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On Hermite-Liouville manifolds

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Abstract. In this paper we study a certain class of Hermitian n-manifolds whose geodesic flows admit n first integrals of certain kind. It is a generalization of Kähler-Liouville manifold in [**3**] and called Hermite-Liouville manifold. We completely determine the local structure of Hermite-Liouville manifolds "of type (A)", and construct global examples over the complex projective space.

1. Introduction.

It is well known that the geodesic flow of the complex projective space $\boldsymbol{CP}^n$ endowed with the standard Kähler metric is integrable in the sense of Hamiltonian mechanics (cf. [**6**]). Actually, the geodesic flow possesses n first integrals which are fiberwise quadratic polynomials and also n first integrals which are fiberwise linear forms, and they are mutually commutative with respect to the Poisson bracket.

The notion of a (proper) Kähler-Liouville manifold was given in [**3**, Part 2] by the second author as a class of Kähler manifolds whose geodesic flows can be integrated in a similar way to that of $\boldsymbol{CP}^n$ (see also [**4**], [**5**]). In another viewpoint, it can be regarded as a complexification (a Hermitian version) of the notion of Liouville manifold. (Liouville manifold is a class of Riemannian manifolds whose metrics are of Liouville-Stäckel type. For the precise definition, see [**3**, Part 1].) The main purpose in [**3**, Part 2] was to investigate global structures of such manifolds. A preceding study for the two-dimensional case was made in [**2**] by the first author.

By definition, a Kähler-Liouville manifold is a pair of a Kähler manifold (M, g, J), $\dim_{\boldsymbol{C}} M = n$, g the metric, J the complex structure, and an n-dimensional real vector space $\mathscr{F}$ of functions on the cotangent bundle T^*M which satisfies the following conditions.

(1) The Poisson bracket $\{F, H\}$ vanishes for every $F, H \in \mathscr{F}$.

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(2) $\mathscr{F}$ contains the Hamiltonian E of the geodesic flow.
(3) For every $F \in \mathscr{F}$ and $p \in M$, $F_p := F|_{T_p^*M}$ is a Hermitian form, i.e., a homogeneous quadratic polynomial which is invariant by the complex structure J.
(4) F_p, $F \in \mathscr{F}$, are simultaneously normalizable for each $p \in M$.
(5) $\mathscr{F}_p := \{F_p; F \in \mathscr{F}\}$ is n-dimensional at some point $p \in M$.

In the above definition, n first integrals which are fiberwise quadratic forms are provided, but first integrals which are fiberwise linear forms are not mentioned. This omission in the definition is justified by the fact that, under a certain nondegeneracy condition, n first integrals which are fiberwise linear forms appear automatically and yield the integrability ([**3**, p. 94]). Those fiberwise-linear first integrals are actually infinitesimal automorphisms of the Kähler manifold M and, if M is compact, then they generate an n torus action and M becomes a toric variety.

Although Kähler-Liouville manifolds provide good examples of Hermitian manifolds with integrable geodesic flows, the Kähler condition itself is *a priori* unrelated to the integrability of the geodesic flows. Moreover, as is easily observed, if E is the Hamiltonian of the geodesic flow and $F \in \mathscr{F}$ is small enough on a Kähler-Liouville manifold, then the metric g' corresponding to $E+F$ is not necessarily Kählerian, but the geodesic flow of (M, g') is still integrable. These facts motivated us to investigate Hermitian manifolds which have similar properties to Kähler-Liouville manifolds. We call such manifolds Hermite-Liouville manifolds. The definition is as follows: An Hermite-Liouville manifold is a pair of a Hermitian manifold (M, g, J) and a real vector space $\mathscr{F}$ of functions on T^*M satisfying the five conditions (1)–(5) stated above. Some previous examples have been described in [**1**].

The aim of this paper is to investigate local structures of Hermite-Liouville manifolds and to construct a family of global non-Kähler examples over complex projective space $\boldsymbol{CP}^n$. We shall mainly treat Hermite-Liouville manifolds with nondegeneracy conditions which are the same as employed with Kähler-Liouville manifolds, called Hermite-Liouville manifolds of type (A). We shall give a complete description of their local structures and then discuss the local integrability of the geodesic flow and the Kähler condition. As a consequence, it will turn out that the following three groups of manifolds are indeed different from one another: Hermite-Liouville manifolds of type (A); those with n fiberwise-linear first integrals; Kähler-Liouville manifolds of type (A). The difference of the latter two groups will also be verified in the global setting.

This paper is organized as follows. In Section 2 we investigate local structures of Hermite-Liouville manifolds of type (A) and present their basic properties. Based on a similar procedure to the case of Liouville-Stäckel's system, we obtain a "canonical form" of the system (Theorem 2.4). Different from the Liouville-Stäckel case, the situation is not completely trivialized in this stage; a matrix-valued function $[\kappa_{ij}]$, a part of the representation matrix of the complex structure, is involved in the formula. It turns out that this function $[\kappa_{ij}]$ plays a key role in determining the local structure. In the subsequent argument, we find a system of partial differential equations which the functions κ_{ij}'s satisfy. In Section 3 we analyze the function $[\kappa_{ij}]$ completely and thus determine the possible forms of $[\kappa_{ij}]$ by solving the system of equations. We also have an argument concerning the complete integrability of the geodesic flow and one concerning the Kähler condition. In the next section, Section 4, first we verify the existence of Hermite-Liouville manifold to each solution $[\kappa_{ij}]$. Then, we concentrate our attention on the case where the geodesic flow possesses n fiberwise-linear first integrals which yield the local integrability and present a slightly modified local construction. Finally, in Section 5, we illustrate how to construct Hermite-Liouville manifolds over $\boldsymbol{CP}^n$ by means of two sets of data for (real) Liouville manifolds defined over $\boldsymbol{RP}^n$. It is shown that those two sets of data parametrize the isomorphism classes of constructed Hermite-Liouville manifolds almost effectively. Also, we show that the constructed Hermite-Liouville manifold is Kählerian if and only if the two sets of data coincide.

Throughout this paper, we assume the differentiability of class C^∞.

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2. Basic properties.

In the present section we shall describe the basic local properties of the Hermite-Liouville manifold. Via a similar argument to the case of Liouville-Stäckel's system, we shall obtain a canonical coordinate system and a canonical form of the system (Theorem 2.4). With them we shall obtain a matrix-valued function $[\kappa_{ij}]$, a key ingredient of the description of the local structure, and find a system of partial differential equations which the functions κ_{ij}'s satisfy (Proposition 2.5 (2)). Throughout this section and the next section, we shall use the convention that indices i, j, k, ℓ, s, t take the integer values $1, \ldots, n$, unless otherwise stated.

Let $(M, g, J; \mathscr{F})$ be an n-dimensional Hermite-Liouville manifold and let $F_1, \ldots, F_n$ a basis of $\mathscr{F}$. For any $F \in \mathscr{F}$ and for any point $p \in M$, we put $F_p = F\,|_{T_p^*M}$, which is regarded as a quadratic form on the cotangent space T_p^*M to p. For each $p \in M$, we set $\mathscr{F}_p = \{F_p; F \in \mathscr{F}\}$, which forms a real vector space of $\dim \leq n$. Let M^0 denote the set of all points $p \in M$ satisfying $\dim(\mathscr{F}_p) = n$, which is an open subset in M. We take an arbitrary point $p_0 \in M^0$ and take a sufficiently small open neighborhood Ω of p_0 in M^0. Let $T\Omega$ and $T^*\Omega$ denote the tangent bundle over Ω and the cotangent bundle over Ω respectively. There then exist an orthonormal frame $V_1, JV_1, \dots, V_n, JV_n$ on Ω, and n^2 functions a_{ij}, $i, j = 1, \dots, n$, on Ω such that, for each i,

$$\sum_{j=1}^{n} a_{ij} F_j = {V_i}^2 + (JV_i)^2 \qquad \text{on } T^*\Omega, \tag{2.1}$$

where V_i, JV_i are regarded as fiberwise linear forms on $T^*\Omega$. Note that the $n \times n$-matrix-valued function $[a_{ij}]$ on Ω is nonsingular at all points in Ω. We also notice that, for each j,

$$\sum_{i=1}^{n} a_{ij} = \text{constant} \qquad \text{on } \Omega. \tag{2.2}$$

In fact, by (2.1) we have

$$2E = \sum_{i=1}^{n} \left({V_i}^2 + (JV_i)^2\right) = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} a_{ij}\right) F_j.$$

The fact that $E \in \mathscr{F}$, which is the condition (2) in the definition of the Hermite-Liouville manifold, implies (2.2).

For each i, we define a complex line bundle D_i over Ω by $(D_i)_p = \mathrm{Span}\{(V_i)_p, (JV_i)_p\}$ for every $p \in \Omega$, where $\mathrm{Span}\{(V_i)_p, (JV_i)_p\}$ means the subspace of the tangent space $T_p\Omega$ spanned by $(V_i)_p$, $(JV_i)_p$. It follows that $T\Omega$ can be written as the direct sum of the complex line bundles $D_1, \dots, D_n$; $T\Omega = D_1 \oplus \cdots \oplus D_n$. Note that the bundles $D_1, \dots, D_n$ are uniquely determined except their ordering; the frame V_i, JV_i is not though.

Let J^* be the complex structure in $T^*\Omega$ defined by

$$\langle J^* u, X\rangle = \langle u, JX\rangle$$

for any 1-form u and for any vector field X on Ω. We can find 1-forms $V_1^*, \dots, V_n^*$ on Ω such that $V_1^*, -J^*V_1^*, \dots, V_n^*, -J^*V_n^*$ forms the coframe on Ω which is dual to the orthonormal frame $V_1, JV_1, \dots, V_n$, JV_n at every point in Ω. For each i, we moreover define the bundle D_i^* over Ω by $V_i^*, J^*V_i^*$ as in the same manner in the definition of D_i, which can also be considered as a complex line bundle with respect to J^*. We note that $T^*\Omega = D_1^* \oplus \cdots \oplus D_n^*$.

Taking certain n^2 real constants r_{ij}, $i,j=1,\dots,n$, we can define n functions $a_1,\dots,a_n$ on Ω by

$$a_i = r_{i1}a_{i1} + r_{i2}a_{i2} + \cdots + r_{in}a_{in}, \quad i=1,\dots,n, \tag{2.3}$$

so that $a_1,\dots,a_n$ are all positive on the whole of Ω. We set

$$b_{ij} = \frac{a_{ij}}{a_i}, \quad i,j=1,\dots,n, \quad \text{on } \Omega \tag{2.4}$$

and

$$W_i = \frac{1}{\sqrt{a_i}}V_i, \quad i=1,\dots,n, \quad \text{on } \Omega. \tag{2.5}$$

It follows that the $n\times n$-matrix-valued function $[b_{ij}]$ on Ω is also nonsingular at all points in Ω, and that the vector fields $W_1, JW_1, \dots, W_n, JW_n$ form an orthogonal frame on Ω. From (2.1) we obtain

$$\sum_{j=1}^{n} b_{ij}F_j = {W_i}^2 + (JW_i)^2, \quad i=1,\dots,n, \quad \text{on } T^*\Omega, \tag{2.6}$$

where W_i, JW_i are also regarded as fiberwise linear forms on $T^*\Omega$. We moreover have the following

Proposition 2.1.

(1) *For any two distinct i,j and for any k,*

$$W_i b_{jk} = (JW_i)b_{jk} = 0 \quad \textit{on } \Omega. \tag{2.7}$$

(2) *For any i,j,*

$$\left\{{W_i}^2 + (JW_i)^2,\ {W_j}^2 + (JW_j)^2\right\} = 0 \quad \textit{on } T^*\Omega. \tag{2.8}$$

Proof. We first verify that there exist $2n^2-2n$ functions λ_{st}, μ_{st}, $s\neq t$, $s,t=1,\dots,n$, on Ω such that, for any two distinct i,j and for any k,

$$V_i a_{jk} = \lambda_{ij}a_{jk}, \quad (JV_i)a_{jk} = \mu_{ij}a_{jk} \quad \text{on } \Omega. \tag{2.9}$$

In fact, define functions $\tilde{\lambda}_{st\ell}$, $\tilde{\mu}_{st\ell}$, $s \neq t$, $s, t, \ell = 1, \dots, n$, on Ω by

$$\tilde{\lambda}_{st\ell} = \sum_{m=1}^{n} \check{a}_{m\ell}(V_s a_{tm}), \quad \tilde{\mu}_{st\ell} = \sum_{m=1}^{n} \check{a}_{m\ell}((JV_s)a_{tm}),$$

where $\check{a}_{m\ell}$ denotes the (m, ℓ)-entry of the inverse matrix of $[a_{ij}]$. It follows that, for any two distinct i, j and for any k,

$$V_i a_{jk} = \sum_{\ell=1}^{n} \tilde{\lambda}_{ij\ell} a_{\ell k}, \quad (JV_i)a_{jk} = \sum_{\ell=1}^{n} \tilde{\mu}_{ij\ell} a_{\ell k} \quad \text{on } \Omega. \tag{2.10}$$

Taking the Poisson Bracket of ${V_i}^2 + (JV_i)^2 = \sum_{s=1}^{n} a_{is} F_s$ and ${V_j}^2 + (JV_j)^2 = \sum_{t=1}^{n} a_{jt} F_t$, both of which represent (2.1), with $i \neq j$, we have

$$\begin{aligned}
&\big\{{V_i}^2 + (JV_i)^2, {V_j}^2 + (JV_j)^2\big\} \\
&\quad = \sum_{s=1}^{n} \sum_{t=1}^{n} \big(a_{it}\{F_t, a_{js}\} - a_{jt}\{F_t, a_{is}\}\big) F_s \\
&\quad = \sum_{s=1}^{n} \big(\big\{{V_i}^2 + (JV_i)^2, a_{js}\big\} - \big\{{V_j}^2 + (JV_j)^2, a_{is}\big\}\big) F_s \\
&\quad = 2 \sum_{\ell=1}^{n} \big(\tilde{\lambda}_{ij\ell} V_i + \tilde{\mu}_{ij\ell}(JV_i) - \tilde{\lambda}_{ji\ell} V_j - \tilde{\mu}_{ji\ell}(JV_j)\big)\big({V_\ell}^2 + (JV_\ell)^2\big)
\end{aligned} \tag{2.11}$$

by virtue of the condition that $\{F_s, F_t\} = 0$, (2.1), and (2.10).

