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Moduli of stable objects in a triangulated category

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Abstract. We introduce the concept of strict ample sequence in a fibered triangulated category and define the stability of the objects in a triangulated category. Then we construct the moduli space of (semi) stable objects by GIT construction.

1. Introduction.

Let $X \to S$ be a projective and flat morphism of noetherian schemes. We consider the functor $\mathrm{Splcpx}_{X/S} : (\mathrm{Sch}/S) \to (\mathrm{Sets})$ defined by

$$\mathrm{Splcpx}_{X/S}(T) = \left\{ E \in D^b(\mathrm{Coh}(X \times_S T)) \;\middle|\; \begin{array}{l} \text{for any geometric point } t \text{ of } T,\ E(t) := \\ E \otimes^L k(t) \text{ is a bounded complex and} \\ \mathrm{Ext}^i(E(t), E(t)) \cong \begin{cases} k(t) & \text{if } i = 0 \\ 0 & \text{if } i = -1 \end{cases} \end{array} \right\} \Big/ \sim,$$

where $E \sim E'$ if there is a line bundle L on T such that $E \cong E' \otimes L$ in $D^b(\mathrm{Coh}(X \times_S T))$. We denote the étale sheafification of $\mathrm{Splcpx}_{X/S}$ by $\mathrm{Splcpx}^{\text{ét}}_{X/S}$. Then the result of [**4**] is that $\mathrm{Splcpx}^{\text{ét}}_{X/S}$ is an algebraic space over S. M. Lieblich extends this result in [**7**] to the case when $X \to S$ is a proper flat morphism of algebraic spaces. So the problem on the construction of the moduli space of objects in a derived category is solved in some sense. However, the moduli space $\mathrm{Splcpx}^{\text{ét}}_{X/S}$ is not separated and it is not a good space in geometric sense. So we want to construct a projective moduli space (or quasi-projective moduli space with a good compactification) as a Zariski open set of $\mathrm{Splcpx}^{\text{ét}}_{X/S}$ such as the moduli space of stable sheaves.

This problem is also motivated by Fourier-Mukai transform. Let X, Y be

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projective varieties over an algebraically closed field k and $\mathscr{P}$ be an object of $D^b(\mathrm{Coh}(X \times Y))$. The functor

$$\Phi : D^b(\mathrm{Coh}(X)) \longrightarrow D^b(\mathrm{Coh}(Y))$$
$$E \mapsto \mathrm{R}(p_Y)_*(p_X^*(E) \otimes^L \mathscr{P})$$

is called a Fourier-Mukai transform if it is an equivalence of categories. Here $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are the projections. Fourier-Mukai transform induces the isomorphisms on moduli spaces and for example the image $\Phi(M_X^P)$ of a moduli space of stable sheaves M_X^P on X by Φ sometimes becomes a moduli space of stable sheaves on Y. The problem on the preservation of stability under Fourier-Mukai transform is investigated by many people and this problem is clearly pointed out by K. Yoshioka in [**11**]. However, the image $\Phi(M_X^P)$ of the moduli space of stable sheaves by the Fourier-Mukai transform may not be contained in the category of coherent sheaves on Y in general and so we must consider certain moduli space of stable objects in the derived category $D^b(\mathrm{Coh}(Y))$.

In this paper we introduce the concept "strict ample sequence" in a triangulated category. "Strict ample sequence" satisfies the condition of ample sequence defined by A. Bondal and D. Orlov in [**2**], but it also satisfies many other conditions because we expect that a "polarization" is determined by strict ample sequence. Indeed we can define stable objects determined by a strict ample sequence and construct the moduli space of stable objects (resp. S-equivalence classes of semistable objects) as a quasi-projective scheme (resp. projective scheme). This is the main result of this paper (Theorem 4.4 and Theorem 4.8). If $\Phi : D^b(\mathrm{Coh}(X)) \to D^b(\mathrm{Coh}(Y))$ is a Fourier-Mukai transform and M_X^P is a moduli space of stable sheaves on X, then the image $\Phi(M_X^P)$ of M_X^P by Φ becomes a moduli space of stable objects in $D^b(\mathrm{Coh}(Y))$ whose stability is determined by some strict ample sequence on $D^b(\mathrm{Coh}(Y))$. So Fourier-Mukai transform always preserves certain stability in our sense (Example 5.3).

T. Bridgeland defined in [**1**] the concept of stability condition on a triangulated category. So we are interested in the relation between the stability condition of Bridgeland and the definition of stability determined by a strict ample sequence. However, it seems rather impossible to expect the construction of a strict ample sequence from the stability condition defined by Bridgeland without any other condition. How to treat the relation between strict ample sequence and stability condition of Brigeland is a problem still unsolved.

2. Definition of fibered triangulated category.

Let S be a noetherian scheme. We denote the category of noetherian schemes over S by (Sch/S) and the derived category of bounded complexes of coherent sheaves on U by $D_c^b(U)$ for $U \in (\mathrm{Sch}/S)$. We denote the derived category of lower bounded complexes of coherent sheaves on U by $D_c^+(U)$ for $U \in (\mathrm{Sch}/S)$. For a noetherian scheme X over S, we denote the full subcategory of $D_c^b(X)$ consisting of the objects of finite Tor-dimension over S by $D^b(\mathrm{Coh}(X/S))$. Then $D^b(\mathrm{Coh}(X/S))$ becomes a triangulated category. For a triangulated category $\mathscr{T}$ and for objects $E, F \in \mathscr{T}$, we write $\mathrm{Ext}^i(E,F) := \mathrm{Hom}_{\mathscr{T}}(E, F[i])$.

DEFINITION 2.1. $p : \mathscr{D} \to (\mathrm{Sch}/S)$ is called a fibered triangulated category if

(1) $\mathscr{D}$ is a category, p is a covariant functor,
(2) for any $U \in (\mathrm{Sch}/S)$, the full subcategory $\mathscr{D}_U := p^{-1}(U)$ of $\mathscr{D}$ is a triangulated category,
(3) for any object $E \in \mathscr{D}_U$ and for any morphism $f : V \to U = p(E)$ in (Sch/S), there exist an object $F \in \mathscr{D}_V$ and a morphism $u : F \to E$ satisfying the condition: For any object $G \in \mathscr{D}_V$ and a morphism $v : G \to E$ with $p(v) = f$, there exists a unique morphism $w : G \to F$ satisfying $p(w) = \mathrm{id}_V$ and $u \circ w = v$, (we denote F by $f^*(E)$ or E_V and we call such morphism u a Cartesian morphism),
(4) any composition of Cartesian morphisms is Cartesian,
(5) for any morphism $V \to U$ in (Sch/S), $\mathscr{D}_U \ni E \mapsto E_V \in \mathscr{D}_V$ is an "exact functor", that is, for any distinguished triangle $E \to F \to G$ in $\mathscr{D}_U$, $E_V \to F_V \to G_V$ is a distinguished triangle in $\mathscr{D}_V$ and for any $E \in \mathscr{D}_U$ and any $i \in \boldsymbol{Z}$, there is an isomorphism $(E[i])_V \cong E_V[i]$ functorial in E.

DEFINITION 2.2. A fibered triangulated category $p : \mathscr{D} \to (\mathrm{Sch}/S)$ has base change property if

(1) for each $U \in (\mathrm{Sch}/S)$, there is a bi-exact bi-functor $\otimes : \mathscr{D}_U \times D^b(\mathrm{Coh}(U/U)) \to \mathscr{D}_U$ such that there is a functorial isomorphism $E[i] \otimes P[j] \cong (E \otimes P)[i+j]$ for $E \in \mathscr{D}_U$, $P \in D^b(\mathrm{Coh}(U/U))$,
(2) for a morphism $\varphi : U \to V$ in (Sch/S), the diagram

$$\begin{array}{ccc} \mathscr{D}_V \times D^b(\mathrm{Coh}(V/V)) & \xrightarrow{\otimes} & \mathscr{D}_V \\ {\scriptstyle \varphi^* \times L\varphi^*}\downarrow & & \downarrow{\scriptstyle \varphi^*} \\ \mathscr{D}_U \times D^b(\mathrm{Coh}(U/U)) & \xrightarrow{\otimes} & \mathscr{D}_U \end{array}$$

is "commutative", precisely, there exists a functorial isomorphism $\varphi^* \circ \otimes \xrightarrow{\sim} \otimes \circ (\varphi^* \times L\varphi^*)$,

(3) for $U \in (\mathrm{Sch}/S)$, there is a bi-exact bi-functor

$$\boldsymbol{R}\operatorname{Hom}_p : \mathscr{D}_U \times \mathscr{D}_U \longrightarrow D^+_c(U)$$

such that for $E_1,\ E_2 \in \mathscr{D}_U$ and for intgers i, j, there is an isomorphism $\boldsymbol{R}\operatorname{Hom}_p(E_1[i], E_2[j]) \cong \boldsymbol{R}\operatorname{Hom}_p(E, F)[j-i]$ functorial in E_1 and E_2 and also for $E_1,\ E_2 \in \mathscr{D}_U$ there is an isomorphism $\operatorname{Hom}_{D(U)}(\mathscr{O}_U, \boldsymbol{R}\operatorname{Hom}_p(E_1, E_2)) \xrightarrow{\sim} \operatorname{Hom}_{\mathscr{D}_U}(E_1, E_2)$ functorial in E_1 and E_2,

