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On Hermitian modular forms mod p

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Abstract. We generalize the notion of modular forms mod p to the case of Hermitian modular forms. Moreover we determine the structure of the algebra of degree 2 Hermitian modular forms mod p in the cases that the corresponding quadratic field is $\boldsymbol{Q}(\sqrt{-1})$ and $\boldsymbol{Q}(\sqrt{-3})$.

1. Introduction.

The theory of modular forms mod p was initially developed by H. P. F. Swinnerton-Dyer [**12**]. Since then the theory has developed into one of the essential tools for studying p-adic and mod p properties of modular forms; for example, it played an essential role when J.-P. Serre defined the notion of p-adic modular forms [**11**]. Consequently, generalization has been attempted by several people. N. M. Katz developed the theory from the viewpoint of algebraic geometry [**7**]. The generalization to the case of modular forms of several variables has been studied by the second author. He determined the structure of the algebra of Siegel modular forms mod p in the case of degree two (cf. [**9**]).

In this paper, we generalize the notion of modular forms mod p to the case of Hermitian modular forms and determine the structure of the algebra of Hermitian modular forms mod p in the cases of degree two over $\boldsymbol{Q}(\sqrt{-1})$ and $\boldsymbol{Q}(\sqrt{-3})$ (Theorem 5.2, Theorem 6.2).

As in the case of Siegel modular forms, the main points of the proof are as follows:

(1) Construction of generators over $\boldsymbol{Z}_{(p)}$ (Theorem 5.1, Theorem 6.1).
(2) Construction of a Hermitian modular form F_{p-1} of weight $p-1$ satisfying

$$F_{p-1} \equiv 1 \pmod{p}.$$

(Proposition 5.1, Proposition 6.1).

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(3) Determination of the Krull dimension of the algebras.

Our results depend strongly on the structure theorem for the graded ring of Hermitian modular forms obtained by T. Dern and A. Krieg [**4**].

2. Hermitian modular forms.

2.1. Definition and notation.

The Hermitian upper half-space of degree n is defined by

$$\boldsymbol{H}_n := \left\{ Z \in M_n(\boldsymbol{C}) \mid \frac{1}{2i}(Z - {}^t\overline{Z}) > 0 \right\}$$

where ${}^t\overline{Z}$ is the transposed complex conjugate of Z. The space $\boldsymbol{H}_n$ contains the Siegel upper half-space of degree n

$$\boldsymbol{S}_n := \boldsymbol{H}_n \cap Sym_n(\boldsymbol{C}).$$

Let $\boldsymbol{K}$ be an imaginary quadratic number field with discriminant $d_{\boldsymbol{K}}$ and ring of integers $\mathscr{O}_{\boldsymbol{K}}$. The Hermitian modular group

$$U_n(\mathscr{O}_{\boldsymbol{K}}) := \left\{ M \in M_{2n}(\mathscr{O}_{\boldsymbol{K}}) \mid {}^t\overline{M} J_n M = J_n,\ J_n = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} \right\}$$

acts on $\boldsymbol{H}_n$ by fractional transformation

$$\boldsymbol{H}_n \ni Z \longmapsto M\langle Z \rangle := (AZ+B)(CZ+D)^{-1}, \quad M = \begin{pmatrix} AB \\ CD \end{pmatrix} \in U_n(\mathscr{O}_{\boldsymbol{K}}).$$

The subgroup $SU_n(\mathscr{O}_{\boldsymbol{K}}) := U_n(\mathscr{O}_{\boldsymbol{K}}) \cap SL_{2n}(\boldsymbol{K})$ coincides with the full group $U_n(\mathscr{O}_{\boldsymbol{K}})$ unless $d_{\boldsymbol{K}} = -3$ or -4.

Let $\Gamma \subset U_n(\mathscr{O}_{\boldsymbol{K}})$ be a subgroup of finite index and ν_k ($k \in \boldsymbol{Z}$) an abelian character of Γ satisfying $\nu_k \cdot \nu_{k'} = \nu_{k+k'}$. We denote by $M_k(\Gamma, \nu_k)$ the space of Hermitian modular forms of weight k and character ν_k with respect to Γ. Namely, it consists of holomorphic functions $F : \boldsymbol{H}_n \longrightarrow \boldsymbol{C}$ satisfying

$$F \mid_k M(Z) := \det(CZ+D)^{-k} F(M\langle Z \rangle) = \nu_k(M) \cdot F(Z),$$

for all $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$.

The subspace $S_k(\Gamma, \nu_k)$ of cusp forms is characterized by the condition

$$\Phi\left(F\mid_k \begin{pmatrix} {}^t\overline{U} & 0 \\ 0 & U \end{pmatrix}\right) \equiv 0 \quad \text{for all } U \in GL_n(\boldsymbol{K})$$

where Φ is the Siegel Φ-operator. A modular form $F \in M_k(\Gamma, \nu_k)$ is called *symmetric* (resp. *skew-symmetric*) if

$$F({}^tZ) = F(Z) \qquad (\text{resp. } F({}^tZ) = -F(Z)).$$

We denote by $M_k(\Gamma, \nu_k)^{sym}$ (resp. $M_k(\Gamma, \nu_k)^{skew}$) the subspace consisting of symmetric (resp. skew-symmetric) modular forms. Moreover

$$S_k(\Gamma, \nu_k)^{sym} := M_k(\Gamma, \nu_k)^{sym} \cap S_k(\Gamma, \nu_k),$$
$$S_k(\Gamma, \nu_k)^{skew} := M_k(\Gamma, \nu_k)^{skew} \cap S_k(\Gamma, \nu_k).$$

2.2. Fourier expansion.

If $F \in M_k(\Gamma, \nu_k)$ satisfies the condition

$$F(Z + B) = F(Z) \quad \text{for all } B \in Her_n(\mathscr{O}_{\boldsymbol{K}}),$$

then F has a Fourier expansion of the form

$$F(Z) = \sum_{0 \le H \in \Lambda_n(\boldsymbol{K})} a_F(H) \exp[2\pi i \mathrm{tr}(HZ)]$$

where

$$\Lambda_n(\boldsymbol{K}) := \left\{H = (h_{kj}) \in Her_n(\boldsymbol{K}) \mid h_{kk} \in \boldsymbol{Z},\ \sqrt{d_{\boldsymbol{K}}}h_{kj} \in \mathscr{O}_{\boldsymbol{K}}\right\}.$$

Put $\omega := (d_{\boldsymbol{K}} + \sqrt{d_{\boldsymbol{K}}})/2$ and define the matrices $\dot{Z} = (\dot{z}_{kj})$ and $\ddot{Z} = (\ddot{z}_{kj})$ by

$$\dot{Z} := \frac{\omega\, {}^tZ - \bar{\omega} Z}{\omega - \bar{\omega}}, \quad \ddot{Z} := \frac{Z - {}^tZ}{\omega - \bar{\omega}}.$$

Then the above F can be considered as a function of the $n(n-1)/2$ complex variables $\ddot{z}_{kj}$ $(k < j)$ in $\ddot{Z}$ and of the $n(n+1)/2$ complex variables $\dot{z}_{kj}$ $(k \le j)$ in $\dot{Z}$. Moreover, F has period 1 for each of these variables. If we define

$$\dot{q}_{kj} := \exp(2\pi i \dot{z}_{kj})\ (k \le j), \quad \ddot{q}_{kj} := \exp(2\pi i \ddot{z}_{kj})\ (k < j)$$

then

$$F = \sum a_F(H)\exp[2\pi i \mathrm{tr}(HZ)] = \sum a_F(H) q^H$$

may be considered as an element of the formal power series ring

$$\boldsymbol{C}\big[\dot{q}_{kj}^{\pm 1}, \ddot{q}_{kj}^{\pm 1} \ (k<j)\big] [\![\dot{q}_{11}, \ldots, \dot{q}_{nn}]\!].$$

Let R be a subring of $\boldsymbol{C}$. We define

$$\begin{aligned} &M_k(U_n(\mathscr{O}_{\boldsymbol{K}}), \nu_k)_R \\ &\quad := \Big\{ F = \sum a_F(H) q^H \in M_k(U_n(\mathscr{O}_{\boldsymbol{K}}), \nu_k) \mid a_F(H) \in R \text{ for all } H \in \Lambda_n(\boldsymbol{K}) \Big\} \end{aligned}$$

and

$$M_k(U_n(\mathscr{O}_{\boldsymbol{K}}), \nu_k)_R^{sym} := M_k(U_n(\mathscr{O}_{\boldsymbol{K}}), \nu_k)_R \cap M_k(U_n(\mathscr{O}_{\boldsymbol{K}}), \nu_k)^{sym}.$$

So we may consider the inclusion:

$$M_k(U_n(\mathscr{O}_{\boldsymbol{K}}), \nu_k)_R \subset R\big[\dot{q}_{kj}^{\pm 1}, \ddot{q}_{kj}^{\pm 1} \ (k<j)\big] [\![\dot{q}_{11}, \ldots, \dot{q}_{nn}]\!].$$

We fix a prime number p. For $F = \sum a_F(H) q^H \in M_k(U_n(\mathscr{O}_{\boldsymbol{K}}), \nu_k)_{\boldsymbol{Q}}$, we define $v_p(F) \in \boldsymbol{Z}$ by

$$v_p(F) := \inf_{H \in \Lambda_n(\boldsymbol{K})} \mathrm{ord}_p(a_F(H)). \tag{2.1}$$

It should be noted that the value $v_p(F)$ is finite.