Both sides of (2.11) are regarded as polynomials in the variables V_m, JV_m, $m = 1, \dots, n$. Since the left-hand side belongs to the ideal generated by $V_i V_j$, $V_i(JV_j)$, $V_j(JV_i)$, and $(JV_i)(JV_j)$, it follows that

$$\tilde{\lambda}_{ij\ell} = \tilde{\mu}_{ij\ell} = 0 \quad \text{on } \Omega \qquad \text{unless } \ell = j.$$

Putting $\lambda_{ij} = \tilde{\lambda}_{ijj}$, $\mu_{ij} = \tilde{\mu}_{ijj}$, we thus obtain (2.9) from (2.10), completing the verification. By the definition (2.3) of a_i and (2.9), we also have, for any two distinct i, j,

$$V_i a_j = \lambda_{ij} a_j, \quad (JV_i)a_j = \mu_{ij} a_j \quad \text{on } \Omega. \tag{2.12}$$

From (2.4), (2.5), (2.9), and (2.12), we obtain (2.7), thus proving (1). Since

$\tilde{\lambda}_{ij\ell} = \delta_{j\ell}\lambda_{ij}$, $\tilde{\mu}_{ij\ell} = \delta_{j\ell}\mu_{ij}$, where $\delta_{j\ell}$ denotes Kronecker's symbol, it follows from (2.11) that, for any two distinct i, j,

$$\begin{aligned}&\bigl\{{V_i}^2 + (JV_i)^2, {V_j}^2 + (JV_j)^2\bigr\}\\ &\quad = -2\bigl(\lambda_{ji}V_j + \mu_{ji}(JV_j)\bigr)\bigl({V_i}^2 + (JV_i)^2\bigr) + 2\bigl(\lambda_{ij}V_i + \mu_{ij}(JV_i)\bigr)\bigl({V_j}^2 + (JV_j)^2\bigr).\end{aligned}$$

The assertion (2) thus follows by direct calculation from this relation and (2.12). (See also [**3**, pp. 84–85].) □

We now introduce nondegenerecy conditions for Hermite-Liouville manifold, which are the counterpart of the condition in [**3**, p. 85] appearing in the definition of type (A) for Kähler-Liouville manifold. We shall denote by $[X, Y]_{D_i}$ the D_i-component of the vector field $[X, Y]$ for any vector fields X, Y on Ω. An Hermite-Liouville manifold $(M, g, J; \mathscr{F})$ is said to be of type (A) if there exists a point p in M^0 at which the following (A-i) and (A-ii) hold:

(A-i) For any i, there exists $k(\neq i)$ such that $([W_k, JW_k]_{D_i})_p \neq 0$;
(A-ii) For any i, there exists ℓ such that $(db_{i\ell})_p \neq 0$.

Note that these conditions do not depend on the choice of the functions $a_1, \dots, a_n$.

Let $(M, g, J; \mathscr{F})$ be an Hermite-Liouville manifold of type (A). We take a basis $F_1, \dots, F_n$ of $\mathscr{F}$ and fix it in the rest of this section and next section, Section 3. We set

$$M^1 = \{p \in M^0;\ \text{Both (A-i) and (A-ii) hold at } p\}. \tag{2.13}$$

Clearly, M^1 is an open subset of M^0 and hence that of M. Let p_0 be an arbitrary point in M^1. We take a sufficiently small neighborhood Ω of p_0 in M^1, the functions b_{ij}, $i, j = 1, \dots, n$, on Ω defined in (2.4), and the vector fields $W_1, \dots, W_n$ on Ω defined in (2.5). Notice that they satisfy the conditions (2.6), (2.7) and (2.8). It follows that $W_1, JW_1, \dots, W_n, JW_n$ form an orthogonal frame on Ω.

Lemma 2.2. *For each i,*

$$\mathrm{Span}\bigl\{([W_k, JW_k]_{D_i})_p\bigr\} = \mathrm{Ker}(db_{i\ell}|_{D_i})_p \quad \text{at every } p \in \Omega,$$

where k and ℓ are the indices taken in the above (A-i) *and* (A-ii) *respectively, and where* $\mathrm{Span}\{([W_k, JW_k]_{D_i})_p\}$ *denotes the real vector space spanned by the vector* $([W_k, JW_k]_{D_i})_p$.

Proof. By Proposition 2.1 (1), we have

$$\mathrm{Span}\big\{([W_k, JW_k]_{D_i})_p\big\} \subseteq \mathrm{Ker}(db_{i\ell}|_{D_i})_p \quad \text{at every } p \in \Omega.$$

From (A-i) and (A-ii) we obtain $\dim(\mathrm{Span}\{([W_k, JW_k]_{D_i})_p\}) = 1$ and $\dim(\mathrm{Ker}(db_{i\ell}|_{D_i})_p) = 1$. These imply the desired equality. □

With the notation of Lemma 2.2, we define real line bundles D_i^-, $i = 1, \dots, n$, over Ω by

$$(D_i^-)_p = \mathrm{Span}\big\{([W_k, JW_k]_{D_i})_p\big\} = \mathrm{Ker}(db_{i\ell}|_{D_i})_p \tag{2.14}$$

for each $p \in \Omega$. It follows that D_i^- is a subbundle of D_i and hence that of $T\Omega$. We moreover define a real vector bundle D^- over Ω by

$$D^- = D_1^- \oplus \cdots \oplus D_n^- \tag{2.15}$$

and a real vector bundle D^+ over Ω by

$$D^+ = JD^-, \tag{2.16}$$

both of which are subbundles of $T\Omega$ with real rank n. It follows that $T\Omega = D^+ \oplus D^-$. For any vector bundle D over Ω, we shall denote by $\Gamma(\Omega, D)$ the vector space of all cross sections of D. We can take an orthogonal frame $W_1, JW_1, \dots, W_n, JW_n$ on Ω so that it satisfies the condition

$$JW_i \in \Gamma(\Omega, D_i^-), \quad i = 1, \dots, n. \tag{2.17}$$

Notice that it also satisfies (2.6), (2.7) and (2.8). We shall call such a frame $\mathscr{F}$-adapted orthogonal frame on Ω.

Proposition 2.3. *Let $W_1, JW_1, \dots, W_n, JW_n$ be an $\mathscr{F}$-adapted orthogonal frame on Ω.*

(1) *For any two distinct i, j and for any k,*

$$W_i b_{jk} = 0 \quad \text{on } \Omega \tag{2.18}$$

and, for any i, j, k,

$$(JW_i) b_{jk} = 0, \quad (JW_i) a_j = 0 \qquad \text{on } \Omega. \tag{2.19}$$

(2) *For any two distinct* i, j,

$$[W_i, W_j] = [JW_i, JW_j] = [W_i, JW_j] = 0 \quad \text{on } \Omega. \tag{2.20}$$

In particular, by regarding D^+ *and* D^- *as distributions, both* D^+ *and* D^- *are involutive.*
(3) *For any* i,

$$[W_i, JW_i] \in \Gamma(\Omega, D^-). \tag{2.21}$$

Proof.
(1) In Proposition 2.1 (1), we have already proved (2.18) and $(JW_i)b_{jk} = 0$ with $i \neq j$. The equality $(JW_i)b_{ik} = 0$ immediately follows from (2.14) and (2.17). From (2.2) and (2.4), we obtain

$$\sum_{t=1}^{n} a_t b_{tm} = \text{constant}, \quad m = 1, \ldots, n.$$

This implies that a_j's are rational functions of b_{tm}, $t, m = 1, \ldots, n$. Therefore we have $(JW_i)a_j = 0$.

(2) Proposition 2.1 (2) is equivalent to the condition that there are $2n^2 - 2n$ functions α_{st}, β_{st}, $s \neq t$, $s, t = 1, \ldots, n$, such that, for any two distinct i, j,

$$\begin{aligned} [W_i, W_j] &= \alpha_{ij} JW_i - \alpha_{ji} JW_j, \\ [W_i, JW_j] &= \beta_{ij} JW_i + \alpha_{ji} W_j, \\ [JW_i, JW_j] &= -\beta_{ij} W_i + \beta_{ji} W_j \quad \text{on } \Omega. \end{aligned}$$

Taking the values of the 1-forms $db_{j\ell}$ and $db_{i\ell}$ on both sides of the second and third equalities respectively, where ℓ is the index taken in (A-ii), we see from (1) and (A-ii) that all α_{st}, β_{st} vanish on Ω, which implies (2).

(3) We see from Proposition 2.1 (1) that, if $t \neq i$, then $[W_i, JW_i]b_{tm} = 0$, $m = 1, \ldots, n$, and hence $([W_i, JW_i]_{D_t})_p$ is a scalar multiple of $(JW_t)_p$ at each $p \in \Omega$. In view of this fact, (2.15), and (2.17), it is sufficient to show that $([W_i, JW_i]_{D_i})_p$ is also a scalar multiple of $(JW_i)_p$ at each $p \in \Omega$. For any j such that $j \neq i$, we consider the vector field $[W_j, JW_j]$ on Ω. By the above arguement, it can be written as

$$[W_j, JW_j] = [W_j, JW_j]_{D_j} + \sum_{s=1}^{j-1} \zeta_{js} JW_s + \sum_{s=j+1}^{n} \zeta_{js} JW_s \quad \text{on } \Omega, \tag{2.22}$$

where ζ_{js}, $s \neq j$, are certain functions on Ω. By (2.20) in (2) we have $[W_i, [W_j, JW_j]] = 0$ on Ω and hence

$$[W_i, [W_j, JW_j]]_{D_i} = 0 \quad \text{on } \Omega.$$

Substituting (2.22) into this relation and again using (2.20) in (2), we obtain, for any $j(\neq i)$,

$$\zeta_{ji}[W_i, JW_i]_{D_i} + (W_i\zeta_{ji})JW_i = 0 \quad \text{on } \Omega. \tag{2.23}$$

In view of (2.22), the condition (A-i) implies that, for each point $p \in \Omega$,

$$\zeta_{ki}(p) \neq 0 \quad \text{for some } k(\neq i). \tag{2.24}$$

The assertion follows from (2.23) and (2.24). □

We now introduce the $\mathscr{F}$-adapted coordinate system to the neighborhood Ω of p_0. Let $W_1, JW_1, \ldots, W_n, JW_n$ be an $\mathscr{F}$-adapted orthogonal frame on Ω. Let S^- be the maximal integral submanifold of D^- in Ω through p_0. We take 1-forms $W_1^*, \ldots, W_n^*$ on Ω so that $W_1^*, -J^*W_1^*, \ldots, W_n^*, -J^*W_n^*$ form a coframe on Ω which is dual to $W_1, JW_1, \ldots, W_n, JW_n$ at every point in Ω. From Proposition 2.3 (2), (3), we see that all the 1-forms $W_1^*, \ldots, W_n^*$ are closed on Ω. We can then construct a coordinate system $(w_1, \ldots, w_n, w_{n+1}, \ldots, w_{2n})$ on Ω such that

$$W_1 = \partial/\partial w_1, \ldots, W_n = \partial/\partial w_n \quad \text{on } \Omega, \tag{2.25}$$

$$JW_1 = \partial/\partial w_{n+1}, \ldots, JW_n = \partial/\partial w_{2n} \ \text{ at all points in } S^-, \tag{2.26}$$

$$\partial/\partial w_{n+1}, \ldots, \partial/\partial w_{2n} \in \Gamma(\Omega, D^-), \tag{2.27}$$

$$\big(w_1(p_0), \ldots, w_n(p_0), w_{n+1}(p_0), \ldots, w_{2n}(p_0)\big) = (0, \ldots, 0, 0, \ldots, 0). \tag{2.28}$$

In fact, let $W_{n+1}, \ldots, W_{2n}$ be the vector fields on Ω such that $W_{n+i} = JW_i$, $i = 1, \ldots, n$, on S^- and that they are invariant under the local $\boldsymbol{R}^n$-action generated by $W_1, \ldots, W_n$. We then have $[W_i, W_j] = 0$ on Ω for every $i, j = 1, \ldots, 2n$. Since D^- is invariant under this action, we also have $W_{n+i} \in \Gamma(\Omega, D^-)$, $i = 1, \ldots, n$. We thus obtain the desired coordinate system $(w_1, \ldots, w_n, w_{n+1}, \ldots, w_{2n})$ on Ω. We shall call such a coordinate system an $\mathscr{F}$-adapted coordinate system on Ω.

In the rest of this section and the next section, we shall assume, without loss of generality, that the neighborhood Ω is given by

$$|w_1| < \Lambda, \dots, |w_n| < \Lambda,\ |w_{n+1}| < \Lambda, \dots, |w_{2n}| < \Lambda, \tag{2.29}$$

where Λ is a certain sufficiently small positive number, and we shall identify Ω with the $2n$-dimensional cube-like domain defined by these inequalities (2.29) in $\boldsymbol{R}^{2n}$. Under this identification, the submanifold S^- in Ω can be written as

$$S^- = \bigl\{(0, \dots, 0, w_{n+1}, \dots, w_{2n}); |w_{n+1}| < \Lambda, \dots, |w_{2n}| < \Lambda\bigr\}. \tag{2.30}$$

By virtue of (2.17), (2.25), and (2.27), we define a set of functions κ_{ij}, $i, j = 1, \dots, n$, on Ω by

$$J(\partial/\partial w_i) = \sum_{j=1}^{n} \kappa_{ij}(\partial/\partial w_{n+j}), \quad i = 1, \dots, n. \tag{2.31}$$

We then define an $n \times n$-matrix-valued function K on Ω by putting

$$K = [\kappa_{ij}], \tag{2.32}$$

which is a nonsingular matrix at every point in Ω. From (2.26) we see that K is the identity matrix at every point in S^-. We denote by K^{-1} the inverse matrix of K, which is also an $n \times n$-matrix-valued function on Ω, and define a set of functions $\check{\kappa}_{ij}$, $i, j = 1, \dots, n$, on Ω as the (i, j)-entries of K^{-1}. It should be noted that, at each point in Ω, the complex structure J is represented by the matrix $\left[\begin{array}{c|c} O & -K^{-1} \\ \hline K & O \end{array}\right]$ with respect to the frame $\partial/\partial w_1, \dots, \partial/\partial w_n, \partial/\partial w_{n+1}, \dots, \partial/\partial w_{2n}$ on Ω.

We are now in a position to state a canonical expression of the system, which is analogous to the one for Liouville-Stäckel's system.

THEOREM 2.4. *Let $(M, g, J; \mathscr{F})$ be an n-dimensional Hermite-Liouville manifold of type* (A) *and let $F_1, \dots, F_n$ a basis of $\mathscr{F}$. Let Ω be a sufficiently small neighborhood of a point in the subset M^1 defined by* (2.13), *let b_{ij}, $i, j = 1, \dots, n$, and κ_{ij}, $i, j = 1, \dots, n$, be functions defined in* (2.4) *and* (2.31) *respectively, and let $(w_1, \dots, w_n, w_{n+1}, \dots, w_{2n})$ an $\mathscr{F}$-adapted coordinate system on Ω.*

Then, for each i, j, the function b_{ij} is that of one variable w_i and the following relation holds on $T^*\Omega$:

$$\sum_{j=1}^{n} b_{ij}(w_i) F_j = (\partial/\partial w_i)^2 + \left(\sum_{s=1}^{n} \kappa_{is}(\partial/\partial w_{n+s})\right)^2, \quad i = 1, \dots, n, \tag{2.33}$$

where $\partial/\partial w_i$, $\partial/\partial w_{n+s}$ are regarded as fiberwise linear forms on $T^*\Omega$.

