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Characterizations of topological dimension by use of normal sequences of finite open covers and Pontrjagin-Schnirelmann theorem

Dedicated to Professor J. Nagata

By Hisao Kato and Masahiro Matsumoto

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Abstract. In 1932, Pontrjagin and Schnirelmann [**15**] proved the classical theorem which characterizes topological dimension by use of box-counting dimensions. They proved their theorem by use of geometric arguments in some Euclidean spaces. In this paper, by use of dimensional theoretical techniques in an abstract topological space, we investigate strong relations between metrics of spaces and box-counting dimensions. First, by use of the numerical information of normal sequences of finite open covers of a space X, we prove directly the following theorem characterizing topological dimension $\dim X$.

Theorem 0.1. Let X be a nonempty separable metric space. Then

$$\begin{aligned}\dim X = \min\Bigg\{ \liminf_{i\to\infty} \frac{\log_3 |\mathscr{U}_i|}{i} \;\Bigg|\; &\{\mathscr{U}_i\}_{i=1}^{\infty} \text{ is a normal star-sequence} \\ &\text{of finite open covers of } X \text{ and a development of } X \Bigg\} \\ = \min\Bigg\{ \liminf_{i\to\infty} \frac{\log_2 |\mathscr{U}_i|}{i} \;\Bigg|\; &\{\mathscr{U}_i\}_{i=1}^{\infty} \text{ is a normal delta-sequence} \\ &\text{of finite open covers of } X \text{ and a development of } X \Bigg\}.\end{aligned}$$

Next, we study box-counting dimensions $\dim_B(X,d)$ by use of Alexandroff-Urysohn metrics d induced by normal sequences. We show that the above theorem implies Pontrjagin-Schnirelmann theorem. The proof is different from the one of Pontrjagin and Schnirelmann (see [**15**]). By use of normal sequences, we can construct freely metrics d which control the values of $\log N(\epsilon,d)/|\log\epsilon|$. In particular, we can construct *chaotic metrics* with respect to the determination of the box-counting dimensions as follows.

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THEOREM 0.2. Let X be an infinite separable metric space. For any $\infty \geq \alpha \geq \dim X$, there is a totally bounded metric d_α on X such that

$$[\alpha, \infty] = \left\{ \liminf_{k\to\infty} \frac{\log N(\epsilon_k, d_\alpha)}{|\log \epsilon_k|} \;\middle|\; \{\epsilon_k\}_{k=1}^{\infty} \text{ is a decreasing sequence of positive numbers with } \lim_{k\to\infty} \epsilon_k = 0 \right\},$$

where $N(\epsilon_k, d_\alpha) = \min\{|\mathscr{U}| \mid \mathscr{U}$ is a finite open cover of X with $mesh_{d_\alpha}(\mathscr{U}) \leq \epsilon_k\}$. In particular, $\dim_B(X, d_\alpha) = \alpha$.

1. Introduction.

Let X be a topological space and let $\mathscr{U}$ be a collection of subsets of X. For any $p \in X$, we mean by the order of $\mathscr{U}$ at p the number of members of $\mathscr{U}$ which contain p, and we denote it by $\operatorname{ord}_p \mathscr{U}$. If there exist infinitely many such members, then $\operatorname{ord}_p \mathscr{U} = \infty$. Let $\mathscr{U}$ and $\mathscr{V}$ be finite open covers of X. If for each $U \in \mathscr{U}$ there exists $V \in \mathscr{V}$ such that $U \subset V$, then we call $\mathscr{U}$ a refinement of $\mathscr{V}$, and we denote this relation by $\mathscr{U} \leq \mathscr{V}$. For a topological space X, we denote by $\dim X$ the topological (covering) dimension of X:

(1) $\dim X \leq n$ ($n = -1, 0, 1, 2, \dots$) if every finite open cover $\mathscr{V}$ of X has a finite open cover $\mathscr{U}$ such that $\mathscr{U} \leq \mathscr{V}$ and $\operatorname{ord} \mathscr{U} \leq n+1$, where $\operatorname{ord} \mathscr{U} = \sup\{\operatorname{ord}_p \mathscr{U} \mid p \in X\}$.
(2) $\dim X = n$ if $\dim X \leq n$ but not $\dim X \leq n-1$.
(3) $\dim X = \infty$ if $\dim X \leq n$ does not hold for any n.

Note that topological dimension is originally defined in terms of local cardinality, order of cover. For topological dimension theory, see [**5**], [**8**], [**10**] and [**11**].

Recently, there has been an increase in the importance of fractal sets in the sciences, and fractal dimension theory has been studied by many scientists and mathematicians (e.g., see [**1**], [**6**], [**9**] and [**14**]). Fractal dimensions depend on the metrics of spaces and hence the analysis of metrics of the spaces is very important. In this paper, we study some properties of topological dimension, metrics and box-counting dimensions of separable metric spaces from a point of view of general topology. In general topology, the notion of normal sequence of open covers is one of the most useful tools for the study (e.g., see [**10**], [**11**], [**12**]). For example, the notion is the essence of metrizability of spaces (see Theorem 2.1). The key word of this paper is "normal sequence" of finite open covers. In this paper, we investigate directly the numerical properties of normal sequences of finite open covers on a given separable metric space X and we give another proof of Pontrjagin-Schnirelmann theorem. Furtheremore, by use of normal sequences we construct metrics d which can control the values of $\log N(\epsilon, d)/|\log \epsilon|$. In particular, we can construct *chaotic metrics* with respect to the determination of the box-counting dimensions. The methods used in this paper are based on dimensional theoretical techniques in an abstract topological space.

In fractal dimension theory, Pontrjagin and Schnirelmann [**15**] proved the following fundamental result involving topological dimension $\dim X$ and (lower) box-counting dimension $\dim_B(X, d)$ for a compact metric space (X, d): For a metric d on X and $\epsilon > 0$, let

$$N(\epsilon, d) = \min\{|\mathscr{U}| \mid \mathscr{U} \text{ is a finite open cover of } X \text{ with } \mathrm{mesh}_d(\mathscr{U}) \le \epsilon\}$$

and

$$\dim_B(X, d) = \sup\left\{ \inf\left\{ \frac{\log N(\epsilon, \rho)}{|\log \epsilon|} \,\middle|\, 0 < \epsilon < \epsilon_0 \right\} \,\middle|\, 0 < \epsilon_0 \right\} \left(= \liminf_{\epsilon \to 0} \frac{\log N(\epsilon, d)}{|\log \epsilon|} \right),$$

where $|A|$ denotes the cardinality of a set A. Then

$$\dim X = \min\{\dim_B(X, d) \mid d \text{ is a metric on } X\}.$$

More generally, Bruijning ([**3**] or [**11**, p. 81, Corollary]) showed that if X is a separable metric space, then

$$\dim X = \min\{\dim_B(X, d) \mid d \text{ is a totally bounded metric on } X\}.$$

Pontrjagin and Schnirelmann proved their theorem by use of geometric arguments in some Euclidean spaces. In fact, such a metric d on X with $\dim X = \dim_B(X, d)$ was obtained by use of geometric arguments (embedding arguments) on polyhedral approximations of n-dimensional sets in the $(2n+1)$-dimensional Euclidean space R^{2n+1} (see [**11**] and [**15**]).

2. Normal sequences of open covers, star-refinements and delta-refinements.

In this paper, we need the following terminology and concepts. Let $\mathscr{U}$ and $\mathscr{V}$ be open covers of a space X. From now we assume that any topological space X is not empty and each element of any open cover of a space is not an empty set.

Suppose that A is a subset of a space X and $\mathscr{U}$ is an open cover of X. Then we denote

$$St(A, \mathscr{U}) = \bigcup\{U \in \mathscr{U} \mid U \cap A \ne \phi\}.$$

Inductively, we define $St^0(A, \mathscr{U}) = A$, $St^1(A, \mathscr{U}) = St(A, \mathscr{U})$ and

$$St^{p+1}(A,\mathscr{U}) = St(St^p(A,\mathscr{U}),\mathscr{U}) = \bigcup\{U \in \mathscr{U} \mid U \cap St^p(A,\mathscr{U}) \neq \phi\} \ (p \geq 1).$$

We put

$$\mathscr{U}^{\star} = \{St(U,\mathscr{U}) \mid U \in \mathscr{U}\} \text{ and } \mathscr{U}^{\Delta} = \{St(x,\mathscr{U}) \mid x \in X\}.$$

Note that if $|\mathscr{U}|$ is finite, then $|\mathscr{U}^{\star}|$ and $|\mathscr{U}^{\Delta}|$ are finite. Also, we put $\mathscr{U}^{\star^0} = \mathscr{U}$, $\mathscr{U}^{\Delta^0} = \mathscr{U}$, $\mathscr{U}^{\star^1} = \mathscr{U}^{\star}$, and $\mathscr{U}^{\Delta^1} = \mathscr{U}^{\Delta}$. Inductively, we define

$$\mathscr{U}^{\star^{p+1}} = (\mathscr{U}^{\star^p})^{\star} = \left\{St(W,\mathscr{U}^{\star^p}) \mid W \in \mathscr{U}^{\star^p}\right\}$$

and

$$\mathscr{U}^{\Delta^{p+1}} = (\mathscr{U}^{\Delta^p})^{\Delta} = \left\{St(x,\mathscr{U}^{\Delta^p}) \mid x \in X\right\}.$$

An open cover $\mathscr{V}$ of X is a *star p-refinement* of an open cover $\mathscr{U}$ of X if $\mathscr{V}^{\star^p} \leq \mathscr{U}$. An open cover $\mathscr{V}$ of X is a *delta p-refinement* of an open cover $\mathscr{U}$ of X if $\mathscr{V}^{\Delta^p} \leq \mathscr{U}$. An open cover $\mathscr{V}$ of X is a *star-refinement* of an open cover $\mathscr{U}$ of X if $\mathscr{V}$ is a *star 1-refinement* of $\mathscr{U}$. An open cover $\mathscr{V}$ of X is a *delta-refinement* of an open cover $\mathscr{U}$ of X if $\mathscr{V}$ is a *delta 1-refinement* of $\mathscr{U}$. Note that $\mathscr{V} \leq \mathscr{V}^{\Delta} \leq \mathscr{V}^{\star} \leq \mathscr{V}^{\Delta^2}$.

Let $\mathscr{U}_i$ $(i = 1, 2, \dots)$ be open covers of X. Then the sequence $\{\mathscr{U}_i\}_{i=1}^{\infty}$ is called a *normal star-sequence* (e.g., see [**10**], [**11**] and [**12**]) if $\mathscr{U}_{i+1}$ is a star-refinement of $\mathscr{U}_i$ $(i = 1, 2, \dots)$. Also, the sequence $\{\mathscr{U}_i\}_{i=1}^{\infty}$ is called a *normal delta-sequence* if $\mathscr{U}_{i+1}$ is a delta-refinement of $\mathscr{U}_i$ $(i = 1, 2, \dots)$. The sequence $\{\mathscr{U}_i\}_{i=1}^{\infty}$ is called a *normal sequence* (e.g., see [**10**], [**11**] and [**12**]) if either $(\star)$ $\{\mathscr{U}_i\}_{i=1}^{\infty}$ is a normal star-sequence or (Δ) $\{\mathscr{U}_i\}_{i=1}^{\infty}$ is a normal delta-sequence. The sequence $\{\mathscr{U}_i\}_{i=1}^{\infty}$ is called a *development* of X if $\{St(x,\mathscr{U}_i) \mid i = 1, 2, \dots\}$ is a neighborhood base for each point x of X.

The following theorem is well known as Alexandroff-Urysohn metrization theorem (e.g., see [**10**], [**11**], [**12**]). In this paper, we need some additional properties of the metrics.

Theorem 2.1 (Alexandroff-Urysohn metrization theorem). *A T_1-space X is metrizable if and only if there exists a sequence $\{\mathscr{U}_i\}_{i=1}^{\infty}$ of open covers of X such that $\{\mathscr{U}_i\}_{i=1}^{\infty}$ is a normal sequence and a development of X.*

For any normal space X $(\neq \phi)$ and natural numbers k and p, we define the following indices:

(1) The index $\star_k^p(X)$ is defined as the least natural number m such that for every open cover $\mathscr{U}$ of X with $|\mathscr{U}| = k$, there is an open cover $\mathscr{V}$ of X such that

$|\mathscr{V}| \leq m$ and $\mathscr{V}^{\star^p} \leq \mathscr{U}$ (see [**13**]).

(2) The index $\Delta_k^p(X)$ is defined as the least natural number m such that for every open cover $\mathscr{U}$ of X with $|\mathscr{U}| = k$, there is an open cover $\mathscr{V}$ of X such that $|\mathscr{V}| \leq m$ and $\mathscr{V}^{\Delta^p} \leq \mathscr{U}$ (see [**13**]).

(3) The index $\tilde{\Delta}_k^p(X)$ is defined as the least natural number m such that for every open covering $\mathscr{U}$ of X with $|\mathscr{U}| = k$, there is an open covering $\mathscr{V}$ of X such that $|\mathscr{V}| \leq m$ and $\{St^p(x, \mathscr{V}) \mid x \in X\} \leq \mathscr{U}$.

(4) The index $\tilde{\star}_k^p(X)$ is defined as the least natural number m such that for every open covering $\mathscr{U}$ of X with $|\mathscr{U}| = k$, there is an open covering $\mathscr{V}$ of X such that $|\mathscr{V}| \leq m$ and $\{St^p(V, \mathscr{V}) \mid V \in \mathscr{V}\} \leq \mathscr{U}$.

By C_m^k, we shall denote the set of all m-element subsets of the set $\{1, 2, \ldots, k\}$ and by $\binom{k}{m}$ its cardinality, i.e.,

$$\binom{k}{m} = \frac{k!}{m!(k-m)!}.$$

For natural numbers k, m and p with $k \geq m$, we define the following indices;

$$\tilde{\Delta}(k; m; p) = \Sigma_{m \geq j_1 \geq j_2 \geq \cdots \geq j_p \geq 1} \binom{k}{j_1}\binom{j_1}{j_2} \cdots \binom{j_{p-1}}{j_p}$$

and

$$\tilde{\star}(k; m; p) = \Sigma_{m \geq j_1 \geq j_2 \geq \cdots \geq j_p \geq 1} \binom{k}{j_1}\binom{j_1}{j_2} \cdots \binom{j_{p-1}}{j_p} j_p.$$

In [**4**], Bruijning and Nagata determined the index $\Delta_k^1(X)$, and in [**7**], Hashimoto and Hattori determined the index $\star_k^1(X)$. In [**2**] Bogatyi and Karpov determined the indices $\tilde{\Delta}_k^p(X)$ and $\tilde{\star}_k^p(X)$ for all k, p. They did not state the next theorem (=Theorem 2.2), but by use of their results, we can easily determine the indices $\Delta_k^p(X)$ and $\star_k^p(X)$ for all k, p. In this paper, we need more detailed properties of the indices. For completeness, in Appendix of Sections 7 we will give the complete proof of Theorem 2.2 and the more detailed information of the indices (see Corollary 7.10). Also we will give other characterizations of topological dimension by use of the indices.

Theorem 2.2 (Corollary 7.10). *Let X be an infinite normal space with $\dim X = m < \infty$ and let k and p be natural numbers. Then*

$$\star_k^p(X) = \begin{cases} \tilde{\star}\left(k; k; \left(\frac{1}{2}\right)(3^p - 1)\right) = k\left[\left(\frac{1}{2}\right)(3^p - 1) + 1\right]^{k-1}, & \text{if } k \leq m+1 \\ \tilde{\star}\left(k; m+1; \left(\frac{1}{2}\right)(3^p - 1)\right), & \text{if } k \geq m+1, \end{cases}$$

and

$$\Delta_k^p(X) = \begin{cases} \tilde{\Delta}(k; k; 2^{p-1}) = (2^{p-1}+1)^k - (2^{p-1})^k, & \text{if } k \leq m+1 \\ \tilde{\Delta}(k; m+1; 2^{p-1}), & \text{if } k \geq m+1. \end{cases}$$

3. Topological dimension and normal sequences of finite open covers.

In this section, we prove Theorem 3.1, which means that topological dimension is characterized in terms of the growth of the global cardinality $|\mathscr{U}_i|$ of members $\mathscr{U}_i$ of normal sequences.

Theorem 3.1. *Let X be a (nonempty) separable metric space. Then*

$$(1)\quad \dim X = \min\left\{ \liminf_{i\to\infty} \frac{\log_3 |\mathscr{U}_i|}{i} \;\middle|\; \{\mathscr{U}_i\}_{i=1}^{\infty} \text{ is a normal star-sequence of finite open covers of } X \text{ and a development of } X \right\}$$

and

$$(2)\quad \dim X = \min\left\{ \liminf_{i\to\infty} \frac{\log_2 |\mathscr{U}_i|}{i} \;\middle|\; \{\mathscr{U}_i\}_{i=1}^{\infty} \text{ is a normal delta-sequence of finite open covers of } X \text{ and a development of } X \right\}.$$

For the proof of Theorem 3.1, we need the followings.

Lemma 3.2. *Let X be a normal space and $m \geq 0$. Then $\dim X \geq m$ if and only if there is an open covering $\mathscr{W}_1 = \{W_1, W_2, \ldots, W_{m+1}\}$ of X such that if $\mathscr{V}$ is any open shrinking of $\mathscr{W}_1$ (i.e., $\mathscr{V} = \{V_1, V_2, \ldots, V_{m+1}\}$ is an open cover of X such that $V_i \subset W_i$ $(i = 1, 2, \ldots, m+1)$), then $\mathscr{V}$ has a non-empty intersection.*

Proof. See Engelking [**5**, Theorem (1.6.9)].

Proposition 3.3. *Let X be an infinite normal space with $\dim X \geq m \geq 0$. Suppose that $\mathscr{W}_1 = \{W_1, W_2, \dots, W_{m+1}\}$ is an open cover of X as in Lemma 3.2.*

(1) *If $\{\mathscr{U}_i\}_{i=1}^{\infty}$ is a normal star-sequence of finite open covers of X and $\mathscr{W}_1 \geq \mathscr{U}_1^{\star}$, then $|\mathscr{U}_i| \geq \tilde{\star}(m+1; m+1; (1/2)(3^i - 1))$ for each i.*
(2) *If $\{\mathscr{U}_i\}_{i=1}^{\infty}$ is a normal delta-sequence of finite open covers of X and $\mathscr{W}_1 \geq \mathscr{U}_1^{\Delta}$, then $|\mathscr{U}_i| \geq \tilde{\Delta}(m+1; m+1; 2^{i-1})$ for each i.*

Proof. By the proofs of Theorem 7.2, Theorem 7.5 and Proposition 7.9 of Appendix, we know that

$$(\star) \quad \star^p\,(X, \mathscr{W}_1) \equiv \min\{|\mathscr{V}| \mid \mathscr{V} \text{ is a finite open cover of } X \text{ such that } \mathscr{V}^{\star^p} \leq \mathscr{W}_1\}$$
$$= \tilde{\star}\left(m+1; m+1; \frac{1}{2}(3^p - 1)\right),$$
$$(\Delta) \quad \Delta^p(X, \mathscr{W}_1) \equiv \min\{|\mathscr{V}| \mid \mathscr{V} \text{ is a finite open cover of } X \text{ such that } \mathscr{V}^{\Delta^p} \leq \mathscr{W}_1\}$$
$$= \tilde{\Delta}(m+1; m+1; 2^{p-1})).$$

We will prove the case (1). The case (2) can be proved similarly to the case (1). Since

$$\mathscr{W}_1 \geq \mathscr{U}_1^{\star} \geq \mathscr{U}_1 \geq \mathscr{U}_2^{\star} \geq \mathscr{U}_2 \geq \mathscr{U}_3^{\star} \geq \mathscr{U}_3 \dots,$$

we see that $\mathscr{U}_i^{\star^i} \leq \mathscr{W}_1$, hence by $(\star)$, we see that

$$|\mathscr{U}_i| \geq \tilde{\star}\left(m+1; m+1; \frac{1}{2}(3^i - 1)\right).$$

Proof of Theorem 3.1. We shall prove the case (1). The case (2) can be proved similarly to the case (1). We may assume that $|X|$ is infinite. Suppose that X is a separable metric space with $\dim X = m < \infty$. The case $\dim X = \infty$ is proved similarly with the aid of Lemma 3.5 below. Let k be a fixed natural number with $k \geq m+1$. Note that

$$\tilde{\star}(k; m+1; p) = \Sigma_{m+1 \geq j_1 \geq j_2 \geq \dots \geq j_p \geq 1} \binom{k}{j_1}\binom{j_1}{j_2} \cdots \binom{j_{p-1}}{j_p} j_p.$$

Since k and $m+1$ are fixed, we can choose a real number $b > 0$ such that for any $p = 1, 2, \dots$ and any j_i $(i = 1, 2, \dots, p)$ with $m+1 \geq j_1 \geq j_2 \geq \dots \geq j_p \geq 1$,

$$1 \leq \binom{k}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{p-1}}{j_p} j_p < b.$$

Note that

$$\Sigma_{m+1 \geq j_1 \geq j_2 \geq \cdots \geq j_p \geq 1} 1 = |{}_{m+1}H_p| = \binom{m+p}{p} = \frac{(m+p)!}{m!p!},$$

where ${}_{m+1}H_p$ denotes the set of the repeated combinations choosing p elements from $m+1$ elements. Hence

$$|{}_{m+1}H_p| \leq \tilde{\star}(k; m+1; p) < |{}_{m+1}H_p| \cdot b.$$

Then we see that

$$\lim_{p\to\infty} \frac{\log \tilde{\star}(k; m+1; p)}{\log p} = \lim_{p\to\infty} \frac{\log |{}_{m+1}H_p|}{\log p} = \lim_{p\to\infty} \frac{\log[(m+p)!/p!m!]}{\log p}$$
$$= \lim_{p\to\infty} \frac{\log(m+p) + \log(m-1+p) + \cdots + \log(1+p) - \log(m!)}{\log p} = m.$$

By Theorem 2.2, we see that

$$\star_k^i(X) = \tilde{\star}\left(k; m+1; \frac{1}{2}(3^i - 1)\right).$$

Then

$$\lim_{i\to\infty} \frac{\log_3 \star_k^i(X)}{i} = \lim_{i\to\infty} \frac{\log_3 \tilde{\star}\left(k; m+1; \frac{1}{2}(3^i-1)\right)}{\log_3 \frac{1}{2}(3^i - 1)} = m.$$

Since X is a separable metric space, we may assume that X is totally bounded and hence we can choose a sequence $\{\mathscr{W}_i\}_{i=1}^{\infty}$ of finite open covers of X such that $\mathscr{W}_{i+1} \leq \mathscr{W}_i$ for each i and $\{\mathscr{W}_i\}_{i=1}^{\infty}$ is a development of X. Also, we may assume that $\mathscr{W}_1 = \{W_1, W_2, \ldots, W_{m+1}\}$ satisfies the condition of Lemma 3.2.