Let $\boldsymbol{Z}_{(p)}$ denote the local ring at p, namely, $\boldsymbol{Z}_{(p)} := \boldsymbol{Q} \cap \boldsymbol{Z}_p$. The following lemma will be needed in later sections.

Lemma 2.1. *If $F \in M_k(U_n(\mathscr{O}_{\boldsymbol{K}}), \nu_k)_{\boldsymbol{Q}}^{sym}$ and $G \in M_{k'}(U_n(\mathscr{O}_{\boldsymbol{K}}), \nu_{k'})_{\boldsymbol{Q}}^{sym}$ satisfy*

$$FG \in M_{k+k'}(U_n(\mathscr{O}_{\boldsymbol{K}}), \nu_{k+k'})_{\boldsymbol{Z}_{(p)}}^{sym} \quad \textit{and} \quad v_p(G) = 0,$$

then

$$F \in M_k(U_n(\mathscr{O}_{\boldsymbol{K}}), \nu_k)_{\boldsymbol{Z}_{(p)}}^{sym}.$$

PROOF. The lemma is an easy consequence of the identity

$$v_p(FG) = v_p(F) + v_p(G). \qquad \square$$

2.3. Hermitian modular forms mod p.

Let $\boldsymbol{Z}_{(p)}$ be as in the previous section. For any element $F = \sum a_F(H)q^H \in M_k(U_n(\mathscr{O}_{\boldsymbol{K}}), \nu_k)_{\boldsymbol{Z}_{(p)}}$, consider the reduction mod p of F:

$$\widetilde{F} := \sum \widetilde{a_F(H)} q^H$$

where $\widetilde{a_F(H)}$ denotes the reduction mod p of $a_F(H) \in \boldsymbol{Z}_{(p)}$. Therefore we may regard $\widetilde{F}$ as follows:

$$\widetilde{F} \in \boldsymbol{F}_p\big[\dot{q}_{kj}^{\pm 1}, \ddot{q}_{kj}^{\pm 1}\ (k < j)\big][\![\dot{q}_{11}, \ldots, \dot{q}_{nn}]\!] =: \boldsymbol{F}_p[\![\boldsymbol{q}]\!].$$

We define subspaces of $\boldsymbol{F}_p[\![\boldsymbol{q}]\!]$:

$$\widetilde{M}_k(U_n(\mathscr{O}_{\boldsymbol{K}}), \nu_k)_p := \Big\{\widetilde{F} = \sum \widetilde{a_F(H)} q^H \mid F \in M_k(U_n(\mathscr{O}_{\boldsymbol{K}}), \nu_k)_{\boldsymbol{Z}_{(p)}}\Big\},$$
$$\widetilde{M}_k(U_n(\mathscr{O}_{\boldsymbol{K}}), \nu_k)_p^{sym} := \Big\{\widetilde{F} = \sum \widetilde{a_F(H)} q^H \mid F \in M_k(U_n(\mathscr{O}_{\boldsymbol{K}}), \nu_k)_{\boldsymbol{Z}_{(p)}}^{sym}\Big\}.$$

The subalgebra

$$\widetilde{M}(U_n(\mathscr{O}_{\boldsymbol{K}}), \nu)_p := \sum_{k \in \boldsymbol{Z}} \widetilde{M}_k(U_n(\mathscr{O}_{\boldsymbol{K}}), \nu_k)_p \subset \boldsymbol{F}_p[\![\boldsymbol{q}]\!]$$

is called *the algebra of Hermitian modular forms* mod p.

REMARK. Later we treat the case where $\nu_k = \det^{k/2}$ or $\det^k$ (cf. (2.2)). We write the sum $\sum_{k \in \boldsymbol{Z}} \widetilde{M}_k(U_n(\mathscr{O}_{\boldsymbol{K}}), \nu_k)_p$ by $\widetilde{M}(U_n(\mathscr{O}_{\boldsymbol{K}}), \nu)_p$ symbolically.

Similarly we can define subalgebras

$$\widetilde{M}(U_n(\mathscr{O}_{\boldsymbol{K}}), \nu)_p^{sym} := \sum_{k \in \boldsymbol{Z}} \widetilde{M}_k(U_n(\mathscr{O}_{\boldsymbol{K}}), \nu_k)_p^{sym},$$
$$\widetilde{M}^{(e)}(U_n(\mathscr{O}_{\boldsymbol{K}}), \nu)_p^{sym} := \sum_{k \in 2\boldsymbol{Z}} \widetilde{M}_k(U_n(\mathscr{O}_{\boldsymbol{K}}), \nu_k)_p^{sym}.$$

The main purpose of this paper is to determine the structure of the algebra

$$\widetilde{M}^{(e)}(U_2(\mathscr{O}_{\boldsymbol{K}}),\nu)_p^{sym}=\sum_{k\in 2\boldsymbol{Z}}\widetilde{M}(U_2(\mathscr{O}_{\boldsymbol{K}}),\nu_k)_p^{sym}$$

for $\boldsymbol{K}=\boldsymbol{Q}(\sqrt{-1})$ and $\boldsymbol{Q}(\sqrt{-3})$ where

$$\nu_k=\begin{cases}\det^{k/2} & \text{for } \boldsymbol{K}=\boldsymbol{Q}(\sqrt{-1}),\\ \det^k & \text{for } \boldsymbol{K}=\boldsymbol{Q}(\sqrt{-3}).\end{cases} \tag{2.2}$$

3. Siegel modular forms.

In this section we introduce some results concerning Siegel modular forms which are needed in later sections.

3.1. Definition and notation.

Let $M_k(\Gamma_n)$ denote the space of Siegel modular forms of weight k ($\in \boldsymbol{Z}$) for the Siegel modular group $\Gamma_n := Sp_n(\boldsymbol{Z})$ and $S_k(\Gamma_n)$ the subspace of cusp forms.

Any Siegel modular form $F(Z)$ in $M_k(\Gamma_n)$ has a Fourier expansion of the form

$$F(Z)=\sum_{0\le T\in\Lambda_n} a_F(T)\exp[2\pi i\mathrm{tr}(TZ)],$$

where

$$\Lambda_n = Sym_n^*(\boldsymbol{Z}) := \{T=(t_{kj})\in Sym_n(\boldsymbol{Q}) \mid t_{kk},\ 2t_{kj}\in\boldsymbol{Z}\}$$

(the lattice in $Sym_n(\boldsymbol{R})$ of half-integral, symmetric matrices).

Taking $q_{kj} := \exp(2\pi i z_{kj})$ with $Z=(z_{kj})\in\boldsymbol{H}_n$, we write

$$q^T := \exp[2\pi i\mathrm{tr}(TZ)] = \prod_{1\le k<j\le n} q_{kj}^{2t_{kj}}\prod_{k=1}^n q_{kk}^{t_{kk}}.$$

Using this notation, we obtain *the generalized q-expansion*:

$$F=\sum_{0\le T\in\Lambda_n} a_F(T)\,q^T=\sum_{t_i}\left(\sum_{t_{kj}} a_F(T)\prod_{k<j} q_{kj}^{2t_{kj}}\right)\prod_{k=1}^n q_{kk}^{t_{kk}}$$
$$\in \boldsymbol{C}\big[q_{kj}^{-1},q_{kj}\big][\![q_{11},\dots,q_{nn}]\!].$$

For any subring $R\subset\boldsymbol{C}$, we adopt the notation,

$$M_k(\Gamma_n)_R := \left\{ F = \sum_{T \in \Lambda_n} a_F(T) q^T \mid a_F(T) \in R \ (\forall T \in \Lambda_n) \right\},$$
$$S_k(\Gamma_n)_R := M_k(\Gamma_n) \cap S_k(\Gamma_n).$$

Any element $F \in M_k(\Gamma_n)_R$ can be regarded as an element of

$$R\big[q_{kj}^{-1}, q_{kj}\big][\![q_{11}, \ldots, q_{nn}]\!].$$

3.2. Siegel modular forms of degree 2.

In this subsection we consider the case of degree 2. A typical example of a Siegel modular form is the Siegel-Eisenstein series

$$G_k(Z) := \sum_{M = \left(\begin{smallmatrix} * & * \\ C & D \end{smallmatrix}\right)} \det(CZ + D)^{-k}, \quad Z \in \boldsymbol{S}_2$$

where $k > 3$ is even and $M = \left(\begin{smallmatrix} * & * \\ C & D \end{smallmatrix}\right)$ runs over a set of representatives $\left\{ \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \right\} \backslash \Gamma_2$. It is known that $G_k \in M_k(\Gamma_2)_{\boldsymbol{Q}}$.