Proof. The fact that the function b_{ij} is that of one variable w_i follows from Proposition 2.3 (1), (2.25), and (2.31). The relation follows from (2.6), (2.25), and (2.31). □

The functions κ_{ij}'s are required some conditions. We then proceed to the argument for them.

Proposition 2.5.
(1) *For each* i, j, *the function* κ_{ij} *is that of two variables* w_i, w_{n+i} *on* $]-\Lambda, \Lambda[^2$;
(2) *The system of partial differential equations*

$$\kappa_{ji}\frac{\partial \kappa_{it}}{\partial w_{n+i}} = \kappa_{ij}\frac{\partial \kappa_{jt}}{\partial w_{n+j}}, \quad i, j, t = 1, \ldots, n,$$

holds on Ω.

Proof.
(1) By (2.25) and (2.31) we have $JW_i = \sum_{j=1}^{n} \kappa_{ij}(\partial/\partial w_{n+j})$ on Ω. We recall Proposition 2.3 (2). From the fact that $[W_k, JW_i] = 0$ on Ω unless $k = i$, we see that, if $k \neq i$, then κ_{ij} does not depend on the variable w_k. From the fact that $[JW_\ell, JW_i] = 0$ on Ω, we obtain

$$\sum_{s=1}^{n}\left(\kappa_{\ell s}\frac{\partial \kappa_{ij}}{\partial w_{n+s}} - \kappa_{is}\frac{\partial \kappa_{\ell j}}{\partial w_{n+s}}\right) = 0, \quad i, j, \ell = 1, \ldots, n, \tag{2.34}$$

on Ω. We here assume that $\ell \neq i$. Since $\kappa_{\ell s}$ does not depend on w_i and since $\kappa_{\ell s} = \delta_{\ell s}$ at every point in S^-, we have $\kappa_{\ell s}(0, \ldots, 0, w_i, 0, \ldots, 0, w_{n+1}, \ldots, w_{2n}) = \delta_{\ell s}$, $s = 1, \ldots, n$, where $\delta_{\ell s}$ denotes Kronecker's symbol. Considering the equation (2.34) at the point $(0, \ldots, 0, w_i, 0, \ldots, 0, w_{n+1}, \ldots, w_{2n})$ in Ω, we thus have $\frac{\partial \kappa_{ij}}{\partial w_{n+\ell}}(0, \ldots, 0, w_i, 0, \ldots, 0, w_{n+1}, \ldots, w_{2n}) = 0$. This means that, if $\ell \neq i$, then κ_{ij} does not depend on the variable $w_{n+\ell}$.
(2) By virtue of (1), we immediately obtain the desired system of equations from (2.34). □

In view of Proposition 2.3 (3), we can define n^2 functions ζ_{ij}, $i, j = 1, \ldots, n$, on Ω by

$$[W_i, JW_i] = \sum_{j=1}^{n} \zeta_{ij} JW_j, \quad i = 1, \ldots, n. \tag{2.35}$$

We then present the condition, say (A-i)$'$, equivalent to (A-i), which appeared in the definition of type (A), as follows:

(A-i)$'$ For each j, there exists $i(\neq j)$ such that $\zeta_{ij}(p) \neq 0$.

We notice that (A-i)$'$ holds at every point $p \in \Omega$ since $\Omega \subset M^1$. From the definition (2.31) of κ_{ij}, we see that the functions ζ_{ij}, $i, j = 1, \ldots, n$, on Ω can be expressed by

$$\zeta_{ij} = \sum_{k=1}^{n} \frac{\partial \kappa_{ik}}{\partial w_i} \breve{\kappa}_{kj} \tag{2.36}$$

in terms of the functions κ_{ij}, $i, j = 1, \ldots, n$, and $\breve{\kappa}_{ij}$, $i, j = 1, \ldots, n$.

PROPOSITION 2.6. *For each j, there exists $i(\neq j)$ such that*

$$\frac{\partial \kappa_{ij}}{\partial w_i}(0,0) \neq 0.$$

PROOF. Since K is the identity matrix at the origin o, we obtain, from (2.36),

$$\zeta_{ij}(0,0) = \frac{\partial \kappa_{ij}}{\partial w_i}(0,0).$$

Thus, the assertion follows from the fact that (A-i)$'$ holds at the origin o. $\square$

We conclude this section by mentioning a property of the distribution D^+. The argument for the functions κ_{ij} will resume at the beginning of Section 3.

From Proposition 2.3 (2) we obtain, for each $p \in \Omega$, the maximal integral submanifold S_p^+ of the distribution D^+ in Ω through p.

PROPOSITION 2.7. *For any point $p \in \Omega$, the submanifold S_p^+ is totally geodesic with respect to the metric g.*

PROOF. By Proposition 2.3 (2), (3), we have

$$\nabla_{W_i} W_j = \begin{cases} -\frac{W_j a_i}{2a_i} W_i - \frac{W_i a_j}{2a_j} W_j & \text{if } i \neq j, \\ -\sum_{k=1}^{n} \frac{a_k (W_k a_i)}{2a_i^2} W_k & \text{if } i = j, \end{cases}$$

where ∇ is the Riemannian connection with respect to the metric g. The fact that $\nabla_{W_i} W_j \in \Gamma(\Omega, D^+)$ implies the assertion. $\square$

3. Analyzing the functions κ_{ij}.

In this section, based on the argument in Section 2, we shall study the functions κ_{ij}, $i, j = 1, \ldots, n$, on Ω introduced in Section 2. Throughout this section, we shall use the same notation as in Section 2 and, in particular, use the same convention that indices i, j, k, ℓ, s, t run from 1 to n, unless otherwise stated.

We here give a brief of the argument of this section as follows. We first introduce an equivalence relation in the index set $I = \{1, \ldots, n\}$. Rearranging the assignment of the indices if necessary, we can express the matrix $K = [\kappa_{ij}]$ by a block-triangular form according to the equivalence relation (Proposition 3.3). After some preparation, we then present the expressions of the functions κ_{ij} in each block in K in terms of the $\mathscr{F}$-adapted coordinate system $(w_1, \ldots, w_n, w_{n+1}, \ldots, w_{2n})$ (Theorem 3.6, Theorem 3.7). We also have an argument concerning the complete integrability of the geodesic flow (Theorem 3.10) and one concerning the Kähler condition (Theorem 3.11).

We begin with recalling the situation in Section 2. We recall that Λ is a positive real number and Ω is considered as the $2n$-dimensional cube-like domain

$$\bigl\{(w_1, \ldots, w_n, w_{n+1}, \ldots, w_{2n}); |w_i| < \Lambda,\ i = 1, \ldots, 2n\bigr\} \tag{3.1}$$

with the coordinate system $(w_1, \ldots, w_n, w_{n+1}, \ldots, w_{2n})$, and also that κ_{ij}, $i, j = 1, \ldots, n$, are the functions on Ω satisfying the following four conditions:

(CK-1) The matrix $K = [\kappa_{ij}]$ is nonsingular at every point in Ω. In particular, it is the identity matrix at every point in the slice $S^- = \{(0, \ldots, 0, w_{n+1}, \ldots, w_{2n}); |w_{n+j}| < \Lambda,\ j = 1, \ldots, n\}$ in Ω.

(CK-2) For each i, j, the function κ_{ij} is that of two variables w_i, w_{n+i} on $]-\Lambda, \Lambda[^2$.

(CK-3) The system of partial differential equations

$$\kappa_{ji} \frac{\partial \kappa_{it}}{\partial w_{n+i}} = \kappa_{ij} \frac{\partial \kappa_{jt}}{\partial w_{n+j}}, \quad i, j, t = 1, \ldots, n, \tag{3.2}$$

holds on Ω.

(CK-4) For each j, there exists an index $i (\neq j)$ such that

$$\frac{\partial \kappa_{ij}}{\partial w_i}(0, 0) \neq 0.$$

In this section, we moreover assume the condition (CK-5) as follows.

(CK-5) The constant Λ is taken sufficiently small so that, for every distinct i, j, the function κ_{ij} has either of the following two properties:

(1) $\kappa_{ij} \equiv 0$ on $]-\Lambda, \Lambda[^2$;
(2) For any $\tilde{\Lambda}$ such that $0 < \tilde{\Lambda} \leq \Lambda$, $\kappa_{ij}|_{]-\tilde{\Lambda},\tilde{\Lambda}[^2} \not\equiv 0$, where $\kappa_{ij}|_{]-\tilde{\Lambda},\tilde{\Lambda}[^2}$ denotes the restriction of κ_{ij} to the domain $]-\tilde{\Lambda}, \tilde{\Lambda}[^2$.

After some preparations we shall actually solve the system of partial differential equations (3.2) under the conditions (CK-1), (CK-2), (CK-4), and (CK-5).

We first introduce two relations $\approx$ and $\sim$, the latter being an equivalence relation, in the index set $I = \{1, 2, \ldots, n\}$ as follows. We write $i \approx j$ if $\kappa_{ij} \not\equiv 0$ and $\kappa_{ji} \not\equiv 0$ on Ω. We then write $i \sim j$ if there exists a finite series $i = s_1, s_2, \ldots, s_\nu = j$ of indices such that $i = s_1 \approx s_2 \approx \cdots \approx s_\nu = j$ with the condition that some of them may coincide. It is easy to verify that the relation $\sim$ is an equivalence relation. It follows that the set I can be decomposed into the disjoint union of its equivalence classes.

Lemma 3.1. *Take $i, j, t \in I$.*
(1) *If* $\dfrac{\partial \kappa_{it}}{\partial w_{n+i}} \not\equiv 0$ *and* $i \sim j$, *then* $\dfrac{\partial \kappa_{jt}}{\partial w_{n+j}} \not\equiv 0$.
(2) *If* $\dfrac{\partial \kappa_{it}}{\partial w_{n+i}} \equiv 0$ *and* $i \sim j$, *then* $\dfrac{\partial \kappa_{jt}}{\partial w_{n+j}} \equiv 0$.

Proof. To prove (1) it is sufficient to show the case where $i \approx j$. Since $\frac{\partial \kappa_{it}}{\partial w_{n+i}}$ is a function of (w_i, w_{n+i}) and κ_{ji} is a function of (w_j, w_{n+j}), and since κ_{ji} is not identically zero, it follows that there is a point $p \in \Omega$ such that $\kappa_{ji}(p) \neq 0$, $\frac{\partial \kappa_{it}}{\partial w_{n+i}}(p) \neq 0$. Then, by the equation (3.2), we obtain $\frac{\partial \kappa_{jt}}{\partial w_{n+j}} \not\equiv 0$. (2) is similar. $\square$

Define two subsets $I^{(*)}$ and $I^{(0)}$ of I by

$$I^{(*)} = \left\{ i \in I; \frac{\partial \kappa_{it}}{\partial w_{n+i}} \not\equiv 0 \text{ for some } t \in I \right\} \tag{3.3}$$

and

$$I^{(0)} = \left\{ i \in I; \frac{\partial \kappa_{it}}{\partial w_{n+i}} \equiv 0 \text{ for all } t \in I \right\}, \tag{3.4}$$

respectively. The set I then can be written as the disjoint union of $I^{(*)}$ and $I^{(0)}$; $I = I^{(*)} \amalg I^{(0)}$. Lemma 3.1 implies that every equivalence class is included alternatively in $I^{(*)}$ or $I^{(0)}$. Thus, $I^{(*)}$ is decomposed into the disjoint unions of the equivalence classes $I_1, \ldots, I_r$ included in $I^{(*)}$; $I^{(*)} = I_1 \amalg \cdots \amalg I_r$. It follows that $I = I_1 \amalg \cdots \amalg I_r \amalg I^{(0)}$.

LEMMA 3.2. *For each $i \in I^{(*)}$, there exists at least one index $j \in I^{(*)} \setminus \{i\}$ such that $i \approx j$. Every equivalence class I_h included in $I^{(*)}$ therefore consists of two or more indices.*

PROOF. From (CK-4) there exists an index $j \in I \setminus \{i\}$ such that $\frac{\partial \kappa_{ji}}{\partial w_j}(0,0) \neq 0$. We then have $\kappa_{ji} \not\equiv 0$. Since $i \in I^{(*)}$, there exists an index $t_1 \in I$ such that $\frac{\partial \kappa_{it_1}}{\partial w_{n+i}} \not\equiv 0$. Observing the equation (3.2) with $t = t_1$, we obtain $\kappa_{ij} \not\equiv 0$. We thus obtain $i \approx j$. $\square$

Let n_h denote the number of all elements of I_h for each $h = 1, \ldots, r$ and let $n^{(0)}$ denote the number of all elements of $I^{(0)}$. It follows that $n_1 + \cdots + n_r + n^{(0)} = n$. By rearranging the assignment of the indices in I if necessary, we may assume, without loss of generality, that I_h, $h = 1, \ldots, r$, and $I^{(0)}$ consist of the consecutive integers in the following forms:

$$I_h = \left\{ \left(\sum_{k=0}^{h-1} n_k \right) + 1, \left(\sum_{k=0}^{h-1} n_k \right) + 2, \ldots, \left(\sum_{k=0}^{h-1} n_k \right) + n_h \right\}, \tag{3.5}$$

$$I^{(0)} = \left\{ \left(\sum_{h=0}^{r} n_h \right) + 1, \left(\sum_{h=0}^{r} n_h \right) + 2, \ldots, n \right\}, \tag{3.6}$$

where $n_0 = 0$. We then observe the matrix-valued function $K = [\kappa_{ij}]$ on Ω.

PROPOSITION 3.3. *The matrix-valued function K on Ω can be expressed in the following block-triangular form*:

$$K = \left[\begin{array}{cccc|c} K_1 & & & O & K_1^{(1)} \\ & K_2 & & & K_2^{(1)} \\ & & \ddots & & \vdots \\ & & & K_r & K_r^{(1)} \\ O & & & & K^{(0)} \end{array}\right], \tag{3.7}$$

where K_h and $K_h^{(1)}$ are an $n_h \times n_h$-matrix and an $n_h \times n^{(0)}$-matrix respectively for each $h = 1, \ldots, r$ and where $K^{(0)}$ is an $n^{(0)} \times n^{(0)}$-matrix.