(4) for any $U \in (\mathrm{Sch}/S)$ and for any objects $E_1, E_2 \in \mathscr{D}_U$, there exist a lower bounded complex $P^\bullet$ of locally free sheaves of finite rank on U and an isomorphism

$$P^\bullet \otimes \mathscr{O}_V \xrightarrow{\sim} \boldsymbol{R}\operatorname{Hom}_p((E_1)_V, (E_2)_V)$$

in $D^+_c(V)$ for any morphism $V \to U$ in (Sch/S), such that the diagram

$$\begin{array}{ccl}
H^0(\Gamma((U, P^\bullet)) & \longrightarrow & \operatorname{Hom}_{D(U)}(\mathscr{O}_U, \boldsymbol{R}\operatorname{Hom}_p(E_1, E_2)) \\
\downarrow & & \\
H^0(\Gamma((V, P^\bullet \otimes \mathscr{O}_V)) & \longrightarrow & \operatorname{Hom}_{D(V)}(\mathscr{O}_V, \boldsymbol{R}\operatorname{Hom}_p((E_1)_V, (E_2)_V)) \\
& \xrightarrow{\sim} & \operatorname{Hom}_{\mathscr{D}_U}(E_1, E_2) \\
& & \downarrow \\
& \xrightarrow{\sim} & \operatorname{Hom}_{\mathscr{D}_V}((E_1)_V, (E_2)_V)
\end{array}$$

is commutative,

(5) for $U \in (\mathrm{Sch}/S)$, $E_1,\ E_2 \in \mathscr{D}_U$ and $F_1,\ F_2 \in D^b(\mathrm{Coh}(U/U))$, there is a functorial isomorphism $\boldsymbol{R}\operatorname{Hom}_p(E_1 \otimes F_1,\ E_2 \otimes_U F_2) \cong \boldsymbol{R}\operatorname{Hom}_p(E_1, E_2) \otimes^L_{\mathscr{O}_U} \boldsymbol{R}\mathscr{H}om(F_1, F_2)$ such that for any morphism $\varphi : V \to U$ in (Sch/S), the diagram

$$\begin{array}{ccc}
\boldsymbol{R}\operatorname{Hom}_p(E_1 \otimes F_1, E_2 \otimes F_2) & \xrightarrow{\sim} & \boldsymbol{R}\operatorname{Hom}_p(E_1, E_2) \otimes^L_{\mathscr{O}_U} \boldsymbol{R}\mathscr{H}om(F_1, F_2) \\
\downarrow & & \downarrow \\
\boldsymbol{R}\varphi_*(\boldsymbol{R}\operatorname{Hom}_p((E_1 \otimes F_1)_V, (E_2 \otimes F_2)_V)) & \xrightarrow{\sim} & \boldsymbol{R}\varphi_*(\boldsymbol{R}\operatorname{Hom}_p((E_1)_V, (E_2)_V) \otimes^L_{\mathscr{O}_V} \boldsymbol{R}\mathscr{H}om((F_1)_V, (F_2)_V))
\end{array}$$

is commutative.

Remark 2.3. For $E, F \in \mathscr{D}_U$, we denote the i-th cohomology $H^i(\boldsymbol{R}\operatorname{Hom}_p(E,F))$ by $R^i \operatorname{Hom}_p(E,F)$. We notice that for three objects $E, F, G \in \mathscr{D}_U$, there is a canonical morphism

$$R^0 \operatorname{Hom}_p(E,F) \times R^0 \operatorname{Hom}_p(F,G) \to R^0 \operatorname{Hom}_p(E,G).$$

Example 2.4. Let $X \to S$ be a flat projective morphism. Then $\{D^b(\operatorname{Coh}(X_U/U))\}_{U \in (\mathrm{Sch}/S)}$ becomes a fibered triangulated category over S which has base change property.

Example 2.5. Let X be a projective scheme over $\boldsymbol{C}$ and G a finite group acting on X. For a scheme $U \in (\mathrm{Sch}/\boldsymbol{C})$, let $D^G(\operatorname{Coh}(X_U/U))$ be the derived category of bounded complexes of G-equivariant coherent sheaves on X_U of finite Tor-dimension over U. Then $\{D^G(\operatorname{Coh}(X_U/U))\}_{U \in (\mathrm{Sch}/\boldsymbol{C})}$ becomes a fibered triangulated category over $\boldsymbol{C}$ which has base change property.

3. Strict ample sequence and stability.

Definition 3.1. Let $p : \mathscr{D} \to (\mathrm{Sch}/S)$ be a fibered triangulated category with base change property. A sequence $\mathscr{L} = \{L_n\}_{n \geq 0}$ of objects of $\mathscr{D}_S$ is said to be a strict ample sequence if it satisfies the following conditions:

(1) $\operatorname{Ext}^i((L_N)_s, (L_n)_s) = 0$ for any $i \neq 0$, $N > n$ and $s \in S$.
(2) There exist isomorphisms

$$\theta_k : R^0 \operatorname{Hom}_p(L_n, L_m) \xrightarrow{\sim} R^0 \operatorname{Hom}_p(L_{n+k}, L_{m+k})$$

for non-negative integers k, m, n with $n \geq m$ such that $\theta_k \circ \theta_l = \theta_{k+l}$ for any k, l and the diagram

$$\begin{array}{ccc} R^0 \operatorname{Hom}_p(L_n, L_m) \otimes R^0 \operatorname{Hom}_p(L_m, L_l) & \xrightarrow{\theta_k \otimes \theta_k} & R^0 \operatorname{Hom}_p(L_{n+k}, L_{m+k}) \otimes R^0 \operatorname{Hom}_p(L_{m+k}, L_{l+k}) \\ \downarrow & & \downarrow \\ R^0 \operatorname{Hom}_p(L_n, L_l) & \xrightarrow{\theta_k} & R^0 \operatorname{Hom}_p(L_{n+k}, L_{l+k}) \end{array}$$

is commutative for non-negative integers k, l, m, n with $n \geq m \geq l$.

(3) There exists a subbundle $V_1 \subset R^0 \operatorname{Hom}_p(L_1, L_0)$ such that the diagram

$$\begin{array}{ccc} V_1 \times R^0 \operatorname{Hom}_p(L_n, L_0) & \xrightarrow{\theta_n \times 1} & R^0 \operatorname{Hom}_p(L_{n+1}, L_n) \times R^0 \operatorname{Hom}_p(L_n, L_0) \\ {\scriptstyle 1\times\theta_1}\downarrow & & \downarrow \\ V_1 \times R^0 \operatorname{Hom}_p(L_{n+1}, L_1) & \longrightarrow & R^0 \operatorname{Hom}_p(L_{n+1}, L_0), \end{array}$$

is commutative for $n \geq 0$, where the right vertical arrow and the bottom horizontal arrow are the canonical composition maps and there exists an integer n_0 such that for any $n \geq n_0$,

$$R^0 \operatorname{Hom}_p(L_n, L_1) \otimes V_1 \longrightarrow R^0 \operatorname{Hom}_p(L_n, L_0)$$

is surjective for any $n \geq n_0$.

(4) For any object $E \in \mathscr{D}_U$ and for any non-negative integer m, there exists a bounded complex $P^\bullet$ of locally free sheaves of finite rank on U such that $\boldsymbol{R} \operatorname{Hom}_p((L_m)_V, E_V) \cong P^\bullet \otimes \mathscr{O}_V$ for any $V \to U$. Moreover, there exists an integer n_0 such that for any $n \geq n_0$, exists an integer N_0 such that for any integers i, N with $N \geq N_0$ and for any $s \in U$,

$$\operatorname{Hom}((L_N)_s, (L_n)_s) \otimes \operatorname{Ext}^i((L_n)_s, E_s) \to \operatorname{Ext}^i((L_N)_s, E_s)$$

is surjective.

(5) If there exist integers i, n_0 and an object $E \in \mathscr{D}_U$ satisfying $\operatorname{Ext}^i((L_n)_s, E_s) = 0$ for any $n \geq n_0$ and for any $s \in U$, then there exist an object $F \in \mathscr{D}_U$ and a morphism $u : E \to F$ such that for any $j > i$, $R^j \operatorname{Hom}_p((L_n)_U, E) \to R^j \operatorname{Hom}_p((L_n)_U, F)$ are isomorphic for $n \gg 0$, and for any $j \leq i$, $R^j \operatorname{Hom}_p((L_n)_U, F) = 0$ for $n \gg 0$.

(6) Take two objects $E, F \in \mathscr{D}_U$ such that for any $i \geq 0$, $R^i \operatorname{Hom}_p((L_n)_U, E) = 0$ for $n \gg 0$ and that for any $i < 0$, $R^i \operatorname{Hom}_p((L_n)_U, F) = 0$ for $n \gg 0$. Then we have $\operatorname{Hom}_{\mathscr{D}_U}(E, F) = 0$.

Proposition 3.2. *Take $E \in \mathscr{D}_U$ such that for any i, $R^i \operatorname{Hom}_p((L_n)_U, E) = 0$ for $n \gg 0$. Then we have $E = 0$.*

Proof. Applying the condition (6) of Definition 3.1, we have $\operatorname{Hom}(E, E) = 0$. In particular $\mathrm{id}_E = 0$. So, for any object $F \in \mathscr{D}_U$ and for any morphism $f \in \operatorname{Hom}(F, E)$ (resp. $g \in \operatorname{Hom}(E, F)$), $f = \mathrm{id}_E \circ f = 0$ (resp. $g = g \circ \mathrm{id}_E = 0$). Thus $E = 0$. □

Remark 3.3. By the condition in Definition 3.1 (2), we can see that $\theta_0 = \mathrm{id}$ and $\theta_k(\mathrm{id}) = \mathrm{id}$. We put $A := \bigoplus_{n\geq 0} R^0 \operatorname{Hom}_p(L_n, L_0)$ and define a multiplication

$$\alpha : R^0 \operatorname{Hom}_p(L_n, L_0) \times R^0 \operatorname{Hom}_p(L_m, L_0) \longrightarrow R^0 \operatorname{Hom}_p(L_{n+m}, L_0)$$

by $\alpha = (\text{composition}) \circ (\theta_m \times \mathrm{id})$. Then A becomes an associative graded ring which is a finitely generated module over $S^*(V_1)$, where $S^*(V_1)$ is the symmetric algebra of V_1 over $\mathscr{O}_S$.