We set

$$\begin{aligned} X_{10} &:= -\frac{43867}{2^{10} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 53}(G_{10} - G_4 G_6), \\ X_{12} &:= -\frac{691 \cdot 1847}{2^{13} \cdot 3^6 \cdot 5^3 \cdot 7^2}\left(G_{12} - \frac{441}{691}G_4^3 - \frac{250}{691}G_6^2\right). \end{aligned} \tag{3.1}$$

Then we have $X_k \in S_k(\Gamma_2)_{\boldsymbol{Z}}$ $(k = 10, 12)$ and

$$a_{X_{10}}\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} = a_{X_{12}}\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} = 1.$$

Theorem 3.1 (Igusa [**6**]). *The graded ring*

$$M^{(e)}(\Gamma_2) := \bigoplus_{k \in 2\boldsymbol{Z}} M_k(\Gamma_2)$$

is generated by four modular forms

$$G_4, \ G_6, \ X_{10}, \ X_{12},$$

which are algebraically independent. Namely,

$$M^{(e)}(\Gamma_2) = \boldsymbol{C}[G_4, G_6, X_{10}, X_{12}].$$

3.3. Siegel modular forms mod *p*.

For any Siegel modular form

$$F = \sum a_F(T) q^T \in M_k(\Gamma_n)_{\boldsymbol{Z}_{(p)}},$$

there exists a formal power series correspondence,

$$\widetilde{F} := \sum \widetilde{a_F(T)} q^T \in \boldsymbol{F}_p\left[q_{kj}^{-1}, q_{kj}\right][\![q_{11}, \ldots, q_{nn}]\!],$$

where $\widetilde{a_F(T)}$ denotes the reduction modulo p of $a_F(T)$. We define

$$\begin{aligned}\widetilde{M}_k(\Gamma_n)_p &:= \left\{\widetilde{F} = \sum \widetilde{a_F(T)} q^T \mid F \in M_k(\Gamma_n)_{\boldsymbol{Z}_{(p)}}\right\}\\ &\subset \boldsymbol{F}_p\left[q_{kj}^{-1}, q_{kj}\right][\![q_{11}, \ldots, q_{nn}]\!].\end{aligned}$$

The algebra

$$\widetilde{M}(\Gamma_n)_p := \sum_{k \in \boldsymbol{Z}} \widetilde{M}_k(\Gamma_n)_p \quad \left(\text{resp. } \widetilde{M}^{(e)}(\Gamma_n)_p := \sum_{k \in 2\boldsymbol{Z}} \widetilde{M}_k(\Gamma_n)_p\right)$$

is called *the algebra of Siegel modular forms* $\bmod p$ (*resp. the algebra of Siegel modular forms* $\bmod p$ *of even weight*).

The structure of $\widetilde{M}(\Gamma_1)_p$ was determined by H. P. F. Swinnerton-Dyer [**12**]. Moreover the structure of $\widetilde{M}(\Gamma_2)_p$ was studied by the second author [**9**]. Here we introduce the structure theorem of $\widetilde{M}^{(e)}(\Gamma_2)_p$ for the cases $p \geq 5$.

Proposition 3.1. *Assume that $p \geq 5$. If $F \in M_k(\Gamma_2)_{\boldsymbol{Z}_{(p)}}$ (k : even), then there exists a unique polynomial $P(x_1, x_2, x_3, x_4) \in \boldsymbol{Z}_{(p)}[x_1, x_2, x_3, x_4]$ such that*

$$F = P(G_4, G_6, X_{10}, X_{12})$$

where G_k ($k = 4, 6$) is the Siegel-Eisenstein series and X_k ($k = 10, 12$) is the cusp form defined in (3.1).

PROPOSITION 3.2. *Assume that $p \geq 5$. There exists a Siegel modular form $F_{p-1} \in M_{p-1}(\Gamma_2)_{\boldsymbol{Z}_{(p)}}$ such that*

$$F_{p-1} \equiv 1 \pmod{p}$$

where the congruence is defined Fourier coefficient-wise.

REMARK. The existence of such a modular form of general degree was studied by Böcherer and the second author [**1**].

THEOREM 3.2 ([**9**]). *Assume that $p \geq 5$. Then we have*

$$\widetilde{M}^{(e)}(\Gamma_2) \cong \boldsymbol{F}_p[x_1, x_2, x_3, x_4]/(\widetilde{A} - 1)$$

where $(\widetilde{A} - 1)$ is the principal ideal generated by $\widetilde{A} - 1$ and $A \in \boldsymbol{Z}_{(p)}[x_1, x_2, x_3, x_4]$ is defined by

$$F_{p-1} = A(G_4, G_6, X_{10}, X_{12}).$$

REMARK. There are many possibilities for the choice of F_{p-1}; however, the polynomial $\widetilde{A} \in \boldsymbol{F}_p[x_1, x_2, x_3, x_4]$ is uniquely determined by p.

4. Hermitian modular forms of degree 2.

4.1. Eisenstein series and cusp forms.

In this section, we deal with Hermitian modular forms of degree 2.

We consider the Hermitian Eisenstein series of degree 2

$$E_k(Z) := \sum_{M = \left(\begin{smallmatrix} * & * \\ C & D \end{smallmatrix}\right)} (\det M)^{k/2} \det(CZ + D)^{-k}, \quad Z \in \boldsymbol{H}_2,$$

where $k > 4$ is even and $M = \left(\begin{smallmatrix} * & * \\ C & D \end{smallmatrix}\right)$ runs over a set of representatives of $\left\{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)\right\} \backslash U_2(\mathscr{O}_{\boldsymbol{K}})$. Then we have

$$E_k \in M_k\bigl(U_2(\mathscr{O}_{\boldsymbol{K}}), \det{}^{-k/2}\bigr)^{sym}.$$

Moreover $E_4 \in M_4(U_2(\mathscr{O}_{\boldsymbol{K}}), \det{}^{-2})^{sym}$ is constructed by the Maass lift ([**10**]).

In the case that the class number of $\boldsymbol{K}$ is one, the Fourier coefficient of E_k is given as follows:

Theorem 4.1 (Krieg [**10**], Dern [**2**], [**3**]). *Assume that the class number of* $\boldsymbol{K}$ *is one. The Fourier coefficient* $a_{E_k}(H)$ *of* E_k *is given as follows.*

$$a_{E_k}(H) = \begin{cases} \dfrac{4k(k-1)}{B_k \cdot B_{k-1,\chi_{\boldsymbol{K}}}} \sum_{0<d|\varepsilon(H)} d^{k-1} G_{\boldsymbol{K}}\left(k-2, \dfrac{|d_{\boldsymbol{K}}|\det(H)}{d^2}\right) & \text{if } \operatorname{rank}(H)=2, \\ -\dfrac{2k}{B_k}\sigma_{k-1}(\varepsilon(H)) & \text{if } \operatorname{rank}(H)=1, \\ 1 & \text{if } H=0, \end{cases}$$

where

B_m *is the* m*-th Bernoulli number,*

$B_{m,\chi_{\boldsymbol{K}}}$ *is the* m*-th generalized Bernoulli number associated with the Kronecker character* $\chi_{\boldsymbol{K}} = \left(\frac{d_{\boldsymbol{K}}}{*}\right)$,

$\varepsilon(H) := \max\{l \in \boldsymbol{N} \mid l^{-1}H \in \Lambda_2(\boldsymbol{K})\}$,

and

$$\begin{aligned} G_{\boldsymbol{K}}(m,N) &:= \frac{1}{1+|\chi_{\boldsymbol{K}}(N)|}\left(\sigma_{m,\chi_{\boldsymbol{K}}}(N) - \sigma^*_{m,\chi_{\boldsymbol{K}}}(N)\right), \\ \sigma_{m,\chi_{\boldsymbol{K}}}(N) &:= \sum_{0<d|N} \chi_{\boldsymbol{K}}(d)d^m, \quad \sigma^*_{m,\chi_{\boldsymbol{K}}}(N) := \sum_{0<d|N} \chi_{\boldsymbol{K}}(N/d)d^m. \end{aligned} \tag{4.1}$$

In the case that the class number of $\boldsymbol{K}$ is 1, we can construct cusp forms by using Hermitian Eisenstein series (cf. [**4**, Corollary 2]).

Proposition 4.1. *Assume that the class number of* $\boldsymbol{K}$ *is* 1. *Then there are symmetric cusp forms*

$$\begin{aligned} f_{10} &:= E_{10} - E_4E_6 \in S_{10}(U_2(\mathscr{O}_{\boldsymbol{K}}), \det{}^{-5})^{sym}, \\ f_{12} &:= E_{12} - \frac{441}{691}E_4^3 - \frac{250}{691}E_6^2 \in S_{12}(U_2(\mathscr{O}_{\boldsymbol{K}}), 1)^{sym}. \end{aligned} \tag{4.2}$$

4.2. The graded ring over $\boldsymbol{Q}(\sqrt{-1})$.

In this section, we deal with the case $\boldsymbol{K} = \boldsymbol{Q}(\sqrt{-1})$. The following result is due to Dern and Krieg.

Theorem 4.2 (Dern-Krieg [**4**]). *Let* $\boldsymbol{K} = \boldsymbol{Q}(\sqrt{-1})$.