Put $k_1 = |\mathscr{W}_1| (= m+1)$. Let $\{\epsilon_i\}_{i=1}^{\infty}$ be a decreasing sequence of positive numbers with $\lim_{i\to\infty} \epsilon_i = 0$. Since

$$\lim_{i\to\infty} \frac{\log_3 \star_{k_1}^i(X)}{i} = m,$$

we can choose a finite open cover $\mathscr{V}_1$ of X and a sufficiently large natural number p_1 such that

$$\mathscr{V}_1^{\star^{p_1}} \leq \mathscr{W}_1 \text{ and } \frac{\log_3 |\mathscr{V}_1|}{p_1} < m + \epsilon_1.$$

For each $j = 1, 2, \ldots, p_1$, we put $\mathscr{U}_j = \mathscr{V}_1^{\star^{p_1 - j}}$. Note that

$$\mathscr{U}_{i+1}^\star = \mathscr{U}_i \ (i = 1, 2, \ldots, p_1 - 1) \text{ and } \frac{\log_3 |\mathscr{U}_{p_1}|}{p_1} = \frac{\log_3 |\mathscr{V}_1|}{p_1} < m + \epsilon_1.$$

Next, we consider the following open cover of X:

$$\mathscr{W}_2 \wedge \mathscr{V}_1 = \{W \cap V \mid W \in \mathscr{W}_2, V \in \mathscr{V}_1, W \cap V \neq \phi\}.$$

Let $k_2 = |\mathscr{W}_2 \wedge \mathscr{V}_1|$. Since

$$\lim_{i \to \infty} \frac{\log_3 \star_{k_2}^i(X)}{i} = m,$$

we can choose a finite open cover $\mathscr{V}_2$ of X and a sufficiently large natural number p_2 such that

$$\mathscr{V}_2^{\star^{p_2}} \leq \mathscr{W}_2 \wedge \mathscr{V}_1 \text{ and } \frac{\log_3 |\mathscr{V}_2|}{p_2} < m + \epsilon_2.$$

For each $j = 1, 2, \ldots, p_2$, we put $\mathscr{U}_{p_1 + j} = \mathscr{V}_2^{\star^{p_2 - j}}$. Then

$$\frac{\log_3 |\mathscr{U}_{p_1 + p_2}|}{p_1 + p_2} \leq \frac{\log_3 |\mathscr{V}_2|}{p_2} < m + \epsilon_2.$$

Also, note that $\mathscr{U}_{i+1}^\star \leq \mathscr{U}_i$ for each $i = 1, 2, \ldots, p_1 + p_2$. If we continue this procedure, we obtain a sequence $\mathscr{V}_1, \mathscr{V}_2, \ldots$ of finite open covers of X and a sequence $p_1, p_2, \ldots$ of natural numbers such that for $i \geq 2$ we have

$$\mathscr{V}_i^{\star^{p_i}} \leq \mathscr{W}_i \wedge \mathscr{V}_{i-1} \text{ and } \frac{\log_3 |\mathscr{V}_i|}{p_i} < m + \epsilon_i.$$

By use of the sequence $\mathscr{V}_1, \mathscr{V}_2, \ldots$, we obtain a sequence $\{\mathscr{U}_i\}_{i=1}^\infty$ of finite open covers of X such that $\{\mathscr{U}_i\}_{i=1}^\infty$ is a normal star-sequence and a development of X

satisfying

$$\liminf_{i\to\infty}\frac{\log_3|\mathscr{U}_i|}{i}\le m=\dim X.$$

By Proposition 3.3, we see that $|\mathscr{U}_i|\ge\star^i_{m+1}(X)$ for each $i\ge 1$. In particular, we can conclude that

$$\liminf_{i\to\infty}\frac{\log_3|\mathscr{U}_i|}{i}=m=\dim X.$$

Furthermore, by Lemma 3.5 (see the next lemma in this section), we see that if $\{\mathscr{U}_i\}_{i=1}^\infty$ is any normal star-sequence of finite open covers of X and a development of X, then there is some i_0 such that $|\mathscr{U}_i|\ge\star^{i-i_0}_{m+1}(X)$ for $i\ge i_0$, in particular,

$$\liminf_{i\to\infty}\frac{\log_3|\mathscr{U}_i|}{i}\ge\liminf_{i\to\infty}\frac{\star^{i-i_0}_{m+1}(X)}{i-i_0}\cdot\frac{i-i_0}{i}=m.$$

Consequently we see that

$$\dim X=\min\left\{\liminf_{i\to\infty}\frac{\log_3|\mathscr{U}_i|}{i}\;\middle|\;\{\mathscr{U}_i\}_{i=1}^\infty \text{ is a normal star-sequence}\right.$$
$$\left.\text{of finite open covers of } X \text{ and a development of } X\right\}.$$

This completes the proof.

For separable metric spaces, we need the Alexandroff-Urysohn metric induced by normal sequences of finite open covers. We also need some additional properties of the metrics in the following sections: Define the functions $D_\star:X\times X\to[0,9]$ and $D_\Delta:X\times X\to[0,4]$ as follows:

($\star$) Let $\{\mathscr{U}_i\}_{i=1}^\infty$ be a normal star-sequence of finite open covers of X and a development of X. For any pair of points x,y of X, we define the function $D_\star:X\times X\to[0,9]$ by
   (1) $D_\star(x,y)=9$ if $\{x,y\}$ is not contained in any element of $\mathscr{U}_1$,
   (2) $D_\star(x,y)=1/3^{i-2}$ if $\{x,y\}$ is contained in an element of $\mathscr{U}_i$ and $\{x,y\}$ is not contained in any element of $\mathscr{U}_j$ for $j>i$,
   (3) $D_\star(x,y)=0$ if $\{x,y\}$ is contained in an element of $\mathscr{U}_i$ for each i.

(Δ) Let $\{\mathscr{U}_i\}_{i=1}^\infty$ be a normal delta-sequence of finite open covers of X and a development of X. For any pair of points x,y of X, we define the function

$D_\Delta : X \times X \to [0, 4]$ by
(1) $D_\Delta(x, y) = 4$ if $\{x, y\}$ is not contained in any element of $\mathscr{U}_1$,
(2) $D_\Delta(x, y) = 1/2^{i-2}$ if $\{x, y\}$ is contained in an element of $\mathscr{U}_i$ and $\{x, y\}$ is not contained in any element of $\mathscr{U}_j$ for $j > i$,
(3) $D_\Delta(x, y) = 0$ if $\{x, y\}$ is contained in an element of $\mathscr{U}_i$ for each i.

Proposition 3.4. *Let X be a T_1-space.*

(1) *If $\{\mathscr{U}_i\}_{i=1}^\infty$ is a normal star-sequence of finite open covers of X and a development of X, then $\{\mathscr{U}_i\}_{i=1}^\infty$ induces a totally bounded metric $d_\star$ on X such that*

$$d_\star(x, y) \le D_\star(x, y) \le 6d_\star(x, y)$$

for any $x, y \in X$. In particular, X is a separable metric space.
(2) *If $\{\mathscr{U}_i\}_{i=1}^\infty$ is a normal delta-sequence of finite open covers of X and a development of X, then $\{\mathscr{U}_i\}_{i=1}^\infty$ induces a totally bounded metric d_Δ on X such that*

$$d_\Delta(x, y) \le D_\Delta(x, y) \le 4d_\Delta(x, y)$$

for any $x, y \in X$. In particular, X is a separable metric space.

Proof. We shall prove the case (1). The proof is slightly different from the one of the case (2). The proof of the case (2) can be found in [**10**, p. 13, Theorem 2.16]. We construct such a metric $d_\star$ as follows: Let $\{\mathscr{U}_i\}_{i=1}^\infty$ be a normal star-sequence of finite open covers of X and a development of X. Put $\mathscr{U}_0 = \{X\}$. Then $D(= D_\star)$ satisfies the following conditions; for any $x, y, u, v \in X$

$$D(x, x) = 0$$
$$D(x, y) = D(y, x)$$
$$D(x, y) \le 3 \max\{D(x, u), D(u, v), D(v, y)\}.$$

We shall prove that

$$D(x, y) \le 3 \max\{D(x, u), D(u, v), D(v, y)\}.$$

Choose $i \ge 0$ such that

$$\max\{D(x, u), D(u, v), D(v, y)\} = 3^{-(i-2)}.$$

We may assume that $i \geq 1$. Then there are $U_1, U_2, U_3 \in \mathscr{U}_i$ such that $x, u \in U_1, u, v \in U_2, v, y \in U_3$. There is $V \in \mathscr{U}_{i-1}$ such that

$$x, y \in U_1 \cup U_2 \cup U_3 \subset St(U_2, \mathscr{U}_i) \subset V \in \mathscr{U}_{i-1}.$$

Then

$$D(x, y) \leq 3^{-(i-3)} = 3 \max\{D(x, u), D(u, v), D(v, y)\}.$$

Set

$$d_\star(x, y)(= d(x, y)) = \inf\{D(x, x_1) + D(x_1, x_2) + \cdots + D(x_n, y) \mid \\ n = 1, 2, \ldots, \text{ and } x_j \in X\}.$$

Since $\{\mathscr{U}_i\}_{i=1}^{\infty}$ is a development of X, $d_\star(= d)$ is a metric on X satisfying $d(x, y) \leq D(x, y)$. Now we shall show that

$$d(x, y) \leq D(x, y) \leq 6d(x, y).$$

First, we prove the following inequality

$$D(x, y) \leq 3D(x, x_1) + 6D(x_1, x_2) + 6D(x_2, x_3) + \cdots + 6D(x_{n-1}, x_n) + 3D(x_n, y).$$

Suppose, on the contrary, that the inequality is not true. Then there is a minimum number N for which

$$D(x, y) > 3D(x, x_1) + 6D(x_1, x_2) + 6D(x_2, x_3) + \cdots + 6D(x_{N-1}, x_N) + 3D(x_N, y).$$

Recall the condition

$$D(x, y) \leq 3 \max\{D(x, u), D(u, v), D(v, y)\}.$$

Then $N > 2$. Put $x_0 = x$, $x_{N+1} = y$. Set

$$k_1 = \min\{r \mid D(x, y) \leq 3D(x, x_r)\}.$$

Then $1 < k_1$. Set

$$k_2 = \max\{r \mid D(x, y) \leq 3D(x_r, y)\}.$$

Then $k_2 < N$. We show that $k_1 \le k_2$. Suppose, on the contrary, that $k_1 > k_2$. Then

$$\begin{aligned}
D(x,y) &> 3D(x, x_{k_1-1}) \\
D(x,y) &> 3D(x_{k_1-1}, x_{k_1}) \\
D(x,y) &> 3D(x_{k_1}, y).
\end{aligned}$$

This contradicts the inequality

$$D(x,y) \le 3\max\{D(x, x_{k_1-1}), D(x_{k_1-1}, x_{k_1}), D(x_{k_1}, y)\}.$$

Hence $k_1 \le k_2$. Since $D(x,y) > 3D(x, x_{k_1-1})$ and $D(x,y) > 3D(x_{k_2+1}, y)$, we have

$$D(x,y) \le 3D(x_{k_1-1}, x_{k_2+1}).$$

Then we have

$$\begin{aligned}
D(x,y) &\le 3D(x, x_{k_1}) \\
D(x,y) &\le 3D(x_{k_1-1}, x_{k_2+1}) \\
D(x,y) &\le 3D(x_{k_2}, y).
\end{aligned}$$

Hence

$$\begin{aligned}
D(x,y) &\le D(x, x_{k_1}) + D(x_{k_1-1}, x_{k_2+1}) + D(x_{k_2}, y) \\
&\le (3D(x, x_1) + 6D(x_1, x_2) + 6D(x_2, x_3) \\
&\qquad + \cdots + 6D(x_{k_1-2}, x_{k_1-1}) + 3D(x_{k_1-1}, x_{k_1})) \\
&\quad + (3D(x_{k_1-1}, x_{k_1}) + 6D(x_{k_1}, x_{k_1+1}) \\
&\qquad + \cdots + 6D(x_{k_2-1}, x_{k_2}) + 3D(x_{k_2}, x_{k_2+1})) \\
&\quad + (3D(x_{k_2}, x_{k_2+1}) + 6D(x_{k_2+1}, x_{k_2+2}) \\
&\qquad + \cdots + 6D(x_{N-1}, x_N) + 3D(x_N, y)) \\
&\le 3D(x, x_1) + 6D(x_1, x_2) + 6D(x_2, x_3) \\
&\qquad + \cdots + 6D(x_{N-1}, x_N) + 3D(x_N, y).
\end{aligned}$$

This is a contradiction. Hence we have

$$D(x,y) \le 3D(x,x_1) + 6D(x_1,x_2) + 6D(x_2,x_3) + \cdots + 6D(x_{n-1},x_n) + 3D(x_n,y).$$

Consequently we have

$$d(x,y) \le D(x,y) \le 6d(x,y).$$

Note that for each $i \ge 3$,

$$St(x, \mathscr{U}_{i+1}) \subset U_d(x, 1/3^{i-2}) \subset St(x, \mathscr{U}_{i-2}),$$

where $U_d(x,\epsilon)$ is the ϵ-neighborhood of x in X. Then we see that d is compatible with the topology of X and $\operatorname{diam}_d(U) \le 1/3^{i-2}$ for each $U \in \mathscr{U}_i$. Hence d $(= d_\star)$ is a totally bounded metric on X.

The proof of the case (2) can be found in [**10**, p. 13, Theorem 2.16]. We will give the outline of the proof. Let $\{\mathscr{U}_i\}_{i=1}^\infty$ be a normal delta-sequence of finite open covers of X and a development of X. Set

$$\begin{aligned} &d_\Delta(x,y)(= d(x,y)) \\ &\quad = \inf\{D(x,x_1) + D(x_1,x_2) + \cdots + D(x_n,y) \mid n = 1,2,\dots, \text{ and } x_j \in X\}. \end{aligned}$$

Then $d_\Delta(= d)$ is a metric on X such that $d(x,y) \le D(x,y) \le 4d(x,y)$ (see [**11**, p. 15]). Clearly, we see that $\operatorname{diam}_d(U) \le 1/2^{i-2}$ for each $U \in \mathscr{U}_i$. Hence d is a totally bounded metric on X.

LEMMA 3.5. *Let X be an infinite separable metric space with* $\dim X \ge m \ge 0$. *Then the followings hold.*

(1) *If $\{\mathscr{U}_i\}_{i=1}^\infty$ is a normal star-sequence of finite open covers of X and a development of X, then there is some i_0 such that*

$$|\mathscr{U}_i| \ge \tilde{\star}\left(m+1; m+1; \frac{1}{2}(3^{i-i_0} - 1)\right)$$

for $i \ge i_0$. In particular,

$$\liminf_{i\to\infty} \frac{\log_3 |\mathscr{U}_i|}{i} \ge m.$$

(2) *If $\{\mathscr{U}_i\}_{i=1}^\infty$ is a normal delta-sequence of finite open covers of X and a development of X, then there is some i_0 such that*

$$|\mathscr{U}_i| \geq \tilde{\Delta}(m+1; m+1; 2^{i-i_0-1})$$

for $i \geq i_0$. In particular,

$$\liminf_{i\to\infty} \frac{\log_2 |\mathscr{U}_i|}{i} \geq m.$$

PROOF. We shall prove the case (1). The case (2) can be proved similarly to the case (1). By (1) of Proposition 3.4, we have the Alexandroff-Urysohn metric d $(= d_\star)$ on X induced by the normal star-sequence $\{\mathscr{U}_i\}_{i=1}^{\infty}$. Then there is the completion $(\tilde{X}, \tilde{d})$ of (X, d) which is obtained by considering all Cauchy sequences $\{x_n\}_{n=1}^{\infty}$ of points of X (e.g., see [Theorem 27, p. 196, J. L. Kelley, General Topology, New York, 1955]). Since d is a totally bounded metric, $(\tilde{X}, \tilde{d})$ is a compact metric space. Note that the natural embedding $i : (X, d) \to (\tilde{X}, \tilde{d})$ is an isometry. Since $\dim \tilde{X} \geq m$, there is an open covering $\mathscr{W}_1 = \{W_1, W_2, \ldots, W_{m+1}\}$ of $\tilde{X}$ such that if $\mathscr{V}$ is any open shrinking of $\mathscr{W}_1$, then $\mathscr{V}$ has a non-empty intersection. Let $\epsilon > 0$ be a Lebesgue number of the open cover $\mathscr{W}_1$. Choose i_0 such that $\epsilon > 1/3^{i_0-2}$. For any cover $\mathscr{V}$ of a space, we set

$$\mathscr{V}^{\star^p} = \left\{St^{(1/2)(3^p-1)}(V, \mathscr{V}) \mid V \in \mathscr{V}\right\}.$$

For each $i \geq i_0$, we consider the closed cover of $\tilde{X}$;

$$\overline{\mathscr{U}_i}^{\star^{i-i_0}} = \left\{St^{(1/2)(3^{i-i_0}-1)}(\overline{U}, \overline{\mathscr{U}_i}) \mid U \in \mathscr{U}_i\right\},$$

where $\overline{\mathscr{U}_i} = \{\overline{U} \mid U \in \mathscr{U}_i\}$ and $\overline{U}$ denotes the closure of U in $\tilde{X}$. Since $\operatorname{diam}_d(U) \leq 1/3^{i-2}$ for each $U \in \mathscr{U}_i$, we see that

$$\operatorname{mesh}_{\tilde{d}}(\overline{\mathscr{U}_i}^{\star^{i-i_0}}) \leq \left(\frac{1}{3^{i-2}}\right) \times 3^{i-i_0} = \frac{1}{3^{i_0-2}} < \epsilon.$$

Since $\overline{\mathscr{U}_i}$ is a closed cover of $\tilde{X}$, for each $\overline{U} \in \overline{\mathscr{U}_i}$ we can choose a sufficiently small open neighborhood U' of $\overline{U} (\in \overline{\mathscr{U}_i})$ in $\tilde{X}$ such that

$$\operatorname{mesh}_{\tilde{d}}(\mathscr{U}'^{\star^{i-i_0}}_i) < \epsilon,$$

where $\mathscr{U}'_i = \{U' \mid \overline{U} \in \overline{\mathscr{U}_i}\}$. Then

$$\mathscr{U}'^{\star^{i-i_0}}_i \leq \mathscr{W}_1.$$

By (1) of Proposition 3.3, we have

$$|\mathscr{U}_i| = |\mathscr{U}'_i| \geq \tilde{\star}\left(m+1; m+1; \frac{1}{2}(3^{i-i_0}-1)\right).$$

Then

$$\liminf_{i\to\infty} \frac{\log_3 |\mathscr{U}_i|}{i} \geq \liminf_{i\to\infty} \frac{\log_3 \tilde{\star}\left(m+1; m+1; \frac{1}{2}(3^{i-i_0}-1)\right)}{i} = m.$$

4. Another proof of Pontrjagin-Schnirelmann theorem.

In this section, we give another proof of Pontrjagin-Schnirelmann theorem. We consider the Alexandroff-Urysohn metrics $d_\star$ and d_Δ which are defined in the proof of Proposition 3.4. From now the metrics $d_\star$ and d_Δ mean the Alexandroff-Urysohn metrics induced by some normal sequences of finite open covers. First, we prove the following.