Proposition 3.4. *Let E_1, E_2 be objects of $\mathscr{D}_U$ and $u : E_1 \to E_2$ be a morphism such that for any integer i the induced morphism $R^i \operatorname{Hom}_p((L_n)_U, E_1) \to R^i \operatorname{Hom}_p((L_n)_U, E_2)$ is isomorphic for $n \gg 0$. Then u is an isomorphism.*

Proof. For any i, there is an exact sequence

$$R^i \operatorname{Hom}_p((L_n)_U, E_1) \xrightarrow{\sim} R^i \operatorname{Hom}_p((L_n)_U, E_2) \longrightarrow R^i \operatorname{Hom}_p((L_n)_U, \operatorname{Cone}(u)) \longrightarrow R^{i+1} \operatorname{Hom}_p((L_n)_U, E_1) \xrightarrow{\sim} R^{i+1} \operatorname{Hom}_p((L_n)_U, E_2)$$

for $n \gg 0$. Thus we have $R^i \operatorname{Hom}_p((L_n)_U, \operatorname{Cone}(u)) = 0$ for $n \gg 0$. By Proposition 3.2 we have $\operatorname{Cone}(u) = 0$, which means that u is an isomorphism. □

Proposition 3.5. *For an integer i and an object $E \in \mathscr{D}_U$ such that for $n \gg 0$, $\operatorname{Ext}^i((L_n)_s, E_s) = 0$ for $s \in U$, the object F given in Definition 3.1 (5) is unique up to an isomorphism.*

Proof. Let $F' \in \mathscr{D}_U$ be another object with a morphism $u' : E \to F'$ having the same property as F. Consider the composite

$$v : \operatorname{Cone}(u)[-1] \longrightarrow E \xrightarrow{u'} F'.$$

Since there is a long exact sequence

$$\cdots \longrightarrow R^j \operatorname{Hom}_p((L_n)_U, E) \longrightarrow R^j \operatorname{Hom}_p((L_n)_U, F) \longrightarrow R^j \operatorname{Hom}_p((L_n)_U, \operatorname{Cone}(u)) \longrightarrow R^{j+1} \operatorname{Hom}_p((L_n)_U, E) \longrightarrow R^{j+1} \operatorname{Hom}_p((L_n)_U, F) \longrightarrow \cdots,$$

we have, for any $j \geq i$, $R^j \operatorname{Hom}_p((L_n)_U, \operatorname{Cone}(u)) = 0$ for $n \gg 0$. Note that for any $j \leq i$, we have $R^j \operatorname{Hom}_p((L_n)_U, F') = 0$ for $n \gg 0$. Then we have $\operatorname{Hom}_{\mathscr{D}_U}(\operatorname{Cone}(u), F') = 0$ and $\operatorname{Hom}_{\mathscr{D}_U}(\operatorname{Cone}(u)[-1], F') = 0$ by condition (6) of Definition 3.1. So we have $v = 0$ and there is a unique morphism $\varphi : F \to F'$

which makes the diagram

$$\begin{array}{ccc} E & \xrightarrow{u} & F \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \varphi} \\ E & \xrightarrow{u'} & F' \end{array}$$

commute. We can see that for any integer j, the morphism $R^j \operatorname{Hom}_p((L_n)_U, F) \to R^j \operatorname{Hom}_p((L_n)_U, F')$ induced by φ is isomorphic for $n \gg 0$. Hence φ is an isomorphism by Proposition 3.4. $\square$

Remark 3.6. In the situation of Definition 3.1 (5), for $n \gg 0$, the induced morphism

$$\operatorname{Ext}^j((L_n)_s, E_s) \to \operatorname{Ext}^j((L_n)_s, F_s)$$

is isomorphic for any $j > i$ and for any $s \in U$, and we have, for $n \gg 0$, $\operatorname{Ext}^j((L_n)_s, F_s) = 0$ for any $j \leq i$ and for any $s \in U$.

Indeed consider the distinguished triangle $E \xrightarrow{u} F \to \operatorname{Cone}(u)$. Note that there is a long exact sequence

$$\begin{aligned} R^j \operatorname{Hom}_p((L_n)_U, E) &\longrightarrow R^j \operatorname{Hom}_p((L_n)_U, F) \longrightarrow R^j \operatorname{Hom}_p((L_n)_U, \operatorname{Cone}(u)) \\ &\longrightarrow R^{j+1} \operatorname{Hom}_p((L_n)_U, E) \longrightarrow R^{j+1} \operatorname{Hom}_p((L_n)_U, F). \end{aligned}$$

Since $R^i \operatorname{Hom}_p((L_n)_U, F) = 0$ for $n \gg 0$, and for any $j > i$, $R^j \operatorname{Hom}_p((L_n)_U, E) \to R^j \operatorname{Hom}_p((L_n)_U, F)$ are isomorphic for $n \gg 0$, we have, for any $j \geq i$, $R^j \operatorname{Hom}_p((L_n)_U, \operatorname{Cone}(u)) = 0$ for $n \gg 0$.

By Definition 3.1 (4), there are integers n_0 and N_0 with $N_0 > n_0$ such that

$$\operatorname{Hom}((L_N)_s, (L_{n_0})_s) \otimes \operatorname{Ext}^j((L_{n_0})_s, \operatorname{Cone}(u)_s) \longrightarrow \operatorname{Ext}^j((L_N)_s, \operatorname{Cone}(u)_s)$$

is suriective for any j, any $N \geq N_0$ and any $s \in U$. By Definition 3.1 (4), there are integers j_0, j_1 such that for $j < j_0$ and $j > j_1$, $\operatorname{Ext}^j((L_{n_0})_s, \operatorname{Cone}(u)_s) = 0$ for any $s \in U$. Then for any $N \geq N_0$, we have $\operatorname{Ext}^j((L_N)_s, \operatorname{Cone}(u)_s) = 0$ for any $j > j_1$ and $s \in U$. For each j with $i \leq j \leq j_1$, there exists an integer $N(j)$ such that for any $N \geq N(j)$, we have $R^j \operatorname{Hom}_p((L_N)_U, \operatorname{Cone}(u)) = 0$. Put

$$\tilde{N} := \max\{N(i), N(i+1), \ldots, N(j_1), N_0\}.$$

By Definition 2.2 (4), we have $\operatorname{Ext}^j((L_N)_s, \operatorname{Cone}(u)_s) = 0$ for any $N \geq \tilde{N}$ and for each j with $i \leq j \leq j_1$ and for any $s \in U$, because $\operatorname{Ext}^{j_1+1}((L_N)_s, \operatorname{Cone}(u)_s) = 0$ for any $s \in U$ and $R^j \operatorname{Hom}_p((L_N)_U, \operatorname{Cone}(u)) = 0$ for $i \leq j \leq j_1$. Thus we have $\operatorname{Ext}^j((L_N)_s, \operatorname{Cone}(u)_s) = 0$ for any $N \geq \tilde{N}$, $j \geq i$ and $s \in U$.

Note that there are integers k_0, k_1 and a positive integer M_0 such that for any $M \geq M_0$ and for any $s \in U$, $\operatorname{Ext}^j((L_M)_s, F_s) = 0$ for $j < k_0$ and $j > k_1$. We may also assume that for any $M \geq M_0$ and for any $s \in U$, $\operatorname{Ext}^i((L_M)_s, E_s) = 0$. From the exact sequence

$$0 = \operatorname{Ext}^i((L_M)_s, E_s) \longrightarrow \operatorname{Ext}^i((L_M)_s, F_s) \longrightarrow \operatorname{Ext}^i((L_M)_s, \operatorname{Cone}(u)_s) = 0,$$

we have $\operatorname{Ext}^i((L_M)_s, F_s) = 0$ for $s \in U$ and $M \geq \max\{M_0, \tilde{N}\}$. By assumption, for each j with $k_0 \leq j \leq i$, there exists an integer $M(j)$ such that $R^j \operatorname{Hom}_p((L_M)_U, F) = 0$ for $M \geq M(j)$. Put

$$\tilde{M} := \max\{\tilde{N}, M_0, M(k_0), M(k_0+1), \dots, M(i)\}.$$

Then we have $\operatorname{Ext}^j((L_M)_s, F_s) = 0$ for $j \leq i$, $s \in U$ and $M \geq \tilde{M}$ by using Definition 2.2 (4), because $\operatorname{Ext}^i((L_M)_s, F_s) = 0$ and $R^j \operatorname{Hom}_p((L_M)_U, F) = 0$ for $k_0 \leq j \leq i$. From the exact sequence

$$\begin{aligned} &\operatorname{Ext}^{j-1}((L_M)_s, \operatorname{Cone}(u)_s) \longrightarrow \operatorname{Ext}^j((L_M)_s, E_s) \\ &\quad \longrightarrow \operatorname{Ext}^j((L_M)_s, F_s) \longrightarrow \operatorname{Ext}^j((L_M)_s, \operatorname{Cone}(u)_s), \end{aligned}$$

we have an isomorphism $\operatorname{Ext}^j((L_M)_s, E_s) \xrightarrow{\sim} \operatorname{Ext}^j((L_M)_s, F_s)$ for $j > i$, $s \in U$ and $M \geq \tilde{M}$.

Lemma 3.7. *If $E \in \mathscr{D}_U$ satisfies $\operatorname{Ext}^i((L_n)_s, E_s) = 0$ for $n \gg 0$, $i \neq 0$ and $s \in U$, then there exist locally free $\mathscr{O}_U$-modules W_0, W_1, W_2, positive integers $n_0 < n_1 < n_2$ and morphisms*

$$(L_{n_2})_U \otimes W_2 \xrightarrow{d^2} (L_{n_1})_U \otimes W_1 \xrightarrow{d^1} (L_{n_0})_U \otimes W_0 \xrightarrow{f} E$$

such that the induced sequence

$$\begin{aligned} &\operatorname{Hom}((L_N)_s, (L_{n_2})_s) \otimes W_2 \longrightarrow \operatorname{Hom}((L_N)_s, (L_{n_1})_s) \otimes W_1 \\ &\quad \longrightarrow \operatorname{Hom}((L_N)_s, (L_{n_0})_s) \otimes W_0 \longrightarrow \operatorname{Hom}((L_N)_s, E_s) \longrightarrow 0 \end{aligned}$$

is exact for $N \gg 0$ and $s \in U$.