Proof. According to the decomposition $I = I^{(*)} \amalg I^{(0)}$, we have the block-decomposed expression

$$K = \left[\begin{array}{c|c} K^{(*)} & K^{(1)} \\ \hline K^{(2)} & K^{(0)} \end{array}\right]$$

such that $K^{(*)}$ and $K^{(0)}$ are a $\left(\sum_{h=0}^{r} n_h\right) \times \left(\sum_{h=0}^{r} n_h\right)$-matrix and an $n^{(0)} \times n^{(0)}$-matrix respectively. We first observe the block $K^{(2)}$. Let κ_{ij} be an arbitrary entry of $K^{(2)}$. Since $i \in I^{(0)}$ and $j \in I^{(*)}$, there exists an index $t_1 \in I$ such that $\frac{\partial \kappa_{it_1}}{\partial w_{n+i}} \equiv 0$ and $\frac{\partial \kappa_{jt_1}}{\partial w_{n+j}} \not\equiv 0$. Observing the equation (3.2) with $t = t_1$, we obtain $\kappa_{ij} \equiv 0$, which means that $K^{(2)} = 0$. We next observe the block $K^{(*)}$. Let I_h and I_m be any two distinct classes in $I^{(*)}$. Take indices $i \in I_h$, $j \in I_m$ and assume that $\kappa_{ij} \not\equiv 0$. Since $i \not\sim j$, we have $\kappa_{ji} \equiv 0$. Observing the equation (3.2), we obtain $\frac{\partial \kappa_{jt}}{\partial w_{n+j}} \equiv 0$ for every $t = 1, \ldots, n$, which contradicts the condition that $j \in I^{(*)}$. We thus have $\kappa_{ij} \equiv 0$ for any $i \in I_h$, $j \in I_m$. This implies that $K^{(*)}$ has the block-diagonal form whose principal diagonal consists of the blocks $K_1, \ldots, K_r$. $\square$

We now proceed to the argument for each block matrix in K described in (3.7). We shall first consider the blocks K_h and $K_h^{(1)}$, $h = 1, \ldots, r$. We notice that K_h, $K_h^{(1)}$ correspond to the class I_h. We consider the following three cases (C1), (C2), and (C3):

(C1) For every $i \in I_h$, there exist two or more indices j's $\in I_h$ such that $\frac{\partial \kappa_{ij}}{\partial w_{n+i}} \not\equiv 0$;

(C2) For every $i \in I_h$, there exists a unique index $\lambda \in I_h$ such that $\frac{\partial \kappa_{i\lambda}}{\partial w_{n+i}} \not\equiv 0$. In particular, the index λ is determined independently of the choice of $i \in I_h$ by Lemma 3.1 (1);

(C3) For every $i, j \in I_h$, $\frac{\partial \kappa_{ij}}{\partial w_{n+i}} \equiv 0$.

Notice that, in view of Lemma 3.1, one of the above three cases must occur for each I_h. Also, notice that the case (C3) occurs only when $I^{(0)} \neq \emptyset$. We here give two preparatory lemmas.

Lemma 3.4. *Let I_h be an equivalence class in $I^{(*)}$ and let i, j distinct indices in I_h such that $\frac{\partial \kappa_{ij}}{\partial w_{n+i}} \not\equiv 0$. Then, $i \approx j$ and there exists a unique non-zero constant C_i^j such that*

$$\kappa_{ij}(w_i, w_{n+i}) = \kappa_{ij}(w_i, 0) e^{C_i^j w_{n+i}} \tag{3.8}$$

and that

$$\frac{\partial \kappa_{jj}}{\partial w_{n+j}} = C_i^j \kappa_{ji}. \tag{3.9}$$

Proof. By the same argument as in the proof of Lemma 3.2 we have $i \approx j$. In view of the equation (3.2) with $t = j$, we define a non-zero constant C_i^j by

$$C_i^j = \frac{1}{\kappa_{ij}(\overline{w}_i, \overline{w}_{n+i})} \cdot \frac{\partial \kappa_{ij}}{\partial w_{n+i}}(\overline{w}_i, \overline{w}_{n+i}) = \frac{1}{\kappa_{ji}(\overline{w}_j, \overline{w}_{n+j})} \cdot \frac{\partial \kappa_{jj}}{\partial w_{n+j}}(\overline{w}_j, \overline{w}_{n+j}),$$

where $(\overline{w}_i, \overline{w}_{n+i})$ and $(\overline{w}_j, \overline{w}_{n+j})$ are points satisfying $\kappa_{ij}(\overline{w}_i, \overline{w}_{n+i}) \neq 0$ and $\kappa_{ji}(\overline{w}_j, \overline{w}_{n+j}) \neq 0$ respectively. Again from the equation (3.2) with $t = j$, we can derive two equations (3.9) and $\frac{\partial \kappa_{ij}}{\partial w_{n+i}} = C_i^j \kappa_{ij}$. From the latter one, we obtain (3.8). □

Lemma 3.5. *Let I_h be an equivalence class in $I^{(*)}$. Let i, j be distinct indices in I_h such that $i \approx j$ and let $t \in I^{(0)}$. Then, there exists a unique constant $C_{ij}^t (= C_{ji}^t)$ such that*

$$\frac{\partial \kappa_{it}}{\partial w_{n+i}} = C_{ij}^t \kappa_{ij}. \tag{3.10}$$

Proof. In view of the equation (3.2), we define a constant $C_{ij}^t (= C_{ji}^t)$ by

$$C_{ij}^t = \frac{1}{\kappa_{ij}(\overline{w}_i, \overline{w}_{n+i})} \cdot \frac{\partial \kappa_{it}}{\partial w_{n+i}}(\overline{w}_i, \overline{w}_{n+i}) = \frac{1}{\kappa_{ji}(\overline{w}_j, \overline{w}_{n+j})} \cdot \frac{\partial \kappa_{jt}}{\partial w_{n+j}}(\overline{w}_j, \overline{w}_{n+j}),$$

where $(\overline{w}_i, \overline{w}_{n+i})$ and $(\overline{w}_j, \overline{w}_{n+j})$ are points satisfying $\kappa_{ij}(\overline{w}_i, \overline{w}_{n+i}) \neq 0$ and $\kappa_{ji}(\overline{w}_j, \overline{w}_{n+j}) \neq 0$ respectively. Again from the equation (3.2), we obtain (3.10). □

We are now in a position to present the expressions for the functions κ_{ij} which are the entries of the block matrices K_h and $K_h^{(1)}$ for each $h = 1, \dots, r$.

Theorem 3.6. *Let I_h be an equivalence class in $I^{(*)}$ and let K_h and $K_h^{(1)}$ be the corresponding block matrices.*

(1) *Assume that I_h is in the case* (C1). *Then there exist functions $\kappa_i(w_i)$, $i \in I_h$, of one variable $w_i \in\,]-\Lambda, \Lambda[$ with $\kappa_i \not\equiv 0$ and $\kappa_i(0) = 0$, constants $B_t^{(h)}$, $t \in I^{(0)}$, and non-zero constants C_i, $i \in I_h$, such that, for each $i, j \in I_h$ and $t \in I^{(0)}$, the functions $\kappa_{ij}(w_i, w_{n+i})$ and $\kappa_{it}(w_i, w_{n+i})$ on $]-\Lambda, \Lambda[^2$ are expressed as follows:*

$$\kappa_{ij}(w_i, w_{n+i}) = \begin{cases} \kappa^0_{ii}(w_i) + \frac{1}{C_i}\kappa_i(w_i)\bigl(e^{C_i w_{n+i}} - 1\bigr) & \text{if } i = j, \\ \frac{1}{C_j}\kappa_i(w_i)\, e^{C_i w_{n+i}} & \text{if } i \neq j; \end{cases} \tag{3.11}$$

$$\kappa_{it}(w_i, w_{n+i}) = \kappa^0_{it}(w_i) + B^{(h)}_t \kappa_i(w_i)\bigl(e^{C_i w_{n+i}} - 1\bigr), \tag{3.12}$$

where $\kappa^0_{ii}(w_i) = \kappa_{ii}(w_i, 0)$ *and* $\kappa^0_{it}(w_i) = \kappa_{it}(w_i, 0)$.

(2) *Assume that* I_h *is in the case* (C2). *Then there exist functions* $\kappa_i(w_i)$, $i \in I_h$, *of one variable* $w_i \in\,]-\Lambda, \Lambda[$ *with* $\kappa_i \not\equiv 0$ *and* $\kappa_i(0) = 0$, *constants* $B^{(h)}_t$, $t \in I^{(0)}$, *and nonzero constants* C_i, $i \in I_h, i \neq \lambda$, *such that, for each* $i, j \in I_h$ *and* $t \in I^{(0)}$, *the functions* $\kappa_{ij}(w_i, w_{n+i})$ *and* $\kappa_{it}(w_i, w_{n+i})$ *on* $]-\Lambda, \Lambda[^2$ *are expressed as follows*:

$$\kappa_{ij}(w_i, w_{n+i}) = \begin{cases} \kappa^0_{ii}(w_i) + \delta_{i\lambda}\kappa_\lambda(w_\lambda) w_{n+\lambda} & \text{if } i = j, \\ \frac{\delta_{i\lambda}}{C_j}\kappa_\lambda(w_\lambda) + \delta_{j\lambda}\kappa_i(w_i) e^{C_i w_{n+i}} & \text{if } i \neq j; \end{cases} \tag{3.13}$$

$$\kappa_{it}(w_i, w_{n+i}) = \begin{cases} \kappa^0_{\lambda t}(w_\lambda) + B^{(h)}_t \kappa_\lambda(w_\lambda) w_{n+\lambda} & \text{if } i = \lambda, \\ \kappa^0_{it}(w_i) + B^{(h)}_t \kappa_i(w_i)\bigl(e^{C_i w_{n+i}} - 1\bigr) & \text{if } i \neq \lambda, \end{cases} \tag{3.14}$$

where $\kappa^0_{ii}(w_i) = \kappa_{ii}(w_i, 0)$ *and* $\kappa^0_{it}(w_i) = \kappa_{it}(w_i, 0)$, *and where* δ_{ij} *is Kronecker's symbol.*

(3) *Assume that* I_h *is in the case* (C3). *Then there exist functions* $\kappa_i(w_i)$, $i \in I_h$, *of one variable* $w_i \in\,]-\Lambda, \Lambda[$ *with* $\kappa_i \not\equiv 0$ *and* $\kappa_i(0) = 0$, *constants* A_{ij}, $i, j \in I_h, i \neq j$, *with* $A_{ij} = A_{ji}$, *and constants* $B^{(h)}_t$, $t \in I^{(0)}$, *such that, for each* $i, j \in I_h$ *and* $t \in I^{(0)}$, *the functions* $\kappa_{ij}(w_i, w_{n+i})$ *and* $\kappa_{it}(w_i, w_{n+i})$ *on* $]-\Lambda, \Lambda[^2$ *are expressed as follows*:

$$\kappa_{ij}(w_i, w_{n+i}) = \begin{cases} \kappa^0_{ii}(w_i) & \text{if } i = j, \\ A_{ij}\kappa_i(w_i) & \text{if } i \neq j; \end{cases} \tag{3.15}$$

$$\kappa_{it}(w_i, w_{n+i}) = \kappa^0_{it}(w_i) + B^{(h)}_t \kappa_i(w_i) w_{n+i}, \tag{3.16}$$

where $\kappa^0_{ii}(w_i) = \kappa_{ii}(w_i, 0)$ *and* $\kappa^0_{it}(w_i) = \kappa_{it}(w_i, 0)$.

PROOF.
(1) In this case, we first have

$$\frac{\partial \kappa_{ij}}{\partial w_{n+i}} \not\equiv 0 \quad \text{for every } i, j \in I_h. \tag{3.17}$$

In fact, for each j, we take two distinct $\ell, s \in I_h$ such that $\frac{\partial \kappa_{j\ell}}{\partial w_{n+j}} \not\equiv 0$, $\frac{\partial \kappa_{js}}{\partial w_{n+j}} \not\equiv 0$. In the case where ℓ or s is equal to j we obtain (3.17) by Lemma 3.1. We then consider the case where both ℓ and s are distinct from j. By Lemma 3.1 we have $\frac{\partial \kappa_{\ell s}}{\partial w_{n+\ell}} \not\equiv 0$ and $\frac{\partial \kappa_{s\ell}}{\partial w_{n+s}} \not\equiv 0$. By Lemma 3.4 we then have $C_j^\ell \kappa_{\ell j} = C_s^\ell \kappa_{\ell s} \big(= \frac{\partial \kappa_{\ell\ell}}{\partial w_{n+\ell}} \big)$ with $C_j^\ell \neq 0$ and $C_s^\ell \neq 0$. We thus have $\frac{\partial \kappa_{\ell j}}{\partial w_{n+\ell}} \not\equiv 0$ and hence have (3.17) by Lemma 3.1.

We shall now prove (3.11). By Lemma 3.4, we see from (3.17) that, if $i \neq j$, then $i \approx j$ and there exists a unique non-zero constant C_i^j such that (3.8) and (3.9) hold. We now verify that, for each i, the constant C_i^j is independent of the choice of $j \in I_h \setminus \{i\}$ as follows. It suffices to verify when I_h consists of three or more indices. Take any distinct $\ell, s \in I_h \setminus \{i\}$. By Lemma 3.4 we have $C_\ell^i \kappa_{i\ell} = C_s^i \kappa_{is} \big(= \frac{\partial \kappa_{ii}}{\partial w_{n+i}} \big)$. Substitute above (3.8) with $j = \ell, s$ into this equation and differentiate both sides of it with respect to w_{n+i}. Comparing these two equations, we obtain $C_i^\ell = C_i^s$, thus verifying the independency.

For each i, we thus put $C_i = C_i^\ell$, where $\ell \in I_h \setminus \{i\}$. It follows from the above (3.8) and (3.9) that, if $i \neq j$, then

$$\kappa_{ij}(w_i, w_{n+i}) = \kappa_{ij}(w_i, 0) e^{C_i w_{n+i}}, \tag{3.18}$$

$$\frac{\partial \kappa_{ii}}{\partial w_{n+i}}(w_i, w_{n+i}) = C_j \kappa_{ij}(w_i, 0) e^{C_i w_{n+i}}. \tag{3.19}$$

For each $i \in I_h$, we can then define a function $\kappa_i(w_i)$ of one variable $w_i \in \,]-\Lambda, \Lambda[$ by

$$\kappa_i(w_i) = \frac{\partial \kappa_{ii}}{\partial w_{n+i}} e^{-C_i w_{n+i}} = C_\ell \kappa_{i\ell}(w_i, 0), \quad \text{where } \ell \in I_h \setminus \{i\}, \tag{3.20}$$

by virtue of (3.19). Since $C_\ell \neq 0$ and $i \approx \ell$, we see that $\kappa_i \not\equiv 0$. We also see that $\kappa_i(0) = 0$ since $\kappa_{i\ell}(0,0) = 0$. From (3.18), (3.19), and (3.20), we thus establish (3.11).