Proposition 4.1. *Let X be a separable metric space. Then*

(1) $\dim X = \min\{\dim_B(X, d_\star) \mid d_\star$ *is the Alexandroff-Urysohn metric on X induced by a sequence $\{\mathscr{U}_i\}_{i=1}^\infty$ which is a normal star-sequence of finite open covers of X and a development of X*$\}$,

(2) $\dim X = \min\{\dim_B(X, d_\Delta) \mid d_\Delta$ *is the Alexandroff-Urysohn metric on X induced by a sequence $\{\mathscr{U}_i\}_{i=1}^\infty$ which is a normal delta-sequence of finite open covers of X and a development of X*$\}$.

Proof. First, we prove the case (1). By Theorem 3.1, there is a normal star-sequence $\{\mathscr{U}_i\}_{i=1}^\infty$ of finite open covers of X and a development of X such that

$$\liminf_{i\to\infty} \frac{\log_3 |\mathscr{U}_i|}{i} = \dim X \ (= m).$$

We may assume that $m < \infty$. The case $m = \infty$ can be proved similarly. Let $d_\star$ be the Alexandroff-Urysohn metric induced by $\{\mathscr{U}_i\}_{i=1}^\infty$. We recall the index

$$N(\epsilon, d_\star) = \min\{|\mathscr{U}| \mid \mathscr{U} \text{ is a finite open cover of } X \text{ with } \mathrm{mesh}_{d_\star}(\mathscr{U}) \leq \epsilon\}.$$

Note that for each i_0,

$$\inf\left\{\frac{\log N(\epsilon,d_\star)}{|\log\epsilon|}\,\middle|\,0<\epsilon\le\frac{1}{3^{i_0-2}}\right\}\le\inf\left\{\frac{\log_3|\mathscr{U}_i|}{\left|\log_3\dfrac{1}{3^{i-2}}\right|}\,\middle|\,i=i_0,i_0+1,\dots\right\}$$

$$=\inf\left\{\frac{\log_3|\mathscr{U}_i|}{i-2}\,\middle|\,i=i_0,i_0+1,\dots\right\}\le\liminf_{i\to\infty}\frac{\log_3|\mathscr{U}_i|}{i}=m.$$

Hence

$$\dim_B(X,d_\star)=\sup\left\{\inf\left\{\frac{\log N(\epsilon,d_\star)}{|\log\epsilon|}\,\middle|\,0<\epsilon<\epsilon_0\right\}\,\middle|\,0<\epsilon_0\right\}\le m=\dim X.$$

Next, let $\{\mathscr{U}_i\}_{i=1}^\infty$ be any normal star-sequence of finite open covers of X and a development of X. Let $d_\star$ be the Alexandroff-Urysohn metric induced by $\{\mathscr{U}_i\}_{i=1}^\infty$. Suppose that $1/3>\epsilon_1>\epsilon_2>\epsilon_3>\cdots$ is a sequence of positive numbers such that $\lim_{k\to\infty}\epsilon_k=0$ and

$$\liminf_{k\to\infty}\frac{\log_3 N(\epsilon_k,d_\star)}{|\log_3\epsilon_k|}=\beta.$$

For each k, we choose the natural number $n(k)$ such that $1/3^{n(k)+1}\le\epsilon_k<1/3^{n(k)}$. Let $\mathscr{W}$ be a finite open cover of X such that $|\mathscr{W}|=N(\epsilon_k,d_\star)$ with $\mathrm{mesh}_{d_\star}(\mathscr{W})\le\epsilon_k$. Let $W\in\mathscr{W}$ and $x\in W$. Choose $V\in\mathscr{U}_{n(k)}$ with $x\in V$. If $y\in W\in\mathscr{W}$, then $1/3^{n(k)}>d_\star(x,y)\ge(1/6)\cdot D_\star(x,y)$ and hence $1/3^{n(k)-2}>D_\star(x,y)$. This implies that there is $U\in\mathscr{U}_{n(k)}$ such that U contains x and y (recall the definition of $D_\star(x,y)$). Then

$$W\subset St(x,\mathscr{U}_{n(k)})\subset St(V,\mathscr{U}_{n(k)})\in\mathscr{U}_{n(k)}^\star\le\mathscr{U}_{n(k)-1}$$

and hence $\mathscr{W}\le\mathscr{U}_{n(k)-1}$. Since $\{\mathscr{U}_i\}_{i=1}^\infty$ is a normal star-sequence of finite open covers and a development of X, there is i_0 as in Lemma 3.5. Put

$$\mathscr{V}_{n(k)-p-1}=\mathscr{W}^{\star^p}$$

for each $p=0,1,2,\dots,n(k)-2$. Then the finite sequence $\{\mathscr{V}_i\}_{i=1}^{n(k)-1}$ satisfies the condition that $\mathscr{V}_{i+1}$ is a star-refinement of $\mathscr{V}_i$. Note that $\mathscr{U}_i\ge\mathscr{V}_i$ and $\mathscr{V}_{n(k)-1}=\mathscr{W}$. By the proof of Proposition 3.3 and Lemma 3.5, we see that

$$N(\epsilon_k,d_\star)=|\mathscr{W}|\ge\star_{m+1}^{n(k)-1-i_0}(X).$$

Then

$$\frac{\log_3 N(\epsilon_k, d_\star)}{|\log_3 \epsilon_k|} = \frac{\log_3 |\mathscr{W}|}{|\log_3 \epsilon_k|} \geq \frac{\log_3 \star_{m+1}^{n(k)-1-i_0}(X)}{|\log_3 3^{n(k)+1}|} = \frac{\log_3 \star_{m+1}^{n(k)-1-i_0}(X)}{n(k)+1}$$
$$= \frac{\log_3 \star_{m+1}^{n(k)-1-i_0}(X)}{n(k)-1-i_0} \cdot \frac{n(k)-1-i_0}{n(k)+1}.$$

We see that

$$\beta = \liminf_{k\to\infty} \frac{\log_3 N(\epsilon_k, d_\star)}{|\log_3 \epsilon_k|} \geq m \cdot 1 = m.$$

Hence $\dim_B(X, d_\star) \geq m$. We have completed the proof of the case (1). The case (2) can be proved similarly to the case (1).

By the proof of Proposition 4.1, we obtain the following.

PROPOSITION 4.2. *Let X be a separable metric space and let $\{\mathscr{U}_i\}_{i=1}^\infty$ be any normal star (resp. delta)-sequence of finite open covers and a development of X. If*

$$\liminf_{i\to\infty} \frac{\log_3 |\mathscr{U}_i|}{i} = \beta \left(\textit{resp.}\ \liminf_{i\to\infty} \frac{\log_2 |\mathscr{U}_i|}{i} = \beta \right)$$

and $d_\star$ (resp. d_Δ) is the Alexandroff-Urysohn metric on X induced by $\{\mathscr{U}_i\}_{i=1}^\infty$, then

$$\dim_B(X, d_\star) \leq \beta \ (\textit{resp.}\ \dim_B(X, d_\Delta) \leq \beta).$$

Let X be a metrizable space and let ρ_1 and ρ_2 be two metrics on X. Then ρ_1 is *Lipschitz equivalent* to ρ_2 if there are positive (real) numbers a and b such that for $x, y \in X$,

$$a \cdot \rho_2(x,y) \leq \rho_1(x,y) \leq b \cdot \rho_2(x,y).$$

PROPOSITION 4.3. *Let (X, ρ) be a metric space. Suppose that $\{\mathscr{U}_i\}_{i=1}^\infty$ is a normal star (resp. delta)-sequence of finite open covers of X and a development of X. Then the followings are equivalent.*

(1) *The Alexandroff-Urysohn metric d ($= d_\star$ or d_Δ) induced by $\{\mathscr{U}_i\}_{i=1}^\infty$ is Lips-*

chitz equivalent to ρ.

(2) *There are positive numbers $c_2 \geq c_1 > 0$ such that for each i,*

$$\left\{U_\rho\left(x, \frac{c_1}{3^i}\right) \,\middle|\, x \in X\right\} \leq \mathscr{U}_i \leq \left\{U_\rho\left(x, \frac{c_2}{3^i}\right) \,\middle|\, x \in X\right\}$$

$$\left(\textit{resp. } \left\{U_\rho\left(x, \frac{c_1}{2^i}\right) \,\middle|\, x \in X\right\} \leq \mathscr{U}_i \leq \left\{U_\rho\left(x, \frac{c_2}{2^i}\right) \,\middle|\, x \in X\right\}\right).$$

Proof. We prove the case that $\{\mathscr{U}_i\}_{i=1}^\infty$ is a normal star-sequence. We use the notations as in the proof of Proposition 3.4. First, we shall show that (1) implies (2). Suppose that $d(= d_\star)$ is Lipschitz equivalent to ρ. Then there are positive numbers a and b such that for $x, y \in X$,

$$a \cdot d(x,y) \leq \rho(x,y) \leq b \cdot d(x,y).$$

Put $c_1 = a/2$ and $c_2 = 9b$. We will show that

$$\left\{U_\rho\left(x, \frac{c_1}{3^i}\right) \,\middle|\, x \in X\right\} \leq \mathscr{U}_i \leq \left\{U_\rho\left(x, \frac{c_2}{3^i}\right) \,\middle|\, x \in X\right\}.$$

Let $x \in X$. We choose $U \in \mathscr{U}_{i+1}$ with $x \in U$. If $y \in X$ with $\rho(x,y) < c_1/3^i$, then $6d(x,y) \leq (6/a) \cdot \rho(x,y) < 6c_1/(a \cdot 3^i) = 1/3^{i-1}$. Since $D_\star(x,y) \leq 6d(x,y) < 1/3^{i-1}$, there is $U_y \in \mathscr{U}_{i+1}$ such that $x, y \in U_y$. Hence there is $U' \in \mathscr{U}_i$ such that if $y \in X$ with $\rho(x,y) < c_1/3^i$, then $y \in U_y \subset St(U, \mathscr{U}_{i+1}) \subset U'$. Then $U_\rho(x, c_1/3^i) \subset U'$. This implies that

$$\left\{U_\rho\left(x, \frac{c_1}{3^i}\right) \,\middle|\, x \in X\right\} \leq \mathscr{U}_i.$$

Let $V \in \mathscr{U}_i$. Choose $x \in V$. If $y \in V$, $d(x,y) \leq D_\star(x,y) \leq 1/3^{i-2}$. Then $\rho(x,y) \leq b \cdot d(x,y) \leq b/3^{i-2} = c_2/3^i$. Hence $V \subset U_\rho(x, c_2/3^i)$. This implies that

$$\mathscr{U}_i \leq \left\{U_\rho\left(x, \frac{c_2}{3^i}\right) \,\middle|\, x \in X\right\}.$$

Next, we show that (2) implies (1). Put $a = c_1/3^3$. Also, choose i_0 such that $c_2/3^{i_0} < c_1/3^2$. Put $b = 2 \cdot 3^{i_0-2} c_1$. Let $x, y \in X$. Suppose that $0 < \rho(x,y) < c_1/3$. Choose i such that $c_1/3^{i+1} \leq \rho(x,y) < c_1/3^i$. Then there is $U \in \mathscr{U}_i$ such that $y \in U_\rho(x, c_1/3^i) \subset U \in \mathscr{U}_i$. Since $x, y \in U \in \mathscr{U}_i$, we have

$$d(x,y) \le D_\star(x,y) \le \frac{1}{3^{i-2}}.$$

Then

$$a \cdot d(x,y) \le \left(\frac{c_1}{3^3}\right)\left(\frac{1}{3^{i-2}}\right) = \frac{c_1}{3^{i+1}} \le \rho(x,y).$$

Now we show that for any $U \in \mathscr{U}_{i+i_0}$, U does not contain both x and y. Suppose, on the contrary, that there is $U \in \mathscr{U}_{i+i_0}$ with $x, y \in U$. By the assumption, there is $z \in X$ such that $x, y \in U \subset U_\rho(z, c_2/3^{i+i_0})$. This implies that $\rho(x,y) \le 2c_2/3^{i+i_0} < 2c_1/3^{i+2} < c_1/3^{i+1}$. This is a contradiction. Hence $D_\star(x,y) \ge 1/3^{i+i_0-3}$. Then

$$b \cdot d(x,y) \ge b\left(\frac{1}{6}\right) D_\star(x,y) \ge b\left(\frac{1}{6}\right)\left(\frac{1}{3^{i+i_0-3}}\right) = \frac{c_1}{3^i} > \rho(x,y).$$

Then we see that if $\rho(x,y) < c_1/3$,

$$a \cdot d(x,y) \le \rho(x,y) \le b \cdot d(x,y).$$

Since $\mathscr{U}_i \le \{U_\rho(x, c_2/3^i) \mid x \in X\}$ for each i, we can choose a positive number $c > 0$ such that if $\rho(x,y) \ge c_1/3$, then $d(x,y) \ge (1/6)D_\star(x,y) \ge c$. Since $\mathscr{U}_i$ is a finite open cover of X and $\mathscr{U}_i \le \{U_\rho(x, c_2/3^i) \mid x \in X\}$ for each i, we see that ρ is totally bounded. Since d and ρ are bounded, we see that there exist a sufficiently small positive number a' and a sufficiently large positive number b' such that for any $x, y \in X$ with $\rho(x,y) \ge c_1/3$,

$$a' \cdot d(x,y) \le \rho(x,y) \le b' \cdot d(x,y).$$

Hence we see that (2) implies (1).

The next proposition implies that for any separable metric space X there is a natural bijection from the set of all totally bounded metrics on X to the set of Alexandroff-Urysohn metrics on X induced by normal sequences of finite open covers which are developments of X, up to Lipschitz equivalence.

Proposition 4.4. *Let X be a separable metric space and let ρ be a totally bounded metric on X. Then there is a normal star (resp. delta)-sequence $\{\mathscr{U}_i\}_{i=1}^\infty$ of finite open covers of X such that $\{\mathscr{U}_i\}_{i=1}^\infty$ is a development of X and ρ is Lipschitz equivalent to d, where d is the Alexandroff-Urysohn metric induced by*

$\{\mathscr{U}_i\}_{i=1}^{\infty}$. *In particular,* $\dim_B(X,\rho) = \dim_B(X,d)$.

Proof. Let $\{\epsilon_i\}_{i=1}^{\infty}$ be a sequence of positive numbers such that $\epsilon_i/4 > \epsilon_{i+1}$ and $1/3^i > \epsilon_i$ for each i. Since ρ is a totally bounded metric on X, for each $i = 1, 2, \ldots$ we can choose a finite subset A_i of X such that if $x \in X$, then there is a point $a \in A_i$ such that $\rho(x,a) < \epsilon_{i+1}$. Put

$$\mathscr{U}_i = \left\{ U\left(a, \frac{1}{3^i} + \epsilon_i\right) \,\middle|\, a \in A_i \right\},$$

where $U(a,\epsilon)$ denotes the open ϵ-neighborhood of a in (X,ρ). Clearly, $\mathscr{U}_i$ is a finite open cover of X. We shall show that $\mathscr{U}_{i+1}^{\star} \leq \mathscr{U}_i$. Let $U = U(a, 1/3^{i+1} + \epsilon_{i+1}) \in \mathscr{U}_{i+1}$. Note that $St(U, \mathscr{U}_{i+1}) \subset U(a, 1/3^i + 3\epsilon_{i+1})$. Then we choose $a' \in A_i$ such that $\rho(a,a') < \epsilon_{i+1}$. This implies that $U(a, 1/3^i + 3\epsilon_{i+1}) \subset U(a', 1/3^i + 4\epsilon_{i+1}) \subset U(a', 1/3^i + \epsilon_i) \in \mathscr{U}_i$ and hence $St(U, \mathscr{U}_{i+1}) \subset U(a', 1/3^i + \epsilon_i) \in \mathscr{U}_i$. Then

$$\mathscr{U}_{i+1}^{\star} \leq \mathscr{U}_i.$$

Note that $\{\mathscr{U}_i\}_{i=1}^{\infty}$ is a development of X. Let $d(= d_\star)$ be the Alexandroff-Urysohn metric induced by $\{\mathscr{U}_i\}_{i=1}^{\infty}$. If we put $c_1 = 1, c_2 = 2$, the normal sequence $\{\mathscr{U}_i\}_{i=1}^{\infty}$ satisfies the condition (2) of Proposition 4.3. By Proposition 4.3, ρ is Lipschitz equivalent to $d(= d_\star)$. Also, we see that $\dim_B(X,\rho) = \dim_B(X,d)$. The proof of the case of normal delta-sequence is similar.

Theorem 4.5 (Pontrjagin-Schnirelmann and Bruijning theorem). *Let X be a separable metric space. Then*

$$\dim X = \min\{\dim_B(X,\rho) \mid \rho \textit{ is a totally bounded metric for } X\}.$$

Proof. By Proposition 4.4, we see that if ρ is any totally bounded metric on X, then there is a normal star-sequence $\{\mathscr{U}_i\}_{i=1}^{\infty}$ of finite open covers of X such that $\{\mathscr{U}_i\}_{i=1}^{\infty}$ is a development of X and $\dim_B(X,\rho) = \dim_B(X,d)$, where d is the Alexandroff-Urysohn metric induced by $\{\mathscr{U}_i\}_{i=1}^{\infty}$. By use of this fact and Proposition 4.1, we complete the proof of Theorem 4.5.

An open cover $\mathscr{U}$ of a space X is *essential* if for any $U \in \mathscr{U}$, $\bigcup\{V \in \mathscr{U} \mid V \neq U\} \neq X$.

