{j,t+1})$, and hence Φ belongs to $\mathrm{rad}^{m+1}(M, \tau_A^- Z_{j,t+1})$, $j \geq t$. This proves our claim. Hence, using additionally the definition of (ad 3), we infer that $\mathrm{Hom}_A(M, \tau_A M) = 0$. Therefore,

$$\dim_K \mathrm{Ext}_A^1(M, M) = \dim_K D\overline{\mathrm{Hom}}_A(M, \tau_A M) \leq \dim_K \mathrm{Hom}_A(M, \tau_A M) = 0,$$

and the required inequality holds. Now, we examine M for some $p \geq t$. Since M has at most two immediate successors in $\mathscr{C}$, we have three cases to consider. Again, if M is an injective A-module, then $\mathrm{Ext}_A^1(M, M) = 0$ and the required inequality holds. Assume that M is a noninjective A-module. If M is the starting vertex of a mesh with exactly one middle term, then we get

$$\begin{tikzcd} M \arrow[dr, "\sigma"'] & & \tau_A^- M \\ & N \arrow[ur, "\varrho"'] & \end{tikzcd}$$

where $N = Z_{p+1,t+1}$. Let

$$\tau_A^- M = \tau_A^- Z_{p,t+1} = N_{p+1} \to N_{p+2} \to \cdots \to N_{p+l} \to \cdots$$

where $l \geq 1$ be the sectional path (finite or infinite) in $\mathscr{C}$ formed by arrows pointing to infinity. Put $v = f_\varrho$ and $u = f_\sigma$. By our assumption, we have uv belongs to $\mathrm{rad}^3(M, \tau_A^- M)$. Similarly, as above, we prove that $\mathrm{Hom}_A(M, N_j) = 0$ for all $j \geq p+1$. Hence, using additionally the definition of (ad 3), we infer that $\mathrm{Hom}_A(M, \tau_A M) = 0$. Therefore,

$$\dim_K \mathrm{Ext}_A^1(M, M) = \dim_K D\overline{\mathrm{Hom}}_A(M, \tau_A M) \leq \dim_K \mathrm{Hom}_A(M, \tau_A M) = 0,$$

and the required inequality holds. If M is the starting vertex of a mesh with exactly two middle terms, then the required inequality follows from the previous considerations. If the n-th operation is of type (ad 3^*), then the proof is dual.

If the n-th admissible operation is of type (ad 4), then, for $r = 0$, $A = B[X \oplus Y]$, and for $r \geq 1$,

$$A = \begin{bmatrix} B & 0 & 0 & \cdots & 0 & 0 \\ Y & F & 0 & \cdots & 0 & 0 \\ Y & F & F & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ Y & F & F & \cdots & F & 0 \\ X \oplus Y & F & F & \cdots & F & F \end{bmatrix}$$

with $r+2$ columns and rows, where X is the pivot of the operation (ad 4) in the generalized multicoil Ω_1, and Y is the starting vertex of a finite sectional path in the generalized multicoil Ω_1 or Ω_2 (see definition of (ad 4)). Note that in this case $l = 1$ or $l = 2$, so $\mathscr{C}$ is obtained from Ω_1 or from the disjoint union of two generalized multicoils Ω_1, Ω_2 by the corresponding translation quiver admissible operation. Let M be an indecomposable A-module in $\mathscr{C}$. If M is a B-module, then M lies in one of the generalized multicoils Ω_1 or Ω_2 of Γ_B. Then, by our inductive assumption, we have $\dim_K \operatorname{Ext}_A^1(M,M) \leq \dim_K \operatorname{End}_A(M)$. If M is not a B-module, then, for $r = 0$, M lies in the infinite rectangle $\mathscr{S}(Z_{01})$ of $\mathscr{C}$ consisting of the A-modules Z_{pq}, for $p \geq 0$, $1 \leq q \leq t+1$, where Z_{01} is the projective A-module and $Z_{p,t+1} = X'_p$. Further, for $r \geq 1$, M lies in the infinite rectangle $\{U_{kl}, Z_{pq}\}_{k,l,p,q}$ (trapezoid) of $\mathscr{C}$ consisting of the A-modules U_{kl}, for $1 \leq k \leq r$, $1 \leq l \leq t+k$, Z_{pq}, for $p \geq 0$, $1 \leq q \leq t+r+1$, where Z_{01}, U_{k1} are the projective A-modules, $Z_{p,t+r+1} = X'_p$ and $t+r$ is the parameter of the operation (ad 4) (see definition of (ad 4)). Again, observe that, for $M = U_{kl}$, $1 \leq k \leq r$, $1 \leq l \leq t+k-1$, and for $M = Z_{pq}$, $p \geq 0$, $1 \leq q \leq t+r$, we have the following mesh