We shall next prove (3.12). Let $\ell \in I_h \setminus \{i\}$. By the above argument, we have $\ell \approx i$. We see from Lemma 3.5 that there exists a constant $C_{i\ell}^t$ such that (3.10) with $j = \ell$ holds. We then put $B_{i\ell}^t = C_{i\ell}^t / C_i C_\ell$, which is symmetric in i and ℓ. The above (3.10) with $j = \ell$ and (3.11) give us

$$\frac{\partial \kappa_{it}}{\partial w_{n+i}}(w_i, w_{n+i}) = B_{i\ell}^t C_i \kappa_i(w_i) e^{C_i w_{n+i}}. \tag{3.21}$$

Since $C_i \neq 0$ and $\kappa_i \not\equiv 0$, we see that $B_{i\ell}^t$ is independent of the choice of ℓ. The symmetry of $B^t_{i\ell}$ in i and ℓ implies that $B^t_{i\ell}$ is independent also of the choice of i and hence of both i and ℓ. We can then define the constant $B^{(h)}_t$ by $B^{(h)}_t = B^t_{ks}$, where $k, s \in I_h$ with $k \neq s$. From (3.21) we thus establish (3.12).

(2) In this case, we first have, for $i, j \in I_h$,

$$\frac{\partial \kappa_{i\lambda}}{\partial w_{n+i}} \not\equiv 0 \quad \text{and} \quad \frac{\partial \kappa_{ij}}{\partial w_{n+i}} \equiv 0 \ \text{ if } j \neq \lambda. \tag{3.22}$$

We shall prove (3.13). The latter equation in (3.22) means that, if $j \neq \lambda$, then

$$\kappa_{ij}(w_i, w_{n+i}) = \kappa_{ij}(w_i, 0). \tag{3.23}$$

Since $\frac{\partial \kappa_{i\lambda}}{\partial w_{n+i}} \not\equiv 0$, we see from Lemma 3.4 that, if $i \neq \lambda$, the relation $i \approx \lambda$ holds and

$$\kappa_{i\lambda}(w_i, w_{n+i}) = \kappa_{i\lambda}(w_i, 0)e^{C_i w_{n+i}}, \tag{3.24}$$

$$\frac{\partial \kappa_{\lambda\lambda}}{\partial w_{n+\lambda}} = C_i \kappa_{\lambda i}, \tag{3.25}$$

where C_i is $C^\lambda_i(\neq 0)$ in Lemma 3.4. By virtue of (3.23) with $i = \lambda$ and (3.25), we define a function $\kappa_\lambda(w_\lambda)$ of one variable $w_\lambda \in\,]-\Lambda, \Lambda[$ by

$$\kappa_\lambda(w_\lambda) = \frac{\partial \kappa_{\lambda\lambda}}{\partial w_{n+\lambda}} = C_\ell \kappa_{\lambda\ell}(w_\lambda, 0), \quad \text{where } \ell \in I_h \setminus \{\lambda\}.$$

We thus have

$$\kappa_{\lambda\lambda}(w_\lambda, w_{n+\lambda}) = \kappa_{\lambda\lambda}(w_\lambda, 0) + \kappa_\lambda(w_\lambda)w_{n+\lambda}, \tag{3.26}$$

$$\kappa_{\lambda j}(w_\lambda, w_{n+\lambda}) = \frac{1}{C_j}\kappa_\lambda(w_\lambda) \quad \text{if } j \neq \lambda. \tag{3.27}$$

For each $i \in I_h \setminus \{\lambda\}$, we moreover define a function $\kappa_i(w_i)$ of one variable $w_i \in\,]-\Lambda, \Lambda[$ by $\kappa_i(w_i) = \kappa_{i\lambda}(w_i, 0)$. It then follows from (3.24) that

$$\kappa_{i\lambda}(w_i, w_{n+i}) = \kappa_i(w_i)e^{C_i w_{n+i}} \quad \text{if } i \neq \lambda. \tag{3.28}$$

By a similar argument as in the proof of (1), we see that $\kappa_i \not\equiv 0, \kappa_i(0) = 0$ for every $i \in I_h$. From the equation (3.2) with $t = \lambda$ and the equation $\frac{\partial^2 \kappa_{j\lambda}}{\partial {w_{n+j}}^2} = C_j \frac{\partial \kappa_{j\lambda}}{\partial w_{n+j}}$ derived from (3.28) or (3.24), we obtain

$$C_j\,\kappa_{ij}\,\frac{\partial\kappa_{j\lambda}}{\partial w_{n+j}}=\frac{\partial\kappa_{ji}}{\partial w_{n+j}}\,\frac{\partial\kappa_{i\lambda}}{\partial w_{n+i}}\quad\text{when } \lambda\neq i\neq j\neq\lambda.$$

From (3.22) and the fact that $C_j \neq 0$, we thus obtain

$$\kappa_{ij}\equiv 0\quad\text{if } \lambda\neq i\neq j\neq\lambda. \tag{3.29}$$

Summarizing (3.23) with $i = j \neq \lambda$, (3.26), (3.27), (3.28), and (3.29), we establish (3.13).

We shall then prove (3.14). Assume $i \neq \lambda$. As in the above arguement, we have $i \approx \lambda$. We then see from Lemma 3.5 that there exists a constant $C^t_{i\lambda}(= C^t_{\lambda i})$ such that

$$\frac{\partial\kappa_{\lambda t}}{\partial w_{n+\lambda}}=C^t_{i\lambda}\kappa_{\lambda i}\quad\text{and}\quad\frac{\partial\kappa_{it}}{\partial w_{n+i}}=C^t_{i\lambda}\kappa_{i\lambda}. \tag{3.30}$$

From (3.27) and the first equation in (3.30), we obtain

$$\frac{\partial\kappa_{\lambda t}}{\partial w_{n+\lambda}}=\frac{C^t_{i\lambda}}{C_i}\kappa_\lambda(w_\lambda). \tag{3.31}$$

This, together with the fact that $\kappa_\lambda \not\equiv 0$, that $C^t_{i\lambda}/C_i$ is independent of the choice of $i \in I_h \setminus \{\lambda\}$. We then put $B^{(h)}_t = C^t_{\ell\lambda}/C_\ell$, where $\ell \in I_h \setminus \{\lambda\}$. From (3.31), the second equation in (3.30), and (3.28), we thus obtain

$$\frac{\partial\kappa_{\lambda t}}{\partial w_{n+\lambda}}=B^{(h)}_t\kappa_\lambda(w_\lambda)\quad\text{and}\quad\frac{\partial\kappa_{it}}{\partial w_{n+i}}=B^{(h)}_t C_i\kappa_i(w_i)e^{C_i w_{n+i}}\quad\text{if } i\neq\lambda,$$

which establish (3.14).

(3) By the definition of the case (C3) we have

$$\kappa_{ij}(w_i, w_{n+i})=\kappa_{ij}(w_i,0)\quad\text{for every } i,j\in I_h. \tag{3.32}$$

By virtue of Lemma 3.1, we can define a subset $I^{(0)}_*$ of $I^{(0)}$ by

$$I^{(0)}_*=\left\{s\in I^{(0)};\,\frac{\partial\kappa_{is}}{\partial w_{n+i}}\not\equiv 0\text{ for all } i\in I_h\right\}. \tag{3.33}$$

We recall that $I^{(0)} \neq \emptyset$. By the definition (3.3) of $I^{(*)}$ and by observing the form of $K = [\kappa_{ij}]$ described in (3.7), we have $I^{(0)}_* \neq \emptyset$.

Let $i, j \in I_h$ and let $t \in I^{(0)}$. We see from Lemma 3.5 that, if $i \neq j$ and $i \approx j$ and if $t \in I_*^{(0)}$, then there exists a constant $C_{ij}^t (= C_{ji}^t)$ such that

$$\frac{\partial \kappa_{it}}{\partial w_{n+i}} = C_{ij}^t \kappa_{ij}. \tag{3.34}$$

By (3.33) we have $C_{ij}^t \neq 0$. We shall then verify that the constant C_{ij}^t can be written as a product of a certain non-zero constant $\tilde{B}_t^{(h)}$ independent of the choice of i, j and a certain non-zero constant C_{ij} independent of the choice of t; $C_{ij}^t = \tilde{B}_t^{(h)} C_{ij}$. Taking $\overline{w}_i$ such that $\kappa_{ij}(\overline{w}_i, 0) \neq 0$, which can be taken independently of the choice of j by virtue of (3.34), we can write

$$C_{ij}^t = \frac{\partial \kappa_{it}}{\partial w_{n+i}}(\overline{w}_i, 0)\,\frac{1}{\kappa_{ij}(\overline{w}_i, 0)} \tag{3.35}$$

by (3.34). We here take an index $m \in I_h$ and fix it in the rest of the proof. Since $i \sim m$, we can take a finite series $i = s_1, s_2, \ldots, s_\tau = m$ of indices such that $i = s_1 \approx s_2 \approx \cdots \approx s_\tau = m$ and $i = s_1 \neq s_2 \neq \cdots \neq s_\tau = m$. For each s_u, $u = 1, \ldots, \tau - 1$, we take $\overline{w}_{s_u}$ such that $\kappa_{s_u s_{u+1}}(\overline{w}_{s_u}, 0) \neq 0$. Using repeatedly the equations (3.2) for $(i, j) = (s_u, s_{u+1}), u = 1, \ldots, \tau - 1$, we obtain

$$\frac{\partial \kappa_{it}}{\partial w_{n+i}}(\overline{w}_i,\, 0) = \beta_{im}\,\frac{\partial \kappa_{mt}}{\partial w_{n+m}}(\overline{w}_m, 0), \tag{3.36}$$

where $\beta_{im} = \prod_{u=1}^{\tau-1} \frac{\kappa_{s_u s_{u+1}}(\overline{w}_{s_u}, 0)}{\kappa_{s_{u+1} s_u}(\overline{w}_{s_{u+1}}, 0)}$. Notice that $\beta_{im} \neq 0$ and that β_{im} is independent of the choice of t. We then put $\tilde{B}_t^{(h)} = \frac{\partial \kappa_{mt}}{\partial w_{n+m}}(\overline{w}_m, 0)$ and $C_{ij} = \beta_{im}/\kappa_{ij}(\overline{w}_i, 0)$. By (3.35) and (3.36), we thus have $C_{ij}^t = \tilde{B}_t^{(h)} C_{ij}$, completing the verification. We notice that $\tilde{B}_t^{(h)} \neq 0$ and $C_{ij} \neq 0$ since $C_{ij}^t \neq 0$.

We therefore see from (3.34) that, if $i \neq j$ and $i \approx j$ and if $t \in I_*^{(0)}$, then

$$\frac{\partial \kappa_{it}}{\partial w_{n+i}} = \tilde{B}_t^{(h)} C_{ij} \kappa_{ij}. \tag{3.37}$$

For each $i \in I_h$, we can define a function $\kappa_i(w_i)$ of one variable $w_i \in\,]-\Lambda, \Lambda[$ by

$$\kappa_i(w_i) = \frac{1}{\tilde{B}_s^{(h)}}\frac{\partial \kappa_{is}}{\partial w_{n+i}} = C_{i\ell}\kappa_{i\ell}(w_i, 0),$$

where $s \in I_*^{(0)}$ and where $\ell \in I_h \setminus \{i\}$ satisfying $i \approx \ell$. By the same argument as in the proof of (1), we see that $\kappa_i \not\equiv 0$ and $\kappa_i(0) = 0$. We moreover define constants A_{ij}, $i \neq j$, $i, j \in I_h$, and $B_t^{(h)}$, $t \in I^{(0)}$, by

$$A_{ij} = \begin{cases} 1/C_{ij} & \text{if } i \approx j, \\ 0 & \text{if } i \not\approx j; \end{cases} \qquad B_t^{(h)} = \begin{cases} \tilde{B}_t^{(h)} & \text{if } t \in I_*^{(0)}, \\ 0 & \text{if } t \in I^{(0)} \setminus I_*^{(0)}. \end{cases}$$

The property that $A_{ij} = A_{ji}$ follows from $C_{ij}^t = C_{ji}^t$.

We now show (3.15) and (3.16). In view of the equation (3.2) with $t \in I_*^{(0)}$, we see that, if $i \not\approx j$, then $\kappa_{ij} \equiv 0$ and $\kappa_{ji} \equiv 0$. From (3.32), the definition of $\kappa_i(w_i)$, and the definition of A_{ij}, we thus establish (3.15). It follows from the definition (3.33) of $I_*^{(0)}$ that, if $t \notin I_*^{(0)}$, then $\frac{\partial \kappa_{it}}{\partial w_{n+i}} \equiv 0$. From (3.37), the definition of $\kappa_i(w_i)$, and the definition of $B_t^{(h)}$, we obtain

$$\frac{\partial \kappa_{it}}{\partial w_{n+i}} = B_t^{(h)} \kappa_i(w_i).$$

Thus (3.16) follows. This completes the proof of Theorem 3.6. □

It remains to observe the entries of $K^{(0)}$. From the very definition (3.4) of $I^{(0)}$, we immediately obtain the following

THEOREM 3.7. *For each $s, t \in I^{(0)}$, the function $\kappa_{st}(w_s, w_{n+s})$ on the domain $]-\Lambda, \Lambda[^2$, which is the (s,t)-entry of $K^{(0)}$, can be written as*

$$\kappa_{st}(w_s, w_{n+s}) = \kappa_{st}^0(w_s), \tag{3.38}$$

where $\kappa_{st}^0(w_s) = \kappa_{st}(w_s, 0)$.

By virtue of the expressions in Theorem 3.6 and Theorem 3.7, we can state properties at the origin $o = (0, \ldots, 0)$ deduced from the condition (A-i)′ in Section 2 as follows.

THEOREM 3.8. *With the same notation as in Theorem 3.6 and Theorem 3.7, the functions $\kappa_i(w_i)$, $i \in I^{(*)}$, and $\kappa_{it}^0(w_i)$, $i \in I, t \in I^{(0)}$, have the following properties:*

(1) *Let I_h be an equivalence class in $I^{(*)}$. If I_h is in the case (C1) or in the case (C3), then, there exist two or more indices i's $\in I_h$ such that $\kappa_i'(0) \neq 0$. If I_h is in the case (C2), then $\kappa_\lambda'(0) \neq 0$ and there exists at least one index $i \in I_h \setminus \{\lambda\}$ such that $\kappa_i'(0) \neq 0$.*

(2) *For each $t \in I^{(0)}$, there exists at least one index $i \in I \setminus \{t\}$ such that* $(\kappa_{it}^0)'(0) \neq 0$.