Corollary 4.6. *Let X be a separable metric space and let ρ be a totally bounded metric on X. Suppose that $\{\mathscr{U}_i\}_{i=1}^{\infty}$ is a normal star (resp. delta)-sequence of finite open covers of X and a development of X and d is the Alexandroff-Urysohn* *metric induced by* $\{\mathscr{U}_i\}_{i=1}^{\infty}$. *If* $\{\mathscr{U}_i\}_{i=1}^{\infty}$ *satisfies the condition* (2) *of Proposition* 4.3 (*see also Proposition* 4.4), *then*

$$\dim_B(X,\rho) = \dim_B(X,d) \leq \liminf_{i\to\infty} \frac{\log_3 |\mathscr{U}_i|}{i}$$

$$\left(\textit{resp.}\ \dim_B(X,\rho) = \dim_B(X,d) \leq \liminf_{i\to\infty} \frac{\log_2 |\mathscr{U}_i|}{i}\right).$$

Moreover, if $\mathscr{U}_i$ *is essential for each* i, *then*

$$\dim_B(X,\rho) = \dim_B(X,d) = \liminf_{i\to\infty} \frac{\log_3 |\mathscr{U}_i|}{i}$$

$$\left(\textit{resp.}\ \dim_B(X,\rho) = \dim_B(X,d) = \liminf_{i\to\infty} \frac{\log_2 |\mathscr{U}_i|}{i}\right).$$

Proof. We assume that $\{\mathscr{U}_i\}_{i=1}^{\infty}$ is a normal star-sequence. Let $d\ (= d_\star)$ be the Alexandroff-Urysohn metric induced by $\{\mathscr{U}_i\}_{i=1}^{\infty}$. By the proof of Proposition 4.1, we see that

$$\dim_B(X,\rho) = \dim_B(X,d) \leq \liminf_{i\to\infty} \frac{\log_3 |\mathscr{U}_i|}{i}.$$

From now we suppose that $\mathscr{U}_i$ is essential for each i. Suppose, on the contrary, that

$$\dim_B(X,d) < \liminf_{i\to\infty} \frac{\log_3 |\mathscr{U}_i|}{i}\ (= \alpha).$$

We assume that $\alpha < \infty$. Put $\dim_B(X,d) = \beta$ and $\delta = \alpha - \beta > 0$. Then we can choose a sequence $1/3 > \epsilon_1 > \epsilon_2 > \epsilon_3 > \cdots$ of positive numbers such that $\lim_{k\to\infty} \epsilon_k = 0$ and for each k,

$$\frac{\log_3 N(\epsilon_k, d)}{\log_3 \dfrac{1}{\epsilon_k}} \leq \beta + \frac{\delta}{2}.$$

For each k, we choose the natural number $n(k)$ such that $1/3^{n(k)+1} \leq \epsilon_k < 1/3^{n(k)}$. Let $\mathscr{W}$ be a finite open cover of X such that $|\mathscr{W}| = N(\epsilon_k, d)$ with $\mathrm{mesh}_d(\mathscr{W}) \leq \epsilon_k$. For any $W \in \mathscr{W}$ and any $x, y \in W$,

$$D_\star(x,y) \le 6 \cdot d(x,y) < \frac{1}{3^{n(k)-2}}.$$

By the proof of Proposition 4.1, we see that $\mathscr{W} \le \mathscr{U}_{n(k)-1}$. Since $\mathscr{U}_{n(k)-1}$ is essential, we can easily see that $|\mathscr{W}| \ge |\mathscr{U}_{n(k)-1}|$. Then

$$\frac{\log_3 N(\epsilon_k, d)}{\log_3 \dfrac{1}{\epsilon_k}} = \frac{\log_3 |\mathscr{W}|}{\log_3 \dfrac{1}{\epsilon_k}} \ge \frac{\log_3 |\mathscr{U}_{n(k)-1}|}{\log_3 3^{n(k)+1}} = \frac{\log_3 |\mathscr{U}_{n(k)-1}|}{n(k)-1} \cdot \frac{n(k)-1}{n(k)+1}.$$

Hence we see that

$$\liminf_{k\to\infty} \frac{\log_3 N(\epsilon_k, d)}{\log_3 \dfrac{1}{\epsilon_k}} \ge \alpha \cdot 1 = \alpha.$$

This is a contradiction. Hence

$$\dim_B(X,\rho) = \dim_B(X,d) = \liminf_{i\to\infty} \frac{\log_3 |\mathscr{U}_i|}{i}.$$

The case $\alpha = \infty$ can be proved similarly.

The next example implies that some normal sequences can be used to calculate $\dim_B(X,\rho)$ of given separable metric spaces (X,ρ).

Example 1. Let F be the von Koch curve [**6**, p. xv] in the plane and let ρ be the usual Euclidean metric. By considering vertices of each stage $E_1, E_2, \dots$ of the construction of F (see [**6**, p. xv]), we have a natural normal star-sequence $\{\mathscr{U}_i\}_{i=1}^\infty$ of finite open covers of F such that

(1) $\{\mathscr{U}_i\}_{i=1}^\infty$ is a development of F,
(2) $|\mathscr{U}_1| = 4+1 = 5$, $|\mathscr{U}_2| = 4^2+1 = 17, \dots$ and in general $|\mathscr{U}_i| = 4^i + 1$ for each i,
(3) $\{\mathscr{U}_i\}_{i=1}^\infty$ satisfies the condition (2) of Proposition 4.3,
(4) $\mathscr{U}_i$ is essential for each i.

By Corollary 4.6, we see that

$$\dim_B(F,\rho) = \liminf_{i\to\infty} \frac{\log_3 |\mathscr{U}_i|}{i} = \log_3 4.$$

Example 2. Let F be the Sierpiński gasket [**6**, p. xvi] in the plane and let

ρ be the usual Euclidean metric. By considering vertices of each stage $E_1, E_2, \ldots$ of the construction of F (see [**6**, p. xvi]), we have a natural normal delta-sequence $\{\mathscr{U}_i\}_{i=1}^\infty$ of finite open covers of F such that

(1) $\{\mathscr{U}_i\}_{i=1}^\infty$ is a development of F,
(2) $|\mathscr{U}_1| = 6$, $|\mathscr{U}_2| = 15, \ldots$ and in general $|\mathscr{U}_i| = 3 + (3/2)(3^i - 1)$ for each i,
(3) $\{\mathscr{U}_i\}_{i=1}^\infty$ satisfies the condition (2) of Proposition 4.3,
(4) $\mathscr{U}_i$ is essential for each i.

By Corollary 4.6, we see that

$$\dim_B(F, \rho) = \liminf_{i \to \infty} \frac{\log_2 |\mathscr{U}_i|}{i} = \log_2 3.$$

REMARK 1. If $\{\mathscr{U}_i\}_{i=1}^\infty$ is a normal star-sequence of finite open covers of X and a development of X such that $\liminf_{i\to\infty}(\log_3 |\mathscr{U}_i|)/i = \dim X$ $(= m)$ and d is the Alexandroff-Urysohn metric induced by the sequence $\{\mathscr{U}_i\}_{i=1}^\infty$, then $\dim \tilde{X} = m$, where $(\tilde{X}, \tilde{d})$ is the compactification of (X, d) defined in Lemma 3.5. In fact, $m = \dim X \le \dim \tilde{X} \le \dim_B(\tilde{X}, \tilde{d}) = \dim_B(X, d) = m$.

5. Chaotic metrics with respect to the determination of the box-counting dimensions.

In this section, we construct *chaotic metrics* with respect to the determination of the box-counting dimensions. By Theorem 3.1, we know that for any separable metric space X, there is a normal star (resp. delta)-sequence $\{\mathscr{U}_i\}_{i=1}^\infty$ of finite open covers of X which is a development of X such that

$$\liminf_{i \to \infty} \frac{\log_3 |\mathscr{U}_i|}{i} = \dim X \left(\text{resp. } \liminf_{i \to \infty} \frac{\log_2 |\mathscr{U}_i|}{i} = \dim X\right).$$

We call such a normal sequence $\{\mathscr{U}_i\}_{i=1}^\infty$ *a fundamental normal sequence* of X. In this section, we consider the case $\dim X \ge 1$. In the next section, we also consider the case $\dim X = 0$ (see Theorem 6.4).

THEOREM 5.1. *Let X be a separable metric space with $\dim X = m \ge 1$. Suppose that $\{\mathscr{U}_i\}_{i=1}^\infty$ is a fundamental normal star-sequence of X (i.e., $\liminf_{i\to\infty}(\log_3 |\mathscr{U}_i|)/i = \dim X$). Let α be any real number with $\alpha \ge m$ $(= \dim X)$ or $\alpha = \infty$. Then there is a subsequence $\{\mathscr{U}_{i_j}\}_{j=1}^\infty$ of $\{\mathscr{U}_i\}_{i=1}^\infty$ such that*

$$[\alpha,\infty] = \left\{ \liminf_{k\to\infty} \frac{\log_3|\mathscr{U}_{i_{n_k}}|}{n_k} \;\middle|\; \{n_k\}_{k=1}^{\infty} \textit{ is an increasing subsequence of natural numbers} \right\}.$$

Also, if d_α $(= d_\star)$ is the Alexandroff-Urysohn metric on X induced by the subsequence $\{\mathscr{U}_{i_j}\}_{j=1}^{\infty}$, then

$$[\alpha,\infty] = \left\{ \liminf_{k\to\infty} \frac{\log N(\epsilon_k, d_\alpha)}{|\log \epsilon_k|} \;\middle|\; \{\epsilon_k\}_{k=1}^{\infty} \textit{ is a decreasing sequence of positive numbers with } \lim_{k\to\infty}\epsilon_k = 0 \right\},$$

where $N(\epsilon, d_\alpha) = \min\{|\mathscr{U}| \mid \mathscr{U}$ is a finite open cover of X with $mesh_{d_\alpha}(\mathscr{U}) \le \epsilon\}$. In particular, $\dim_B(X, d_\alpha) = \alpha$.

Proof. We assume that α is a real number. The case $\alpha = \infty$ can be proved similarly. Let $\{\delta_k\}_{k=1}^{\infty}$ be a decreasing sequence of positive numbers with $\lim_{k\to\infty}\delta_k = 0$. Take a countable subset $T = \{\alpha_k \mid k = 1, 2, \dots\}$ of $[\alpha,\infty)$ such that $\overline{T} = [\alpha,\infty)$. Note that $\alpha_k/m \ge 1$. Since

$$\liminf_{i\to\infty} \frac{\log_3|\mathscr{U}_i|}{i} = m,$$

for δ_1 and α_1 we choose a finite sequence $\{i_j\}_{j=1}^{j_1}$ of natural numbers such that

$$\frac{\alpha_1}{m}\cdot j \le i_j < \frac{\alpha_1}{m}\cdot (j+1)$$

and

$$\left| \frac{\log_3|\mathscr{U}_{i_{j_1}}|}{j_1} - \alpha_1 \right| = \left| \frac{\log_3|\mathscr{U}_{i_{j_1}}|}{i_{j_1}} \cdot \frac{i_{j_1}}{j_1} - \alpha_1 \right| \le \delta_1.$$

For δ_2 and α_2, we choose a finite sequence $\{i_j\}_{j=j_1+1}^{j_2}$ of natural numbers such that

$$\frac{\alpha_2}{m}\cdot (j - j_1) + i_{j_1} \le i_j < \frac{\alpha_2}{m}\cdot (j - j_1 + 1) + i_{j_1}$$

and

$$\left| \frac{\log_3 |\mathscr{U}_{i_{j_2}}|}{j_2} - \alpha_2 \right| = \left| \frac{\log_3 |\mathscr{U}_{i_{j_2}}|}{i_{j_2}} \cdot \frac{i_{j_2}}{j_2} - \alpha_2 \right| \leq \delta_2.$$

We continue this procedure. For each δ_k and α_k, we choose a finite sequence $\{i_j\}_{j=j_{k-1}+1}^{j_k}$ of natural numbers such that

$$\frac{\alpha_k}{m} \cdot (j - j_{k-1}) + i_{j_{k-1}} \leq i_j < \frac{\alpha_k}{m} \cdot (j - j_{k-1} + 1) + i_{j_{k-1}}$$

and

$$\left| \frac{\log_3 |\mathscr{U}_{i_{j_k}}|}{j_k} - \alpha_k \right| = \left| \frac{\log_3 |\mathscr{U}_{i_{j_k}}|}{i_{j_k}} \cdot \frac{i_{j_k}}{j_k} - \alpha_k \right| \leq \delta_k.$$

In this procedure we choose the sequence $\{j_k\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty} \frac{\log_3 |\mathscr{U}_{i_{j_k}}|}{i_{j_k}} = m (= \dim X) \text{ and } \lim_{k\to\infty} (j_{k+1} - j_k) = \infty.$$

Consequently, we obtain a sequence $\{j_k\}_{k=1}^{\infty}$ of natural numbers and a subsequence $\{\mathscr{U}_{i_j}\}_{j=1}^{\infty}$ of $\{\mathscr{U}_i\}_{i=1}^{\infty}$. Note that $i_j/j \geq \alpha/m$ for each j. In fact, $i_j/j \geq \alpha/m$ for each $1 \leq j \leq j_1$. If we assume that $i_{j_{k-1}}/j_{k-1} \geq \alpha/m$, then for $j_{k-1}+1 \leq j \leq j_k$

$$\frac{\alpha}{m} \cdot j = \frac{\alpha}{m} \cdot (j - j_{k-1}) + \frac{\alpha}{m} \cdot j_{k-1} \leq \frac{\alpha_k}{m} \cdot (j - j_{k-1}) + i_{j_{k-1}} \leq i_j.$$

Hence $i_j/j \geq \alpha/m$ for each j. Then

$$\liminf_{j\to\infty} \frac{\log_3 |\mathscr{U}_{i_j}|}{j} = \liminf_{j\to\infty} \frac{\log_3 |\mathscr{U}_{i_j}|}{i_j} \cdot \frac{i_j}{j} \geq m \cdot \frac{\alpha}{m} = \alpha.$$

By the construction, we see that

$$[\alpha, \infty] = \left\{ \liminf_{k\to\infty} \frac{\log_3 |\mathscr{U}_{i_{n_k}}|}{n_k} \;\middle|\; \{n_k\}_{k=1}^{\infty} \text{ is an increasing subsequence of natural numbers} \right\}.$$

Let d_α $(= d_\star)$ be the Alexandroff-Urysohn metric on X induced by the subsequence $\{\mathscr{U}_{i_j}\}_{j=1}^{\infty}$. First, we show that

$$\dim_B(X, d_\star) \geq \alpha.$$

Suppose that $1/3 > \epsilon_1 > \epsilon_2 > \epsilon_3 > \cdots$ is a sequence of positive numbers such that $\lim_{k\to\infty} \epsilon_k = 0$. For each k, we choose the natural number $n(k)$ such that $1/3^{n(k)+1} \leq \epsilon_k < 1/3^{n(k)}$. Let $\mathscr{W}$ be a finite open cover of X such that $|\mathscr{W}| = N(\epsilon_k, d_\star)$ with $\mathrm{mesh}_{d_\star}(\mathscr{W}) \leq \epsilon_k$. Let $W \in \mathscr{W}$ and $x \in W$. Choose $V \in \mathscr{U}_{i_{n(k)}}$ with $x \in V$. If $y \in W \in \mathscr{W}$, then $1/3^{n(k)} > d_\star(x, y) \geq (1/6) \cdot D_\star(x, y)$ and hence $1/3^{n(k)-2} \geq D_\star(x, y)$. This implies that there is $U \in \mathscr{U}_{i_{n(k)}}$ such that U contains x and y. Then

$$W \subset St(x, \mathscr{U}_{i_{n(k)}}) \subset St(V, \mathscr{U}_{i_{n(k)}}) \in \mathscr{U}^\star_{i_{n(k)}} \leq \mathscr{U}_{i_{n(k)-1}}$$

and hence $\mathscr{W} \leq \mathscr{U}_{i_{n(k)-1}}$. Since $\{\mathscr{U}_i\}_{i=1}^\infty$ is a normal star-sequence of open covers, there is i_0 as in Lemma 3.5. Similarly to the proof of Proposition 4.1, we see that

$$N(\epsilon_k, d_\star) = |\mathscr{W}| \geq \star_{m+1}^{i_{n(k)-1} - i_0}(X).$$

Then

$$\frac{\log_3 N(\epsilon_k, d_\star)}{|\log_3 \epsilon_k|} = \frac{\log_3 |\mathscr{W}|}{|\log_3 \epsilon_k|} \geq \frac{\log_3 \star_{m+1}^{i_{n(k)-1}-i_0}(X)}{|\log_3 3^{n(k)+1}|} = \frac{\log_3 \star_{m+1}^{i_{n(k)-1}-i_0}(X)}{n(k)+1}$$
$$= \frac{\log_3 \star_{m+1}^{i_{n(k)-1}-i_0}(X)}{i_{n(k)-1} - i_0} \cdot \frac{i_{n(k)-1} - i_0}{n(k)-1} \cdot \frac{n(k)-1}{n(k)+1}.$$

We have

$$\liminf_{k\to\infty} \frac{\log_3 N(\epsilon_k, d_\star)}{|\log_3 \epsilon_k|} \geq m \cdot \frac{\alpha}{m} = \alpha.$$

Hence

$$\dim_B(X, d_\star) \geq \alpha.$$

Let $\beta \geq \alpha$ be any positive number. The case $\beta = \infty$ can be proved similarly. By the construction of the subsequence $\{\mathscr{U}_{i_j}\}_{j=1}^\infty$, we can choose a subsequence $\{n_k\}_{k=1}^\infty$ of natural numbers such that

$$\lim_{k\to\infty} \frac{\log_3 |\mathscr{U}_{i_{n_k}}|}{i_{n_k}} = m \quad \text{and} \quad \lim_{k\to\infty} \frac{i_{n_k}}{n_k} = \frac{\beta}{m}.$$

Since $\lim_{k\to\infty}(j_{k+1}-j_k)=\infty$, we may assume that

$$\lim_{k\to\infty}\frac{i_{n_k-3}}{n_k-3}=\frac{\beta}{m}.$$

Put $\epsilon_k=1/3^{n_k-2}$ for each $k\geq 1$. We shall show that

$$\lim_{k\to\infty}\frac{\log N(\epsilon_k,d_\star)}{|\log\epsilon_k|}=\beta.$$

Since $d_\star(x,y)\leq D_\star(x,y)$ for $x,y\in X$, we see that $\mathrm{mesh}_{d_\star}(\mathscr{U}_{i_{n_k}})\leq 1/3^{n_k-2}$. Hence we have

$$\limsup_{k\to\infty}\frac{\log N(\epsilon_k,d_\star)}{|\log\epsilon_k|}\leq\lim_{k\to\infty}\frac{\log_3|\mathscr{U}_{i_{n_k}}|}{n_k-2}=\lim_{k\to\infty}\frac{\log_3|\mathscr{U}_{i_{n_k}}|}{i_{n_k}}\cdot\frac{i_{n_k}}{n_k-2}=\beta.$$

Let $\mathscr{W}$ be a finite open cover of X such that $|\mathscr{W}|=N(\epsilon_k,d_\star)$ with $\mathrm{mesh}_{d_\star}(\mathscr{W})\leq\epsilon_k$. Suppose that $W\in\mathscr{W}$ and $x\in W$. Choose $V\in\mathscr{U}_{i_{n_k-2}}$ containing x. Since $D_\star(x,y)\leq 6d_\star(x,y)$ for $x,y\in X$, we see that if $y\in W$, then

$$D_\star(x,y)\leq 6d_\star(x,y)\leq\frac{6}{3^{n_k-2}}<\frac{1}{3^{n_k-4}}.$$

Then we have $U\in\mathscr{U}_{i_{n_k-2}}$ which contains x and y. Then

$$W\subset St(x,\mathscr{U}_{i_{n_k-2}})\subset St(V,\mathscr{U}_{i_{n_k-2}})\in\mathscr{U}^\star_{i_{n_k-2}}\leq\mathscr{U}_{i_{n_k-3}}$$

and hence

$$\mathscr{W}\leq\mathscr{U}_{i_{n_k-3}}.$$

Now we will recall Lemma 3.5. Let i_0 be a natural number as in Lemma 3.5. Then we have

$$|\mathscr{W}|\geq\star_{m+1}^{i_{n_k-3}-i_0}(X).$$

This implies that

$$N(\epsilon_k,d_\star)=|\mathscr{W}|\geq\star_{m+1}^{i_{n_k-3}-i_0}(X).$$

Then

$$\frac{\log_3 N(\epsilon_k, d_\star)}{|\log_3 \epsilon_k|} = \frac{\log_3 |\mathscr{W}|}{|\log_3 \epsilon_k|} \geq \frac{\log_3 \star_{m+1}^{i_{n_k-3}-i_0}(X)}{\log_3 3^{n_k-2}} = \frac{\log_3 \star_{m+1}^{i_{n_k-3}-i_0}(X)}{n_k - 2}$$
$$= \frac{\log_3 \star_{m+1}^{i_{n_k-3}-i_0}(X)}{i_{n_k-3} - i_0} \cdot \frac{i_{n_k-3} - i_0}{n_k - 3} \cdot \frac{n_k - 3}{n_k - 2}.$$

Then

$$\liminf_{k\to\infty} \frac{\log_3 N(\epsilon_k, d_\alpha)}{|\log_3 \epsilon_k|} \geq m \cdot \frac{\beta}{m} = \beta.$$

Hence

$$\lim_{k\to\infty} \frac{\log_3 N(\epsilon_k, d_\star)}{|\log_3 \epsilon_k|} = \beta.$$

Consequently we can conclude that

$$[\alpha, \infty] = \left\{ \liminf_{k\to\infty} \frac{\log N(\epsilon_k, d_\star)}{|\log \epsilon_k|} \,\middle|\, \{\epsilon_k\}_{k=1}^\infty \text{ is a decreasing sequence} \right.$$
$$\left. \text{of positive numbers with } \lim_{k\to\infty} \epsilon_k = 0 \right\}.$$

REMARK 2. Let X be a separable metric space with $\dim X = m \geq 1$. Suppose that $\{\mathscr{U}_i\}_{i=1}^\infty$ is a fundamental normal star-sequence of X. Then

$$[\dim X, \infty] = \left\{ \liminf_{j\to\infty} \frac{\log_3 |\mathscr{U}_{i_j}|}{j} \,\middle|\, \{\mathscr{U}_{i_j}\}_{j=1}^\infty \text{ is a subsequence of } \{\mathscr{U}_i\}_{i=1}^\infty \right\}$$
$$= \big\{ \dim_B(X, d_\star) \mid d_\star \text{ is the Alexandroff-Urysohn metric on } X$$
$$\text{induced by a subsequence } \{\mathscr{U}_{i_j}\}_{j=1}^\infty \text{ of } \{\mathscr{U}_i\}_{i=1}^\infty \big\}.$$

In other words, all box-counting dimensions of X are generated by fundamental normal star-sequences of X.