(1) *There exists a skew-symmetric Hermitian modular form* $\phi_4 \in S_4(U_2(\mathscr{O}_{\boldsymbol{K}}), \chi_{\boldsymbol{K}} \det)^{skew}$ *such that*

$$\phi_4 \mid_{\boldsymbol{S}_2} \equiv 0.$$

(2) *The graded ring*

$$\bigoplus_{k \in 2\boldsymbol{Z}} M_k(U_2(\mathscr{O}_{\boldsymbol{K}}), \det^{k/2})^{sym}$$

is generated by

$$E_4,\ E_6,\ \phi_4^2,\ E_{10} \ \textit{and}\ E_{12}$$

which are algebraically independent.

Remark. The form ϕ_4 is constructed by the Borcherds product. Namely, it has an infinite product expression and the divisor can be specified exactly.

For later purposes, we replace some of the above generators by modular forms with integral Fourier coefficients.

4.2.1. Form of weight 4 over $\boldsymbol{Q}(\sqrt{-1})$.

Lemma 4.1.

(1) $E_4 \in M_4(U_2(\mathscr{O}_{\boldsymbol{K}}), \det^2)_{\boldsymbol{Z}}^{sym}$.
(2) $E_4 \mid_{\boldsymbol{S}_2} = G_4$ *where* G_4 *is the Siegel-Eisenstein series of weight* 4.

Proof.

(1) If $\mathrm{rank}(H) = 2$,

$$\frac{4k(k-1)}{B_k \cdot B_{k-1,\chi_{\boldsymbol{K}}}} = \frac{4 \cdot 4 \cdot 3}{B_4 \cdot B_{3,\chi_{-4}}} = -960 \in \boldsymbol{Z}.$$

Hence we have

$$a_{E_4}(H) \in \boldsymbol{Z} \quad \text{if } \mathrm{rank}(H) = 2.$$

If $\mathrm{rank}(H) = 1$, then $a_{E_4}(H) = 240\sigma_3(\varepsilon(H))$. These facts imply integrality:

$$E_4 \in M_4(U_2(\mathscr{O}_{\boldsymbol{K}}), \det^{-2})_{\boldsymbol{Z}}^{sym} = M_4(U_2(\mathscr{O}_{\boldsymbol{K}}), \det^2)_{\boldsymbol{Z}}^{sym}.$$

(2) Noting that $E_4 \mid_{\boldsymbol{S}_2} \in M_4(\Gamma_2)$ and $\dim M_4(\Gamma_2) = 1$, we have $E_4 \mid_{\boldsymbol{S}_2} = G_4$. $\square$

4.2.2. Form of weight 6 over $\boldsymbol{Q}(\sqrt{-1})$.

Lemma 4.2.

(1) $E_6 \in M_6(U_2(\mathscr{O}_{\boldsymbol{K}}), \det^3)_{\boldsymbol{Z}}^{sym}$.

(2) $E_6 \mid_{\boldsymbol{S}_2} = G_6$ *where* G_6 *is the Siegel-Eisenstein series of weight* 6.

Proof.

(1) If $\mathrm{rank}(H) = 2$,

$$a_{E_6}(H) = \frac{4 \cdot 6 \cdot 5}{B_6 \cdot B_{5,\chi_{\boldsymbol{K}}}} \sum_{0<d|\varepsilon(H)} d^5 G_{\boldsymbol{K}}\left(4, \frac{4\det(H)}{d^2}\right)$$
$$= -\frac{2016}{5} \sum_{0<d|\varepsilon(H)} d^5 G_{\boldsymbol{K}}\left(4, \frac{4\det(H)}{d^2}\right), \quad \text{(cf. (4.1))}.$$

Noting that $4\det(H) \not\equiv 1 \pmod 4$ and Fermat's congruence, we obtain

$$G_{\boldsymbol{K}}\left(4, \frac{4\det(H)}{d^2}\right) \equiv 0 \pmod 5.$$

This implies $a_{E_6}(H) \in \boldsymbol{Z}$ if $\mathrm{rank}(H) = 2$. In the case that $\mathrm{rank}(H) = 1$, we have

$$a_{E_6}(H) = -\frac{2 \cdot 6}{B_6} \sigma_5(\varepsilon(H)) = -504\sigma_5(\varepsilon(H)) \in \boldsymbol{Z}.$$

These statements imply that $a_{E_6}(H) \in \boldsymbol{Z}$. Namely, we have

$$E_6 \in M_6(U_2(\mathscr{O}_{\boldsymbol{K}}), \det^{-3})_{\boldsymbol{Z}}^{sym} = M_6(U_2(\mathscr{O}_{\boldsymbol{K}}), \det^3)_{\boldsymbol{Z}}^{sym}.$$

(2) The proof is similar to the case of weight 4. □

4.2.3. Form of weight 8 over $\boldsymbol{Q}(\sqrt{-1})$.

Lemma 4.3. *We define*

$$\chi_8 := \phi_4^2,$$

where ϕ_4 *is the skew-symmetric Hermitian modular form given in Theorem* 4.2. *Then we have the following results*:

(1) $\chi_8 \in S_8(U_2(\mathscr{O}_{\boldsymbol{K}}), 1)_{\boldsymbol{Z}}^{sym}$.

(2) $a_{\chi_8}\left(\begin{smallmatrix} 1 & (1+i)/2 \\ (1-i)/2 & 1 \end{smallmatrix}\right) = 1$, *namely* $v_p(\chi_8) = 0$ (*cf.* (2.1)).
(3) $\chi_8 \mid_{\boldsymbol{S}_2} \equiv 0$.
(4) $\chi_8 = -(61/(2^{10} \cdot 3^2 \cdot 5^2))(E_8 - E_4^2) = -(61/230400)(E_8 - E_4^2)$.

Proof. The facts (1) and (2) are consequences of [**4**, Corollary 4]. Namely, they come from the fact that ϕ_4 is constructed by the Borcherds product. Here we give another proof.

Let $M_k(\Gamma_0(4), \chi_{-4}^k)$ be the space of modular forms of weight k on the congruence subgroup $\Gamma_0(4) \subset SL_2(\boldsymbol{Z})$ with character χ_{-4}^k. We define the subspace $M_k^*(\Gamma_0(4), \chi_{-4}^k)$ by

$$\begin{aligned} &M_k^*\big(\Gamma_0(4), \chi_{-4}^k\big) \\ &\quad := \Big\{ f = \sum a_f(n) q^n \in M_k(\Gamma_0(4), \chi_{-4}^k) \mid a_f(n) = 0, \text{ if } \chi_{-4}(n) = 1 \Big\}. \end{aligned}$$

This is an analogue of Kohnen's plus space (cf. [**10**, p. 670]). Krieg constructed an isomorphism

$$\Omega : \mathscr{M}_k(U_2(\mathscr{O}_{\boldsymbol{K}}), 1) \longrightarrow M_{k-1}^*\big(\Gamma_0(4), \chi_{-4}^{k-1}\big) \tag{4.3}$$

(cf. [**10**, p. 676, Theorem]), where $\mathscr{M}_k(U_2(\mathscr{O}_{\boldsymbol{K}}), 1) \subset M_k(U_2(\mathscr{O}_{\boldsymbol{K}}), 1)^{sym}$ is the Hermitian Maass space defined in [**10**, p. 667]. Moreover, $F \in \mathscr{M}_k(U_2(\mathscr{O}_{\boldsymbol{K}}), 1)$ is a cusp form if and only if $\Omega(F)$ is a cusp form.

We know that

$$\theta(z) := 1 + 2\sum_{n=1}^{\infty} q^{n^2} \quad \text{and} \quad F_2(z) := \sum_{\substack{n \geq 1 \\ n:\text{odd}}} \sigma_1(n) q^n$$

are generators of the graded ring

$$\bigoplus_{k \in \boldsymbol{Z}} M_k\big(\Gamma_0(4), \chi_{-4}^k\big).$$

If we set

$$\begin{aligned} h_7(z) &:= \theta^6(z) F_2^2(z) - 16\theta^2(z) F_2^3(z) \\ &= \sum_{n=1}^{\infty} a_7(n) q^n, \end{aligned}$$

then $h_7 \in S_7^*(\Gamma_0(4), \chi_{-4})$ (the space of cusp forms). We see that

$$\chi_8 \in S_8(U_2(\mathscr{O}_{\boldsymbol{K}}), 1)^{sym}$$

and, moreover

$$\Omega(\chi_8) = -\frac{2}{i} h_7 \quad \text{(cf. (4.3))}.$$

From this, we obtain

$$a_{\chi_8}(H) = \sum_{0<d|\varepsilon(H)} d^7 \frac{1}{1+\left|\chi_{-4}\left(\dfrac{4\det(H)}{d^2}\right)\right|} a_7\left(\frac{4\det(H)}{d^2}\right).$$

We must show the integrality of χ_8. We shall prove

$$\frac{1}{1+\left|\chi_{-4}\left(\dfrac{4\det(H)}{d^2}\right)\right|} a_7\left(\frac{4\det(H)}{d^2}\right) \in \boldsymbol{Z}. \tag{4.4}$$

To prove this, we note the q-expansion of $F_2(z)$. We see that

$$h_7 \equiv F_2^2 \pmod{2\boldsymbol{Z}[\![q]\!]}.$$

This means that, if $a_7(n)$ is odd, then n must be even. This implies (4.4) and proves (1). Since $a_7(2) = 1$, we have (2). The fact (3) comes from $\phi_4 \mid_{\boldsymbol{S}_2} \equiv 0$. The identity (4) is obtained by calculations of the Fourier coefficients of E_8 and E_4^2. □

4.2.4. Form of weight 10 over $\boldsymbol{Q}(\sqrt{-1})$.

Lemma 4.4. *We define*

$$F_{10} := -\frac{277}{2^9 \cdot 3^3 \cdot 5^2 \cdot 7} f_{10} = -\frac{277}{2419200}(E_{10} - E_4 E_6),$$

where f_{10} is the cusp form of weight 10 *defined in* (4.2)*. Then we have the following results*:

(1) $F_{10} \in S_{10}(U_2(\mathscr{O}_{\boldsymbol{K}}), \det^5)_{\boldsymbol{Z}}^{sym}$.
(2) $F_{10} \mid_{\boldsymbol{S}_2} = 6X_{10}$, *where $X_{10} \in S_{10}(\Gamma_2)$ is Igusa's cusp form of weight* 10 *defined in* (3.1).