$$\begin{tikzcd} & L \arrow[dr] & \\ M \arrow[ur] \arrow[dr] & & \tau_A^- M \\ & N \arrow[ur] & \end{tikzcd}$$

where $L = U_{k,l+1}$, $N = U_{k+1,l}$, $\tau_A^- M = U_{k+1,l+1}$ for $1 \leq k \leq r-1$, $L = U_{r,l+1}$, $N = Z_{0,l}$, $\tau_A^- M = Z_{0,l+1}$ for $k = r$ or $L = Z_{p,q+1}$, $N = Z_{p+1,q}$, $\tau_A^- M = Z_{p+1,q+1}$. Since we can lengthen any nonzero path in $\mathscr{C}$ from $\tau_A^- M$ to M to a nonzero path in $\mathscr{C}$ from M to M (by a path $M \to N \to \tau_A^- M$ of length two), the required inequality follows from the previous considerations. Additionally, we know that Z_{01}, U_{k1}, $1 \leq k \leq r$, are projective A-modules, and so $\operatorname{Ext}_A^1(Z_{01}, Z_{01}) = 0$ and $\operatorname{Ext}_A^1(U_{k1}, U_{k1}) = 0$. From the definition of (ad 4) we know that for any $M = U_{k,t+k}$, $1 \leq k \leq r$, we have the following mesh in $\mathscr{C}$

$$\begin{tikzcd}[column sep=small] U_{k,t+k} \arrow[dr, "\eta"'] & & U_{k+1,t+k+1} \\ & U_{k+1,t+k} \arrow[ur, "\xi"'] & \end{tikzcd}$$

starting at $U_{k,t+k}$, where $U_{r+1,t+r} = Z_{0,t+r}$, $U_{r+1,t+r+1} = Z_{0,t+r+1}$. Put $w = f_\eta$ and $h = f_\xi$. By our assumption, wh belongs to $\operatorname{rad}^3(U_{k,t+k}, U_{k,t+k+1})$. Moreover, for any M in $\{U_{kl}, Z_{pq}\}_{k,l,p,q}$, there exists an infinite sectional path Σ in $\mathscr{C}$ of the form

$$\tau_A^- M = \tau_A^- U_{k,t+k} = U_{k+1,t+k+1} \to \cdots \to Z_{0,t+k+1} \to Z_{1,t+k+1} \to \cdots$$

Let X be an arbitrary A-module on Σ. Then, from the above remarks, we have $\operatorname{Hom}_A(M, X) = 0$. Hence, using the definition of (ad 4), we infer that $\operatorname{Hom}_A(M, \tau_A M) = 0$. Therefore,

$$\dim_K \operatorname{Ext}^1_A(M, M) = \dim_K D\overline{\operatorname{Hom}}_A(M, \tau_A M) \le \dim_K \operatorname{Hom}_A(M, \tau_A M) = 0,$$

and the required inequality holds. Let $M = Z_{p,t+r+1}$, for some $p \ge 0$. Since M has at most two immediately successors in $\mathscr{C}$, we have three cases to consider. Again, if M is an injective A-module, then $\operatorname{Ext}^1_A(M, M) = 0$, and the required inequality holds. Assume that M is a noninjective A-module. If M is the starting vertex of a mesh with exactly one middle term, then we get

$$\begin{tikzcd}[column sep=small] M \arrow[dr, "\sigma"'] & & \tau_A^- M \\ & N \arrow[ur, "\varrho"'] & \end{tikzcd}$$

where $N = Z_{p+1,t+r+1}$. Let

$$\tau_A^- M = \tau_A^- Z_{p,t+r+1} = N_{p+1} \to N_{p+2} \to \cdots \to N_{p+l} \to \cdots$$