In case of normal delta-sequence of finite open covers of X, we also obtain the following theorem. The proof is similar to the one of Theorem 5.1.

THEOREM 5.2. *Let X be a separable metric space with* $\dim X = m \geq 1$. *Suppose that $\{\mathscr{U}_i\}_{i=1}^\infty$ is a fundamental normal delta-sequence of X (i.e.,*

$\liminf_{i\to\infty}(\log_2|\mathscr{U}_i|)/i = \dim X$). *Let* α *be any real number with* $\alpha \geq m$ $(= \dim X)$ *or* $\alpha = \infty$. *Then there is a subsequence* $\{\mathscr{U}_{i_j}\}_{j=1}^\infty$ *of* $\{\mathscr{U}_i\}_{i=1}^\infty$ *such that*

$$[\alpha, \infty] = \left\{ \liminf_{k\to\infty} \frac{\log_2|\mathscr{U}_{i_{n_k}}|}{n_k} \;\middle|\; \{n_k\}_{k=1}^\infty \text{ is an increasing subsequence of natural numbers} \right\}.$$

Also, if $d_\alpha (= d_\Delta)$ *is the Alexandroff-Urysohn metric on* X *induced by the subsequence* $\{\mathscr{U}_{i_j}\}_{j=1}^\infty$, *then*

$$[\alpha, \infty] = \left\{ \liminf_{k\to\infty} \frac{\log N(\epsilon_k, d_\alpha)}{|\log \epsilon_k|} \;\middle|\; \{\epsilon_k\}_{k=1}^\infty \text{ is a decreasing sequence of positive numbers with } \lim_{k\to\infty}\epsilon_k = 0 \right\}.$$

In particular, $\dim_B(X, d_\alpha) = \alpha$.

PROPOSITION 5.3. *Suppose that* X *is a separable metric space. Let* $\{\mathscr{U}_i\}_{i=1}^\infty$ *be a normal star* (*resp. delta*)*-sequence of finite open covers and a development of* X, *and let* $\{\mathscr{U}_{i_j}\}_{j=1}^\infty$ *be a subsequence of* $\{\mathscr{U}_i\}_{i=1}^\infty$. *If* d_1 (*resp.* ρ_1) *is the Alexandroff-Urysohn metric on* X *induced by* $\{\mathscr{U}_i\}_{i=1}^\infty$ *and* d_2 (*resp.* ρ_2) *is the Alexandroff-Urysohn metric on* X *induced by the subsequence* $\{\mathscr{U}_{i_j}\}_{j=1}^\infty$, *then* $\dim_B(X, d_1) \leq \dim_B(X, d_2)$ (*resp.* $\dim_B(X, \rho_1) \leq \dim_B(X, \rho_2)$).

PROOF. We will prove $\dim_B(X, d_1) \leq \dim_B(X, d_2)$. Choose a sequence $1/3 > \epsilon_1 > \epsilon_2 > \epsilon_3 > \cdots$ of positive numbers such that $\lim_{k\to\infty}\epsilon_k = 0$ and

$$\lim_{k\to\infty} \frac{\log_3 N(\epsilon_k, d_2)}{|\log_3 \epsilon_k|} = \dim_B(X, d_2).$$

For each k, we choose the natural number $n(k)$ such that $1/3^{n(k)+1} \leq \epsilon_k < 1/3^{n(k)}$. Let $\mathscr{W}$ be a finite open cover of X such that $|\mathscr{W}| = N(\epsilon_k, d_2)$ with $\operatorname{mesh}_{d_2}(\mathscr{W}) \leq \epsilon_k$. Similarly to the proof of Theorem 5.1, we see that $\mathscr{W} \leq \mathscr{U}_{i_{n(k)-1}}$. Then we see that $\operatorname{mesh}_{d_1}(\mathscr{W}) \leq 1/3^{i_{n(k)-1}-2}$. Hence

$$N\left(\frac{1}{3^{i_{n(k)-1}-2}}, d_1\right) \leq |\mathscr{W}| = N(\epsilon_k, d_2).$$

Since $1/\epsilon_k < 3^{n(k)+1}$ and $n(k)+1 = (n(k)-1)+2 \le i_{n(k)-1}+2$, we have

$$\begin{aligned}
&\dim_B(X,d_1)\\
&\le \liminf_{k\to\infty} \frac{\log_3 N\left(\dfrac{1}{3^{i_{n(k)-1}-2}}, d_1\right)}{\left|\log_3 \dfrac{1}{3^{i_{n(k)-1}-2}}\right|} = \liminf_{k\to\infty} \frac{\log_3 N\left(\dfrac{1}{3^{i_{n(k)-1}-2}}, d_1\right)}{i_{n(k)-1}-2}\\
&= \liminf_{k\to\infty} \frac{\log_3 N\left(\dfrac{1}{3^{i_{n(k)-1}-2}}, d_1\right)}{i_{n(k)-1}+2} \le \liminf_{k\to\infty} \frac{\log_3 N(\epsilon_k, d_2)}{\log_3 \dfrac{1}{\epsilon_k}}\\
&= \dim_B(X,d_2).
\end{aligned}$$

Hence

$$\dim_B(X,\rho_1) \le \dim_B(X,\rho_2).$$

The rest of proof is similar. We omit the proof.

6. Upper box-counting dimension $\overline{\dim}_B(X,d)$ and normal sequences of finite open covers.

In this sectin, we study some relations between upper box-counting dimension and normal sequence of finite open covers. For a separable metric space (X,ρ), we consider the upper box-counting dimension of (X,d) (e.g., see [**6**] and [**14**]):

$$\overline{\dim}_B(X,d) = \limsup_{\epsilon\to 0} \frac{\log N(\epsilon,\rho)}{|\log\epsilon|}.$$

PROPOSITION 6.1. *Let X be a separable metric space and let $\{\mathscr{U}_i\}_{i=1}^\infty$ be a normal star (resp. delta)-sequence of finite open covers and a development of X. If d (resp. ρ) is the Alexandroff-Urysohn metric on X induced by $\{\mathscr{U}_i\}_{i=1}^\infty$ and $\limsup_{i\to\infty}(\log_3|\mathscr{U}_i|)/i = \beta$ (resp. $\limsup_{i\to\infty}(\log_2|\mathscr{U}_i|)/i = \beta$), then $\overline{\dim}_B(X,d) \le \beta$ (resp. $\overline{\dim}_B(X,\rho) \le \beta$).*

PROOF. We give the proof of the case of normal star-sequence. Choose a sequence $1/3 > \epsilon_1 > \epsilon_2 > \cdots$ of positive numbers such that $\lim_{k\to\infty}\epsilon_k = 0$ and

$$\lim_{k\to\infty} \frac{\log N(\epsilon_k,d)}{|\log\epsilon_k|} = \overline{\dim}_B(X,d).$$

For each k, let $n(k)$ be the natural number such that $1/3^{n(k)+1} \leq \epsilon_k < 1/3^{n(k)}$. Then $n(k) \leq n(k+1)$. Since $\mathrm{mesh}_d(\mathscr{U}_{n(k)+3}) \leq 1/3^{n(k)+1} \leq \epsilon_k$, we see that

$$\limsup_{k\to\infty} \frac{\log_3 N(\epsilon_k, d)}{\log_3 \dfrac{1}{\epsilon_k}} \leq \limsup_{k\to\infty} \frac{\log_3 |\mathscr{U}_{n(k)+3}|}{\log_3 3^{n(k)}}$$
$$= \limsup_{k\to\infty} \frac{\log_3 |\mathscr{U}_{n(k)+3}|}{n(k)+3} \cdot \frac{n(k)+3}{n(k)} \leq \beta.$$

Hence we have

$$\overline{\dim}_B(X,d) \leq \beta.$$

By modifying the proof of Theorem 5.1, we can prove the following.

Theorem 6.2. *Let X be a separable metric space with $\dim X = m \geq 1$. Suppose that there is a sequence $\{\mathscr{U}_i\}_{i=1}^\infty$ which is a normal star (resp. delta)-sequence of finite open covers of X and a development of X such that*

$$\lim_{i\to\infty} \frac{\log_3 |\mathscr{U}_i|}{i} = m \left(\textit{resp. } \lim_{i\to\infty} \frac{\log_2 |\mathscr{U}_i|}{i} = m \right).$$

Then for any α, β with $m \leq \alpha \leq \beta \leq \infty$, there is a totally bounded metric $d_{\alpha,\beta}$ on X such that

$$[\alpha,\beta] = \left\{ \liminf_{k\to\infty} \frac{\log N(\epsilon_k, d_{\alpha,\beta})}{|\log \epsilon_k|} \,\middle|\, \{\epsilon_k\} \textit{ is a decreasing sequence of positive numbers with } \lim_{k\to\infty} \epsilon_k = 0 \right\}.$$

In particular, $\underline{\dim}_B(X, d_{\alpha,\beta}) = \alpha \leq \beta = \overline{\dim}_B(X, d_{\alpha,\beta})$.

Corollary 6.3. *Let $I = [0,1]$ be the unit interval and let $X = I^m$ be the m-cube $(m \geq 1)$. Then there is a sequence $\{\mathscr{U}_i\}_{i=1}^\infty$ which is a normal star (resp. delta)-sequence of finite open covers and a development of X such that*

$$\lim_{i\to\infty} \frac{\log_3 |\mathscr{U}_i|}{i} = m \left(\textit{resp. } \lim_{i\to\infty} \frac{\log_2 |\mathscr{U}_i|}{i} = m \right).$$

Moreover, for any α, β with $m \leq \alpha \leq \beta \leq \infty$, there is a metric $d_{\alpha,\beta}$ on X such *that*

$$[\alpha, \beta] = \left\{ \liminf_{k\to\infty} \frac{\log N(\epsilon_k, d_{\alpha,\beta})}{|\log \epsilon_k|} \;\middle|\; \{\epsilon_k\} \textit{ is a decreasing sequence of positive numbers with } \lim_{k\to\infty} \epsilon_k = 0 \right\}.$$

In particular, $\underline{\dim}_B(X, d_{\alpha,\beta}) = \alpha \le \beta = \overline{\dim}_B(X, d_{\alpha,\beta})$.

Proof. For each $i = 1, 2, \dots$, consider the family

$$\mathscr{A} = \left\{ \left[\frac{k}{3^i}, \frac{k+1}{3^i}\right] \;\middle|\; k = 0, 1, 2, \dots, 3^i - 1 \right\}$$

of closed subintervals of I. Put

$$\mathscr{W}(i) = \left\{ \mathrm{Int}_I\left(St\left(\left\{ \frac{k}{3^i} \right\}, \mathscr{A} \right) \right) \;\middle|\; k = 0, 1, 2, \dots, 3^i \right\}.$$

Then $\mathscr{W}(i)$ is an open cover of I. For an open cover of I^m, we consider the following set:

$$\begin{aligned} \mathscr{U}_i &= \mathscr{W}(i) \times \mathscr{W}(i) \times \cdots \times \mathscr{W}(i) \\ &= \{U_1 \times \cdots \times U_m \mid U_k \in \mathscr{W}(i) \text{ for } k = 1, 2, \dots, m\}. \end{aligned}$$

Then $\{\mathscr{U}_i\}_{i=1}^\infty$ is a normal star-sequence of finite open covers and a development of X such that $|\mathscr{U}_i| = (3^i + 1)^m$ for each i, in particular $\lim_{i\to\infty}(\log_3 |\mathscr{U}_i|)/i = m$. Also, note that each $\mathscr{U}_i$ is essential for each i and if ρ_1 is the usual Euclidean metric on I^m, $\{\mathscr{U}_i\}_{i=1}^\infty$ satisfies the condition (2) of Proposition 4.3 (see also Corollary 4.6). For the case of normal delta-sequence, we consider the family

$$\mathscr{B} = \left\{ \left[\frac{k}{2^i}, \frac{k+1}{2^i}\right] \;\middle|\; k = 0, 1, 2, \dots, 2^i - 1) \right\}.$$

The proof is similar. We omit the proof.

For the case of $\dim X = 0$, we have the following theorem.

Theorem 6.4. *Let X be an infinite 0-dimensional separable metric space. Then*

(1) *there is a sequence* $\{\mathscr{U}_i\}_{i=1}^{\infty}$ *of disjoint finite clopen covers of* X *such that* $\{\mathscr{U}_i\}_{i=1}^{\infty}$ *is a development of* X, $\mathscr{U}_{i+1} \leq \mathscr{U}_i$ *and* $|\mathscr{U}_i| = i$ *for each* i,
(2) *for any* α, β *with* $0 \leq \alpha \leq \beta \leq \infty$, *there is a totally bounded metric* $d_{\alpha,\beta}$ *on* X *such that*

$$[\alpha, \beta] = \left\{ \liminf_{k\to\infty} \frac{\log N(\epsilon_k, d_{\alpha,\beta})}{|\log \epsilon_k|} \,\middle|\, \{\epsilon_k\} \text{ is a decreasing sequence of positive numbers with } \lim_{k\to\infty} \epsilon_k = 0 \right\}.$$

In particular, $\underline{\dim}_B(X, d_{\alpha,\beta}) = \alpha \leq \beta = \overline{\dim}_B(X, d_{\alpha,\beta})$.

Proof. Since X is an infinite 0-dimensional separable metric space, we can easily construct a sequence $\{\mathscr{U}_i\}_{i=1}^{\infty}$ of disjoint clopen covers of X such that $|\mathscr{U}_i| = i$, $\mathscr{U}_{i+1} \leq \mathscr{U}_i$ for each i and $\{\mathscr{U}_i\}_{i=1}^{\infty}$ is a development of X. Note that $\mathscr{U}_i^{\star} = \mathscr{U}_i$. Hence $\{\mathscr{U}_i\}_{i=1}^{\infty}$ is a normal star-sequence of finite open covers of X and

$$\lim_{i\to\infty} \frac{\log |\mathscr{U}_i|}{i} = \lim_{i\to\infty} \frac{\log i}{i} = 0.$$

We will prove that for any $0 \leq \alpha \leq \beta \leq \infty$, there exist a subsequence $\{\mathscr{U}_{i_j}\}_{j=1}^{\infty}$ of $\{\mathscr{U}_i\}_{i=1}^{\infty}$ such that

$$[\alpha, \beta] = \left\{ \liminf_{k\to\infty} \frac{\log_3 |\mathscr{U}_{i_{n_k}}|}{n_k} \,\middle|\, \{n_k\}_{k=1}^{\infty} \text{ is an increasing subsequence of natural numbers} \right\}.$$

First, we prove the case $0 < \alpha = \beta < \infty$. We choose a natural number p such that $(3^\alpha)^{p+1} - (3^\alpha)^p > 1$ and $(3^\alpha)^p > p$. Put $i_j = j$ for $j = 1, 2, \ldots, p-1$. For $j \geq p$, we can choose a natural number i_j such that $(3^\alpha)^j \leq i_j < (3^\alpha)^{j+1}$. Then

$$\alpha \leq \frac{\log_3 i_j}{j} < \alpha \cdot \frac{j+1}{j}.$$

Consider the subsequence $\{\mathscr{U}_{i_j}\}_{j=1}^{\infty}$ of $\{\mathscr{U}_i\}_{i=1}^{\infty}$. Then

$$\lim_{j\to\infty} \frac{\log_3 |\mathscr{U}_{i_j}|}{j} = \alpha.$$

Next, we prove the case $0 < \alpha < \beta < \infty$. We choose a sufficiently large natural number j_1 such that

$$(3^\alpha)^{j_1+1} - (3^\alpha)^{j_1} > 1,\ (3^\alpha)^{j_1} > j_1 \text{ and } (3^\alpha)^{j_1+1} < (3^\beta)^{j_1}.$$

Put $i_j = j$ for $j = 1, 2, \dots, j_1 - 1$ and choose the natural number i_{j_1} such that

$$(3^\beta)^{j_1} \leq i_{j_1} < (3^\beta)^{j_1} + 1 < (3^\beta)^{j_1+1}.$$

Then

$$\beta \leq \frac{\log_3 i_{j_1}}{j_1} < \beta \cdot \frac{j_1 + 1}{j_1}.$$

Put

$$j_2 = \min\{x \in \{j_1, j_1+1, \dots\} \mid x - j_1 + i_{j_1} < (3^\alpha)^x\}.$$

Note that $j_2 > j_1$. Then $i_{j_1} + (j_2 - 1) - j_1 \geq (3^\alpha)^{j_2-1}$ and $i_{j_1} + j_2 - j_1 < (3^\alpha)^{j_2}$. Put $i_j = i_{j_1} + (j - j_1)$ for $j_1 \leq j \leq j_2 - 1$ and choose the natural number i_{j_2} such that

$$(3^\beta)^{j_2} \leq i_{j_2} < (3^\beta)^{j_2} + 1 < (3^\beta)^{j_2+1}.$$

Then

$$\beta \leq \frac{\log_3 i_{j_2}}{j_2} < \beta \cdot \frac{j_2 + 1}{j_2}.$$

Put

$$j_3 = \min\{x \in \{j_2, j_2+1, \dots\} \mid x - j_2 + i_{j_2} < (3^\alpha)^x\}.$$

Then $j_3 > j_2$, $i_{j_2} + (j_3 - 1) - j_2 \geq (3^\alpha)^{j_3-1}$ and $i_{j_2} + j_3 - j_2 < (3^\alpha)^{j_3}$. Put $i_j = i_{j_2} + (j - j_2)$ for $j_2 \leq j \leq j_3 - 1$ and choose the natural number i_{j_3} such that

$$(3^\beta)^{j_3} \leq i_{j_3} < (3^\beta)^{j_3} + 1 < (3^\beta)^{j_3+1}.$$

If we continue this procedure, we have increasing sequences $\{j_k\}_{k=1}^\infty$ and $\{i_j\}_{j=1}^\infty$ of natural numbers. Now we will show the following claim (*):

(*) If $j \geq j_1$ and $(\log_3 M)/j \geq \alpha$ for some $M > 0$, then $(\log_3 M)/j > (\log_3(M+1))/(j+1)$.

Put $\gamma = (\log_3 M)/j$. Then $M = (3^\gamma)^j$. Since $\gamma \geq \alpha$ and $j \geq j_1$, we see that $(3^\gamma)^{j+1} > (3^\gamma)^j + 1$. Hence

$$\frac{\log_3 M}{j} = \gamma = \frac{\log_3 (3^\gamma)^{j+1}}{j+1} > \frac{\log_3((3^\gamma)^j + 1)}{j+1} = \frac{\log_3(M+1)}{j+1}.$$

Hence the claim (*) is true.
By the construction, we see that

$$\lim_{k\to\infty} \frac{\log_3 i_{j_k}}{j_k} = \beta.$$

Let $j_k \leq j \leq j_{k+1} - 2$. Then

$$\frac{\log_3 i_j}{j} > \frac{\log_3 i_{j+1}}{j+1}.$$

In fact, $(\log_3 i_j)/j \geq \alpha$ and by the claim (*),

$$\frac{\log_3 i_j}{j} - \frac{\log_3 i_{j+1}}{j+1} = \frac{\log_3(i_{j_k} + j - j_k)}{j} - \frac{\log_3(i_{j_k} + (j+1) - j_k)}{j+1} > 0.$$

Also, note that for $j_k \leq j \leq j_{k+1} - 1$,

$$\begin{aligned} &\left| \frac{\log_3(i_{j_k} + j - j_k)}{j} - \frac{\log_3(i_{j_k} + (j+1) - j_k)}{j+1} \right| \\ &\leq \left| \frac{\log_3(i_{j_k} + j - j_k)}{j} - \frac{\log_3(i_{j_k} + j - j_k)}{j+1} \right| \\ &\leq \frac{\log_3(i_{j_k} + j - j_k)}{j(j+1)} = \frac{\log_3 i_j}{j} \cdot \frac{1}{j+1} \leq 2\beta \cdot \frac{1}{j+1}. \end{aligned}$$

Hence we know that for $j_k \leq j \leq j_{k+1} - 2$, $|(\log_3 i_j)/j - (\log_3 i_{j+1})/(j+1)|$ is sufficiently small if k is sufficiently large. Also, since $(\log_3 i_{j_{k+1}-1})/(j_{k+1} - 1) \geq \alpha$ and $(\log_3(i_{j_k} + j_{k+1} - j_k))/(j_{k+1}) < \alpha$, we see that

$$\left| \frac{\log_3 i_{j_{k+1}-1}}{j_{k+1} - 1} - \alpha \right| \leq 2\beta \cdot \frac{1}{j_{k+1}}.$$

Hence

$$\lim_{k\to\infty} \frac{\log_3 i_{j_{k+1}-1}}{j_{k+1}-1} = \alpha.$$

Consider the subsequence $\{\mathscr{U}_{i_j}\}_{j=1}^{\infty}$ of $\{\mathscr{U}_i\}_{i=1}^{\infty}$. By use of the above facts, we see that

$$[\alpha, \beta] = \left\{ \liminf_{k\to\infty} \frac{\log_3 |\mathscr{U}_{i_{n_k}}|}{n_k} \;\middle|\; \{n_k\}_{k=1}^{\infty} \text{ is an increasing subsequence of natural numbers} \right\}.$$

The case $\alpha = 0$ or $\beta = \infty$ can be proved similarly.