Proof. By Definition 3.1 (4), there exist integers n_0, N_0 with $N_0 > n_0$ such that for any $s \in U$,

$$\mathrm{Hom}((L_N)_s, (L_{n_0})_s) \otimes \mathrm{Hom}((L_{n_0})_s, E_s) \to \mathrm{Hom}((L_N)_s, E_s)$$

is surjective for $N \geq N_0$ and $\mathrm{Ext}^i((L_n)_s, E_s) = 0$ for $n \geq n_0$, $i \neq 0$ and $s \in U$. There is a canonical morphism

$$f : (L_{n_0})_U \otimes R^0 \mathrm{Hom}_p((L_{n_0})_U, E) \longrightarrow E$$

and we put $F^1 := \mathrm{Cone}(f)[-1]$. Then we can see that $\mathrm{Ext}^i((L_N)_s, (F^1)_s) = 0$ for $N \geq N_0$, $i \neq 0$ and $s \in U$. We can find integers n_1, N_1 with $N_1 > n_1$ such that for any $s \in U$,

$$\mathrm{Hom}((L_N)_s, (L_{n_1})_s) \otimes \mathrm{Hom}((L_{n_1})_s, (F^1)_s) \longrightarrow \mathrm{Hom}((L_N)_s, (F^1)_s)$$

is surjective for $N \geq N_1$ and $\mathrm{Ext}^i((L_n)_s, (F^1)_s) = 0$ for $n \geq n_1$, $i \neq 0$ and $s \in U$. Consider the canonical morphism

$$g : (L_{n_1})_U \otimes R^0 \mathrm{Hom}_p((L_{n_1})_U, F^1) \longrightarrow F^1$$

and put $F^2 := \mathrm{Cone}(g)[-1]$. We can find again integers n_2, N_2 with $N_2 > n_2$ such that for any $s \in U$,

$$\mathrm{Hom}((L_N)_s, (L_{n_2})_s) \otimes \mathrm{Hom}((L_{n_2})_s, (F^2)_s) \longrightarrow \mathrm{Hom}((L_N)_s, (F^2)_s)$$

is surjective for $N \geq N_2$ and $\mathrm{Ext}^i((L_n)_s, (F^2)_s) = 0$ for $n \geq n_2$, $i \neq 0$ and $s \in U$. There is a canonical morphism

$$h : (L_{n_2})_U \otimes R^0 \mathrm{Hom}_p((L_{n_2})_U, F^2) \longrightarrow F^2$$

and we obtain a sequence of morphisms

$$\begin{aligned}(L_{n_2})_U \otimes R^0 \mathrm{Hom}_p((L_{n_2})_U, F^2) &\longrightarrow (L_{n_1})_U \otimes R^0 \mathrm{Hom}_p((L_{n_1})_U, F^1) \\ &\longrightarrow (L_{n_0})_U \otimes R^0 \mathrm{Hom}_p((L_{n_0})_U, E) \longrightarrow E\end{aligned}$$

such that for $N \geq \max\{N_0, N_1, N_2\}$, the induced sequence

$$\begin{aligned}
&\operatorname{Hom}((L_N)_s,(L_{n_2})_s)\otimes R^0\operatorname{Hom}_p((L_{n_2})_U,F^2)\\
&\quad\longrightarrow \operatorname{Hom}((L_N)_s,(L_{n_1})_s)\otimes R^0\operatorname{Hom}_p((L_{n_1})_U,F^1)\\
&\quad\longrightarrow \operatorname{Hom}((L_N)_s,(L_{n_0})_s)\otimes R^0\operatorname{Hom}_p((L_{n_0})_U,E)\\
&\quad\longrightarrow \operatorname{Hom}((L_N)_s,E_s)\longrightarrow 0
\end{aligned}$$

is exact for any $s \in U$. If we put $W_0 = R^0 \operatorname{Hom}_p((L_{n_0})_U, E)$ and $W_i = R^0 \operatorname{Hom}_p((L_{n_i})_U, F^i)$ for $i = 1, 2$, then we can see by Definition 2.2 (4) that W_i are locally free $\mathscr{O}_U$-modules and have the desired property. □

PROPOSITION 3.8. *Let E_1, E_2 be objects of $\mathscr{D}_U$ such that $\operatorname{Ext}^i((L_n)_s, (E_j)_s) = 0$ for $j = 1, 2$, $n \gg 0$, $i \neq 0$ and $s \in U$. If $f : E_1 \to E_2$ is a morphism in $\mathscr{D}_U$ such that the induced morphisms $R^0 \operatorname{Hom}_p((L_n)_U, E_1) \to R^0 \operatorname{Hom}_p((L_n)_U, E_2)$ are zero for $n \gg 0$, then $f = 0$.*

PROOF. By assumption, there is an integer N_0 such that for any $N \geq N_0$, the morphism

$$R^0 \operatorname{Hom}_p((L_N)_U, E_1) \to R^0 \operatorname{Hom}_p((L_N)_U, E_2)$$

induced by f is zero and $\operatorname{Ext}^i((L_N)_s, (E_j)_s) = 0$ for $j = 1, 2$, $i \neq 0$ and $s \in U$. By Lemma 3.7, there are locally free sheaves W_0, W_1, W_2, integers $n_0 < n_1 < n_2$ and morphisms

$$(L_{n_2})_U \otimes W_2 \longrightarrow (L_{n_1})_U \otimes W_1 \longrightarrow (L_{n_0})_U \otimes W_0 \xrightarrow{\varphi} E_1$$

such that the induced sequence

$$\begin{aligned}
&\operatorname{Hom}((L_N)_s,(L_{n_2})_s)\otimes W_2\longrightarrow \operatorname{Hom}((L_N)_s,(L_{n_1})_s)\otimes W_1\\
&\quad\longrightarrow \operatorname{Hom}((L_N)_s,(L_{n_0})_s)\otimes W_0\longrightarrow \operatorname{Hom}((L_N)_s,(E_1)_s)\longrightarrow 0
\end{aligned}$$

is exact for $N \gg 0$ and $s \in U$. We can take n_0 so that $n_0 \geq N_0$. Consider the distinguished triangle

$$(L_{n_0})_U \otimes W_0 \longrightarrow E_1 \longrightarrow \operatorname{Cone}(\varphi).$$

We can see that $\operatorname{Ext}^i((L_n)_s, \operatorname{Cone}(\varphi)_s) = 0$ for $n \gg 0$, $i \neq -1$ and $s \in U$. So we have $\operatorname{Hom}_{\mathscr{D}_U}(\operatorname{Cone}(\varphi), E_2) = 0$ by (6) of Definition 3.1 and the homomorphism

$$\mathrm{Hom}_{\mathscr{D}_U}(E_1, E_2) \to \mathrm{Hom}_{\mathscr{D}_U}((L_{n_0})_U \otimes W_0, E_2) \tag{†}$$

induced by φ is injective. On the other hand, the homomorphism

$$R^0 \mathrm{Hom}_p((L_{n_0})_U \otimes W_0, E_1) \longrightarrow R^0 \mathrm{Hom}_p((L_{n_0})_U \otimes W_0, E_2)$$

induced by f is zero. So we have $f \circ \varphi = 0$. By the injectivity of (†), we have $f = 0$. □

Since $A = \bigoplus_{n \geq 0} R^0 \mathrm{Hom}_p(L_n, L_0)$ becomes a finite algebra over $S^*(V_1)$, the associated sheaf $\mathscr{A} := \tilde{A}$ becomes a coherent sheaf of algebras on $\boldsymbol{P}(V_1)$. For each object $E \in \mathscr{D}_U$ satisfying $\mathrm{Ext}^i((L_n)_s, E_s) = 0$ for $n \gg 0$, $i \neq 0$ and $s \in U$, the associated sheaf $\left(\bigoplus_{n \geq 0} R^0 \mathrm{Hom}_p((L_n)_U, E)\right)^{\sim}$ on $\boldsymbol{P}(V_1)_U = \boldsymbol{P}(V_1) \times_S U$ becomes a coherent $\mathscr{A}_U$-module flat over U.

Proposition 3.9. *The correspondence* $E \mapsto \left(\bigoplus_{n \geq 0} R^0 \mathrm{Hom}_p((L_n)_U, E)\right)^{\sim}$ *gives an equivalence of categories between the full subcategory of* $\mathscr{D}_U$ *consisting of the objects* E *of* $\mathscr{D}_U$ *satisfying* $\mathrm{Ext}^i((L_n)_s, E_s) = 0$ *for* $n \gg 0$, $i \neq 0$ *and* $s \in U$ *and the category of coherent* $\mathscr{A}_U$*-modules flat over* U.

Proof. First we will prove that the functor

$$\psi : E \mapsto \left(\bigoplus_{n \geq 0} R^0 \mathrm{Hom}_p((L_n)_U, E)\right)^{\sim}$$

is fully faithful. Take any objects E, F of $\mathscr{D}_U$ which satisfy $\mathrm{Ext}^i((L_n)_s, E_s) = 0$, $\mathrm{Ext}^i((L_n)_s, F_s) = 0$ for $n \gg 0$, $i \neq 0$ and $s \in U$. By Proposition 3.8,

$$\mathrm{Hom}(E, F) \longrightarrow \mathrm{Hom}(\psi(E), \psi(F)) \tag{†}$$

is injective. Take any homomorphism $f \in \mathrm{Hom}(\psi(E), \psi(F))$. There exists an integer n_0 such that for any $n \geq n_0$, $\mathrm{Ext}^i((L_n)_s, E_s) = 0$, $\mathrm{Ext}^i((L_n)_s, F_s) = 0$ for $i \neq 0$ and $s \in U$ and the homomorphisms

$$\mathrm{Hom}((L_N)_s, (L_{n_0})_s) \otimes \mathrm{Hom}((L_{n_0})_s, E_s) \longrightarrow \mathrm{Hom}((L_N)_s, E_s)$$
$$\mathrm{Hom}((L_N)_s, (L_{n_0})_s) \otimes \mathrm{Hom}((L_{n_0})_s, F_s) \longrightarrow \mathrm{Hom}((L_N)_s, F_s)$$