Proof. We recall from Proposition 2.6 in Section 2 that the condition (A-i)$'$ at the origin o means that, for each j, there exists $i \in I \setminus \{j\}$ such that $\frac{\partial \kappa_{ij}}{\partial w_i}(0,0) \neq 0$. We first consider the case where I_h is in the case (C1) or in the case (C3). We take an index $j \in I_h$ and observe the j-th column in the matrix (3.7) in Proposition 3.3. By (A-i)$'$ we can find an index $i_1 \in I_h \setminus \{j\}$ such that $\frac{\partial \kappa_{i_1 j}}{\partial w_{i_1}}(0,0) \neq 0$. From the expressions (3.11), (3.15) of κ_{ij} in Theorem 3.6, we obtain $\kappa_{i_1}'(0) \neq 0$. Moreover, also for i_1 found above, we can find an index $i_2 \in I_h \setminus \{i_1\}$ such that $\kappa_{i_2}'(0) \neq 0$ by the same way, which implies the first assertion. We next consider the case where I_h is in the case (C2). We take an index $j \in I_h \setminus \{\lambda\}$ and observe the j-th column in the matrix (3.7). By (A-i)$'$ and by the expression (3.13) of κ_{ij} in Theorem 3.6, we have $\frac{\partial \kappa_{\lambda j}}{\partial w_\lambda}(0,0) \neq 0$ and hence $\kappa_\lambda'(0) \neq 0$. We observe also the λ-column. By (A-i)$'$ and by the expression (3.13) of κ_{ij}, there exists $i \in I_h \setminus \{\lambda\}$ such that $\frac{\partial \kappa_{i\lambda}}{\partial w_i}(0,0) \neq 0$ and hence that $\kappa_i'(0) \neq 0$. These prove (1).

By the expressions (3.12), (3.14), and (3.16) of κ_{ij} in Theorem 3.6 and by the expression (3.38) of κ_{ij} in Theorem 3.7, we have, for any $i \in I$ and for any $t \in I^{(0)}$,

$$\frac{\partial \kappa_{it}}{\partial w_i}(0,0) = (\kappa_{it}^0)'(0).$$

The assersion (2) thus follows from the condition (A-i)$'$. □

We now proceed to the argument for the complete integrability of the geodesic flow and that for the metric g to be Kählerian.

Proposition 3.9. *For any $t \in I^{(0)}$, regarding the vector field $\partial/\partial w_{n+t}$ as a fiberwise-linear function on $T^*\Omega$, we have*

$$\{F_i, \partial/\partial w_{n+t}\} = 0, \quad i = 1, \dots, n, \quad \text{on } T^*\Omega.$$

Proof. Let $t \in I^{(0)}$. From (CK-2) and the definition (3.4) of $I^{(0)}$, we see that $\frac{\partial \kappa_{ij}}{\partial w_{n+t}} = 0$ for all $i, j \in I$. This implies that

$$[\partial/\partial w_{n+t}, J(\partial/\partial w_i)] = 0, \quad i = 1, \dots, n, \quad \text{on } \Omega.$$

On the other hand, from the fact that, for each i, j, the function b_{ij} is that of one variable w_i, which was stated in Theorem 2.4 in Section 2, we obtain

$$\frac{\partial b_{ij}}{\partial w_{n+t}} = 0, \quad i, j = 1, \ldots, n, \quad \text{on } \Omega.$$

From the formula (2.33) in Theorem 2.4 in Section 2, we thus obtain the assertion. $\square$

Theorem 3.10. *Let $(M, g, J; \mathscr{F})$ be an n-dimensional Hermite-Liouville manifold of type* (A) *and let $F_1, \ldots, F_n$ a basis for $\mathscr{F}$. Let Ω be a sufficiently small neighborhood of a point in the subset M^1 defined by* (2.13) *in Section* 2, *let $(w_1, \ldots, w_n, w_{n+1}, \ldots, w_{2n})$ an $\mathscr{F}$-adapted coordinate system on Ω, and let K and $K^{(0)}$ the matrix-valued functions defined in* (2.32) *in Section* 2 *and* (3.7) *respectively.*

Assume that $K = K^{(0)}$. Then

$$\{F_i, \partial/\partial w_{n+j}\} = 0 \quad \textit{for all } i, j = 1, \ldots, n$$

on $T^*\Omega$. In particular, the geodesic flow is completely integrable on $T^*\Omega$.

Conversely, if there exist n vector fields $U_1, \ldots, U_n$ on Ω such that

(U1) *For any i, j, $\{F_i, U_j\} = 0$ on $T^*\Omega$;*

(U2) *$U_1, \ldots, U_n$ are linearly independent at all points in Ω,*

then $K = K^{(0)}$ and U_j, $j = 1, \ldots, n$, are equal to linear combinations of $\partial/\partial w_{n+1}, \ldots, \partial/\partial w_{2n}$ with constant coefficients.

Proof. The former half immediately follows from Proposition 3.9. The proof of the latter half is as follows.

We first show that the condition (U1) is equivalent to both of the following two conditions:

(U1-i) For any i, j, k, $U_j b_{ik} = 0$ on Ω;

(U1-ii) For any i, j, $[W_i, U_j] = [JW_i, U_j] = 0$ on Ω.

In fact, recall the relation (2.6) in Section 2:

$$\sum_{k=1}^{n} b_{ik} F_k = {W_i}^2 + (JW_i)^2, \quad i = 1, \ldots, n, \quad \text{on } T^*\Omega,$$

which is the previous form of (2.33) in Theorem 2.4. We notice that $W_i = \partial/\partial w_i$ and that W_i, JW_i are regarded as fiberwise linear forms on $T^*\Omega$. Taking the Poisson bracket of both sides and U_j, we have, for any i, j,

$$-\sum_{k=1}^{n} (U_j b_{ik}) F_k - \sum_{k=1}^{n} b_{ik} \{F_k, U_j\} = 2[W_i, U_j] W_i + 2[JW_i, U_j](JW_i) \tag{3.39}$$

on $T^*\Omega$. It is then obvious that (U1-i) and (U1-ii) imply (U1). Then, we shall verify that (U1) implies (U1-i) and (U1-ii). Assume (U1). It then follows that, for any i, j,

$$-\sum_{k=1}^{n}(U_j b_{ik})F_k = 2[W_i, U_j]W_i + 2[JW_i, U_j](JW_i) \quad \text{on } T^*\Omega. \tag{3.40}$$

Taking the values of both sides of this equation (3.40) at the covectors W_s^* and $-J^*W_s^*$ respectively, we obtain, for any i, j, s,

$$\sum_{k=1}^{n}(U_j b_{ik})\check{b}_{ks} = -2\delta_{is}\big\langle [W_i, U_j], W_i^*\big\rangle = -2\delta_{is}\big\langle [JW_i, U_j], -J^*W_i^*\big\rangle \quad \text{on } \Omega,$$

where $\check{b}_{ks}$ is (k, s)-entry of the inverse matrix of $[b_{ij}]$ and where δ_{is} denotes Kronecker's symbol. We put

$$\sigma_{ij} = \big\langle [W_i, U_j], W_i^*\big\rangle = \big\langle [JW_i, U_j], -J^*W_i^*\big\rangle, \quad i, j = 1, \dots, n. \tag{3.41}$$

It then follows that, for any i, j, k,

$$U_j b_{ik} = -2\sigma_{ij} b_{ik} \quad \text{on } \Omega. \tag{3.42}$$

Substituting (3.42) into (3.40), we obtain, for any i, j,

$$\sigma_{ij}\big(W_i^2 + (JW_i)^2\big) = [W_i, U_j]W_i + [JW_i, U_j](JW_i) \quad \text{on } T^*\Omega. \tag{3.43}$$

Recalling (2.3) and (2.4) in Section 2, we have

$$\sum_{k=1}^{n} r_{ik} b_{ik} \equiv 1 \quad \text{on } \Omega, \quad i = 1, \dots, n,$$

where r_{ik} are constants. Differentiating both sides with respect to the vector field U_j and using (3.42), we see that all σ_{ij} vanish on Ω. Thus, we obtain (U1-i) from (3.42) and also obtain, for any i, j,

$$\big\langle [W_i, U_j], W_i^*\big\rangle = \big\langle [JW_i, U_j], -J^*W_i^*\big\rangle = 0 \quad \text{on } \Omega, \tag{3.44}$$

$$[W_i, U_j]W_i + [JW_i, U_j](JW_i) = 0 \quad \text{on } T^*\Omega \tag{3.45}$$

from (3.41), (3.43). The property (3.45) together with (3.44) implies that, for any i, j,

$$[W_i, U_j] = \tau_{ij}(JW_i), \quad [JW_i, U_j] = -\tau_{ij} W_i \quad \text{on } \Omega, \tag{3.46}$$

where τ_{ij} are functions on Ω. Taking the values of the 1-form $db_{i\ell}$ on both sides of the second equation in (3.46), where ℓ is the index taken in (A-ii) in Section 2, we see from the condition (A-ii) in Section 2, Proposition 2.3 (1) in Section 2, and (U1-i) that all τ_{ij} vanish on Ω. We thus obtain (U1-ii), verifying the equivalence.

We now observe the vector fields $U_1, \ldots, U_n$ on Ω. We put $U_j = \sum_{k=1}^n \eta_{jk} W_k + \sum_{k=1}^n \xi_{jk}(JW_k)$, $j = 1, \ldots, n$, where η_{jk}, ξ_{jk} are functions on Ω. It obviously follows from (U1-i) that $Ub_{i\ell} = 0$ on Ω, where ℓ is the index taken in (A-ii) in Section 2. This together with the condition (A-ii) in Section 2 and Proposition 2.3 (1) in Section 2 that all η_{jk} vanish on Ω and hence that $U_j = \sum_{k=1}^n \xi_{jk}(JW_k)$. The condition (U2) then means that the matrix $[\xi_{ij}]$ is nonsingular at all points in Ω. From (U1-ii), Proposition 2.3 (2) in Section 2, and (2.35) in Section 2, we see that a system of partial differential equations

$$\frac{\partial \xi_{jk}}{\partial w_i} = -\zeta_{ik}\xi_{ji}, \quad \frac{\partial \xi_{jk}}{\partial w_{n+i}} = 0, \quad i, j, k = 1, \ldots, n \tag{3.47}$$

holds on Ω. The second equation in (3.47) implies that all ξ_{ij} are independent of the values of $w_{n+1}, \ldots, w_{2n}$. We then see from the first equation in (3.47) that all ζ_{ij} can be written only in terms of ξ_{st}, $s, t = 1, \ldots, n$, and hence are also independent of the values of $w_{n+1}, \ldots, w_{2n}$. From (2.36) in Section 2 and (CK-2), we obtain

$$\frac{\partial}{\partial w_i}\left(\frac{\partial \kappa_{ij}}{\partial w_{n+i}}\right) = \zeta_{ii}\,\frac{\partial \kappa_{ij}}{\partial w_{n+i}}, \quad i, j = 1, \ldots, n.$$

Since $K = [\kappa_{ij}]$ is the identity matrix at every point in S^-, we have $\kappa_{ij}(0, w_{n+i}) = \delta_{ij}$ for all $w_{n+i} \in \,]-\mathit{\Lambda}, \mathit{\Lambda}[$ and hence $\frac{\partial \kappa_{ij}}{\partial w_{n+i}}(0, w_{n+i}) = 0$ for all $w_{n+i} \in \,]-\mathit{\Lambda}, \mathit{\Lambda}[$. These imply that all $\frac{\partial \kappa_{ij}}{\partial w_{n+i}}$ vanish on Ω and hence that all κ_{ij} are independent of the values of $w_{n+1}, \ldots, w_{2n}$, which means $K = K^{(0)}$. Moreover, from (2.36) in Section 2 and (3.47), we have

$$\frac{\partial}{\partial w_i}\sum_{k=1}^n \xi_{jk}\kappa_{ks} = 0, \quad i, j, s = 1, \ldots, n.$$

Since $U_j = \sum_{k,s=1}^{n} \xi_{jk}\kappa_{ks}(\partial/\partial w_{n+s})$, the last assertion holds. $\square$

THEOREM 3.11. *Let $(M, g, J; \mathscr{F})$ be an n-dimensional Hermite-Liouville manifold of type*(A)*. Let Ω and $(w_1, \ldots, w_n, w_{n+1}, \ldots, w_{2n})$ be the neighborhood and the coordinate system as in Theorem* 3.10*, respectively. Then, the Riemannian metric g is Kählerian on Ω if and only if the following system of partial differential equations holds on Ω :*

$$\frac{a_j}{(a_i)^2}\frac{\partial a_i}{\partial w_j} = \sum_{k=1}^{n} \frac{\partial \kappa_{ik}}{\partial w_i}\check{\kappa}_{kj}, \qquad i \neq j, \quad i, j = 1, \ldots, n, \tag{3.48}$$

where the functions a_i and κ_{ij} are those defined in (2.3) *and* (2.31) *in Section* 2 *respectively, and where $\check{\kappa}_{ij}$ denotes the function defined as the (i, j)-entry of the inverse matrix K^{-1} of $K = [\kappa_{ij}]$.*

In particular, if the Riemannian metric g is Kählerian on Ω, then the geodesic flow is completely integrable on $T^*\Omega$.

PROOF. The Kähler form ω on Ω, which is defined by $\omega(X, Y) = g(X, JY)$ for any vector $X, Y \in T_p\Omega$ at each point $p \in \Omega$, can be written by

$$\omega = \sum_{j=1}^{n} \frac{1}{a_j} dw_j \wedge J^*(dw_j) \quad \text{on } \Omega.$$

From (2.19) in Proposition 2.3 (1), we see that $a_1, \ldots, a_n$ are all independent of the values of $w_{n+1}, \ldots, w_{2n}$. By (2.35) in Section 2, we have $d(J^*(dw_j)) = -\sum_{i=1}^{n} \zeta_{ij}(dw_i) \wedge J^*(dw_i)$ on Ω for each j. We thus obtain

$$d\omega = \sum_{i,j=1}^{n} \left(\frac{1}{a_j}\zeta_{ij} - \frac{1}{(a_i)^2}\frac{\partial a_i}{\partial w_j}\right) dw_j \wedge dw_i \wedge J^*(dw_i) \quad \text{on } \Omega.$$

This together with (2.36) in Section 2 implies that g is Kählerian on Ω if and only if (3.48) holds on Ω.

We then consider the case where g is Kählerian on Ω. In this case all ζ_{ij} are independent of the values of $w_{n+1}, \ldots, w_{2n}$. In fact, we first see from (3.48) that if $i \neq j$, then ζ_{ij} are independent of the values because so are $a_1, \ldots, a_n$. We can verify the independency for ζ_{jj}, $j = 1, \ldots, n$, as follows. From the relation $[W_j, [W_i, JW_i]] = 0$ on Ω for each $i \neq j$, which was already appeared in the proof of Proposition 2.3 (3) in Section 2, and from (2.35) in Section 2, we obtain

$$W_j\zeta_{ik} + \zeta_{ij}\zeta_{jk} = 0, \quad i \neq j,\ i, j, k = 1, \ldots, n.$$

By setting $k = j$, we have

$$\frac{\partial \zeta_{ij}}{\partial w_j} + \zeta_{ij}\zeta_{jj} = 0, \quad i \neq j,\ i, j = 1, \ldots, n.$$

Recalling (A-i)$'$ in Section 2, for each j and for each point p in Ω, we can take an index i such that $\zeta_{ij} \neq 0$ on some neighborhood of p. These imply that ζ_{jj}, $j = 1, \ldots, n$, are written by some ζ_{ij} with $i \neq j$ on some neighborhood of each point in Ω and hence are independent of the values of $w_{n+1}, \ldots, w_{2n}$, which completes the verification. By the same argument as in the proof of Theorem 3.10, we see that all κ_{ij} are independent of the values of $w_{n+1}, \ldots, w_{2n}$ and hence that $K = K^{(0)}$, where K and $K^{(0)}$ are the matrix-valued functions on Ω defined in (2.32) in Section 2 and (3.7) respectively. By Theorem 3.10, we conclude that the geodesic flow is completely integrable on $T^*\Omega$. $\square$

4. Local construction.

In the previous section we have completely solved the system of equations (3.2) under the conditions (CK-1), (CK-2), (CK-4), and (CK-5). In this section, we first show that there exists a corresponding Hermite-Liouville manifold to each solution $K = [\kappa_{ij}]$ described in Theorem 3.6 and Theorem 3.7. After that we shall give a bit finer description for the case $K = K^{(0)}$, where K and $K^{(0)}$ are as in Section 3.