Let $d_{\alpha,\beta}$ $(= d_\star)$ be the Alexandroff-Urysohn metric on X induced by the normal star-sequence $\{\mathscr{U}_{i_j}\}_{j=1}^{\infty}$. Then we show that for each $x, y \in X$,

$$d_\star(x, y) = D_\star(x, y).$$

Recall

$$d_\star(x, y) = \inf\{D_\star(x, x_1) + D_\star(x_1, x_2) + \cdots + D_\star(x_n, y) \mid n = 1, 2, \ldots, \text{ and } x_k \in X\}.$$

Note that $\mathscr{U}_{i_1} = \mathscr{U}_1 = \{X\}$. Suppose that $x \neq y$. Put $D_\star(x, y) = 1/3^{j-2}$. Then we have $U \in \mathscr{U}_{i_j}$ such that $x, y \in U$. Also we have V_x, $V_y \in \mathscr{U}_{i_{j+1}}$ such that $V_x \cap V_y = \phi$ and $x \in V_x$, $y \in V_y$. Let $x_k \in X$ $(k = 1, 2, \ldots, n)$. Put $x_0 = x$, $x_{n+1} = y$. Since $\mathscr{U}_{i_{j+1}}$ is a disjoint cover of X, we can choose two points x_k, x_{k+1} such that there exists no element V of $\mathscr{U}_{i_{j+1}}$ that contains x_k and x_{k+1}. Then $D_\star(x_k, x_{k+1}) \geq 1/3^{j-2}$, which implies that $d_\star(x, y) \geq D_\star(x, y)$. Since $d_\star(x, y) \leq D_\star(x, y)$, we see that

$$d_\star(x, y) = D_\star(x, y).$$

Then

$$N\left(\frac{1}{3^{j-2}}, d_\star\right) = |\mathscr{U}_{i_j}| = i_j.$$

Note that for any $1/3^{j-2} \leq \epsilon < 1/3^{j-3}$, $d_\star(x,y) \leq \epsilon$ if and only if $d_\star(x,y) \leq 1/3^{j-2}$, and hence

$$N(\epsilon, d_\star) = N\left(\frac{1}{3^{j-2}}, d_\star\right) = |\mathscr{U}_{i_j}| = i_j.$$

By use of these facts, we see that $d_\star = d_{\alpha,\beta}$ is a desired metric on X.

Remark 3. The metric $d_\star$ in the proof of Theorem 6.4 is an ultrametric on X, i.e., for any $x, y, z \in X$

$$d_\star(x,y) \leq \max\{d_\star(x,z), d_\star(z,y)\}.$$

7. Appendix.

In this appendix, we will give the complete proof of Theorem 2.2 and we will give other characterizations of dimension by use of $\Delta^p(X, \mathscr{U})$ and $\star^p(X, \mathscr{U})$.

7.1. Delta-indices and star-indices.

Recall the following indices:

(1) The index $\tilde{\Delta}^p_k(X)$ is defined as the least natural number m such that for every open covering $\mathscr{U}$ of X with $|\mathscr{U}| = k$, there is an open covering $\mathscr{V}$ of X such that $|\mathscr{V}| \leq m$ and $\{St^p(x, \mathscr{V}) \mid x \in X\} \leq \mathscr{U}$.
(2) The index $\tilde{\star}^p_k(X)$ is defined as the least natural number m such that for every open covering $\mathscr{U}$ of X with $|\mathscr{U}| = k$, there is an open covering $\mathscr{V}$ of X such that $|\mathscr{V}| \leq m$ and $\{St^p(V, \mathscr{V}) \mid V \in \mathscr{V}\} \leq \mathscr{U}$.
(3) The index $\Delta^p_k(X)$ is defined as the least natural number m such that for every open covering $\mathscr{U}$ of X with $|\mathscr{U}| = k$, there is an open covering $\mathscr{V}$ of X such that $|\mathscr{V}| \leq m$ and $\mathscr{V}^{\Delta^p} \leq \mathscr{U}$.
(4) The index $\star^p_k(X)$ is defined as the least natural number m such that for every open covering $\mathscr{U}$ of X with $|\mathscr{U}| = k$, there is an open covering $\mathscr{V}$ of X such that $|\mathscr{V}| \leq m$ and $\mathscr{V}^{\star^p} \leq \mathscr{U}$.

For an open cover $\mathscr{U}$ of X, we define the following indices.

(5) $\tilde{\Delta}^p(X, \mathscr{U}) = \min\{|\mathscr{V}| \mid \mathscr{V}$ is a finite open covering of X such that $\{St^p(x, \mathscr{V}) \mid x \in X\} \leq \mathscr{U}\}$.
(6) $\tilde{\star}^p(X, \mathscr{U}) = \min\{|\mathscr{V}| \mid \mathscr{V}$ is a finite open covering of X such that $\{St^p(V, \mathscr{V}) \mid V \in \mathscr{V}\} \leq \mathscr{U}\}$.
(7) $\Delta^p(X, \mathscr{U}) = \min\{|\mathscr{V}| \mid \mathscr{V}$ is a finite open covering of X such that $\mathscr{V}^{\Delta^p} \leq \mathscr{U}\}$.
(8) $\star^p(X, \mathscr{U}) = \min\{|\mathscr{V}| \mid \mathscr{V}$ is a finite open covering of X such that $\mathscr{V}^{\star^p} \leq \mathscr{U}\}$.

For natural numbers k, m and p with $k \geq m$, we define the following indices;

$$\tilde{\Delta}(k;m;p) = \Sigma_{m \geq j_1 \geq j_2 \geq \cdots \geq j_p \geq 1} \binom{k}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{p-1}}{j_p}$$

and

$$\tilde{\star}(k;m;p) = \Sigma_{m \geq j_1 \geq j_2 \geq \cdots \geq j_p \geq 1} \binom{k}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{p-1}}{j_p} j_p.$$

PROPOSITION 7.1. *Let k and p be natural numbers. Then*

(1) $\tilde{\Delta}(k;k;p) = (p+1)^k - p^k$,
(2) $\tilde{\star}(k;k;p) = k(p+1)^{k-1}$,
(3) $\Sigma_{k \geq j_1 \geq j_2 \geq \cdots \geq j_p \geq 1} 1 = |{}_k H_p| = \binom{k+p-1}{p}$.

PROOF. By induction on p, we shall prove (1). If $p = 1$, then (1) is true. We assume that (1) is true for $p-1$ $(p \geq 2)$. Then

$$\begin{aligned}
\tilde{\Delta}(k;k;p) &= \Sigma_{k \geq j_1 \geq j_2 \geq \cdots \geq j_p \geq 1} \binom{k}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{p-1}}{j_p} \\
&= \Sigma_{j_1=1}^{k} \left[\binom{k}{j_1} \Sigma_{j_1 \geq j_2 \geq \cdots \geq j_p \geq 1} \binom{j_1}{j_2}\binom{j_2}{j_3}\cdots\binom{j_{p-1}}{j_p} \right] \\
&= \Sigma_{j_1=1}^{k} \left[\binom{k}{j_1} \tilde{\Delta}(j_1;j_1;p-1) \right] \\
&= \Sigma_{j_1=1}^{k} \left[\binom{k}{j_1} (p^{j_1} - (p-1)^{j_1}) \right] \\
&= \Sigma_{j_1=1}^{k} \binom{k}{j_1} p^{j_1} - \Sigma_{j_1=1}^{k} \binom{k}{j_1} (p-1)^{j_1} \\
&= (p+1)^k - p^k.
\end{aligned}$$

Next, we shall prove (2) by induction on p. The fact of the case $p = 1$ has been given in [**7**]. For completeness, we give the proof.

$$\Sigma_{k \geq j \geq 1} \binom{k}{j} j = k\left[\Sigma_{k \geq j \geq 1} \binom{k-1}{j-1}\right] = k\left[\Sigma_{i=0}^{k-1} \binom{k-1}{i}\right] = k \cdot 2^{k-1}.$$

We assume that (2) is true for $p-1$ $(p \geq 2)$.

$$\tilde{\star}(k;k;p) = \Sigma_{k \geq j_1 \geq j_2 \geq \cdots \geq j_p \geq 1} \binom{k}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{p-1}}{j_p} j_p$$

$$= \Sigma_{j_1=1}^{k} \binom{k}{j_1}\left[\Sigma_{j_1 \geq j_2 \geq \cdots \geq j_p \geq 1} \binom{j_1}{j_2}\binom{j_2}{j_3}\cdots\binom{j_{p-1}}{j_p} j_p\right]$$

$$= \Sigma_{j_1=1}^{k}\left[\binom{k}{j_1}\tilde{\star}(j_1;j_1;p-1)\right] = \Sigma_{j_1=1}^{k}\binom{k}{j_1}(j_1 p^{j_1-1})$$

$$= k\left[\Sigma_{j_1=1}^{k}\binom{k-1}{j_1-1}p^{j_1-1}\right] = k\left[\Sigma_{i=0}^{k-1}\binom{k-1}{i}p^{i}\right] = k(p+1)^{k-1}.$$

7.2. The indices $\tilde{\Delta}_k^p(X)$ and $\tilde{\star}_k^p(X)$.

Let X be a topological space. By a *swelling* of the family $\{A_s\}_{s\in S}$ of subsets of X, we mean any family $\{B_s\}_{s\in S}$ of subsets of X such that $A_s \subset B_s$ $(s \in S)$ and for every finite set of indices $s_1, s_2, \dots, s_m \in S$,

$$\bigcap_{i=1}^{m} A_{s_i} \neq \phi \ \text{ if and only if } \ \bigcap_{i=1}^{m} B_{s_i} \neq \phi.$$

Theorem 7.2. *Let X be an infinite normal space with* $\dim X = n$ *and let k and p be natural numbers. Then*

$$\tilde{\Delta}_k^p(X) = \begin{cases} \tilde{\Delta}(k;k;p) = (p+1)^k - p^k, & \text{if } k \leq n+1 \\ \tilde{\Delta}(k;n+1;p), & \text{if } k \geq n+1. \end{cases}$$

To prove Theorem 7.2, we need the following lemmata.

Lemma 7.3. *Every finite family $\{F_i \mid i = 1,2,\dots,k\}$ of closed subsets of a normal space X has an open swelling $\{U_i \mid i = 1,2,\dots,k\}$.*

Proof. See Engelking [**5**, Theorem (3.1.1)].

Lemma 7.4. *Suppose that X is an infinite normal space. Let $\mathscr{U} = \{U_j \mid j \in \alpha\}$ be an open covering of X such that $|\alpha| < \infty$ and every open shrinking $\mathscr{V}$ of $\mathscr{U}$ has a non-empty intersection. If $\mathscr{P} = \{P_j \mid j \in \alpha\}$ is a closed shrinking of $\mathscr{U}$ and $\mathscr{F} = \{F_j \mid j \in \alpha\}$ is a closed covering of X such that $F_j \subset Int(P_j)$ $(j \in \alpha)$, then for any nonempty subset β of α,*

$$\phi \neq \bigcap\{(X - F_j) \mid j \in \alpha - \beta\} \cap \bigcap\{F_j \mid j \in \beta\} \cap \bigcap\{P_j \mid j \in \alpha\}$$
$$\Big(= \bigcap\{(P_j - F_j) \mid j \in \alpha - \beta\} \cap \bigcap\{F_j \mid j \in \beta\}\Big)$$

Proof. For each $i \in \alpha$, choose an open set Q_i such that

$$F_i \subset Q_i \subset \overline{Q}_i \subset Int(P_i).$$

We shall show that the family

$$\mathscr{T} = \{P_j \mid j \in \alpha - \beta\} \cup \left\{ \left[F_j \cap \bigcap_{i \in \alpha - \beta} (X - Q_i) \right] \middle| j \in \beta \right\}$$

is a closed covering of X. Let $x \in X - \bigcup\{P_j \mid j \in \alpha - \beta\}$. Since $\mathscr{F}$ is a covering of X, there is $j_0 \in \alpha$ such that $x \in F_{j_0} \in \mathscr{F}$. If $j_0 \notin \beta$, we see that

$$x \notin X - F_{j_0} \supset X - Q_{j_0} \supset X - \bigcup_{j \in \alpha - \beta} P_j.$$

This is a contradiction. Hence $j_0 \in \beta$. Also we see that $x \in \bigcap_{i \in \alpha - \beta}(X - Q_i)$ and hence $x \in F_{j_0} \cap \bigcap_{i \in \alpha - \beta}(X - Q_i)$. Then $\mathscr{T}$ is a closed covering of X and a shrinking of $\mathscr{U}$. If $\mathscr{T}$ has an empty intersection, by Lemma 7.3 we have an open swelling $\mathscr{V}$ such that $\mathscr{V}$ is a shrinking of $\mathscr{U}$ and $\mathscr{V}$ has an empty intersection. Hence we see that $\mathscr{T}$ has a non-empty intersection:

$$\bigcap\{(X - Q_j) \mid j \in \alpha - \beta\} \cap \bigcap\{F_j \mid j \in \beta\} \cap \bigcap\{P_j \mid j \in \alpha\} \neq \phi.$$

Consequently we see that

$$\bigcap\{(X - F_j) \mid j \in \alpha - \beta\} \cap \bigcap\{F_j \mid j \in \beta\} \cap \bigcap\{P_j \mid j \in \alpha\}$$
$$\supset \bigcap\{(X - Q_j) \mid j \in \alpha - \beta\} \cap \bigcap\{F_j \mid j \in \beta\} \cap \bigcap\{P_j \mid j \in \alpha\} \neq \phi.$$

Proof of Theorem 7.2. Let k, p be any natural numbers. We give the proof only for the case of $k \geq n+1$. The case of $k < n+1$ can be proved similarly to the case of $k \geq n+1$. Suppose that $\mathscr{U} = \{U_1, U_2, \ldots, U_k\}$ is an open covering of X with $|\mathscr{U}| = k$. Since $\dim X = n$, for each $i = 1, 2, \ldots, p$, we can choose a finite open shrinking $\mathscr{V}^i = \{V_1^i, V_2^i, \ldots, V_k^i\}$ of $\mathscr{U}$ and a finite closed shrinking $\mathscr{F}^i = \{F_1^i, F_2^i, \ldots, F_k^i\}$ of $\mathscr{U}$ such that $\mathrm{ord}(\mathscr{V}^1) \leq n+1$ and

$$U_j \supset V_j^1 \supset F_j^1 \supset V_j^2 \supset F_j^2 \supset \cdots \supset V_j^p \supset F_j^p \quad (j = 1, 2, \ldots, k).$$

For a finite sequence $\{1, 2, \ldots, k\} \supset A_1 \supset A_2 \supset \cdots \supset A_p \neq \phi$ of nonempty subsets of $\{1, 2, \ldots, k\}$, we define the following open set:

$$\begin{aligned} W(A_1, A_2, \ldots, A_p) = & \bigcap \{V_j^1 \mid j \in A_1\} \cap \bigcap \{X - F_j^1 \mid j \notin A_1\} \\ & \cap \bigcap \{V_j^2 \mid j \in A_2\} \cap \bigcap \{X - F_j^2 \mid j \notin A_2\} \\ & \cap \bigcap \{V_j^3 \mid j \in A_3\} \cap \bigcap \{X - F_j^3 \mid j \notin A_3\} \\ & \cdots\cdots \\ & \cap \bigcap \{V_j^p \mid j \in A_p\} \cap \bigcap \{X - F_j^p \mid j \notin A_p\}. \end{aligned}$$

Put

$$\begin{aligned} \mathscr{W} = \{W(A_1, A_2, \ldots, A_p) \mid \{1, 2, \ldots, k\} \supset A_1 \supset A_2 \supset \cdots \supset A_p \neq \phi \\ \text{and } W(A_1, A_2, \ldots, A_p) \neq \phi\}. \end{aligned}$$

Since $\operatorname{ord}(\mathscr{V}^1) \leq n+1$, we see that $W(A_1, A_2, \ldots, A_p) = \phi$ if $|A_1| > n+1$. Then we see

$$|\mathscr{W}| \leq \Sigma_{n+1 \geq j_1 \geq j_2 \geq \cdots \geq j_p \geq 1} \binom{k}{j_1} \binom{j_1}{j_2} \cdots \binom{j_{p-1}}{j_p} = \tilde{\Delta}(k; n+1; p).$$

We shall show that $\mathscr{W}$ is a finite open covering of X. Let $x \in X$. Put

$$A_i = \{j \in \{1, 2, \ldots, k\} \mid x \in F_j^i\} \quad (i = 1, 2, \ldots, p).$$

Then we see that

$$\{1, 2, \ldots, k\} \supset A_1 \supset A_2 \supset \cdots \supset A_p \neq \phi$$

and $x \in W(A_1, A_2, \ldots, A_p)$. This implies that $\mathscr{W}$ is a finite open covering of X.

Since $\mathscr{F}^p$ is a covering of X, for $x \in X$ we can choose $j_0 \in \{1, 2, \ldots, k\}$ with $x \in F_{j_0}^p$. Then we shall show that

$$St^p(x, \mathscr{W}) \subset V_{j_0}^1 \subset U_{j_0}.$$

Let $W_1, W_2, \ldots, W_p \in \mathscr{W}$ such that

$$x \in W_1, W_i \cap W_{i+1} \neq \phi \quad (i = 1, 2, \ldots, p-1).$$

Put $W_i = W(A_1^i, A_2^i, \ldots, A_p^i)$ $(i = 1, 2, \ldots, p)$. Since

$$x \in W_1 = W\big(A_1^1, A_2^1, \ldots, A_p^1\big) \subset \bigcap\big\{V_j^p \mid j \in A_p^1\big\} \cap \bigcap\big\{X - F_j^p \mid j \notin A_p^1\big\},$$

we see that $j_0 \in A_p^1$. Since

$$\phi \neq W_1 \cap W_2 \subset \bigcap\big\{V_j^p \mid j \in A_p^1\big\} \cap \bigcap\big\{X - F_j^{p-1} \mid j \notin A_{p-1}^2\big\},$$

we see that $A_p^1 \subset A_{p-1}^2$. If not, there is some $j' \in A_p^1 - A_{p-1}^2$. Then

$$\phi = V_{j'}^p \cap \big(X - F_{j'}^{p-1}\big) \supset \bigcap\big\{V_j^p \mid j \in A_p^1\big\} \cap \bigcap\big\{X - F_j^{p-1} \mid j \notin A_{p-1}^2\big\} \supset W_1 \cap W_2.$$

This is a contradiction. Also, by use of $W_2 \cap W_3 \neq \phi$, we see that $A_{p-1}^2 \subset A_{p-2}^3$. By induction on $i = 1, 2, \ldots, p$, we see that

$$j_0 \in A_p^1 \subset A_{p-1}^2 \subset A_{p-2}^3 \subset \cdots \subset A_1^p.$$

This implies that

$$W_p = W\big(A_1^p, A_2^p, \ldots, A_p^p\big) \subset \bigcap\big\{V_j^1 \mid j \in A_1^p\big\} \subset V_{j_0}^1.$$

Hence

$$St^p(x, \mathscr{W}) \subset V_{j_0}^1 \subset U_{j_0}.$$

This implies that $\{St^p(x, \mathscr{W}) \mid x \in X\} \leq \mathscr{U}$. Hence we conclude that

$$\tilde{\Delta}_k^p(X) \leq \tilde{\Delta}(k; n+1; p).$$

Next, we shall show the converse of the inequality. Let $\{C(\alpha) \mid \alpha \in C_{n+1}^k\}$ be a family of pairwise disjoint closed subsets of X such that $\dim C(\alpha) = n$ for each $\alpha \in C_{n+1}^k$ (see [**4**, Lemma 2]). For each $\alpha \in C_{n+1}^k$, choose an open covering $\mathscr{G}(\alpha) = \{G_i^\alpha \mid i \in \alpha\}$ of $C(\alpha)$ such that every open shrinking of $\mathscr{G}(\alpha)$ has a non-empty intersection. We choose an open shrinking $\mathscr{U}(\alpha) = \{U_i^\alpha \mid i \in \alpha\}$ of

$\mathscr{G}(\alpha)$ such that $\overline{\mathscr{U}(\alpha)} = \{\overline{U_i^\alpha} \mid i \in \alpha\}$ is also a shrinking of $\mathscr{G}(\alpha)$. For each $i \in \{1, 2, \ldots, k\}$, let

$$U_i = \Big[X - \bigcup\{C(\alpha) \mid \alpha \in C_{n+1}^k\}\Big] \cup \bigcup\{U_i^\alpha \mid i \in \alpha\}.$$

Then $\mathscr{U} = \{U_1, U_2, \ldots, U_k\}$ is an open covering of X. Let $\mathscr{V}'$ be a finite open covering of X such that $\{St^p(x, \mathscr{V}') \mid x \in X\} \leq \mathscr{U}$. We shall show that

$$|\mathscr{V}'| \geq \tilde{\Delta}(k; n+1; p).$$

Since X is a normal space, there is a closed shrinking $\mathscr{K}$ of $\mathscr{V}'$. By Lemma 7.3, there is an open swelling $\mathscr{V}$ of $\mathscr{K}$ such that $\mathscr{V} \leq \mathscr{V}'$. Moreover, we may assume that $\overline{\mathscr{V}} = \{\overline{V} \mid V \in \mathscr{V}\}$ is also a swelling of $\mathscr{K}$. Then $\mathscr{V}$ satisfies the following property; if $V, W \in \mathscr{V}$, then

$$(\sharp) \quad V \cap W = \phi \ \text{ if and only if } \ \overline{V} \cap \overline{W} = \phi.$$

Note that $|\mathscr{V}'| = |\mathscr{V}|$ and $\{St^p(x, \mathscr{V}) \mid x \in X\} \leq \mathscr{U}$.