where $l \ge 1$, be the sectional path (finite or infinite) in $\mathscr{C}$ formed by arrows pointing to infinity. Put $v = f_\varrho$ and $u = f_\sigma$. By our assumption, we know that uv belongs to $\operatorname{rad}^3(M, \tau_A^- M)$. Similarly, as above we prove that $\operatorname{Hom}_A(M, N_j) = 0$ for all $j \ge p+1$. Hence, using additionally the definition of (ad 4), we infer that $\operatorname{Hom}_A(M, \tau_A M) = 0$. Therefore,

$$\dim_K \operatorname{Ext}^1_A(M, M) = \dim_K D\overline{\operatorname{Hom}}_A(M, \tau_A M) \le \dim_K \operatorname{Hom}_A(M, \tau_A M) = 0,$$

and the required inequality holds. If M is the starting vertex of a mesh with exactly two middle terms, then the required inequality follows from the previous considerations. If the n-th operation is of type (ad 4*), then the proof is dual.

If the n-th admissible operation is of type (ad 5) then $\mathscr{C}$ is obtained from the disjoint union of the finite family of generalized multicoils $\Omega_1, \Omega_2, \dots, \Omega_l$, $1 \le l \le s$, which are generalized standard. Since in the definition of admissible operation (ad 5) we use the finite versions (fad 1), (fad 2), (fad 3), (fad 4) of the admissible operations (ad 1), (ad 2), (ad 3), (ad 4) and the admissible operation (ad 4), the required inequality follows from the above considerations. If the n-th operation is of type (ad 5*), then the proof is dual.

(ii) Since id_M does not belong to $\mathrm{rad}(M, M)$, it follows from the proof of (i) that $\dim_K \mathrm{Ext}^1_A(M, M) = \dim_K \mathrm{End}_A(M)$ if and only if M belongs to one of the stable tubes $\mathscr{T}_i$, $1 \le i \le s$, and $\mathrm{ql}(M)$ is divisible by the rank of $\mathscr{T}_i$ (see Corollary 2.3).

5. Proof of Theorem 1.1.

Let A be a finite dimensional K-algebra over a field K with a separating family $\mathscr{C}_A$ of almost cyclic coherent components in Γ_A, and $\Gamma_A = \mathscr{P}_A \vee \mathscr{C}_A \vee \mathscr{Q}_A$ be the induced decomposition of Γ_A. Then it follows from Theorem 3.2 that there exists s concealed canonical factor algebra C (not necessarily connected) of A such that A is a generalized multicoil enlargement of C using modules of a separating family $\mathscr{T}_C$ of stable tubes of Γ_C. Moreover, applying Theorem 3.3, we infer that there exists a unique factor algebra A_l of A which is a quasitilted algebra of canonical type with a separating family $\mathscr{T}_{A_l}$ of coray tubes such that $\Gamma_{A_l} = \mathscr{P}_{A_l} \vee \mathscr{T}_{A_l} \vee \mathscr{Q}_{A_l}$ and $\mathscr{P}_{A_l} = \mathscr{P}_A$, and a unique factor algebra A_r of A which is a quasitilted algebra of canonical type with a separating family $\mathscr{T}_{A_r}$ of ray tubes such that $\Gamma_{A_r} = \mathscr{P}_{A_r} \vee \mathscr{T}_{A_r} \vee \mathscr{Q}_{A_r}$ and $\mathscr{Q}_{A_r} = \mathscr{Q}_A$. In fact, it follows from the proof of [**18**, Theorem C] that A_l is a branch coextension of C and A_r is a branch extension of C, the both using modules from $\mathscr{T}_C$.

Let M be an indecomposable module in $\mathrm{mod}\, A$. We claim that $\mathrm{Ext}^r_A(M, M) = 0$ for any $r \ge 2$. Since, by Theorem 3.3, $\mathrm{pd}_A M \le 2$ and $\mathrm{id}_A M \le 2$, we obtain $\mathrm{Ext}^r_A(M, M) = 0$ for any $r \ge 3$. Further, if M belongs to $\mathscr{P}_A$, then $\mathrm{pd}_A M \le 1$, and consequently $\mathrm{Ext}^2_A(M, M) = 0$. Similarly, if M belongs to $\mathscr{Q}_A$, then $\mathrm{id}_A M \le 1$, and so $\mathrm{Ext}^2_A(M, M) = 0$. Assume M belongs to $\mathscr{C}_A$. Consider the projective cover $\pi : P(M) \to M$ of M in $\mathrm{mod}\, A$ and $\Omega(M) = \mathrm{Ker}\, \pi$. Then we have an exact sequence