are surjective for $N \gg 0$ and $s \in U$. For a coherent $\mathscr{A}_U$-module $\mathscr{E}$, we denote $\mathscr{E} \otimes \mathscr{O}_{\boldsymbol{P}(V_1)_U}(n)$ simply by $\mathscr{E}(n)$. We denote the structure morphism $\boldsymbol{P}(V_1)_U \to U$

by π. Then we may assume that $R^i\pi_*(\psi(E)(n_0)) = 0$, $R^i\pi_*(\psi(F)(n_0)) = 0$ for $i > 0$ and that the homomorphisms

$$\pi_*(\psi(E)(n_0)) \otimes \mathscr{A}(-n_0) \longrightarrow \psi(E)$$
$$\pi_*(\psi(F)(n_0)) \otimes \mathscr{A}(-n_0) \longrightarrow \psi(F)$$

are surjective. We may also assume that

$$R^0 \operatorname{Hom}((L_{n_0})_U, E) \longrightarrow \pi_*(\psi(E)(n_0))$$
$$R^0 \operatorname{Hom}((L_{n_0})_U, F) \longrightarrow \pi_*(\psi(F)(n_0))$$

are isomorphic. Consider the distinguished triangles

$$\operatorname{Cone}(v)[-1] \xrightarrow{\iota_1} (L_{n_0})_U \otimes R^0 \operatorname{Hom}_p((L_{n_0})_U, E) \xrightarrow{v} E$$
$$\operatorname{Cone}(w)[-1] \xrightarrow{\iota_2} (L_{n_0})_U \otimes R^0 \operatorname{Hom}_p((L_{n_0})_U, F) \xrightarrow{w} F.$$

Then we can see that $\operatorname{Ext}^i((L_N)_s, \operatorname{Cone}(v)[-1]_s) = 0$, $\operatorname{Ext}^i((L_N)_s, \operatorname{Cone}(w)[-1]_s) = 0$ for $N \gg 0$, $i \neq 0$ and $s \in U$. The homomorphism $f : \psi(E) \to \psi(F)$ induces a homomorphism

$$f(n_0) : R^0 \operatorname{Hom}_p((L_{n_0})_U, E) \cong \pi_*(\psi(E)(n_0))$$
$$\longrightarrow \pi_*(\psi(F)(n_0)) \cong R^0 \operatorname{Hom}_p((L_{n_0})_U, F).$$

Then $f(n_0)$ induces a homomorphism

$$\tilde{f} : (L_{n_0})_U \otimes R^0 \operatorname{Hom}_p((L_{n_0})_U, E) \longrightarrow (L_{n_0})_U \otimes R^0 \operatorname{Hom}_p((L_{n_0})_U, F).$$

Consider the composite

$$w \circ \tilde{f} \circ \iota_1 : \operatorname{Cone}(v)[-1] \xrightarrow{\iota_1} (L_{n_0})_U \otimes R^0 \operatorname{Hom}_p((L_{n_0})_U, E)$$
$$\xrightarrow{\tilde{f}} (L_{n_0})_U \otimes R^0 \operatorname{Hom}_p((L_{n_0})_U, F) \xrightarrow{w} F.$$

Then we have $\psi(w \circ \tilde{f} \circ \iota_1) = \psi(w) \circ \psi(\tilde{f}) \circ \psi(\iota_1) = f \circ \psi(v) \circ \psi(\iota_1) = f \circ \psi(v \circ \iota_1) = 0$. Since

$$\mathrm{Hom}(\mathrm{Cone}(v)[-1], F) \longrightarrow \mathrm{Hom}(\psi(\mathrm{Cone}(v)[-1]), \psi(F))$$

is injective, we have $w \circ \tilde{f} \circ \iota_1 = 0$. So there is a morphism $f' : E \to F$, which makes the diagram

$$\begin{array}{ccccc} \mathrm{Cone}(v)[-1] & \xrightarrow{\iota_1} & (L_{n_0})_U \otimes R^0 \mathrm{Hom}_p((L_{n_0})_U, E) & \xrightarrow{v} & E \\ & & \downarrow \tilde{f} & & \downarrow f' \\ \mathrm{Cone}(w)[-1] & \xrightarrow{\iota_2} & (L_{n_0})_U \otimes R^0 \mathrm{Hom}_p((L_{n_0})_U, F) & \xrightarrow{w} & F \end{array}$$

commute. This commutative diagram induces a commutative diagram

$$\begin{array}{ccc} \pi_*(\psi(E)(n_0)) \otimes \mathscr{A}(-n_0) & \xrightarrow{\psi(v)} & \psi(E) \\ \downarrow & & \downarrow \psi(f') \\ \pi_*(\psi(F)(n_0)) \otimes \mathscr{A}(-n_0) & \longrightarrow & \psi(F). \end{array}$$

Since $(\psi(f') - f) \circ \psi(v) = \psi(f') \circ \psi(v) - f \circ \psi(v) = \psi(w) \circ \psi(\tilde{f}) - \psi(w) \circ \psi(\tilde{f}) = 0$, we have $\psi(f') - f = 0$ because $\psi(v)$ is surjective. So we have $\psi(f') = f$. Thus (†) is surjective and ψ becomes a fully faithful functor.

Take any coherent $\mathscr{A}_U$-module $\mathscr{E}$ flat over U. There is an exact sequence of coherent $\mathscr{A}_U$-modules

$$W_2 \otimes \mathscr{A}(-n_2) \xrightarrow{\delta^2} W_1 \otimes \mathscr{A}(-n_1) \xrightarrow{\delta^1} W_0 \otimes \mathscr{A}(-n_0) \longrightarrow \mathscr{E} \longrightarrow 0,$$

where W_0, W_1, W_2 are locally free sheaves on U and $n_2 \gg n_1 \gg n_0 \gg 0$. The above sequence induces a sequence of morphisms

$$(L_{n_2})_U \otimes W_2 \xrightarrow{d^2} (L_{n_1})_U \otimes W_1 \xrightarrow{d^1} (L_{n_0})_U \otimes W_0.$$

By construction we have $d^1 \circ d^2 = 0$. So there is a morphism $u : \mathrm{Cone}(d^2) \to (L_{n_0})_U \otimes W_0$ such that the diagram

$$\begin{array}{ccccc} (L_{n_1})_U \otimes W_1 & & \xrightarrow{\quad d^1 \quad} & & (L_{n_0})_U \otimes W_0 \\ & \searrow & & \nearrow_{u} & \\ & & \mathrm{Cone}(d^2) & & \end{array}$$

is commutative. Note that $\mathrm{Ext}^i((L_N)_s, \mathrm{Cone}(d^2)_s) = 0$ for $N \gg 0$, $i \neq -1, 0$ and $s \in U$. So we have $\mathrm{Ext}^i((L_N)_s, \mathrm{Cone}(u)_s) = 0$ for $N \gg 0$, $i \neq -2, -1, 0$ and $s \in U$. Since $\mathscr{E}$ is flat over U, the sequence

$$W_2 \otimes \mathscr{A}(-n_2) \otimes k(s) \longrightarrow W_1 \otimes \mathscr{A}(-n_1) \otimes k(s)$$
$$\longrightarrow W_0 \otimes \mathscr{A}(-n_0) \otimes k(s) \longrightarrow \mathscr{E} \otimes k(s) \longrightarrow 0$$

is exact for any $s \in U$. So we obtain the exact commutative diagram

$$\begin{array}{ccccc}
H^0(W_2 \otimes \mathscr{A}(N-n_2) \otimes k(s)) & \longrightarrow & H^0(W_1 \otimes \mathscr{A}(N-n_1) \otimes k(s)) & \longrightarrow & H^0(W_0 \otimes \mathscr{A}(N-n_0) \otimes k(s)) \\
\cong \downarrow & & \cong \downarrow & & \cong \downarrow \\
\mathrm{Hom}((L_N)_s, (L_{n_2})_s \otimes (W_2)_s) & \longrightarrow & \mathrm{Hom}((L_N)_s, (L_{n_1})_s \otimes (W_1)_s) & \longrightarrow & \mathrm{Hom}((L_N)_s, (L_{n_0})_s \otimes (W_0)_s)
\end{array}$$

for $N \gg 0$ and $s \in U$. Here we denote $W_i \otimes k(s)$ by $(W_i)_s$ for $i = 0, 1, 2$. We have a factorization

$$\begin{array}{ccc}
\mathrm{Hom}((L_N)_s, (L_{n_1})_s \otimes (W_1)_s) & \longrightarrow & \mathrm{Hom}((L_N)_s, (L_{n_0})_s \otimes (W_0)_s) \\
& \searrow \quad \nearrow & \\
& \mathrm{Hom}((L_N)_s, \mathrm{Cone}(d^2)_s) &
\end{array}$$

for $N \gg 0$ and $s \in U$, and the homomorphism $\mathrm{Hom}((L_N)_s, (L_{n_1})_s \otimes (W_1)_s) \longrightarrow \mathrm{Hom}((L_N)_s, \mathrm{Cone}(d^2)_s)$ is surjective for $N \gg 0$ and $s \in U$, because $\mathrm{Ext}^1((L_N)_s, (L_{n_2})_s \otimes (W_2)_s) = 0$ for $N \gg 0$ and $s \in U$. So we can see that the homomorphism

$$\mathrm{Hom}((L_N)_s, \mathrm{Cone}(d^2)_s) \longrightarrow \mathrm{Hom}((L_N)_s, (L_{n_0})_s \otimes (W_0)_s)$$

is injective for $N \gg 0$ and $s \in U$. Since there is an exact sequence

$$0 = \mathrm{Ext}^{-1}((L_N)_s, (L_{n_0})_s \otimes (W_0)_s) \longrightarrow \mathrm{Ext}^{-1}((L_N)_s, \mathrm{Cone}(u)_s)$$
$$\xrightarrow{0} \mathrm{Hom}((L_N)_s, \mathrm{Cone}(d^2)_s) \longrightarrow \mathrm{Hom}((L_N)_s, (L_{n_0})_s \otimes (W_0)_s)$$

for $N \gg 0$ and $s \in U$, we have $\mathrm{Ext}^{-1}((L_N)_s, \mathrm{Cone}(u)_s) = 0$ for $N \gg 0$ and $s \in U$.