We shall show the above fact in the following way: For each sequence $\{1, 2, \ldots, k\} \supset \beta_1 \supset \beta_2 \supset \cdots \supset \beta_p$ of subsets of $\{1, 2, \ldots, k\}$ with $1 \leq |\beta_p| \leq |\beta_1| \leq n+1$, we choose an element $V(\beta_1, \beta_2, \ldots, \beta_p) \in \mathscr{V}$ such that

$$\begin{aligned}
\beta_1 &= \{j \in \{1, 2, \ldots, k\} \mid V(\beta_1, \beta_2, \ldots, \beta_p) \subset U_j\},\\
\beta_2 &= \{j \in \{1, 2, \ldots, k\} \mid St(V(\beta_1, \beta_2, \ldots, \beta_p), \mathscr{V}) \subset U_j\},\\
\beta_3 &= \{j \in \{1, 2, \ldots, k\} \mid St^2(V(\beta_1, \beta_2, \ldots, \beta_p), \mathscr{V}) \subset U_j\},\\
&\cdots\cdots\\
\beta_p &= \{j \in \{1, 2, \ldots, k\} \mid St^{p-1}(V(\beta_1, \beta_2, \ldots, \beta_p), \mathscr{V}) \subset U_j\}.
\end{aligned}$$

In this way we can assign in a one-to-one manner an element $V(\beta_1, \beta_2, \ldots, \beta_p) \in \mathscr{V}$ to each sequence $\{1, 2, \ldots, k\} \supset \beta_1 \supset \beta_2 \supset \cdots \supset \beta_p$ with $1 \leq |\beta_p| \leq |\beta_1| \leq n+1$ and hence we see that

$$\tilde{\Delta}_k^p(X) \geq \Sigma_{n+1 \geq j_1 \geq j_2 \geq \cdots \geq j_p \geq 1} \binom{k}{j_1}\binom{j_1}{j_2} \cdots \binom{j_{p-1}}{j_p} = \tilde{\Delta}(k; n+1; p).$$

To prove this, let $\{1, 2, \ldots, k\} \supset \beta_1 \supset \beta_2 \supset \cdots \supset \beta_p$ be a sequence of subsets of $\{1, 2, \ldots, k\}$ such that $1 \leq |\beta_p| \leq |\beta_1| \leq n+1$. Choose $\gamma \subset \{1, 2, \ldots, k\}$ such that $\beta_1 \cap \gamma = \phi$ and $\beta_1 \cup \gamma \in C_{n+1}^k$. Let $\alpha = \beta_1 \cup \gamma$.

For $i \in \alpha$, we choose a closed set $H_i \subset G_i^\alpha$ such that $\overline{U_i^\alpha} \subset Int_{C(\alpha)}(H_i)$ $(i \in \alpha)$. For each $i \in \alpha$, consider the following subsets of $C(\alpha)$:

$$\begin{aligned}
P_i^0 &= H_i,\\
F_i^0 &= \overline{U_i^\alpha},\\
P_i^1 &= C(\alpha) - St^1((C(\alpha) - U_i), \mathscr{V}),\\
F_i^1 &= \overline{C(\alpha) \cap \bigcup\{V \in \mathscr{V} \mid V \cap St^1((C(\alpha) - U_i), \mathscr{V}) = \phi\}},\\
P_i^2 &= C(\alpha) - St^2((C(\alpha) - U_i), \mathscr{V}),\\
F_i^2 &= \overline{C(\alpha) \cap \bigcup\{V \in \mathscr{V} \mid V \cap St^2((C(\alpha) - U_i), \mathscr{V}) = \phi\}},\\
&\cdots\cdots\\
P_i^{p-1} &= C(\alpha) - St^{p-1}((C(\alpha) - U_i), \mathscr{V}),\\
F_i^{p-1} &= \overline{C(\alpha) \cap \bigcup\{V \in \mathscr{V} \mid V \cap St^{p-1}((C(\alpha) - U_i), \mathscr{V}) = \phi\}},\\
P_i^p &= C(\alpha) - St^p((C(\alpha) - U_i), \mathscr{V}).
\end{aligned}$$

By $(\sharp)$, we see that

$$P_i^p \subset Int_{C(\alpha)} F_i^{p-1} \subset F_i^{p-1} \subset Int_{C(\alpha)} P_i^{p-1} \subset \cdots \subset F_i^0 \subset Int_{C(\alpha)} P_i^0$$

Since $\{St^p(x, \mathscr{V}) \mid x \in X\} \leq \mathscr{U}$, we see that the family $\{P_i^p \mid i \in \alpha\}$ is a closed covering of $C(\alpha)$ and also it is a shrinking of $\mathscr{U}(\alpha)$. In general, we can not conclude that the family $\{Int_{C(\alpha)} P_i^p \mid i \in \alpha\}$ is a covering of $C(\alpha)$. Put

$$\begin{aligned}
&W(\alpha, \beta_1, \beta_2, \ldots, \beta_p)\\
&\quad = \bigcap\{P_j^0 \mid j \in \alpha\} \cap \Big[\bigcap\{P_j^1 \mid j \in \beta_1\} - \bigcup\{F_j^0 \mid j \in \alpha - \beta_1\}\Big]\\
&\quad \cap \Big[\bigcap\{P_j^2 \mid j \in \beta_2\} - \bigcup\{F_j^1 \mid j \in \alpha - \beta_2\}\Big]\\
&\quad \cap \Big[\bigcap\{P_j^3 \mid j \in \beta_3\} - \bigcup\{F_j^2 \mid j \in \alpha - \beta_3\}\Big]\\
&\qquad\qquad \cdots\cdots\\
&\quad \cap \Big[\bigcap\{P_j^{p-1} \mid j \in \beta_{p-1}\} - \bigcup\{F_j^p \mid j \in \alpha - \beta_p\}\Big]\\
&\quad \cap \bigcap\{P_j^p \mid j \in \beta_p\}.
\end{aligned}$$

Note that

$$
\begin{aligned}
&W(\alpha, \beta_1, \beta_2, \dots, \beta_p) \\
&\quad = \bigcap \{P_j^0 \mid j \in \alpha\} \cap \Big[\bigcap \{P_j^1 \mid j \in \beta_1\}) - \bigcup \{F_j^0 \mid j \in \alpha - \beta_1\}\Big] \\
&\qquad \cap \Big[\bigcap \{P_j^2 \mid j \in \beta_2\} - \bigcup \{F_j^1 \mid j \in \beta_1 - \beta_2\}\Big] \\
&\qquad \cap \Big[\bigcap \{P_j^3 \mid j \in \beta_3\} - \bigcup \{F_j^2 \mid j \in \beta_2 - \beta_3\}\Big] \\
&\qquad\qquad\qquad\qquad \cdots\cdots \\
&\qquad \cap \Big[\bigcap \{P_j^{p-1} \mid j \in \beta_{p-1}\} - \bigcup \{F_j^p \mid j \in \beta_{p-1} - \beta_p\}\Big] \\
&\qquad \cap \bigcap \{P_j^p \mid j \in \beta_p\} \\
&\quad = \bigcap \{(P_j^0 - F_j^0) \mid j \in \alpha - \beta_1\} \\
&\qquad \cap \bigcap \{(P_j^1 - F_j^1) \mid j \in \beta_1 - \beta_2\} \\
&\qquad \cap \bigcap \{(P_j^2 - F_j^2) \mid j \in \beta_2 - \beta_3\} \\
&\qquad\qquad\qquad \cdots\cdots \\
&\qquad \cap \bigcap \{(P_j^{p-1} - F_j^{p-1}) \mid j \in \beta_{p-1} - \beta_p\} \\
&\qquad \cap \bigcap \{P_j^p \mid j \in \beta_p\}.
\end{aligned}
$$

To apply Lemma 7.4 to our proof, we put $\beta = \beta_p$ and consider two families of closed subsets of $C(\alpha)$:

$$
\begin{aligned}
\mathscr{P} &= \{P_j^0 \mid j \in \alpha - \beta_1\} \cup \{P_j^1 \mid j \in \beta_1 - \beta_2\} \\
&\quad \cup \dots \cup \{P_j^{p-2} \mid j \in \beta_{p-2} - \beta_{p-1}\} \cup \{P_j^{p-1} \mid j \in \beta_{p-1}\} = \{P_j \mid j \in \alpha\},
\end{aligned}
$$

and

$$
\begin{aligned}
\mathscr{F} &= \{F_j^0 \mid j \in \alpha - \beta_1\} \cup \{F_j^1 \mid j \in \beta_1 - \beta_2\} \\
&\quad \cup \dots \cup \{F_j^{p-2} \mid j \in \beta_{p-2} - \beta_{p-1}\} \cup \{F_j^{p-1} \mid j \in \beta_{p-1} - \beta_p\} \\
&\quad \cup \{F_j^p \mid j \in \beta_p\} = \{F_j \mid j \in \alpha\}.
\end{aligned}
$$

By Lemma 7.4, we see that

$$W(\alpha, \beta_1, \beta_2, \dots, \beta_p) \neq \phi.$$

Hence we can choose $x \in W(\alpha, \beta_1, \beta_2, \dots, \beta_p)$ and $V = V(\beta_1, \beta_2, \dots, \beta_p) \in \mathscr{V}$ with $x \in V$. We shall show that for each $i = 1, \dots, p$,

$$\beta_i = \{j \in \{1, 2, \dots, k\} \mid St^{i-1}(V(\beta_1, \beta_2, \dots, \beta_p), \mathscr{V}) \subset U_j\}.$$

Let $j \in \beta_i$. We choose $i' \geq i$ such that $j \in \beta_{i'} - \beta_{i'+1}$, where we put $\beta_{p+1} = \phi$. Then

$$x \in W(\alpha, \beta_1, \beta_2, \dots, \beta_p) \subset P_j^{i'} - F_j^{i'} \subset P_j^{i'} = C(\alpha) - St^{i'}((C(\alpha) - U_j), \mathscr{V}).$$

Since $x \notin St^{i'}((C(\alpha) - U_j), \mathscr{V})$, we see that $St^{i'}(x, \mathscr{V}) \subset U_j$. Since $i' \geq i$ and $x \in V(\beta_1, \beta_2, \dots, \beta_p)$,

$$St^{i-1}(V(\beta_1, \beta_2, \dots, \beta_p), \mathscr{V}) \subset St^{i'-1}(V(\beta_1, \beta_2, \dots, \beta_p), \mathscr{V}) \subset St^{i'}(x, \mathscr{V}) \subset U_j.$$

This implies that

$$\beta_i \subset \big\{j \in \{1, 2, \dots, k\} \mid St^{i-1}(V(\beta_1, \beta_2, \dots, \beta_p), \mathscr{V}) \subset U_j\big\}.$$

Next, suppose that $j \notin \beta_i$. If $j \notin \alpha$, we see that $C(\alpha) \cap U_j = \phi$ and $x \in V(\beta_1, \beta_2, \dots, \beta_p) \cap C(\alpha)$. Then $St^{i-1}(V(\beta_1, \beta_2, \dots, \beta_p), \mathscr{V})$ is not contained in U_j. If $j \in \alpha$, we choose $i' < i$ such that $j \in \beta_{i'} - \beta_{i'+1}$ and we put $\beta_0 = \alpha$. Then

$$x \in W(\alpha, \beta_1, \beta_2, \dots, \beta_p) \subset P_j^{i'} - F_j^{i'}.$$

Hence

$$x \notin F_j^{i'} = \overline{C(\alpha) \cap \bigcup \{V \in \mathscr{V} \mid V \cap St^{i'}((C(\alpha) - U_j), \mathscr{V}) = \phi\}}.$$

This implies that $V(\beta_1, \beta_2, \dots, \beta_p) \cap St^{i'}((C(\alpha) - U_j), \mathscr{V}) \neq \phi$. Hence

$$St^{i'}(V(\beta_1, \beta_2, \dots, \beta_p), \mathscr{V}) \cap (C(\alpha) - U_j) \neq \phi.$$

Since

$$St^{i-1}(V(\beta_1, \beta_2, \dots, \beta_p), \mathscr{V}) \supset St^{i'}(V(\beta_1, \beta_2, \dots, \beta_p), \mathscr{V}),$$

we see that $St^{i-1}(V(\beta_1,\beta_2,\dots,\beta_p),\mathscr{V})$ is not contained in U_j. This implies that

$$\beta_i \supset \{j \in \{1,2,\dots,k\} \mid St^{i-1}(V(\beta_1,\beta_2,\dots,\beta_p),\mathscr{V}) \subset U_j\}.$$

Hence

$$\beta_i = \{j \in \{1,2,\dots,k\} \mid St^{i-1}(V(\beta_1,\beta_2,\dots,\beta_p),\mathscr{V}) \subset U_j\}.$$

Finally, we can conclude that

$$\tilde{\Delta}^p_k(X) = \tilde{\Delta}(k;n+1;p).$$

This completes the proof.

Theorem 7.5. *Let X be an infinite normal space with* $\dim X = n$ *and let k and p be natural numbers. Then*

$$\tilde{\star}^p_k(X) = \begin{cases} \tilde{\star}(k;k;p) = k(p+1)^{k-1}, & \text{if } k \le n+1 \\ \tilde{\star}(k;n+1;p), & \text{if } k \ge n+1. \end{cases}$$

Proof. The proof is similar to that of Theorem 7.2. For completeness, we give the proof. Let k, p be any natural numbers. We give the proof only for the case of $k \ge n+1$. The case of $k < n+1$ can be proved similarly to the case of $k \ge n+1$. Suppose that $\mathscr{U} = \{U_1,U_2,\dots,U_k\}$ is an open covering of X with $|\mathscr{U}| = k$. Since $\dim X = n$, for each $i = 1,2,\dots,p$, we can choose a finite open shrinking $\mathscr{V}^i = \{V^i_1,V^i_2,\dots,V^i_k\}$ of $\mathscr{U}$ and a finite closed shrinking $\mathscr{F}^i = \{F^i_1,F^i_2,\dots,F^i_k\}$ of $\mathscr{U}$ such that $\mathrm{ord}(\mathscr{V}^1) \le n+1$ and

$$U_j \supset V^1_j \supset F^1_j \supset V^2_j \supset F^2_j \supset \cdots \supset V^p_j \supset F^p_j \ (j = 1,2,\dots,k).$$

We may assume that $\{Int(F^p_i) \mid i = 1,2,\dots,k\}$ is an open covering of X. Let $\mathscr{G} = \{G_1,G_2,\dots,G_k\}$ be an open covering of X with $\overline{G}_j \subset F^p_j$ $(j = 1,2,\dots,k)$. For a finite sequence $\{1,2,\dots,k\} \supset A_1 \supset A_2 \supset \cdots \supset A_p \neq \phi$ of nonempty subsets of $\{1,2,\dots,k\}$ and $t \in A_p$, we define the following open set:

$$\begin{aligned} W(A_1,A_2,\dots,A_p : t) = & \bigcap\{V^1_j \mid j \in A_1\} \cap \bigcap\{X - F^1_j \mid j \notin A_1\} \\ & \cap \bigcap\{V^2_j \mid j \in A_2\} \cap \bigcap\{X - F^2_j \mid j \notin A_2\} \end{aligned}$$

$$\cap \bigcap \{V_j^3 \mid j \in A_2\} \cap \bigcap \{X - F_j^3 \mid j \notin A_3\}$$
$$\cdots\cdots$$
$$\cap \bigcap \{V_j^p \mid j \in A_p\} \cap \bigcap \{X - F_j^p \mid j \notin A_p\}$$
$$\cap\, G_t.$$

Put

$$\mathscr{W} = \{W(A_1, A_2, \ldots, A_p : t) \mid \{1, 2, \ldots, k\} \supset A_1 \supset A_2 \supset \cdots \supset A_p \ni t \text{ and } W(A_1, A_2, \ldots, A_p : t) \neq \phi\}.$$

Since $\mathrm{ord}(\mathscr{V}^1) \leq n+1$, we see that $W(A_1, A_2, \ldots, A_p : t) = \phi$ if $|A_1| > n+1$. Then we see that

$$|\mathscr{W}| \leq \Sigma_{n+1 \geq j_1 \geq j_2 \geq \cdots \geq j_p \geq 1} \binom{k}{j_1} \binom{j_1}{j_2} \cdots \binom{j_{p-1}}{j_p} j_p = \tilde{\star}(k; n+1; p).$$

Let $x \in X$. Put

$$A_i = \{j \in \{1, 2, \ldots, k\} \mid x \in F_j^i\} \ (i = 1, 2, \ldots, p).$$

Also, we choose $t \in A_p$ with $x \in G_t$. Then we see that

$$\{1, 2, \ldots, k\} \supset A_1 \supset A_2 \supset \cdots \supset A_p \ni t$$

and $x \in W(A_1, A_2, \ldots, A_p : t)$. This implies that $\mathscr{W}$ is a finite open covering of X. Also, by the similar argument to the proof of Theorem 7.2, we see that

$$St^p(W(A_1, A_2, \ldots, A_p : t), \mathscr{W}) \subset U_t.$$

Hence

$$\tilde{\star}_k^p(X) \leq \tilde{\star}(k; n+1; p).$$

Next, we shall show the converse of the inequality. Let $\{C(\alpha) \mid \alpha \in C_{n+1}^k\}$ be a family of pairwise disjoint closed subsets of X such that $\dim C(\alpha) = n$ for each $\alpha \in C_{n+1}^k$. For each $\alpha \in C_{n+1}^k$, we also choose an open covering $\mathscr{G}(\alpha) = \{G_i^\alpha \mid i \in \alpha\}$ of $C(\alpha)$ such that every open shrinking of $\mathscr{G}(\alpha)$ has a non-empty intersection. We choose an open shrinking $\mathscr{U}(\alpha) = \{U_i^\alpha \mid i \in \alpha\}$ of $\mathscr{G}(\alpha)$ such that $\overline{\mathscr{U}(\alpha)} = \{\overline{U_i^\alpha} \mid i \in \alpha\}$ is also a shrinking of $\mathscr{G}(\alpha)$. For each $i \in \{1, 2, \dots, k\}$, let

$$U_i = \Big[X - \bigcup\{C(\alpha) \mid \alpha \in C_{n+1}^k\}\Big] \cup \bigcup\{U_i^\alpha \mid i \in \alpha\}.$$

Then $\mathscr{U} = \{U_1, U_2, \dots, U_k\}$ is an open covering of X. Let $\mathscr{V}$ be a finite open covering of X such that $\{St^p(V, \mathscr{V}) \mid V \in \mathscr{V}\} \leq \mathscr{U}$. As in the proof of Theorem 7.2, we may assume that if $V, W \in \mathscr{V}$, then

$$(\sharp) \quad V \cap W = \phi \ \text{ if and only if } \ \overline{V} \cap \overline{W} = \phi.$$