$$0 \to \Omega(M) \to P(M) \to M \to 0$$

in $\mathrm{mod}\, A$, and consequently $\mathrm{Ext}^2_A(M, M) \cong \mathrm{Ext}^1_A(\Omega(M), M)$. Moreover, we showed in the proof of [**18**, Theorem E] that $\Omega(M) = M_1 \oplus M_2$, where M_1 is a projective A-module and M_2 is a module from $\operatorname{add}(\mathscr{P}_A)$. Since $\mathscr{C}_A$ separates $\mathscr{P}_A$ from $\mathscr{Q}_A$, we have $\operatorname{Hom}_A(\mathscr{C}_A, \mathscr{P}_A) = 0$. Applying the Auslander-Reiten formula, we obtain K-linear isomorphisms

$$\operatorname{Ext}^1_A(\Omega(M), M) \cong D\overline{\operatorname{Hom}}_A(M, \tau_A\Omega(M)) \cong D\overline{\operatorname{Hom}}_A(M, \tau_A M_2) = 0,$$

because M belongs to $\mathscr{C}_A$ and $\tau_A M_2$ belongs to $\operatorname{add}(\mathscr{P}_A)$. Therefore, we obtain $\operatorname{Ext}^2_A(M, M) = 0$.

This shows that the statements (ii) and (iii) are equivalent.

Observe also that the separating family $\mathscr{C}_A$ consists of pairwise orthogonal generalized standard almost cyclic coherent components of Γ_A. Therefore, it follows from Theorem 1.3 that $\dim_K \operatorname{Ext}^1_A(M, M) \leq \dim_K \operatorname{End}_A(M)$ for any indecomposable module M in $\mathscr{C}_A$.

Assume now that $g(A) \leq 1$. Then the quasitilted algebras A_l and A_r are products of tilted algebras of Euclidean type or tubular algebras (see [**13**] and [**32**]). In particular, every component of the family $\mathscr{P}_A = \mathscr{P}_{A_l}$ is either a preprojective component of Euclidean type or a generalized standard ray tube. Similarly, every component of the family $\mathscr{Q}_A = \mathscr{Q}_{A_r}$ is either a preinjective component of Euclidean type or a generalized standard coray tube. It is well known that every indecomposable module M in a preprojective component or preinjective component of Γ_A is directing, and then $\operatorname{Ext}^1_A(M, M) = 0$ (see [**25**, (2.4)(8)] or [**1**, Proposition IX. 1.4]). Moreover, every ray tube of $\mathscr{P}_A$ and every coray tube of $\mathscr{Q}_A$ is a generalized standard almost cyclic coherent component of Γ_A. Therefore, applying Theorem 1.3, we conclude that

$$\chi_A([M]) = \dim_K \operatorname{End}_A(M) - \dim_K \operatorname{Ext}^1_A(M, M) \geq 0$$

for any indecomposable module M in $\operatorname{mod} A$. This shows that (i) implies (ii).

Assume now that $g(A) > 1$. Then, by [**18**, Theorem F], one of the quasitilted algebras A_l and A_r is wild. Applying then results on the structure of module categories of quasitilted algebras of wild canonical type proved in [**11**], [**12**], [**13**], we conclude Γ_A admits a component Γ which is preprojective or preinjective and the factor algebra $B = A/\operatorname{ann}_A(\Gamma)$ is a wild tilted algebra. Then it follows from [**10**, Theorem 6.2] that there is an indecomposable B-module M such that $\dim_K \operatorname{Ext}^1_B(M, M) > \dim_K \operatorname{End}_B(M)$. Observe that $\operatorname{End}_B(M) = \operatorname{End}_A(M)$ and $\dim_K \operatorname{Ext}^1_A(M, M) \geq \dim_K \operatorname{Ext}^1_B(M, M)$. Therefore, we obtain the inequality

$$\chi_A([M]) = \dim_K \operatorname{End}_A(M) - \dim_K \operatorname{Ext}^1_A(M, M) < 0,$$

because $\mathrm{Ext}_A^r(M,M) = 0$ for $r \geq 2$, as we proved in the first part of our proof. This shows that (ii) implies (i).