By Definition 3.1 (5) and Remark 3.6, there is an object $E \in \mathscr{D}_U$ and a morphism $\alpha : \mathrm{Cone}(u) \to E$ such that $R^0 \operatorname{Hom}_p((L_N)_U, \mathrm{Cone}(u)) \to R^0 \operatorname{Hom}_p((L_N)_U, E)$ is isomorphic for $N \gg 0$ and that $\operatorname{Ext}^j((L_N)_s, E_s) = 0$ for $N \gg 0$, $j \neq 0$ and $s \in U$. We can see that the sequence

$$R^0 \operatorname{Hom}_p((L_N)_U, (L_{n_2})_U \otimes W_2) \longrightarrow R^0 \operatorname{Hom}_p((L_N)_U, (L_{n_1})_U \otimes W_1) \longrightarrow$$
$$R^0 \operatorname{Hom}_p((L_N)_U, (L_{n_0})_U \otimes W_0) \longrightarrow R^0 \operatorname{Hom}_p((L_N)_U, \mathrm{Cone}(u)) \longrightarrow 0$$

is exact. Since $R^0 \operatorname{Hom}_p((L_N)_U, \mathrm{Cone}(u)) \cong R^0 \operatorname{Hom}_p((L_N)_U, E)$ for $N \gg 0$, there is an integer N_0 such that for any $N \geq N_0$, there is a unique isomorphism $R^0 \operatorname{Hom}_p((L_N)_U, E) \xrightarrow{\sim} \pi_*(\mathscr{E}(N))$ which makes the diagram

$$\begin{array}{ccccccc}
R^0 \operatorname{Hom}_p((L_N)_U, (L_{n_1})_U \otimes W_1) & \longrightarrow & R^0 \operatorname{Hom}_p((L_N)_U, (L_{n_0})_U \otimes W_0) & \longrightarrow & R^0 \operatorname{Hom}_p((L_N)_U, E) & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
\pi_*(\mathscr{A}(N-n_1) \otimes W_1) & \longrightarrow & \pi_*(\mathscr{A}(N-n_0) \otimes W_0) & \longrightarrow & \pi_*(\mathscr{E}(N)) & \longrightarrow & 0
\end{array}$$

commute. Note that there is a canonical commutative diagram

$$\begin{array}{ccc}
R^0 \operatorname{Hom}_p((L_{N+m})_U, (L_N)_U) \otimes R^0 \operatorname{Hom}_p((L_N)_U, E) & \longrightarrow & R^0 \operatorname{Hom}_p((L_{N+m})_U, E) \\
\downarrow & & \downarrow \\
\pi_*(\mathscr{A}(m)) \otimes \pi_*(\mathscr{E}(N)) & \longrightarrow & \pi_*(\mathscr{E}(N+m))
\end{array}$$

for $N \geq N_0$ and a non-negative integer m. Then we have an isomorphism

$$\bigoplus_{n \geq N_0} R^0 \operatorname{Hom}_p((L_n)_U, E) \xrightarrow{\sim} \bigoplus_{n \geq N_0} \pi_*(\mathscr{E}(n))$$

of graded A_U-modules. So we obtain an isomorphism

$$\psi(E) = \left(\bigoplus_{n \geq N_0} R^0 \operatorname{Hom}_p((L_n)_U, E) \right)^{\sim} \xrightarrow{\sim} \left(\bigoplus_{n \geq N_0} \pi_*(\mathscr{E}(n)) \right)^{\sim} \cong \mathscr{E}.$$

Thus ψ becomes an equivalence of categories. □

Definition 3.10. For a geometric point $\operatorname{Spec} k \to S$, an object $E \in \mathscr{D}_k$ is said to be $\mathscr{L}$-stable (resp. $\mathscr{L}$-semistable) if $\operatorname{Ext}^i((L_n)_k, E) = 0$ for $n \gg 0$ and $i \neq 0$ and the inequality

$$\frac{\dim \operatorname{Hom}((L_m)_k, F)}{\dim \operatorname{Hom}((L_n)_k, F)} < \frac{\dim \operatorname{Hom}((L_m)_k, E)}{\dim \operatorname{Hom}((L_n)_k, E)}$$
$$\left(\text{resp. } \frac{\dim \operatorname{Hom}((L_m)_k, F)}{\dim \operatorname{Hom}((L_n)_k, F)} \leq \frac{\dim \operatorname{Hom}((L_m)_k, E)}{\dim \operatorname{Hom}((L_n)_k, E)}\right)$$

holds for $n \gg m \gg 0$ and for any non-zero object $F \in \mathscr{D}_k$ satisfying $\operatorname{Ext}^i((L_N)_k, F) = 0$ for $N \gg 0$ and $i \neq 0$ with a morphism $\iota : F \to E$ such that ι is not isomorphic and $\operatorname{Hom}((L_n)_k, F) \to \operatorname{Hom}((L_n)_k, E)$ is injective for $n \gg 0$.

Remark 3.11. Let $\operatorname{Spec} k \to S$ be a geometric point and E an object of $\mathscr{D}_k$ satisfying $\operatorname{Ext}^i((L_n)_k, E) = 0$ for $i \neq 0$ and $n \gg 0$. Let $\mathscr{E}$ be the coherent $\mathscr{A}_k$-module corresponding to E as in Proposition 3.9. Then E is $\mathscr{L}$-stable (resp. $\mathscr{L}$-semistable) if and only if for any coherent $\mathscr{A}_k$-submodule $\mathscr{F}$ of $\mathscr{E}$ with $0 \neq \mathscr{F} \subsetneq \mathscr{E}$, the inequality

$$\frac{\chi(\mathscr{F}(m))}{\chi(\mathscr{F}(n))} < \frac{\chi(\mathscr{E}(m))}{\chi(\mathscr{E}(n))} \quad \left(\text{resp. } \frac{\chi(\mathscr{F}(m))}{\chi(\mathscr{F}(n))} \leq \frac{\chi(\mathscr{E}(m))}{\chi(\mathscr{E}(n))}\right) \tag{1}$$

holds for $n \gg m \gg 0$. We say a coherent $\mathscr{A}_k$-module $\mathscr{E}$ stable (resp. semistable) if the corresponding object E of $\mathscr{D}_k$ is $\mathscr{L}$-stable (resp. $\mathscr{L}$-semistable).

Remark 3.12. For a field K with a morphism $\operatorname{Spec} K \to S$ and an object $E \in \mathscr{D}_K$, we say that E is $\mathscr{L}$-stable (resp. $\mathscr{L}$-semistable) if $E_{\bar{K}}$ is $\mathscr{L}$-stable (resp. $\mathscr{L}$-semistable), where $\bar{K}$ is the algebraic closure of K.

4. Existence of the moduli space of stable objects.

Definition 4.1. Let $p : \mathscr{D} \to (\mathrm{Sch}/S)$ be a fibered triangulated category with base change property and $\mathscr{L} = \{L_n\}_{n \geq 0}$ be a strict ample sequence. For a numerical polynomial $P(t) \in \boldsymbol{Q}[t]$, we define a moduli functor $\mathscr{M}_{\mathscr{D}}^{P,\mathscr{L}} : (\mathrm{Sch}/S) \to (\mathrm{Sets})$ by

$$\mathscr{M}_{\mathscr{D}}^{P,\mathscr{L}}(T) := \left\{ E \in \mathscr{D}_T \;\middle|\; \begin{array}{l} \text{for any geometric point } s \text{ of } T, \text{ for } n \gg 0, \\ \operatorname{Ext}^i((L_n)_s, E_s) = 0 \text{ for } i \neq 0 \text{ and} \\ \operatorname{Hom}((L_n)_s, E_s) = P(n) \text{ and } E_s \text{ is } \mathscr{L}\text{-stable} \end{array} \right\} \Big/ \sim,$$

where $E \sim E'$ if there exists a line bundle L on T and an isomorphism $E \xrightarrow{\sim} E' \otimes L$.

We also define a moduli functor $\overline{\mathscr{M}_{\mathscr{D}}^{P,\mathscr{L}}} : (\mathrm{Sch}/S) \to (\mathrm{Sets})$ by

$$\overline{\mathscr{M}_{\mathscr{D}}^{P,\mathscr{L}}}(T) := \left\{ E \in \mathscr{D}_T \,\middle|\, \begin{array}{l} \text{for any geometric point } s \text{ of } T, \text{ for } n \gg 0, \\ \mathrm{Ext}^i((L_n)_s, E_s) = 0 \text{ for } i \neq 0 \text{ and} \\ \mathrm{Hom}((L_n)_s, E_s) = P(n) \text{ and } E_s \text{ is } \mathscr{L}\text{-semistable} \end{array} \right\} \Big/ \sim,$$

where $E \sim E'$ if there exists a line bundle L on T such that $E \cong E' \otimes L$ or there exist sequences $0 = E_0 \to E_1 \to \cdots \to E_\alpha = E$ and $0 = E'_0 \to E'_1 \to \cdots \to E'_\alpha = E'$ such that $\mathrm{Ext}^i((L_n)_s, (E_j)_s) = \mathrm{Ext}^i((L_n)_s, (E'_j)_s) = 0$ for $n \gg 0$, $i \neq 0$ and $s \in T$, $\mathrm{Hom}((L_n)_s, (E_j)_s) \to \mathrm{Hom}((L_n)_s, (E_{j+1})_s)$ and $\mathrm{Hom}((L_n)_s, (E'_j)_s) \to \mathrm{Hom}((L_n)_s, (E'_{j+1})_s)$ are injective for $n \gg 0$ and $s \in T$ and $\bigoplus_{j=1}^{\alpha} F_j \cong \bigoplus_{j=1}^{\alpha} F'_j \otimes L$, where $F_j = \mathrm{Cone}(E_{j-1} \to E_j)$, $F'_j = \mathrm{Cone}(E'_{j-1} \to E'_j)$ and for any geometric point s of T, $(F_j)_s$ and $(F'_j)_s$ are $\mathscr{L}$-stable such that

$$\frac{\dim \mathrm{Hom}((L_m)_s, (F_j)_s)}{\dim \mathrm{Hom}((L_n)_s, (F_j)_s)} = \frac{P(m)}{P(n)} = \frac{\dim \mathrm{Hom}((L_m)_s, (F'_j)_s)}{\dim \mathrm{Hom}((L_n)_s, (F'_j)_s)}$$

for $n \gg m \gg 0$ and for $j = 1, 2, \ldots, \alpha$.