PROOF. We set

$$h_9(z) := \theta^{10}(z)F_2^2(z) - 12\theta^6(z)F_2^3(z) - 64\theta^2(z)F_2^4(z) = \sum_{n=1}^{\infty} a_9(n)q^n \in M_9^*(\Gamma_0(4), \chi_{-4}).$$

If we consider Krieg's isomorphism Ω (cf. (4.3)), then we have

$$F_{10} \in S_{10}(U_2(\mathscr{O}_{\boldsymbol{K}}), \det^5)^{sym}$$

and

$$\Omega(F_{10}) = -\frac{2}{i} h_9.$$

From this, we have

$$a_{F_{10}}(H) = \sum_{0<d|\varepsilon(H)} d^9 \frac{1}{1+\left|\chi_{-4}\left(\frac{4\det(H)}{d^2}\right)\right|} a_9\left(\frac{4\det(H)}{d^2}\right).$$

By the similar argument to that of Lemma 4.3, we can prove the integrality of F_{10}. This proves (1). The Siegel modular form $F_{10} \mid_{\boldsymbol{S}_2}$ is a cusp form of weight 10. Since $\dim S_{10}(\Gamma_2) = 1$, $F_{10} \mid_{\boldsymbol{S}_2}$ is a constant multiple of X_{10}. If we note that

$$\begin{aligned} a_{F_{10}|\boldsymbol{S}_2}\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} &= a_{F_{10}}\begin{pmatrix} 1 & \frac{1-i}{2} \\ \frac{1+i}{2} & 1 \end{pmatrix} + a_{F_{10}}\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} + a_{F_{10}}\begin{pmatrix} 1 & \frac{1+i}{2} \\ \frac{1-i}{2} & 1 \end{pmatrix} \\ &= 1+4+1 = 6, \end{aligned}$$

we have

$$F_{10} \mid_{\boldsymbol{S}_2} = 6X_{10}. \qquad \square$$

4.2.5. Form of weight 12 over $\boldsymbol{Q}(\sqrt{-1})$.

LEMMA 4.5. *We define*

$$F_{12} := -\frac{19 \cdot 691 \cdot 2659}{2^{11} \cdot 3^7 \cdot 5^3 \cdot 7^2 \cdot 73}\left(f_{12} + \frac{2^9 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 6791}{19 \cdot 691 \cdot 2659} E_4\chi_8\right)$$

$$
\begin{aligned}
&= -\frac{34910011}{2002662144000}E_{12} - \frac{34801}{1009152000}E_4^3 + \frac{414251}{9082368000}E_4E_8 \\
&\quad + \frac{50521}{8010648576}E_6^2,
\end{aligned}
$$

where f_{12} *is the cusp form of weight* 12 *defined in* (4.2). *Then we have the following results*:

(1) $F_{12} \in S_{12}(U_2(\mathscr{O}_{\boldsymbol{K}}),1)^{sym}_{\boldsymbol{Z}}$.
(2) $F_{12} \mid_{\boldsymbol{S}_2} = X_{12}$, *where* $X_{12} \in S_{12}(\Gamma_2)$ *is Igusa's cusp form of weight* 12 *defined in* (3.1).

Proof. We set

$$
\begin{aligned}
h_{11}(z) &:= 2\theta^{10}(z)F_2^3(z) - 32\theta^6(z)F_2^4(z) \\
&= \sum_{n=1}^{\infty} a_{11}(n)q^n \in M_{11}^*(\Gamma_0(4),\chi_{-4}).
\end{aligned}
$$

If we consider Krieg's isomorphism Ω (cf. (4.3), then we have

$$F_{12} \in S_{12}(U_2(\mathscr{O}_{\boldsymbol{K}}),1)^{sym}$$

and

$$a_{F_{12}}(H) = \sum_{0<d\mid\varepsilon(H)} d^{11}\frac{1}{1+\left|\chi_{-4}\left(\dfrac{4\det(H)}{d^2}\right)\right|}\,a_{11}\left(\frac{4\det(H)}{d^2}\right).$$

This implies (1). The Siegel modular form $F_{12} \mid_{\boldsymbol{S}_2}$ is a cusp form of weight 12. Since $\dim S_{12}(\Gamma_2) = 1$, $F_{12} \mid_{\boldsymbol{S}_2}$ is a constant multiple of X_{12}. If we note that

$$
\begin{aligned}
a_{F_{12}|\boldsymbol{S}_2}\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} &= a_{F_{12}}\begin{pmatrix} 1 & \frac{1-i}{2} \\ \frac{1+i}{2} & 1 \end{pmatrix} + a_{F_{12}}\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} + a_{F_{12}}\begin{pmatrix} 1 & \frac{1+i}{2} \\ \frac{1-i}{2} & 1 \end{pmatrix} \\
&= 0+1+0 = 1,
\end{aligned}
$$

we obtain

$$F_{12} \mid_{\boldsymbol{S}_2} = X_{12}. \qquad \square$$

Summarizing these results, we obtain the following theorem.

THEOREM 4.3. *Let $\boldsymbol{K}=\boldsymbol{Q}(\sqrt{-1})$.*

(1) *The graded ring*

$$\bigoplus_{k\in 2\boldsymbol{Z}} M_k\big(U_2(\mathscr{O}_{\boldsymbol{K}}),\det^{k/2}\big)^{sym}$$

is generated by

$$E_4,\ E_6,\ \chi_8,\ F_{10} \text{ and } F_{12}$$

which are algebraically independent. Moreover, all of these forms have integral Fourier coefficients.

(2) $E_4\mid_{\boldsymbol{S}_2}=G_4$, $E_6\mid_{\boldsymbol{S}_2}=G_6$, $\chi_8\mid_{\boldsymbol{S}_2}\equiv 0$, $F_{10}\mid_{\boldsymbol{S}_2}=6X_{10}$, $F_{12}\mid_{\boldsymbol{S}_2}=X_{12}$, *where G_k is the Siegel-Eisenstein series and X_k is Igusa's cusp form defined in* (3.1).

4.3. The graded ring over $\boldsymbol{Q}(\sqrt{-3})$.

In this section, we deal with the case $\boldsymbol{K}=\boldsymbol{Q}(\sqrt{-3})$. The following result is due to Dern and Krieg.

THEOREM 4.4 (Dern-Krieg [**4**]). *Let $\boldsymbol{K}=\boldsymbol{Q}(\sqrt{-3})$.*

(1) *There exists a skew-symmetric Hermitian modular form $\phi_9 \in S_9(U_2(\mathscr{O}_{\boldsymbol{K}}),1)^{skew}$ such that*

$$\phi_9\mid_{\boldsymbol{S}_2}\equiv 0.$$

(2) *The graded ring*

$$\bigoplus_{k\in 2\boldsymbol{Z}} M_k(U_2(\mathscr{O}_{\boldsymbol{K}}),\det^{k})^{sym}$$

is generated by

$$E_4,\ E_6,\ E_{10}\ E_{12} \text{ and } \phi_9^2$$

which are algebraically independent.

REMARK. The form ϕ_9 is constructed by the Borcherds product.

As in the case that $\boldsymbol{Q}(\sqrt{-1})$, we replace some of the generators.

4.3.1. Form of weight 4 over $\boldsymbol{Q}(\sqrt{-3})$.

Lemma 4.6.

(1) $E_4 \in M_4(U_2(\mathscr{O}_{\boldsymbol{K}}), \det^4)^{sym}_{\boldsymbol{Z}}$.

(2) $E_4 \mid_{\boldsymbol{S}_2} = G_4$.

Proof.