Let $\Omega = \,]-\varLambda, \varLambda\,[^{2n}$ $(\varLambda > 0)$ be a small cube-like domain with the coordinate system $(w_1, \ldots, w_n, w_{n+1}, \ldots, w_{2n})$. Take any solution $K = [\kappa_{ij}(w_i, w_{n+i})]$ described in Theorem 3.6 and Theorem 3.7. Notice that it is an $n \times n$-matrix-valued function defined on Ω. We then define a complex structure J on Ω by

$$J(\partial/\partial w_i) = \sum_{j=1}^{n} \kappa_{ij}(\partial/\partial w_{n+i}), \quad i = 1, \ldots, n.$$

The integrability of J follows from the fact that Nijenhuis' tensor of J vanishes identically on Ω. We thus obtain a complex manifold (Ω, J). Take n constants $c_1, \ldots, c_n$ and n^2 functions $b_{ij}(w_i)$, $i, j = 1, \ldots, n$, of one variable such that (i) the matrix $B = [b_{ij}]$ is non-singular at every point in Ω; (ii) for each j, $\sum_{i=1}^{n} c_i\check{b}_{ij}$ is a positive function on Ω, where $\check{b}_{ij}$ denotes the (i, j)-entry of the inverse matrix B^{-1} of B. We next define an Hermitian metric g on Ω by

$$g = \sum_{i=1}^{n} \frac{1}{\sum_{j=1}^{n} c_j \check{b}_{ji}} \left[(dw_i)^2 + \left(\sum_{k=1}^{n} \check{\kappa}_{ki} dw_{n+k} \right)^2 \right],$$

where $\check{\kappa}_{ij}$ denotes the (i,j)-entry of the inverse matrix K^{-1} of K. We thus obtain an Hermitian manifold (Ω, g, J). We finally define n fiberwise homogenous polynomial functions $F_1, \dots, F_n$ on $T^*\Omega$ by

$$\sum_{j=1}^{n} b_{ij} F_j = (\partial/\partial w_i)^2 + \left(\sum_{s=1}^{n} \kappa_{is} (\partial/\partial w_{n+s}) \right)^2, \quad i = 1, \dots, n,$$

and set $\mathscr{F}$ to be the vector space spanned by $F_1, \dots, F_n$. Thus, we obtain an Hermite-Liouville manifold $(\Omega, g, J; \mathscr{F})$.

In particular, if the taken solution $K = [\kappa_{ij}]$ has the properties (1) and (2) in Theorem 3.8 and if $B = [b_{ij}]$ has the property that, for each i, $b'_{ij}(0) \neq 0$ for some j, then the constructed Hermite-Liouville manifold is of type (A).

In the rest of this section, we present a bit finer description of the local constructions corresponding to the case $K = K^{(0)}$, which will be useful for the comparison with the global constructions in the next section. Let $\Omega^+ =]-\Lambda, \Lambda[^n$ $(\Lambda > 0)$ be a cube-like domain with the coordinate system $(w_1, \dots, w_n)$. Take n^2 functions $b_{ij}(w_i)$, $i, j = 1, \dots, n$, of one variable $w_i \in]-\Lambda, \Lambda[$ such that (b-i) $\det[b_{ij}] \neq 0$ on Ω^+; (b-ii) Every entry of the n-th row of the inverse matrix $[b_{ij}]^{-1}$ of $[b_{ij}]$ is positive on Ω^+. We set $W_i = \partial/\partial w_i$, $i = 1, \dots, n$, on Ω^+. We define the functions $H_1, \dots, H_n$ on the cotangent bundle $T^*(\Omega^+)$ by

$$\sum_{j=1}^{n} b_{ij}(w_i) H_j = W_i^2, \quad i = 1, \dots, n.$$

It follows from (b-ii) that H_n is positive definite at all points in Ω^+ and defines a Riemannian metric of Liouville-Stäckel type on Ω^+. Thus, $H_1, \dots, H_n$ give a structure of (real) Liouville manifold with the Hamiltonian $H_n/2$ of the geodesic flow (cf. [**3**, Part I]).

Now, take another family $\{\kappa_{ij}(w_i)\}_{i,j=1,\dots,n}$ of n^2 functions of one variable such that $\det[\kappa_{ij}] \neq 0$ on Ω^+. We then define the vector fields $X_1, \dots, X_n$ on Ω^+ by

$$\sum_{j=1}^{n} \kappa_{ij}(w_i) X_j = W_i, \quad i = 1, \dots, n.$$

We see that $X_1, \dots, X_n$ are commutative and hence obtain a coordinate system

$(x_1, \dots, x_n)$ on Ω^+ such that $X_i = \partial/\partial x_i$, $i = 1, \dots, n$.

Let Ω^- be an open subset of $\boldsymbol{R}^n$ with the coordinate system $(y_1, \dots, y_n)$ and set $\Omega = \Omega^+ \times \Omega^-$. Putting $z_i = x_i + \sqrt{-1}\, y_i$, $i = 1, \dots, n$, we regard Ω as an open subset of $\boldsymbol{C}^n = \{(z_1, \dots, z_n)\}$. The vector fields W_i and X_i naturally extend onto Ω. Denote by J the natural complex structure on Ω; $J(\partial/\partial x_i) = \partial/\partial y_i$. We then define the functions $F_1, \dots, F_n$ on the cotangent bundle $T^*\Omega$ by

$$\sum_{j=1}^{n} b_{ij}(w_i)\, F_j = W_i^2 + (JW_i)^2, \quad i = 1, \dots, n,$$

and put $\mathscr{F} = \mathrm{Span}\{F_1, \dots, F_n\}$. Taking the Hermitian metric g so that $F_n/2$ is the Hamiltonian of the geodesic flow, we thus obtain an Hermite-Liouville manifold $(\Omega, g, J; \mathscr{F})$.

We now put $Y_i = \partial/\partial y_i$, $i = 1, \dots, n$. It is easy to see that Y_i preserve g, J, and commute with F_j's. In particular, the geodesic flow of (Ω, g) is integrable with the first integrals $F_1, \dots, F_n, Y_1 \dots, Y_n$. Notice that if for each j, there is $i (\neq j)$ such that $\kappa'_{ij}(0) \neq 0$, and for each i there is j such that $b'_{ij}(0) \neq 0$, then the constructed Hermite-Liouville manifold is of type (A) and corresponds to the case $K = K^{(0)}$ around a point $(o, p) \in \Omega$, where $p \in \Omega^-$ is any point.

5. Global construction.

In the present section we shall construct global examples of Hermite-Liouville manifolds biholomorphic to the complex projective space $\boldsymbol{CP}^n$. It is known that any Kähler-Liouville manifold (proper, type (A)) defined over $\boldsymbol{CP}^n$ is given by complexifying a certain Liouville manifold defined over the real projective space $\boldsymbol{RP}^n$, and the latter is constructed from a circle and suitable $n-1$ functions on it, called the core of type (B) (see [**3**, Part 2, Section 7, Part 1, Sections 3.2–3.4], and [**4**]).

In this section we shall prepare two sets of cores of type (B), with one of which it is possible to make a Kähler-Liouville manifold and the other is not necessarily so. First, we shall construct a Liouville manifold diffeomorphic to $\boldsymbol{RP}^n$ using the latter core, and then "complexify" it by using a scheme given by the former core.

By definition, a (possible) core of type (B) is a pair of a circle $\boldsymbol{R}/l\boldsymbol{Z}$ $(l > 0)$ with the standard metric dt^2 and a set $\{[f_1(t)], \dots, [f_{n-1}(t)]\}$ of projective classes of $n-1$ functions on it satisfying the following conditions.

(1) There are constants $0 < \beta_1 < \cdots < \beta_{n-1} < l/2$ such that $f_m(\pm\beta_m) = 0$, $f_m(t) > 0$ for $-\beta_m < t < \beta_m$, and $f_m(t) < 0$ for $\beta_m < t < l - \beta_m$.
(2) $f'_m(\beta_m) < 0$.

(3) $f_m(t) = f_m(-t)$ for any $t \in \boldsymbol{R}/l\boldsymbol{Z}$.
(4) $f_1(t) < \cdots < f_{n-1}(t)$ for any $t \in \boldsymbol{R}/l\boldsymbol{Z}$.

From a core of type (B) one can construct a Liouville manifold as follows. Put $\beta_0 = 0$, $\beta_n = l/2$, and define positive numbers $\alpha_1, \dots, \alpha_n$ by

$$\int_{\beta_{i-1}}^{\beta_i} \frac{dt}{\sqrt{(-1)^{i-1} f_1(t) \dots f_{n-1}(t)}} = \frac{\alpha_i}{4}.$$

Define the mapping $\boldsymbol{R}/\alpha_i\boldsymbol{Z} \to [\beta_{i-1}, \beta_i]$ $(w_i \mapsto t)$ by

$$\left(\frac{dt}{dw_i}\right)^2 = (-1)^{i-1} f_1(t) \dots f_{n-1}(t),$$
$$t(w_i) = t(-w_i) = t(\alpha_i/2 - w_i), \quad t(0) = \beta_i, \quad t(\alpha_i/4) = \beta_{i-1}.$$

Put

$$R = \prod_{i=1}^{n} (\boldsymbol{R}/\alpha_i\boldsymbol{Z}) = \{(w_1, \dots, w_n)\},$$

and define the involutions σ_i, $1 \le i \le n-1$, and τ on R by

$$\sigma_i(x) = \left(w_1, \dots, w_{i-1}, -w_i, \frac{\alpha_{i+1}}{2} - w_{i+1}, w_{i+2}, \dots, w_n\right),$$
$$\tau(x) = \left(w_1 + \frac{\alpha_1}{2}, -w_2, \dots, -w_n\right).$$

It is easily seen that they are mutually commutative and generate a group G isomorphic to $(\boldsymbol{Z}/2\boldsymbol{Z})^n$. Then the quotient space $N = R/G$ is diffeomorphic to the real projective space with a natural differentiable structure.

Define the functions $f_{ik} \in C^\infty(\boldsymbol{R}/\alpha_i\boldsymbol{Z})$ by

$$f_{ik}(w_i) = f_k(t(w_i)), \quad 1 \le k \le n-1, \ 1 \le i \le n,$$

and the matrix-valued function $[b_{ij}(w_i)]_{1 \le i,j \le n}$ by

$$b_{ij} = b_{ij}(w_i) = \begin{cases} (-1)^i \prod_{k \ne j} f_{ik}(w_i) & (1 \le j \le n-1), \\ (-1)^{i+1} \prod_k f_{ik}(w_i) & (j = n). \end{cases}$$

Then by the formula

$$\sum_{j=1}^{n} b_{ij}(w_i)F_j = (\partial/\partial w_i)^2, \quad 1 \le i \le n,$$

one obtains well-defined symmetric 2-tensor fields $F_1, \dots, F_n$ on N. Also, F_n turns out to be positive definite at any point. Thus, putting $\mathscr{F} = \mathrm{Span}\{F_1, \dots, F_n\}$, one gets a Liouville manifold $(N, g; \mathscr{F})$ over N whose energy function is equal to $F_n/2$.

With a special kind of core of type (B) one can construct a Kähler-Liouville manifold over the complex projective space. Let $v(t)$ be a function on $\boldsymbol{R}/l\boldsymbol{Z}$ and let $0 < \beta_1 < \cdots < \beta_{n-1} < l/2$ and $c_* > 0$ be a constant which satisfy the following conditions.

(1) $v(-t) = v(t)$.
(2) $v(0) = 1$, $v(l/2) = 0$.
(3) $v'(t) < 0$ if $0 < t < l/2$.
(4) $v'(\beta_i) = -\sqrt{2c_* c_i(1 - c_i)}$, $1 \le i \le n-1$, where $c_i = v(\beta_i)$.
(5) $-v''(0) = v''(l/2) = c_*$.

Then, clearly $\boldsymbol{R}/l\boldsymbol{Z}$ and $\{[v - c_1], \dots, [v - c_{n-1}]\}$ form a core of type (B) and yield objects explained above: a torus $\tilde{R} = \prod_{i=1}^{n}(\boldsymbol{R}/\tilde{\alpha}_i\boldsymbol{Z}) = \{(\tilde{w}_1, \dots, \tilde{w}_n)\}$, the functions $\tilde{b}_{ij}(\tilde{w}_i)$, a manifold $\tilde{N}$ and the branched covering $\tilde{R} \to \tilde{N}$, the symmetric 2-tensor fields $\tilde{F}_1, \dots, \tilde{F}_n$ defined by

$$\sum_{j=1}^{n} \tilde{b}_{ij}(\tilde{w}_i)\tilde{F}_j = (\partial/\partial \tilde{w}_i)^2, \quad 1 \le i \le n, \tag{5.1}$$

and the Riemannian metric $\tilde{g}$ on $\tilde{N}$. As above, $(\tilde{N}, \tilde{g}; \tilde{\mathscr{F}})$ is a Liouville manifold, where $\tilde{\mathscr{F}} = \mathrm{Span}\{\tilde{F}_1, \dots, \tilde{F}_n\}$. Putting $v_i(\tilde{w}_i) = v(t(\tilde{w}_i))$, we have

$$\tilde{g} = \sum_{i=1}^{n}(-1)^{n-i}\left(\prod_{k \ne i}\bigl(v_k(\tilde{w}_k) - v_i(\tilde{w}_i)\bigr)\right)(d\tilde{w}_i)^2.$$

Putting $c_0 = 1$ and $c_n = 0$, we define vector fields $X_0, \dots, X_n$ on $\tilde{N}$ by the formula

$$X_i = \frac{\mathrm{grad}\bigl(\prod_k (v_k - c_i)\bigr)}{c_* \prod_{\substack{0 \le m \le n \\ m \ne i}}(c_m - c_i)}, \quad 0 \le i \le n,$$

where $\operatorname{grad} f$ denotes the gradient vector field of f with respect to the metric $\tilde{g}$. They satisfy $[X_i, X_j] = 0$ for any i, j, and

$$\sum_{i=0}^{n} X_i = 0,$$
$$c_* \sum_{i=0}^{n} \prod_{k \neq i} \bigl(v_j(\tilde{w}_j) - c_k\bigr) X_i = (-1)^j v_j'(\tilde{w}_j)(\partial/\partial \tilde{w}_j), \quad 1 \leq j \leq n. \tag{5.2}$$

Note that they are also determined by the above formulas.