Proposition 4.2. *For any numerical polynomial $P(t) \in \boldsymbol{Q}[t]$, the family*

$$\left\{ E \,\middle|\, \begin{array}{l} E \in \mathscr{D}_k \text{ for some geometric point } \mathrm{Spec}\, k \to S, \\ E \text{ is } \mathscr{L}\text{-semistable and } \mathrm{Hom}((L_n)_k, E) = P(n) \text{ for } n \gg 0 \end{array} \right\}$$

is bounded.

Proof. It suffices to show that the corresponding family of coherent $\mathscr{A}$-modules on the fibers of $\boldsymbol{P}(V_1)$ over S is bounded. For a coherent sheaf $\mathscr{G}$ on $\boldsymbol{P}(V_1)$, we can write

$$\chi(\mathscr{G}(n)) = \sum_{i=0}^{d} a_i(\mathscr{G}) \binom{n+d-i}{d-i}$$

with $a_i(\mathscr{G})$ integers and we write $\mu(G) = a_1(\mathscr{G})/a_0(\mathscr{G})$. Let $\mathscr{E}$ be a coherent $\mathscr{A}_k$-module such that $\chi(\mathscr{E}(n)) = P(n)$ and the corresponding object of $\mathscr{D}_k$ is $\mathscr{L}$-semistable. Note that $\mathscr{E}$ is of pure dimension. We can take the slope maximal destabilizer $\mathscr{F}$ of $\mathscr{E}$ as a sheaf on $\boldsymbol{P}(V_1)$. Let $\tilde{\mathscr{F}}$ be the image of $\mathscr{F} \otimes \mathscr{A} \to \mathscr{E}$.

Note that there exists a locally free sheaf W of finite rank on S, positive integer N and a surjection

$$W \otimes \mathscr{O}(-N) \longrightarrow \mathscr{A}$$

Then we obtain a surjection

$$W \otimes \mathscr{F}(-N) \longrightarrow \mathscr{F} \otimes \mathscr{A} \longrightarrow \tilde{\mathscr{F}}.$$

Since $W \otimes \mathscr{F}(-N)$ is slope semistable, we have

$$\mu(\mathscr{F}) - N = \mu(W \otimes \mathscr{F}(-N)) \leq \mu(\tilde{\mathscr{F}}) \leq \mu(\mathscr{E}).$$

So the maximal slope $\mu(\mathscr{F})$ is bounded by $N + \mu(\mathscr{E})$. Then we obtain the boundedness by [**6**, Theorem 4.2]. □

PROPOSITION 4.3. *Assume that $U \in (\mathrm{Sch}/S)$ and $E \in \mathscr{D}_U$ are given. Then the subsets*

$$U^s = \{x \in U \mid E_x \text{ is } \mathscr{L}\text{-stable}\}$$
$$U^{ss} = \{x \in U \mid E_x \text{ is } \mathscr{L}\text{-semistable}\}$$

of U are open.

PROOF. First we will show that

$$U' = \left\{x \in U \mid \mathrm{Ext}^i((L_n)_x, E_x) = 0 \text{ for } n \gg 0 \text{ and } i \neq 0\right\}$$

is open in U. By Definition 3.1 (4), there exists a positive integer n_0 such that for any $n \geq n_0$, exists an integer N_n with $N_n > n$ such that for any $N \geq N_n$,

$$\mathrm{Hom}((L_N)_s, (L_n)_s) \otimes \mathrm{Ext}^i((L_n)_s, E_s) \longrightarrow \mathrm{Ext}^i((L_N)_s, E_s)$$

is surjective for any i and $s \in U$. By Definition 3.1 (4), there are integers k_1, k_2 with $k_1 < k_2$ such that $\mathrm{Ext}^i((L_{n_0})_s, E_s) = 0$ for any $s \in U$ except for $k_1 \leq i \leq k_2$. Then we have $\mathrm{Ext}^i((L_N)_s, E_s) = 0$ for $N \geq N_{n_0}$ and $s \in U$, except for $k_1 \leq i \leq k_2$. Now take any point $x \in U'$. For each $i \neq 0$ with $k_1 \leq i \leq k_2$, there is an integer m_i with $m_i \geq n_0$ such that $\mathrm{Ext}^i((L_{m_i})_x, E_x) = 0$. For any $N \geq N_{m_i}$,

$$\mathrm{Hom}((L_N)_s,(L_{m_i})_s)\otimes \mathrm{Ext}^i((L_{m_i})_s,E_s)\longrightarrow \mathrm{Hom}((L_N)_s,E_s)$$

is surjective for any $s\in U$. By using Definition 2.2 (4), we can see that there exists an open neighborhood U_i of x such that $\mathrm{Ext}^i((L_{m_i})_y,E_y)=0$ for any $y\in U_i$. Then we have $\mathrm{Ext}^i((L_N)_y,E_y)=0$ for $N\geq N_{m_i}$. If we put

$$V:=\bigcap_{k_1\leq i\leq k_2, i\neq 0} U_i$$

then V is an open neighborhood of x. Put

$$\tilde{N}:=\max\bigl(\{N_{m_i}\mid k_1\leq i\leq k_2, i\neq 0\}\cup\{N_{n_0}\}\bigr).$$

Then we have $\mathrm{Ext}^i((L_N)_y,E_y)=0$ for any $y\in V$, $i\neq 0$ and $N\geq\tilde{N}$, which means $V\subset U'$. Thus U' is an open subset of U.

By Proposition 3.9, $E_{U'}$ corresponds to a coherent $\mathscr{A}_{U'}$-module $\mathscr{E}$ flat over U'. We can see that U^s coincides with

$$\{x\in U'\mid \mathscr{E}\otimes k(x) \text{ is a stable } \mathscr{A}_x\text{-module}\}.$$

We can see by the argument similar to that of [**3**, Proposition 2.3.1], that this subset is open in U'. By the same argument we can also see the openness of U^{ss}. □

THEOREM 4.4. *There exists a coarse moduli scheme* $\overline{M^{P,\mathscr{L}}_{\mathscr{D}}}$ *of* $\overline{\mathscr{M}^{P,\mathscr{L}}_{\mathscr{D}}}$ *and an open subscheme* $M^{P,\mathscr{L}}_{\mathscr{D}}$ *of* $\overline{M^{P,\mathscr{L}}_{\mathscr{D}}}$ *which is a coarse moduli scheme of* $\mathscr{M}^{P,\mathscr{L}}_{\mathscr{D}}$.

Before constructing the moduli space, we first note the following lemma:

LEMMA 4.5. *Let $P(x)$ be a numerical polynomial. Then there exists an integer m_0 such that for any $m\geq m_0$, any geometric point s of S, any semi-stable $\mathscr{A}_s$-module $\mathscr{E}$ with $\chi(\mathscr{E}(n))=P(n)$,*

(1) *$\mathscr{E}(m)$ is generated by global sections and $H^i(\mathscr{E}(m))=0$ for $i>0$,*
(2) *for any nonzero coherent $\mathscr{A}_s$-submodule $\mathscr{F}\subset\mathscr{E}$, the inequality*

$$\dim H^0(\mathscr{F}(m))\leq\frac{a_0(\mathscr{F})}{a_0(\mathscr{E})}\dim H^0(\mathscr{E}(m))$$

holds, where

$$\chi(\mathscr{E}(n)) = \sum_{i=0}^{d} a_i(\mathscr{E}) \binom{n+d-i}{d-i}, \quad \chi(\mathscr{F}(n)) = \sum_{i=0}^{d} a_i(\mathscr{F}) \binom{n+d-i}{d-i}.$$

Moreover the equality holds if and only if $\chi(\mathscr{E}(n))/a_0(\mathscr{E}) = \chi(\mathscr{F}(n))/a_0(\mathscr{F})$ *as polynomials in* n.

Proof. Proof is essentially the same as [**8**, Proposition 4.10]. □

Take m_0 as in Lemma 4.5. Replacing S by its connected component, we may assume that S is connected. Replacing m_0 if necessary, we may assume by Proposition 4.2 that for any geometric point $E \in \overline{\mathscr{M}_{\mathscr{D}}^{P,\mathscr{L}}}(k)$ and for any $m \geq m_0$, $\mathrm{Ext}^i((L_m)_k, E) = 0$ for $i \neq 0$ and

$$\mathrm{Hom}((L_n)_k, (L_m)_k) \otimes \mathrm{Hom}((L_m)_k, E) \longrightarrow \mathrm{Hom}((L_n)_k, E)$$

is surjective for $n \gg 0$. For a geometric point $E \in \overline{\mathscr{M}_{\mathscr{D}}^{P,\mathscr{L}}}(k)$, we consider the canonical morphism

$$u : (L_{m_0})_k \otimes \mathrm{Hom}((L_{m_0})_k, E) \longrightarrow E$$

and put $E_1 := \mathrm{Cone}(u)[-1]$. We can take $m_1 \gg m_0$ such that for any such E and for any $m \geq m_1$, $\mathrm{Ext}^i((L_m)_k, E_1) = 0$ for $i \neq 0$ and

$$\mathrm{Hom}((L_n)_k, (L_m)_k) \otimes \mathrm{Hom}((L_m)_k, E_1) \longrightarrow \mathrm{Hom}((L_n)_k, E_1)$$

is surjective for $n \gg 0$. We consider the canonical morphism

$$v : (L_{m_1})_k \otimes \mathrm{Hom}((L_{m_1})_k, E_1) \longrightarrow E_1$$

and put $E_2 := \mathrm{Cone}(v)[-1]$. We can take $m_2 \gg 0$ such that for any E and for any $m \geq m_2$, $\mathrm{Ext}^i((L_m)_k, E_2) = 0$ for $i \neq 0$ and

$$\mathrm{Hom}((L_n)_k, (L_m)_k) \otimes \mathrm{Hom}((L_m)_k, E_2) \longrightarrow \mathrm{Hom}((L_n)_k, E_2)$$

is surjective for $n \gg 0$. We put

$$r_0 := \dim_k \mathrm{Hom}((L_{m_0})_k, E), \quad r_1 := \dim_k((L_{m_1})_k, E_1), \quad r_2 := \dim_k((L_{m_2})_k, E_2)$$

and

$$W_0 := \mathscr{O}_S^{\oplus r_0}, \quad W_1 := \mathscr{O}_S^{\oplus r_1}, \quad W_2 := \mathscr{O}_S^{\oplus r_2}.$$