(1) In this case, we have

$$\frac{4k(k-1)}{B_k \cdot B_{k-1,\chi_{\boldsymbol{K}}}} = \frac{4 \cdot 4 \cdot 3}{B_4 \cdot B_{3,\chi_{-3}}} = -2160 \in \boldsymbol{Z}.$$

Hence, by Theorem 4.1, we have

$$a_{E_4}(H) \in \boldsymbol{Z} \quad \text{if } \operatorname{rank}(H) = 2.$$

If $\operatorname{rank}(H) = 1$, then $a_{E_4}(H) = 240\sigma_3(\varepsilon(H))$. These facts imply

$$E_4 \in M_4(U_2(\mathscr{O}_{\boldsymbol{K}}), \det^{-2})^{sym}_{\boldsymbol{Z}} = M_4(U_2(\mathscr{O}_{\boldsymbol{K}}), \det^4)^{sym}_{\boldsymbol{Z}}.$$

A direct calculation shows (2). □

4.3.2. Form of weight 6 over $\boldsymbol{Q}(\sqrt{-3})$.

Lemma 4.7.

(1) $E_6 \in M_6(U_2(\mathscr{O}_{\boldsymbol{K}}), \det^6)^{sym}_{\boldsymbol{Z}}$.

(2) $E_6 \mid_{\boldsymbol{S}_2} = G_6$.

Proof.

(1) If $k = 6$, then

$$\frac{4k(k-1)}{B_k \cdot B_{k-1,\chi_{\boldsymbol{K}}}} = \frac{4 \cdot 6 \cdot 5}{B_6 \cdot B_{5,\chi_{-3}}} = -1512 \in \boldsymbol{Z}.$$

Hence, by the similar argument to the weight 4 case, we have

$$E_6 \in M_6(U_2(\mathscr{O}_{\boldsymbol{K}}), \det^{-3})^{sym}_{\boldsymbol{Z}} = M_6(U_2(\mathscr{O}_{\boldsymbol{K}}), \det^6)^{sym}_{\boldsymbol{Z}}.$$

We have (2) by a direct calculation. □

4.3.3. Form of weight 10 over $\boldsymbol{Q}(\sqrt{-3})$.

Lemma 4.8. *We define*

$$F_{10} := -\frac{809}{2^9 \cdot 3^5 \cdot 5^2 \cdot 7} f_{10} = -\frac{809}{21772800}(E_{10} - E_4 E_6).$$

Then we have

(1) $F_{10} \in S_{10}(U_2(\mathscr{O}_{\boldsymbol{K}}), \det^{10})_{\boldsymbol{Z}}^{sym}$.
(2) $F_{10} \mid_{\boldsymbol{S}_2} = 2X_{10}$.

Proof.
(1) We give an explicit formula for the Fourier coefficient of F_{10}. Note that

$$E_1(z) := 1 + 6\sum_{n=1}^{\infty}\Big(\sum_{0<d|n}\chi_{-3}(d)\Big)q^n \in M_1(\Gamma_0(3), \chi_{-3})$$

and

$$\Delta_3(z) := \sum_{n=1}^{\infty}\bigg(\sum_{0<d|n}\chi_{-3}(d)\Big(\frac{n}{d}\Big)^2\bigg)q^n \in M_3(\Gamma_0(3), \chi_{-3})$$

generate the graded ring

$$\bigoplus_{k\in\boldsymbol{Z}} M_k\big(\Gamma_0(3), \chi_{-3}^k\big).$$

If we set

$$h_9(z) := E_1^3(z)\Delta_3^2(z) - 27\Delta_3^3(z) = \sum_{n=1}^{\infty} a_9(n)q^n,$$

then $h_9 \in M_9^*(\Gamma_0(3), \chi_{-3})$ (the Kohnen plus-subspace), where

$$\begin{aligned} &M_k^*(\Gamma_0(3), \chi_{-3}) \\ &\quad := \Big\{ f = \sum a_f(n)q^n \in M_k(\Gamma_0(3), \chi_{-3}) \mid a_f(n) = 0, \text{ if } \chi_{-3}(n) = 1 \Big\}. \end{aligned}$$

If we consider Krieg's isomorphism (cf. (4.3)), then we have

$$F_{10} \in S_{10}(U_2(\mathscr{O}_{\boldsymbol{K}}), \det^{10})^{sym}$$

and

$$a_{F_{10}}(H) = \sum_{0<d|\varepsilon(H)} d^9 \frac{2}{1 + \left|\chi_{-3}\left(\frac{3\det(H)}{d^2}\right)\right|} a_9\left(\frac{3\det(H)}{d^2}\right).$$

A similar calculation in Lemma 4.4 shows (2). □

4.3.4. Form of weight 12 over $\boldsymbol{Q}(\sqrt{-3})$.

LEMMA 4.9. *We define*

$$F_{12} := -\frac{691 \cdot 1847}{2^{13} \cdot 3^6 \cdot 5^3 \cdot 7^2} f_{12} = -\frac{1276277}{36578304000}\left(E_{12} - \frac{441}{691}E_4^3 - \frac{250}{691}E_6^2\right).$$

Then we have

(1) $F_{12} \in S_{12}(U_2(\mathscr{O}_{\boldsymbol{K}}),1)_{\boldsymbol{Z}}^{sym}$.
(2) $F_{12} \mid_{\boldsymbol{S}_2} = 2X_{12}$.

PROOF.
(1) If we define

$$h_{11}(z) := E_1^5(z)\Delta_3^2(z) - 27E_1^2(z)\Delta_3^3(z) = \sum_{n=1}^{\infty} a_{11}(n)q^n,$$

then

$$h_{11} \in M_{11}^*(\Gamma_0(3), \chi_{-3}).$$

If we consider Krieg's isomorphism, then

$$F_{12} \in S_{12}(U_2(\mathscr{O}_{\boldsymbol{K}}),1)^{sym}$$

and

$$a_{F_{12}}(H) = \sum_{0<d|\varepsilon(H)} d^{11} \frac{2}{1 + \left|\chi_{-3}\left(\frac{3\det(H)}{d^2}\right)\right|} a_{11}\left(\frac{3\det(H)}{d^2}\right).$$

Hence we get (1).

A similar calculation in Lemma 4.5 shows (2). □

4.3.5. Form of weight 18 over $\boldsymbol{Q}(\sqrt{-3})$.

Lemma 4.10. *We define*

$$\chi_{18} := \phi_9^2.$$

Then we have the following results:

(1) $\chi_{18} \in S_{18}(U_2(\mathscr{O}_{\boldsymbol{K}}), 1)_{\boldsymbol{Z}}^{sym}$.

(2) $a_{\chi_{18}}\begin{pmatrix} 2 & 2i/\sqrt{3} \\ -2i/\sqrt{3} & 2 \end{pmatrix} = 1$, *namely* $v_p(\chi_{18}) = 0$ (*cf.* (2.1)).

(3) $\chi_{18} \mid_{\boldsymbol{S}_2} \equiv 0$.

Proof. The facts (1) and (2) come from [**4**, Corollary 3]. The fact (3) is a consequence of $\phi_9 \mid_{\boldsymbol{S}_2} \equiv 0$. □

From these results, we have the following theorem.

Theorem 4.5. *Let $\boldsymbol{K} = \boldsymbol{Q}(\sqrt{-3})$.*

(1) *The graded ring*

$$\bigoplus_{k \in 2\boldsymbol{Z}} M_k(U_2(\mathscr{O}_{\boldsymbol{K}}), \det^k)^{sym}$$

is generated by

$$E_4,\ E_6,\ F_{10},\ F_{12} \text{ and } \chi_{18}$$

which are algebraically independent. Moreover all of these forms have integral Fourier coefficients.

(2) $E_4 \mid_{\boldsymbol{S}_2} = G_4$, $E_6 \mid_{\boldsymbol{S}_2} = G_6$, $F_{10} \mid_{\boldsymbol{S}_2} = 2X_{10}$, $F_{12} \mid_{\boldsymbol{S}_2} = 2X_{12}$, $\chi_{18} \mid_{\boldsymbol{S}_2} \equiv 0$ *where G_k is the Siegel-Eisenstein series and X_k is Igusa's cusp form given in* (3.1).

5. Main theorem for the case of $\boldsymbol{K} = \boldsymbol{Q}(\sqrt{-1})$.

Throughout this section we assume that $\boldsymbol{K} = \boldsymbol{Q}(\sqrt{-1})$ and determine the structure of the Hermitian modular forms mod p. To do this, we begin with some results in the first two subsections.

5.1. Graded ring over $\boldsymbol{Z}_{(p)}$ for $\boldsymbol{Q}(\sqrt{-1})$.

In this subsection, we determine the structure of the graded ring of Hermitian modular forms with p-integral Fourier coefficients in the case $\boldsymbol{K}=\boldsymbol{Q}(\sqrt{-1})$.

We shall show the following.