Now, let $[u_0, \dots, u_n]$ be the homogeneous coordinate system of $\boldsymbol{RP}^n$ and let $\pi : \boldsymbol{R}^{n+1} \setminus \{0\} \to \boldsymbol{RP}^n$ be the natural projection. By integrating the vector fields X_i, it turns out that there is a diffeomorphism $\phi : \tilde{N} \to \boldsymbol{RP}^n$ such that

$$\phi_*(X_i) = \pi_*(u_i(\partial/\partial u_i)), \quad 0 \leq i \leq n.$$

Let $\boldsymbol{CP}^n$ be the complex projective space with the homogeneous coordinates $[u_0, \dots, u_n]$ whose real part is $\boldsymbol{RP}^n$. The torus $U(1)^n = U(1)^{n+1}/U(1)$ naturally acts on $\boldsymbol{CP}^n$:

$$\bigl((\lambda_0, \dots, \lambda_n), [u_0, \dots, u_n]\bigr) \mapsto [\lambda_0 u_0, \dots, \lambda_n u_n], \quad |\lambda_i| = 1.$$

Then the vector fields X_i extends to $\boldsymbol{CP}^n$ so that they are invariant under the torus action. Clearly, $Y_i = JX_i$, $0 \leq i \leq n$, generate the torus action. We denote by $\mathscr{Y}$ the abelian Lie algebra spanned by Y_i's. Also, each $\tilde{F}_i$ is extended to the whole $\boldsymbol{CP}^n$ in the following way: First, we identify $\tilde{F}_i$, a section of $S^2(T\boldsymbol{RP}^n)$, with a symmetric 2-form on $\boldsymbol{RP}^n$ by using the natural identification of tangent and cotangent bundles. Then, we extend it as a symmetric 2-form on $\boldsymbol{CP}^n$ by the conditions (1) it is Hermitian at any point; (2) $\tilde{F}_i(X, JY) = 0$ for any vectors X, Y tangent to $\boldsymbol{RP}^n$; (3) it is invariant under the torus action; (4) the restriction of $\tilde{F}_i$ to $T\boldsymbol{RP}^n$ coincides with the original one. Finally we identify it with a contravariant symmetric 2-tensor field on $\boldsymbol{CP}^n$ (see [**3**, p. 138 Lemma 7.8]). Let $\mathscr{F}$ be the vector space spanned by the extended $\tilde{F}_i$'s. Then, with the Kähler metric g determined by $\tilde{F}_n$, $\mathscr{F}$ provides a structure of Kähler-Liouville manifold over $\boldsymbol{CP}^n$, and with $\mathscr{F}$ and $\mathscr{Y}$ the geodesic flow of $(\boldsymbol{CP}^n, g)$ becomes integrable.

Remark. Putting $v(t) = (\cos t)^2$, $l = \pi$, $c^* = 2$, one obtains the Fubini-Study metric.

Now, we shall construct an Hermite-Liouville manifold over $\boldsymbol{CP}^n$ from given two cores of type (B): one is a general kind, $\{[f_1(t)], \dots, [f_{n-1}(t)])\}$, and the other is a special kind $\{[v-c_1],\dots,[v-c_{n-1}]\}$. We assume the constants $l>0$ (the length of the core circle) and β_i's (zeros of the core functions) are the same for the above two cores. Also, we use the same symbols as in the above explanation.

Since l and β_i's are common, we have a diffeomorphism $\boldsymbol{R}/\alpha_i\boldsymbol{Z} \to \boldsymbol{R}/\tilde{\alpha}_i\boldsymbol{Z}$ for each i so that $w_i=0 \leftrightarrow \tilde{w}_i=0$, $d\tilde{w}_i/dw_i>0$ and the following diagram is commutative:

$$\begin{tikzcd} \boldsymbol{R}/\alpha_i\boldsymbol{Z} \arrow[r] \arrow[d] & \boldsymbol{R}/\tilde{\alpha}_i\boldsymbol{Z} \arrow[d] \\ {[\beta_{i-1},\beta_i]} \arrow[r, equal] & {[\beta_{i-1},\beta_i].} \end{tikzcd}$$

This gives the diffeomorphism $R\to\tilde{R}$ and hence the diffeomorphism $\psi: N\to\tilde{N}$. Put $H_i=\psi_*F_i$, which are symmetric 2-tensor fields on $\tilde{N}$. Identifying $\tilde{N}$ with $\boldsymbol{RP}^n \subset \boldsymbol{CP}^n$ as above, we extend H_i to the whole $\boldsymbol{CP}^n$ in the same way as explained above. Then the Hermite metric g on $\boldsymbol{CP}^n$ determined by H_n is not Kählerian in general. Let $\mathscr{H}$ be the vector space spanned by the extended H_i's. The pair $(g,\mathscr{H})$ provides a structure of Hermite-Liouville manifold over $\boldsymbol{CP}^n$, and the geodesic flow of $(\boldsymbol{CP}^n,g)$ is again integrable with $\mathscr{H}$ and $\mathscr{Y}$. In view of the criterion described at the end of Section 4, it is easily verified that the Hermite-Liouville manifolds constructed here are of type (A) and correspond to the case $K=K^{(0)}$ in Section 3. Note that (A-i) is satisfied on the open dense subset defined by $u_i\neq 0$ for any i, but (A-ii) is not necessary so. It is only clear that (A-ii) is satisfied near the torus-orbits through the branch points of the covering $\boldsymbol{R}\to\tilde{N}=\boldsymbol{RP}^n\subset\boldsymbol{CP}^n$.

The following theorem clarifies which one is Kählerian among the constructed Hermite-Liouville manifolds.

Theorem 5.1. *Let $(\boldsymbol{CP}^n,g,\mathscr{H})$ be an Hermite-Liouville manifold constructed from two cores*

$$\big(\boldsymbol{R}/l\boldsymbol{Z};[f_1],\dots,[f_{n-1}]\big),\quad \big(\boldsymbol{R}/l\boldsymbol{Z};[v-c_1],\dots,[v-c_{n-1}]\big).$$

Then it is Kählerian if and only if $[f_i]=[v-c_i]$ for every i.

Proof. Suppose that $(\boldsymbol{CP}^n,g)$ is Kählerian. Then, by the construction, it is of type (A) in the meaning of [**3**, p. 85], and the associated partially ordered set $\mathscr{A}$ consists of one element (see [**3**, p. 88]). Then, the theorem follows from Theorem 7.2 in [**3**] and its proof. □

Let us give a simple example of Hermite-Liouville manifold which is not Kählerian. Let $(\boldsymbol{CP}^n, g, \mathscr{H})$ be a Kähler-Liouville manifold constructed from a core

$$(\boldsymbol{R}/l\boldsymbol{Z}; [v - c_1], \dots, [v - c_{n-1}]),$$

as explained above. Let $H_i \in \mathscr{H}$, $1 \le i \le n$, be as above. Then the metric g corresponds to H_n via Legendre transformation. Now, take small constants ϵ_i, $1 \le i \le n-1$, and put

$$\tilde{H}_n = H_n + \sum_{i=1}^{n-1} \epsilon_i H_i.$$

If ϵ_i are small enough, then $\tilde{H}_n$ is still positive definite, and one obtains the corresponding Hermite metric $\tilde{g}$ on $\boldsymbol{CP}^n$. Clearly, $(\boldsymbol{CP}^n, \tilde{g}, \mathscr{H})$ is an Hermite-Liouville manifold. Observing the real Liouville manifold obtained by restricting to $\boldsymbol{RP}^n$, one can easily see that it is constructed from two cores

$$(\boldsymbol{R}/l\boldsymbol{Z}; [f_1], \dots, [f_{n-1}]), \quad (\boldsymbol{R}/l\boldsymbol{Z}; [v - c_1], \dots, [v - c_{n-1}]),$$

where

$$f_i(t) = \frac{v(t) - c_i}{1 + \epsilon_i (v(t) - c_i)}, \quad 1 \le i \le n-1.$$

Therefore it is not Kählerian by the previous theorem.

Finally, let us state a theorem which will answer to the isomorphism problem on the constructed Hermite-Liouville manifolds.

THEOREM 5.2. *Let $(\boldsymbol{CP}^n, g_\nu, \mathscr{H}_\nu)$ be an Hermite-Liouville manifold constructed with cores*

$$\big(\boldsymbol{R}/l_\nu\boldsymbol{Z}; [f_{\nu,1}], \dots, [f_{\nu,n-1}]\big), \quad \big(\boldsymbol{R}/l_\nu\boldsymbol{Z}; [v_\nu - c_{\nu,1}], \dots, [v_\nu - c_{\nu,n-1}]\big)$$

of type (B) *as above* ($\nu = 1, 2$). *Then, there is a holomorphic isometry $\Phi : (\boldsymbol{CP}^n, g_1) \to (\boldsymbol{CP}^n, g_2)$ which maps $\mathscr{H}_1$ to $\mathscr{H}_2$ if and only if $l_1 = l_2$ and either*

$$[f_{2,i}(t)] = [f_{1,i}(t)], \ c_{2,i} = c_{1,i}, \ 1 \le i \le n-1, \quad v_2(t) = v_1(t)$$

or

$$[f_{2,i}(t)] = [-f_{1,n-i}(l_1/2 - t)], \; c_{2,i} = 1 - c_{1,n-i}, \; 1 \le i \le n-1,$$
$$v_2(t) = 1 - v_1(l_1/2 - t).$$

Proof. Let $[u_0, \dots, u_n]$ be the homogeneous coordinates of $\boldsymbol{CP}^n$ as above. Let $\tilde{L}_i$ be the hyperplane in $\boldsymbol{CP}^n$ defined by $u_i = 0$, and let $L_i = \tilde{L}_i \cap \boldsymbol{RP}^n$, $i = 0, \dots, n$. Put $\boldsymbol{CP}_1^n = \boldsymbol{CP}^n \setminus \cup_{i=0}^n \tilde{L}_i$. As stated before, the condition (A-i) is satisfied on $\boldsymbol{CP}_1^n$, and (A-ii) is satisfied on a certain open subset $\boldsymbol{CP}_{2,\nu}^n$ of it, which is invariant under the torus action generated by $\mathscr{Y}$, for each $\nu = 1, 2$. Since the vector space $\mathscr{Y}$ of vector fields, restricted to $\boldsymbol{CP}_{2,\nu}^n$, is determined by the Hermite-Liouville structure $(g_\nu, \mathscr{H}_\nu)$ (Theorem 3.10), it is determined on the whole $\boldsymbol{CP}^n$ by $(g_\nu, \mathscr{H}_\nu)$ as a vector space of infinitesimal holomorphic transformations. Thus Φ preserves $\mathscr{Y}$. Since $\boldsymbol{CP}_1^n$ is determined by $\mathscr{Y}$, it is also preserved by Φ.

Now, take a point $p \in \boldsymbol{RP}^n \cap \boldsymbol{CP}_{2,1}^n$ and fix it. Then $\boldsymbol{RP}^n$ is the totally geodesic submanifolds whose tangent space at p is equal to D_p^+. Composing Φ with a transformation of $\boldsymbol{CP}^n$ generated by an element of $\mathscr{Y}$ if necessary, we may assume that $\Phi(p) \in \boldsymbol{RP}^n \cap \boldsymbol{CP}_{2,2}^n$. Since Φ maps D_p^+ to $D_{\Phi(p)}^+$, it follows that Φ preserves the submanifold $\boldsymbol{RP}^n$.

Putting

$$\hat{g}_\nu = g_\nu|_{\boldsymbol{RP}^n}, \quad \hat{\mathscr{H}_\nu} = \{\hat{H}; H \in \mathscr{H}_\nu\}, \quad \hat{H} = H|_{T^*\boldsymbol{RP}^n},$$

we obtain (real) Liouville manifolds $(\boldsymbol{RP}^n, \hat{g}_\nu, \hat{\mathscr{H}_\nu})$, and we see that

$$\Phi : \left(\boldsymbol{RP}^n, \hat{g}_1, \hat{\mathscr{H}_1}\right) \to \left(\boldsymbol{RP}^n, \hat{g}_2, \hat{\mathscr{H}_2}\right)$$

is an isomorphism of Liouville manifolds. Therefore, by Theorem 3.4.1 in [**3**], it follows that their cores are mutually isomorphic, i.e., $l_1 = l_2$ $(= l)$ and either

$$[f_{2,i}(t)] = [f_{1,i}(t)], \quad 1 \le i \le n-1, \tag{5.3}$$

or

$$[f_{2,i}(t)] = [-f_{1,n-i}(l/2 - t)], \quad 1 \le i \le n-1, \tag{5.4}$$

hold on $\boldsymbol{R}/l\boldsymbol{Z}$. More precisely, the map Φ preserves the core submanifold $\cap_{i=1}^{n-1} L_i$, which is isometric to $\boldsymbol{R}/l\boldsymbol{Z}$, and the subset consisting of the two points

$$L_0 \bigcap \bigcap_{i=1}^{n-1} L_i \quad \text{and} \quad L_n \bigcap \bigcap_{i=1}^{n-1} L_i$$

on it, which correspond to $0, l/2 \in \boldsymbol{R}/l\boldsymbol{Z}$. Hence Φ, viewed as a map on $\boldsymbol{R}/l\boldsymbol{Z}$, is either the map $t \mapsto \pm t$ or the map $t \mapsto l/2 \pm t$. In the first case, we have (5.3), and in the second case, we have (5.4).

Next, we shall observe the abelian Lie algebra $J\mathscr{Y}$ of vector fields on $\boldsymbol{RP}^n$ generated by $X_0, \ldots, X_n$ described above, which is preserved by Φ. Since $X_i = 0$, $1 \le i \le n-1$, on the core submanifold, the formula (5.2) turns out to be

$$c_{\nu,*}X_n = v_\nu'(t)\frac{d}{dt}$$

on the core submanifold $\cap_{i=1}^{n-1} L_i = \boldsymbol{R}/l\boldsymbol{Z}$. Since X_n is mapped to its scalar multiple by Φ on the core submanifold, so is the derivatives of the functions $v_\nu(t)$. Thus we have $v_2(t) = v_1(t)$ if Φ on $\boldsymbol{R}/l\boldsymbol{Z}$ is given by $t \mapsto \pm t$, and $v_2(t) = 1 - v_1(l/2 - t)$ if Φ on $\boldsymbol{R}/l\boldsymbol{Z}$ is given by $t \mapsto l/2 \pm t$. Since $c_{\nu,i}$ is given by $v_\nu(\beta_i)$, the theorem therefore follows. □

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