6. Proof of Corollary 1.2.

Let A be a tame finite dimensional K-algebra with a separating family $\mathscr{C}_A$ of almost cyclic coherent components in Γ_A, and $\Gamma_A = \mathscr{P}_A \vee \mathscr{C}_A \vee \mathscr{Q}_A$ be the induced decomposition of Γ_A. Then by Theorem 3.2 and [**18**, Theorem F], A is a tame generalized multicoil enlargement of a product C of tame concealed algebras (concealed canonical algebras of Euclidean type). Moreover, the left quasitilted algebra A_l and the right quasitilted algebra A_r of A are products of tilted algebras of Euclidean type or tubular algebras. We also note that the tubular algebras are tame concealed canonical algebras with infinitely many families of sincere generalized standard stable tubes.

Let M be an indecomposable module in $\mathrm{mod}\, A$. It follows from Theorem 1.1 that $\chi_A([M]) \geq 0$.

Assume M belongs to $\mathscr{C}_A$. Then it follows from Theorem 1.3 that $\chi_A([M]) = 0$ if and only if M is a module lying in a generalized standard stable tube $\mathscr{T}$ of Γ_C and the quasi-length $\mathrm{ql}(M)$ of M in $\mathscr{T}$ is divisible by the rank of $\mathscr{T}$.

Assume M belongs to $\mathscr{P}_A$. If M belongs to a preprojective component of $\mathscr{P}_A$, then M is a directing module, and hence $\chi_A([M]) = \dim_K \mathrm{End}_A(M) > 0$. Suppose M does not belong to a preprojective component of $\mathscr{P}_A$. Then M belongs to a generalized standard ray tube $\mathscr{T}$ of a tubular factor algebra B of A_l. By general theory, B is a tubular (branch) extension of a tame concealed algebra Λ, which is clearly a factor algebra of B, and hence of A. In case $\mathscr{T}$ is a sincere stable tube of Γ_B, then $\chi_A([M]) = \chi_B([M]) = 0$ if and only if the quasi-length $\mathrm{ql}(M)$ of M in $\mathscr{T}$ is divisible by the rank of $\mathscr{T}$ (see Corollary 2.3). In case $\mathscr{T}$ is not a sincere stable tube of Γ_B, then $\mathscr{T}$ is either a ray tube containing at least one projective module or a sincere stable tube of Γ_Λ. Then it follows from Theorem 1.3 that $\chi_A([M]) = 0$ if and only if M is a module of a stable tube Γ of Γ_Λ and the quasi-length $\mathrm{ql}(M)$ of M in Γ is divisible by the rank of Γ.

Assume M belongs to $\mathscr{Q}_A$. If M belongs to a preinjective component of $\mathscr{Q}_A$ then M is a directing module, and hence $\chi_A([M]) = \dim_K \mathrm{End}_A(M) > 0$. Suppose M does not belong to a preinjective component of $\mathscr{Q}_A$. Then M belongs to a generalized standard coray tube $\mathscr{T}^*$ of a tubular factor algebra B^* of A_r. By general theory, B^* is a tubular (branch) coextension of a tame concealed algebra Λ^*, which is obviously a factor algebra of B^*, and hence of A. In case $\mathscr{T}^*$ is a sincere stable tube of Γ_{B^*}, then, by Corollary 2.3, $\chi_A([M]) = \chi_{B^*}([M]) = 0$ if and only if the quasi-length $\mathrm{ql}(M)$ of M in $\mathscr{T}^*$ is divisible by the rank of $\mathscr{T}^*$. In case $\mathscr{T}^*$ is not a sincere stable tube of Γ_{B^*}, then $\mathscr{T}^*$ is either a coray tube containing at least one injective module or a sincere stable tube of Γ_{Λ^*}. Then it follows from Theorem 1.3 that $\chi_A([M]) = 0$ if and only if M is a module in a stable tube Γ^* of Γ_{Λ^*} and the quasi-length $\mathrm{ql}(M)$ of M in Γ^* is divisible by the rank of Γ^*.

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