Theorem 5.1. *Let $\boldsymbol{K}=\boldsymbol{Q}(\sqrt{-1})$. Assume that $p\geq 5$. If $F\in M_k(U_2(\mathscr{O}_{\boldsymbol{K}}),\nu_k)^{sym}_{\boldsymbol{Z}_{(p)}}$ (k : even), then there exists a polynomial $P(x_1,x_2,x_3,x_4,x_5)\in \boldsymbol{Z}_{(p)}[x_1,x_2,x_3,x_4,x_5]$ such that*

$$F=P(E_4,E_6,\chi_8,F_{10},F_{12}),$$

in other words,

$$\bigoplus_{0\leq k\in 2\boldsymbol{Z}} M_k(U_2(\mathscr{O}_{\boldsymbol{K}}),\nu_k)^{sym}_{\boldsymbol{Z}_{(p)}}=\boldsymbol{Z}_{(p)}[E_4,E_6,\chi_8,F_{10},F_{12}].$$

Proof. Let $F\in M_k(U_2(\mathscr{O}_{\boldsymbol{K}}),\nu_k)^{sym}_{\boldsymbol{Z}_{(p)}}$. By Theorem 4.3, there exist two polynomials $P_1\in \boldsymbol{C}[x_1,x_2,x_3,x_4]$ and $P_2\in \boldsymbol{C}[x_1,x_2,x_3,x_4,x_5]$ such that

$$F=P_1(E_4,E_6,F_{10},F_{12})+\chi_8\cdot P_2(E_4,E_6,\chi_8,F_{10},F_{12}).$$

If we restrict both sides to $\boldsymbol{S}_2$, then we obtain

$$F\mid_{\boldsymbol{S}_2}=P_1(G_4,G_6,6X_{10},X_{12})$$

because of Theorem 4.3, (2). Since $F\mid_{\boldsymbol{S}_2}\in M_k(\Gamma_2)_{\boldsymbol{Z}_{(p)}}$, there exists a unique polynomial $Q\in \boldsymbol{Z}_{(p)}[x_1,x_2,x_3,x_4]$ such that

$$F\mid_{\boldsymbol{S}_2}=P_1(G_4,G_6,6X_{10},X_{12})=Q(G_4,G_6,X_{10},X_{12}).$$

We note that the modular forms G_4, G_6, X_{10}, and X_{12} are algebraically independent (cf. Theorem 4.3) and 6^{-1} is p-integral. Therefore we see that

$$P_1\in \boldsymbol{Z}_{(p)}[x_1,x_2,x_3,x_4].$$

This implies that

$$\begin{aligned}&\chi_8\cdot P_2(E_4,E_6,\chi_8,F_{10},F_{12})\\&\quad=F-P_1(E_4,E_6,F_{10},F_{12})\in M_k(U_2(\mathscr{O}_{\boldsymbol{K}}),\nu_k)^{sym}_{\boldsymbol{Z}_{(p)}}.\end{aligned}$$

If we apply Lemma 2.1 of Section 2.2 to the left-hand side, then

$$P_2(E_4, E_6, \chi_8, F_{10}, F_{12}) \in M_{k-8}(U_2(\mathscr{O}_{\boldsymbol{K}}), \nu_{k-8})^{sym}_{\boldsymbol{Z}_{(p)}}.$$

(Note that $v_p(\chi_8) = 0$.) Using an inductive argument on the weight, we see that

$$P_2 \in \boldsymbol{Z}_{(p)}[x_1, x_2, x_3, x_4, x_5].$$

This completes the proof of Theorem 5.1. □

5.2. Existence of some modular form in the case $\boldsymbol{K} = \boldsymbol{Q}(\sqrt{-1})$.
In [**8**], the authors showed the existence of a Hermitian modular form with trivial character which is congruent to 1 modulo p under the condition that $\boldsymbol{K} = \boldsymbol{Q}(\sqrt{-1})$ or $\boldsymbol{Q}(\sqrt{-3})$. H. Hentschel and G. Nebe [**5**] have constructed such modular forms in a more general setting.

In the case of degree 2 and $p \geq 5$, we can construct such a modular form in another way.

PROPOSITION 5.1. *Let $\boldsymbol{K} = \boldsymbol{Q}(\sqrt{-1})$. Assume that $p \geq 5$. Then there exists a Hermitian modular form $F_{p-1} \in M_{p-1}(U_2(\mathscr{O}_{\boldsymbol{K}}), \nu_{p-1})^{sym}_{\boldsymbol{Z}_{(p)}}$ such that*

$$F_{p-1} \equiv 1 \pmod{p}.$$

PROOF. Let $\phi_{4,1}$ and $\phi_{6,1}$ be the normalized Hermitian Jacobi-Eisenstein series of index 1 and respective weights 4, 6. In the case of $\boldsymbol{K} = \boldsymbol{Q}(\sqrt{-1})$, all of the Fourier coefficients of ϕ_k $(k = 4, 6)$ are rational integral and the constant term is equal to 1 (e.g. cf. [**8**, Section 6]). We set

$$\psi_{p-1,1} := \begin{cases} g_4^{(p-5)/4} \cdot \phi_{4,1} & \text{if } p \equiv 1 \pmod{4}, \\ g_4^{(p-7)/4} \cdot \phi_{6,1} & \text{if } p \equiv 3 \pmod{4}, \end{cases}$$

where g_4 is the normalized Eisenstein series of weight 4 for $SL_2(\boldsymbol{Z})$. Then we have

$$\psi_{p-1,1} \in J_{p-1,1}(U_1(\mathscr{O}_{\boldsymbol{K}}), \nu_{p-1}),$$

where $J_{k,1}(U_1(\mathscr{O}_{\boldsymbol{K}}), \nu_k)$ is the space of the Jacobi forms of weight k and index 1 with character ν_k. (For the precise definition, we refer to [**2**, Section 1.3].) Moreover all of the Fourier coefficients of $\psi_{p-1,1}$ are rational integral and the constant term is equal to 1. We now take the Maass lift $\mathscr{M}$ from $J_{k,1}(U_1(\mathscr{O}_{\boldsymbol{K}}), \nu_k)$

to $M_k(U_2(\mathscr{O}_{\boldsymbol{K}}),\nu_k)$ as in [**2**, p. 80, (4.3)]. Then

$$F_{p-1} := -\frac{2(p-1)}{B_{p-1}}\mathscr{M}(\psi_{p-1,1})$$

satisfies

$$F_{p-1} \in M_{p-1}(U_2(\mathscr{O}_{\boldsymbol{K}}),\nu_{p-1})^{sym}_{\boldsymbol{Z}_{(p)}} \quad \text{and} \quad F_{p-1} \equiv 1 \pmod{p}. \qquad \square$$

5.3. Structure of the algebra of mod p Hermitian modular forms over $\boldsymbol{Q}(\sqrt{-1})$.

In this subsection, we determine the structure of the ring of Hermitian modular forms mod p. The following theorem is one of our main results.

Theorem 5.2. *Let $\boldsymbol{K} = \boldsymbol{Q}(\sqrt{-1})$ and $p \geq 5$. We take a modular form*

$$F_{p-1} \in M_{p-1}(U_2(\mathscr{O}_{\boldsymbol{K}}),\nu_{p-1})^{sym}_{\boldsymbol{Z}_{(p)}} \quad \textit{such that } F_{p-1} \equiv 1 \pmod{p}.$$

(*The existence of such a form is guaranteed by Proposition* 5.1.)

If $B(x_1,x_2,x_3,x_4,x_5) \in \boldsymbol{Z}_{(p)}[x_1,x_2,x_3,x_4,x_5]$ is the polynomial defined by

$$F_{p-1} = B(E_4,E_6,\chi_8,F_{10},F_{12}),$$

then the polynomial $\widetilde{B}-1$ is irreducible in $\boldsymbol{F}_p[x_1,x_2,x_3,x_4,x_5]$ and

$$\widetilde{M}^{(e)}(U_2(\mathscr{O}_{\boldsymbol{K}}),\nu)^{sym}_p \cong \boldsymbol{F}_p[x_1,x_2,x_3,x_4,x_5]/(\widetilde{B}-1). \tag{5.1}$$

Proof. First we show the irreducibility of $\widetilde{B}-1$. We restrict the Hermitian modular form F_{p-1} to $\boldsymbol{S}_2$. Then it still satisfies the congruence

$$F_{p-1}\mid_{\boldsymbol{S}_2} \equiv 1 \pmod{p}.$$

By Theorem 3.2, there exists a polynomial $A \in \boldsymbol{Z}_{(p)}[x_1,x_2,x_3,x_4]$ such that $F_{p-1}\mid_{\boldsymbol{S}_2} = A(G_4,G_6,X_{10},X_{12})$ and $\widetilde{A}-1$ is irreducible in $\boldsymbol{F}_p[x_1,x_2,x_3,x_4]$. Seeking a contradiction, we suppose that $\widetilde{B}-1$ is decomposed in $\boldsymbol{F}_p[x_1,x_2,x_3,x_4,x_5]$:

$$\widetilde{B}(x_1,x_2,x_3,x_4,x_5)-1 = \widetilde{P_1}(x_1,x_2,x_3,x_4,x_5)\widetilde{P_2}(x_1,x_2,x_3,x_4,x_5).$$

This implies that

$$\begin{aligned}\widetilde{A}(x_1,x_2,x_3,x_4)-1&=\widetilde{B}(x_1,x_2,0,\widetilde{6}x_3,x_4)-1\\&=\widetilde{P_1}(x_1,x_2,0,\widetilde{6}x_3,x_4)\widetilde{P_2}(x_1,x_2,0,\widetilde{6}x_3,x_4).\end{aligned}$$

This contradicts the irreducibility of $\widetilde{A}-1$.