We shall show that

$$|\mathscr{V}| \geq \tilde{\star}(k; n+1; p).$$

We show this fact in the following way: For each sequence $\beta_1 \supset \beta_2 \supset \cdots \supset \beta_p$ of subsets of $\{1, 2, \dots, k\}$ with $1 \leq |\beta_p| \leq |\beta_1| \leq n+1$ and $t \in \beta_p$, we choose an element $V(\beta_1, \beta_2, \dots, \beta_p : t) \in \mathscr{V}$ such that if $t,\ t' \in \beta_p$ and $t \neq t'$, then

$$V(\beta_1, \beta_2, \dots, \beta_p : t) \neq V(\beta_1, \beta_2, \dots, \beta_p : t'),$$

and

$$\begin{aligned}
&\beta_1 = \{j \in \{1, 2, \dots, k\} \mid V(\beta_1, \beta_2, \dots, \beta_p : t) \subset U_j\},\\
&\beta_2 = \{j \in \{1, 2, \dots, k\} \mid St(V(\beta_1, \beta_2, \dots, \beta_p : t), \mathscr{V}) \subset U_j\},\\
&\beta_3 = \{j \in \{1, 2, \dots, k\} \mid St^2(V(\beta_1, \beta_2, \dots, \beta_p : t), \mathscr{V}) \subset U_j\},\\
&\qquad\qquad \cdots\cdots\\
&\beta_p = \{j \in \{1, 2, \dots, k\} \mid St^{p-1}(V(\beta_1, \beta_2, \dots, \beta_p : t), \mathscr{V}) \subset U_j\},\\
&\qquad\qquad St^p(V(\beta_1, \beta_2, \dots, \beta_p : t), \mathscr{V}) \subset U_t.
\end{aligned}$$

In this way we can assign in a one-to-one manner an element $V(\beta_1, \beta_2, \dots, \beta_p : t) \in \mathscr{V}$ to each sequence $\{1, 2, \dots, k\} \supset \beta_1 \supset \beta_2 \supset \cdots \supset \beta_p$ with $1 \leq |\beta_p| \leq |\beta_1| \leq n+1$ and $t \in \beta_p$ and hence we see that

$$\tilde{\star}_k^p(X) \geq \Sigma_{n+1 \geq j_1 \geq j_2 \geq \cdots \geq j_p \geq 1} \binom{k}{j_1}\binom{j_1}{j_2} \cdots \binom{j_{p-1}}{j_p} j_p = \tilde{\star}(k; n+1; p).$$

To prove this fact, let $\{1,2,\dots,k\} \supset \beta_1 \supset \beta_2 \supset \cdots \supset \beta_p$ be a sequence of subsets of $\{1,2,\dots,k\}$ such that $1 \leq |\beta_p| \leq |\beta_1| \leq n+1$. Choose $\gamma \subset \{1,2,\dots,k\}$ such that $\beta_1 \cap \gamma = \phi$ and $\beta_1 \cup \gamma \in C^k_{n+1}$. Let $\alpha = \beta_1 \cup \gamma$. For $i \in \alpha$, we choose a closed set $H_i \subset G_i^\alpha$ such that $\overline{U_i^\alpha} \subset Int_{C(\alpha)}(H_i)$ $(i \in \alpha)$. As in the proof of Theorem 7.2, for each $i \in \alpha$, consider the following subsets of $C(\alpha)$:

$$\begin{aligned}
P_i^0 &= H_i,\\
F_i^0 &= \overline{U_i^\alpha},\\
P_i^1 &= C(\alpha) - St^1((C(\alpha) - U_i), \mathscr{V}),\\
F_i^1 &= \overline{C(\alpha) \cap \bigcup \{V \in \mathscr{V} \mid V \cap St^1((C(\alpha) - U_i), \mathscr{V}) = \phi\}},\\
P_i^2 &= C(\alpha) - St^2((C(\alpha) - U_i), \mathscr{V}),\\
F_i^2 &= \overline{C(\alpha) \cap \bigcup \{V \in \mathscr{V} \mid V \cap St^2((C(\alpha) - U_i), \mathscr{V}) = \phi\}},\\
&\cdots\cdots\\
P_i^{p-1} &= C(\alpha) - St^{p-1}((C(\alpha) - U_i), \mathscr{V}),\\
F_i^{p-1} &= \overline{C(\alpha) \cap \bigcup \{V \in \mathscr{V} \mid V \cap St^{p-1}((C(\alpha) - U_i), \mathscr{V}) = \phi\}},\\
P_i^p &= C(\alpha) - St^p((C(\alpha) - U_i), \mathscr{V}).
\end{aligned}$$

Since $\{St^p(V, \mathscr{V}) \mid x \in X\} \leq \mathscr{U}$, we see that the family $\{Int_{C(\alpha)} P_i^p \mid i \in \alpha\}$ is a covering of $C(\alpha)$. Also, put

$$\begin{aligned}
W(\alpha, \beta_1, \beta_2, \dots, \beta_p) = &\bigcap \{(P_j^0 - F_j^0) \mid j \in \alpha - \beta_1\}\\
&\cap \bigcap \{(P_j^1 - F_j^1) \mid j \in \beta_1 - \beta_2\}\\
&\cap \bigcap \{(P_j^2 - F_j^2) \mid j \in \beta_2 - \beta_3\}\\
&\cdots\cdots\\
&\cap \bigcap \{(P_j^{p-1} - F_j^{p-1}) \mid j \in \beta_{p-1} - \beta_p\}\\
&\cap \bigcap \{P_j^p \mid j \in \beta_p\}.
\end{aligned}$$

By Lemma 7.4, we see that $W(\alpha, \beta_1, \beta_2, \dots, \beta_p) \neq \phi$. For each $V \in \mathscr{V}$, we choose $f(V) \in \{1,2,\dots,k\}$ such that $St^p(V, \mathscr{V}) \subset U_{f(V)}$. Note that $f : \mathscr{V} \to \{1,2,\dots,k\}$ is a function. Let $\mathscr{V}_\alpha = \{V \in \mathscr{V} \mid St^p(V, \mathscr{V}) \subset U_i$ for some $i \in \alpha\}$. Then for each $i \in \alpha$, we put $\mathscr{V}_i = \{V \in \mathscr{V} \mid f(V) = i\}$. Note that $\{\mathscr{V}_i \mid i \in \alpha\}$ is a decomposition of $\mathscr{V}_\alpha$. Let $W_i' = \bigcup\{V \in \mathscr{V}_i\} \cap C(\alpha)$ $(i \in \alpha)$. Since $\{W_i' \mid i \in \alpha\}$ is an open covering of $C(\alpha)$ such that $W_i' \subset Int_{C(\alpha)} P_i^p$ $(i \in \alpha)$, we can choose a closed shrinking $\{W_i \mid i \in \alpha\}$ of $\{W_i' \mid i \in \alpha\}$. Applying Lemma 7.4 as in the proof of Theorem 7.2, we can conclude that

$$W(\alpha, \beta_1, \beta_2, \ldots, \beta_p) \cap \bigcap\{W_t \mid t \in \beta_p\} \neq \phi.$$

Choose a point $x \in W(\alpha, \beta_1, \beta_2, \ldots, \beta_p) \cap \bigcap\{W_t' \mid t \in \beta_p\}$. For each $t \in \beta_p$, we can choose $V(\beta_1, \beta_2, \ldots, \beta_p : t) \in \mathscr{V}_t$ containing the point x. Then we see that $V(\beta_1, \beta_2, \ldots, \beta_p : t)$ satisfies the desired conditions. Finally, we can conclude that

$$\tilde{\star}_k^p(X) = \tilde{\star}(k; n+1; p).$$

7.3. Characterizations of dimension by use of $\boldsymbol{\Delta^p(X, \mathscr{U})}$ and $\boldsymbol{\star^p(X, \mathscr{U})}$.

In Theorem 7.11, we give other characterizations of dimension by use of $\Delta^p(X, \mathscr{U})$ and $\star^p(X, \mathscr{U})$.

Theorem 7.6. *Let X be a normal space with* $\dim X = n$.

(1) *If $\mathscr{U}$ is any finite open covering of X with $|\mathscr{U}| = k$, then*

$$\tilde{\Delta}^p(X, \mathscr{U}) \leq \tilde{\Delta}(k; n+1; p).$$

(2) *For any $k \geq n+1$, there is a finite open covering $\mathscr{U}$ of X such that $|\mathscr{U}| = k$ and*

$$\tilde{\Delta}^p(X, \mathscr{U}) = \tilde{\Delta}(k; n+1; p).$$

Hence

$$\lim_{p \to \infty} \frac{\log \tilde{\Delta}^p(X, \mathscr{U})}{\log p} = n.$$

Proof. Note that

$$\tilde{\Delta}(k; n+1; p) = \Sigma_{n+1 \geq j_1 \geq j_2 \geq \cdots \geq j_p \geq 1} \binom{k}{j_1}\binom{j_1}{j_2} \cdots \binom{j_{p-1}}{j_p}$$

and

$$\tilde{\star}(k;n+1;p) = \Sigma_{n+1\geq j_1\geq j_2\geq\cdots\geq j_p\geq 1}\binom{k}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{p-1}}{j_p}j_p.$$

Since k and $n+1$ are fixed, we can choose a real number $0 < b$ such that for any $p = 1, 2, \dots,$ and j_i $(i = 1, 2, \dots, p)$ with $n+1 \geq j_1 \geq j_2 \geq \cdots \geq j_p \geq 1$,

$$1 \leq \binom{k}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{p-1}}{j_p} < b$$

and

$$1 \leq \binom{k}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{p-1}}{j_p}j_p < b.$$

Hence

$$|{}_{n+1}H_p| \leq \tilde{\Delta}(k;n+1;p) < |{}_{n+1}H_p|\cdot b$$

and

$$|{}_{n+1}H_p| \leq \tilde{\star}(k;n+1;p) < |{}_{n+1}H_p|\cdot b.$$

Then we see that

$$\begin{aligned}\lim_{p\to\infty}\frac{\log\tilde{\Delta}(k;n+1;p)}{\log p} &= \lim_{p\to\infty}\frac{\log|{}_{n+1}H_p|}{\log p} = \lim_{p\to\infty}\frac{\log[(n+p)!/p!n!]}{\log p}\\ &= \lim_{p\to\infty}\frac{\log(n+p)+\log(n-1+p)+\cdots+\log(1+p)}{\log p} = n.\end{aligned}$$

Similarly, we have

$$\lim_{p\to\infty}\frac{\log\tilde{\star}(k;n+1;p)}{\log p} = n.$$

By use of the proof of Theorem 7.6, we can also prove the following theorem.

THEOREM 7.7. *Let X be a normal space. Then*

$$\dim X = \sup\left\{\limsup_{p\to\infty}\frac{\log\tilde{\Delta}^p(X,\mathscr{U})}{\log p}\;\middle|\;\mathscr{U}\text{ is a finite open covering of }X\right\}.$$

Similarly, we have

Theorem 7.8. *Let X be a normal space with* $\dim X = n$.

(1) *If $\mathscr{U}$ is any finite open covering of X with $|\mathscr{U}| = k$, then*

$$\tilde{\star}^p(X, \mathscr{U}) \leq \tilde{\star}(k; n+1; p).$$

(2) *For any $k \geq n+1$, there is a finite open covering $\mathscr{U}$ of X such that $|\mathscr{U}| = k$ and*

$$\tilde{\star}^p(X, \mathscr{U}) = \tilde{\star}(k; n+1; p).$$

Hence

$$\lim_{p\to\infty} \frac{\log \tilde{\star}^p(X, \mathscr{U})}{\log p} = n.$$

Consequently, if X is a normal space with $\infty \geq \dim X \geq 0$,

$$\dim X = \sup\left\{ \limsup_{p\to\infty} \frac{\log \tilde{\star}^p(X, \mathscr{U})}{\log p} \;\middle|\; \mathscr{U} \textit{ is a finite open covering of } X \right\}.$$

Proposition 7.9. *For any natural number p and any finite open covering $\mathscr{U}$ of an infinite normal space X, we have*

$$\begin{aligned} \Delta^p(X, \mathscr{U}) &= \tilde{\Delta}^{2^{p-1}}(X, \mathscr{U}), \\ \star^p(X, \mathscr{U}) &= \tilde{\star}^{(1/2)(3^p-1)}(X, \mathscr{U}), \\ \Delta_k^p(X) &= \tilde{\Delta}_k^{2^{p-1}}(X) \quad \textit{and} \\ \star_k^p(X) &= \tilde{\star}_k^{(1/2)(3^p-1)}(X). \end{aligned}$$

Proof. First, we shall prove that for any open covering $\mathscr{V}$ of X and a natural number $p \geq 1$,

$$(1) \quad \mathscr{V}^{\Delta^p} (= \{St(x, \mathscr{V}^{\Delta^{p-1}}) \mid x \in X\}) = \{St^{2^{p-1}}(x, \mathscr{V}) \mid x \in X\}.$$

In fact, we prove that for each $p = 1, 2, \ldots$ and $x \in X$,

$$(1)' \quad St(x, \mathscr{V}^{\Delta^{p-1}}) = St^{2^{p-1}}(x, \mathscr{V}).$$

If $p=1$, by definitions we can easily see that (1)' is true. By induction on p, we shall prove (1)'. We assume that (1)' is true for p. Since (1) is true for p, for each $x \in X$, we have

$$\begin{aligned} St(x, \mathscr{V}^{\Delta^p}) &= \cup\{T \in \mathscr{V}^{\Delta^p} \mid x \in T\} \\ &= \cup\bigl\{St^{2^{p-1}}(y, \mathscr{V}) \mid y \in X \text{ and } x \in St^{2^{p-1}}(y, \mathscr{V})\bigr\} = St^{2^p}(x, \mathscr{V}). \end{aligned}$$

This implies that (1)' is true for $p+1$ and hence (1) is also true for $p+1$.

Next, we shall prove that for any open covering $\mathscr{V}$ of X and a natural number $p \geq 1$,

$$(2) \quad \mathscr{V}^{\star^p} = \bigl\{St^{(1/2)(3^p-1)}(V, \mathscr{V}) \mid V \in \mathscr{V}\bigr\}.$$

By induction on p, we shall prove that (2) is true. If $p=1$, by definitions (2) is true. We assume that (2) is true for p. If $W \in \mathscr{V}^{\star^{p+1}}$, there is $U \in \mathscr{V}^{\star^p}$ such that $W = St(U, \mathscr{V}^{\star^p})$. By the assumption, we see that there is $V \in \mathscr{V}$ such that $U = St^{(1/2)(3^p-1)}(V, \mathscr{V})$. Hence

$$\begin{aligned} W &= \bigcup\{U' \in \mathscr{V}^{\star^p} \mid U \cap U' \neq \phi\} \\ &= \bigcup\bigl\{St^{(1/2)(3^p-1)}(V', \mathscr{V}) \mid \\ &\qquad St^{(1/2)(3^p-1)}(V, \mathscr{V}) \cap St^{(1/2)(3^p-1)}(V', \mathscr{V}) \neq \phi \text{ for } V' \in \mathscr{V}\bigr\} \\ &= St^{3\times(1/2)(3^p-1)+1}(V, \mathscr{V}) = St^{(1/2)(3^{p+1}-1)}(V, \mathscr{V}). \end{aligned}$$

This implies that $\mathscr{V}^{\star^{p+1}} \subset \bigl\{St^{(1/2)(3^{p+1}-1)}(V, \mathscr{V}) \mid V \in \mathscr{V}\bigr\}$. To prove the converse inclusion, for each $V \in \mathscr{V}$ we put $U = St^{(1/2)(3^p-1)}(V, \mathscr{V})$. By induction, we see that $U \in \mathscr{V}^{\star^p}$ and

$$St^{(1/2)(3^{p+1}-1)}(V, \mathscr{V}) = St(U, \mathscr{V}^{\star^p}) \in \mathscr{V}^{\star^{p+1}}.$$

This implies that (2) is true. By use of the facts (1) and (2), we can easily complete the proof.

We have the following corollary which is Theorem 2.2 in Section 2.

COROLLARY 7.10. *Let X be a normal space and $\dim X = n$ and let k and p be natural numbers. Then*

$$\Delta_k^p(X) = \begin{cases} \tilde{\Delta}(k;k;2^{p-1}) = (2^{p-1}+1)^k - (2^{p-1})^k, & \textit{if } \ k \leq n+1 \\ \tilde{\Delta}(k;n+1;2^{p-1}), & \textit{if } \ k \geq n+1 \end{cases}$$

and

$$\star_k^p(X) = \begin{cases} \tilde{\star}\left(k;k;\dfrac{1}{2}(3^p-1)\right) = k\left[\dfrac{1}{2}(3^p-1)+1\right]^{k-1}, & \textit{if } \ k \leq n+1 \\ \tilde{\star}\left(k;n+1;\dfrac{1}{2}(3^p-1)\right), & \textit{if } \ k \geq n+1. \end{cases}$$

Now we give other characterizations of topological dimension by use of the indices $\Delta_k^p(X,\mathscr{U})$ and $\star_k^p(X,\mathscr{U})$ as follows.

Theorem 7.11. *Let X be a normal space. Then*

$$\dim X = \sup\left\{ \limsup_{p\to\infty} \frac{\log_2 \Delta^p(X,\mathscr{U})}{p} \,\middle|\, \mathscr{U} \textit{ is a finite open covering of } X \right\}$$

and

$$\dim X = \sup\left\{ \limsup_{p\to\infty} \frac{\log_3 \star^p(X,\mathscr{U})}{p} \,\middle|\, \mathscr{U} \textit{ is a finite open covering of } X \right\}.$$

Proof. By Theorem 7.7 and Theorem 7.8, we have

$$\begin{aligned} \dim X &= \sup\left\{ \limsup_{p\to\infty} \frac{\log_2 \tilde{\Delta}^{2^{p-1}}(X,\mathscr{U})}{\log_2 2^{p-1}} \,\middle|\, \mathscr{U} \text{ is a finite open covering of } X \right\} \\ &= \sup\left\{ \limsup_{p\to\infty} \frac{\log_2 \Delta^p(X,\mathscr{U})}{p-1} \,\middle|\, \mathscr{U} \text{ is a finite open covering of } X \right\} \\ &= \sup\left\{ \limsup_{p\to\infty} \frac{\log_2 \Delta^p(X,\mathscr{U})}{p} \,\middle|\, \mathscr{U} \text{ is a finite open covering of } X \right\}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \dim X &= \sup\left\{ \limsup_{p\to\infty} \frac{\log_3 \tilde{\star}^{(1/2)(3^p-1)}(X,\mathscr{U})}{\log_3 \dfrac{1}{2}(3^p-1)} \,\middle|\, \mathscr{U} \text{ is a finite open covering of } X \right\} \\ &= \sup\left\{ \limsup_{p\to\infty} \frac{\log_3 \star^p(X,\mathscr{U})}{p} \,\middle|\, \mathscr{U} \text{ is a finite open covering of } X \right\}. \end{aligned}$$

Corollary 7.12. *Let X be an infinite normal space with $\dim X = n$.*

(1) *If $\mathscr{U}_1$ and $\mathscr{U}_2$ are finite open coverings of X with $\mathscr{U}_1 \le \mathscr{U}_2$, then*

$$\tilde{\Delta}^p(X, \mathscr{U}_1) \ge \tilde{\Delta}^p(X, \mathscr{U}_2), \quad \tilde{\star}^p(X, \mathscr{U}_1) \ge \tilde{\star}^p(X, \mathscr{U}_2),$$
$$\Delta^p(X, \mathscr{U}_1) \ge \Delta^p(X, \mathscr{U}_2), \quad \star^p(X, \mathscr{U}_1) \ge \star^p(X, \mathscr{U}_2).$$

(2) *There is a finite open covering $\mathscr{U}$ of X such that if $\mathscr{U}'$ is any finite open covering of X with $\mathscr{U}' \le \mathscr{U}$, then*

$$\lim_{p\to\infty} \frac{\log \tilde{\Delta}^p(X, \mathscr{U}')}{\log p} = n, \quad \lim_{p\to\infty} \frac{\log \tilde{\star}^p(X, \mathscr{U}')}{\log p} = n,$$
$$\lim_{p\to\infty} \frac{\log_2 \Delta^p(X, \mathscr{U}')}{p} = n, \quad \lim_{p\to\infty} \frac{\log_3 \star^p(X, \mathscr{U}')}{p} = n.$$

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Hisao Kato
Institute of Mathematics
University of Tsukuba
Ibaraki, 305-8571 Japan
E-mail: hisakato@sakura.cc.tsukuba.ac.jp

Masahiro Matsumoto
Institute of Mathematics
University of Tsukuba
Ibaraki, 305-8571 Japan