Secondly, we show the isomorphism (5.1). We consider the following diagram:

$$\begin{tikzcd}
\boldsymbol{F}_p[x_1,x_2,x_3,x_4,x_5] \arrow[r, "\varphi_H"] \arrow[d, "\rho"'] & \widetilde{M}^{(e)}(U_2(\mathscr{O}_{\boldsymbol{K}}),\nu)_p^{sym} \arrow[d, "\psi"] \\
\boldsymbol{F}_p[x_1,x_2,x_3,x_4] \arrow[r, "\varphi_S"] & \widetilde{M}^{(e)}(\Gamma_2)_p
\end{tikzcd}$$

where

$$\begin{aligned}\varphi_H\big(\widetilde{P}(x_1,x_2,x_3,x_4,x_5)\big)&:=\widetilde{P}\big(\widetilde{E}_4,\widetilde{E}_6,\widetilde{\chi}_8,\widetilde{F}_{10},\widetilde{F}_{12}\big),\\ \varphi_S\big(\widetilde{Q}(x_1,x_2,x_3,x_4)\big)&:=\widetilde{Q}\big(\widetilde{G}_4,\widetilde{G}_6,\widetilde{X}_{10},\widetilde{X}_{12}\big),\\ \rho\big(\widetilde{P}(x_1,x_2,x_3,x_4,x_5)\big)&:=\widetilde{P}\big(x_1,x_2,0,\widetilde{6}x_3,x_4\big),\\ \psi\big(\widetilde{F}\big)&:=\widetilde{F\mid_{\boldsymbol{S}_2}}\in\widetilde{M}^{(e)}(\Gamma_2)_p.\end{aligned}$$

We shall show that

$$\operatorname{Ker}\varphi_H=\big(\widetilde{B}-1\big). \tag{5.2}$$

The inclusion $(\widetilde{B}-1)\subset\operatorname{Ker}\varphi_H$ is a consequence of $F_{p-1}\equiv 1 \pmod p$. Assume that

$$\big(\widetilde{B}-1\big)\subsetneq\operatorname{Ker}\varphi_H. \tag{5.3}$$

Since the map φ_H is surjective, we have

$$\boldsymbol{F}_p[x_1,x_2,x_3,x_4,x_5]/\operatorname{Ker}\varphi_H\cong\widetilde{M}^{(e)}(U_2(\mathscr{O}_{\boldsymbol{K}}),\nu)_p^{sym}.$$

By the assumption (5.3), we have

$$\begin{aligned}&\text{Krull dim}\,\widetilde{M}^{(e)}(U_2(\mathscr{O}_{\boldsymbol{K}}),\nu)_p^{sym}\\&\quad=\text{Krull dim}\,\boldsymbol{F}_p[x_1,x_2,x_3,x_4,x_5]/\text{Ker}\,\varphi_H\le 3.\end{aligned}\tag{5.4}$$

On the other hand, the map ψ is surjective and

$$\widetilde{M}^{(e)}(U_2(\mathscr{O}_{\boldsymbol{K}}),\nu)_p^{sym}/\text{Ker}\,\psi\cong\widetilde{M}^{(e)}(\Gamma_2)_p.$$

We note that $\text{Ker}\,\psi\ne 0$. In fact $\widetilde{\chi}_8$ is a non-zero element of $\text{Ker}\,\psi$. Since

$$\text{Krull dim}\,\widetilde{M}^{(e)}(U_2(\mathscr{O}_{\boldsymbol{K}}),\nu)_p^{sym}/\text{Ker}\,\psi=3\quad\text{(cf. Theorem 3.2)},$$

we have

$$\begin{aligned}3&=\text{Krull dim}\,\widetilde{M}^{(e)}(U_2(\mathscr{O}_{\boldsymbol{K}}),\nu)_p^{sym}/\text{Ker}\,\psi\\&<\text{Krull dim}\,\widetilde{M}^{(e)}(U_2(\mathscr{O}_{\boldsymbol{K}}),\nu)_p^{sym}.\end{aligned}$$

This contradicts (5.4) and completes the proof of (5.1). □

6. Main theorem in the case $\boldsymbol{K}=\boldsymbol{Q}(\sqrt{-3})$.

In this section, we assume that $\boldsymbol{K}=\boldsymbol{Q}(\sqrt{-3})$ and determine the structure of the corresponding algebra of Hermitian modular forms mod p. The proof is carried out by similar argument as in the case $\boldsymbol{Q}(\sqrt{-1})$.

6.1. Graded ring over $\boldsymbol{Z}_{(p)}$ for $\boldsymbol{Q}(\sqrt{-3})$.

As in Section 5.1, we determine the structure of the graded ring of Hermitian modular forms with p-integral Fourier coefficients in the case $\boldsymbol{K}=\boldsymbol{Q}(\sqrt{-3})$.

Theorem 6.1. *Assume that $p\ge 5$. If $F\in M_k(U_2(\mathscr{O}_{\boldsymbol{K}}),\nu_k)_{\boldsymbol{Z}_{(p)}}^{sym}$ (k : even), then there exists a polynomial $P(x_1,x_2,x_3,x_4,x_5)\in\boldsymbol{Z}_{(p)}[x_1,x_2,x_3,x_4,x_5]$ such that*

$$F=P(E_4,E_6,F_{10},F_{12},\chi_{18}),$$

in other words

$$\bigoplus_{0\le k\in 2\boldsymbol{Z}}M_k(U_2(\mathscr{O}_{\boldsymbol{K}}),\nu_k)_{\boldsymbol{Z}_{(p)}}^{sym}=\boldsymbol{Z}_{(p)}[E_4,E_6,F_{10},F_{12},\chi_{18}].$$

Here $\nu_k := \det^k$ (*cf.* (2.2)).

PROOF. In the argument in Theorem 5.1, we replace χ_8 by χ_{18}. □

6.2. Existence of some modular form in the case $\boldsymbol{K} = \boldsymbol{Q}(\sqrt{-3})$. We present the corresponding result to Proposition 5.1.

PROPOSITION 6.1. *Let $\boldsymbol{K} = \boldsymbol{Q}(\sqrt{-3})$ and $p \geq 5$. Then there exists a Hermitian modular form $F_{p-1} \in M_{p-1}(U_2(\mathscr{O}_{\boldsymbol{K}}), \nu_{p-1})^{sym}_{\boldsymbol{Z}_{(p)}}$ such that*

$$F_{p-1} \equiv 1 \pmod{p}.$$

PROOF. The proof of Proposition 5.1 is essentially valid in this case after making a minor change. Let $\phi_{4,1}$ and $\phi_{6,1}$ be the normalized Hermitian Jacobi-Eisenstein series of index 1 and respective weight 4, 6. In the case of $\boldsymbol{K} = \boldsymbol{Q}(\sqrt{-3})$, all of the Fourier coefficients of ϕ_k $(k = 4, 6)$ are rational integral and the constant term is equal to 1. Following the argument in Proposition 5.1, we set

$$\psi_{p-1,1} := \begin{cases} g_6^{(p-7)/6} \cdot \phi_{6,1} & \text{if } p \equiv 1 \pmod{6}, \\ g_6^{(p-5)/6} \cdot \phi_{4,1} & \text{if } p \equiv 5 \pmod{6}, \end{cases}$$

where g_6 is the normalized Eisenstein series of weight 6 for $SL_2(\boldsymbol{Z})$. Then we can construct F_{p-1} by taking the Maass lift as in the proof of Proposition 5.1. □

6.3. Structure of the algebra of mod p Hermitian modular forms over $\boldsymbol{Q}(\sqrt{-3})$.
We state the structure theorem of Hermitian modular forms mod p in the case $\boldsymbol{Q}(\sqrt{-3})$.

THEOREM 6.2. *Let $\boldsymbol{K} = \boldsymbol{Q}(\sqrt{-3})$ and $p \geq 5$. We take a modular form*

$$F_{p-1} \in M_{p-1}(U_2(\mathscr{O}_{\boldsymbol{K}}), \nu_{p-1})^{sym}_{\boldsymbol{Z}_{(p)}} \quad \text{such that } F_{p-1} \equiv 1 \pmod{p}.$$

(*The existence of such form is guaranteed by Proposition* 6.1.)

If $B(x_1, x_2, x_3, x_4, x_5) \in \boldsymbol{Z}_{(p)}[x_1, x_2, x_3, x_4, x_5]$ is the polynomial defined by

$$F_{p-1} = B(E_4, E_6, F_{10}, F_{12}, \chi_{18}),$$

then the polynomial $\widetilde{B} - 1$ is irreducible in $\boldsymbol{F}_p[x_1, x_2, x_3, x_4, x_5]$ and

$$\widetilde{M}^{(e)}(U_2(\mathscr{O}_{\boldsymbol{K}}),\nu)_p^{sym} \cong \boldsymbol{F}_p[x_1,x_2,x_3,x_4,x_5]/(\widetilde{B}-1).$$

Proof. The proof is similar to that of Theorem 5.2. □

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