Note that r_0, r_1, r_2 are independent of the choice of E and only depend on P and $\mathscr{L}$. We set

$$Z := \boldsymbol{V}\big(R^0 \operatorname{Hom}_p(L_{m_2}, L_{m_1})^\vee \otimes W_2 \otimes W_1^\vee\big) \times \boldsymbol{V}\big(R^0 \operatorname{Hom}_p(L_{m_1}, L_{m_0})^\vee \otimes W_1 \otimes W_0^\vee\big).$$

Let

$$(L_{m_2})_Z \otimes W_2 \xrightarrow{\tilde{v}} (L_{m_1})_Z \otimes W_1 \xrightarrow{\tilde{u}} (L_{m_0})_Z \otimes W_0$$

be the universal family. There exists a closed subscheme $Y \subset Z$ such that

$$Y(T) = \{g \in Z(T) \mid g^*(\tilde{u} \circ \tilde{v}) = 0\}.$$

for any $T \in (\mathrm{Sch}/S)$. Since the sequence

$$\operatorname{Hom}(\operatorname{Cone}(\tilde{v}_Y), (L_{m_0})_Y \otimes W_0) \xrightarrow{\beta} \operatorname{Hom}((L_{m_1})_Y \otimes W_1, (L_{m_0})_Y \otimes W_0)$$
$$\xrightarrow{\tilde{v}^*} \operatorname{Hom}((L_{m_2})_Y \otimes W_2, (L_{m_0})_Y \otimes W_0)$$

is exact and $\tilde{v}^*(\tilde{u}_Y) = \tilde{u}_Y \circ \tilde{v}_Y = 0$, there exists a morphism $\tilde{w} : \operatorname{Cone}(\tilde{v}_Y) \to (L_{m_0})_Y \otimes W_0$ such that $\beta(\tilde{w}) = \tilde{u}_Y$. We put $\tilde{B} := \operatorname{Cone}(\tilde{w})$ and set

$$Y' := \big\{x \in Y \mid \operatorname{Ext}^{-1}((L_n)_x, \tilde{B}_x) = 0 \text{ for } n \gg 0\big\}$$

Then we can see that Y' is an open subset of Y. Note that for any $x \in Y'$, $\operatorname{Ext}^i((L_n)_x, \tilde{B}_x) = 0$ for $n \gg 0$ except for $i = -2, 0$. By Definition 3.1 (5), there exist an object $\tilde{E} \in \mathscr{D}_{Y'}$ and a morphism $\tilde{B}_{Y'} \to \tilde{E}$ such that $\operatorname{Ext}^i((L_n)_x, \tilde{E}_x) = 0$ for $n \gg 0$, $x \in Y'$ and $i \neq 0$ and $\operatorname{Hom}((L_n)_x, \tilde{B}_x) \to \operatorname{Hom}((L_n)_x, \tilde{E}_x)$ is isomorphic for $n \gg 0$ and $x \in Y'$. If we set

$$\tilde{E}_1 := \operatorname{Cone}((L_{m_0})_{Y'} \otimes W_0 \to \tilde{E})[-1],$$

$\operatorname{Cone}(\tilde{v})_{Y'} \to (L_{m_0})_{Y'} \otimes W_0$ factors through $\tilde{E}_1$. Moreover, for any $x \in Y'$, $\operatorname{Ext}^i((L_n)_x, (\tilde{E}_1)_x) = 0$ for $i \neq 0$ and $\operatorname{Hom}((L_n)_x, \operatorname{Cone}(\tilde{v})_x) \to \operatorname{Hom}((L_n)_x, (\tilde{E}_1)_x)$ is isomorphic for $n \gg 0$. If we set

$$\tilde{E}_2 := \mathrm{Cone}((L_{m_1})_{Y'} \otimes W_1 \to \tilde{E}_1)[-1],$$

then $\tilde{v}_{Y'}$ factors through $\tilde{E}_2$. Now we put

$$Y^{ss} := \left\{ x \in Y' \,\middle|\, \begin{array}{l} W_0 \otimes k(x) \to \mathrm{Hom}((L_{m_0})_x, \tilde{E}_x) \text{ is isomorphic,} \\ W_j \otimes k(x) \to \mathrm{Hom}((L_{m_j})_x, (\tilde{E}_j)_x) \text{ are isomorphic for } j = 1, 2, \\ \mathrm{Hom}((L_n)_x, \tilde{E}_x) = P(n) \text{ for } n \gg 0 \text{ and } \tilde{E}_x \text{ is } \mathscr{L}\text{-semistable} \end{array} \right\}$$

and

$$Y^s := \left\{ x \in Y^{ss} \mid \tilde{E}_x \text{ is } \mathscr{L}\text{-stable} \right\}.$$

Then we can check that Y^s, Y^{ss} are open subsets of Y'. If we put

$$G := GL(W_0) \times GL(W_1) \times GL(W_2),$$

then there is a canonical action of G on Z and Y, Y', Y^{ss}, Y^s are preserved by this action. For a sufficiently large integer N, we put

$$\begin{aligned} \alpha_0 &:= \operatorname{rank} W_2 + N \operatorname{rank} W_1 \\ \alpha_1 &:= -N \operatorname{rank} W_0 \\ \alpha_2 &:= -\operatorname{rank} W_0 \end{aligned}$$

and consider the character

$$\chi : G \longrightarrow \boldsymbol{G}_m; \quad (g_0, g_1, g_2) \mapsto \det(g_0)^{\alpha_0} \det(g_1)^{\alpha_1} \det(g_2)^{\alpha_2}.$$

Let us consider the quiver consisting of three vertices v_2, v_1, v_0 and $\mathrm{rank}_{\mathscr{O}_S} R^0 \mathrm{Hom}_p(L_{m_2}, L_{m_1})$-arrows from v_2 to v_1 and $\mathrm{rank}_{\mathscr{O}_S} R^0 \mathrm{Hom}_p(L_{m_1}, L_{m_0})$-arrows from v_1 to v_0. Then the points of Z correspond to the representations of this quiver (see [**5**] for the definition of quiver and its representation).

Lemma 4.6. *If we take $N \gg m_2 \gg m_1 \gg m_0 \gg 0$, Y^{ss} is contained in the set $Z^{ss}(\chi)$ of χ-semistable points of Z in the sense of* [**5**]. *Moreover, Y^s is contained in the set $Z^s(\chi)$ of χ-stable points of Z.*

Proof. Take any geometric point x of Y^{ss} and vector subspaces $W_i' \subset (W_i)_x$ $(0 \le i \le 2)$ which induce commutative diagrams

$$\begin{array}{ccc} W_2' & \longrightarrow & W_1' \otimes R^0 \operatorname{Hom}_p(L_{m_2}, L_{m_1})_x \\ \downarrow & & \downarrow \\ (W_2)_x & \longrightarrow & (W_1)_x \otimes R^0 \operatorname{Hom}_p(L_{m_2}, L_{m_1})_x \end{array}$$

$$\begin{array}{ccc} W_1' & \longrightarrow & W_0' \otimes R^0 \operatorname{Hom}_p(L_{m_1}, L_{m_0})_x \\ \downarrow & & \downarrow \\ (W_1)_x & \longrightarrow & (W_0)_x \otimes R^0 \operatorname{Hom}_p(L_{m_1}, L_{m_0})_x. \end{array}$$

From [**5**], we should say that

$$\alpha_0 \dim W_0' + \alpha_1 \dim W_1' + \alpha_2 \dim W_2' \geq 0.$$

Let $\mathscr{E}$ be the Y^{ss}-flat $\mathscr{A}_{Y^{ss}}$-module corresponding to $\tilde{E}_{Y^{ss}}$ by Proposition 3.9. Then a morphism $\mathscr{A}(-m_0) \otimes W_0' \to \mathscr{E}_x$ is induced and we denote its image by $\mathscr{E}(W_0')$. Note that $\mathscr{E}_x$ is of pure dimension and so $\mathscr{E}(W_0')$ is also of pure dimension. Since the family

$$\left\{\mathscr{E}(W_0') \mid W_0' \subset (W_0)_x,\ x \text{ is a geometric point of } Y^{ss}\right\}$$

is bounded, we can find an integer $m_1 \gg m_0$ such that for $K_1' := \ker(W_0' \otimes \mathscr{A}(-m_0) \to \mathscr{E}(W_0'))$, $K_1'(m_1)$ is generated by global sections and $H^i(K_1'(m_1)) = 0$, $H^i(\mathscr{A}_x(m_1 - m_0)) = 0$ for $i > 0$. Moreover we can find an integer $m_2 \gg m_1$ such that for $K_2' := \ker(H^0(K_1'(m_1)) \otimes \mathscr{A}(-m_1) \to K_1')$, $K_2'(m_2)$ is generated by global sections and $H^i(K_2'(m_2)) = 0$, $H^i(\mathscr{A}_x(m_2 - m_1)) = 0$, $H^i(\mathscr{A}_x(m_2 - m_0)) = 0$ and $H^i(K_1'(m_2)) = 0$ for $i > 0$. If we put $\tilde{W}_1' := H^0(K_1'(m_1))$ and $\tilde{W}_2' := H^0(K_2'(m_2))$, then we have

$$\begin{aligned} \dim H^0\big(\mathscr{E}(W_0')(m_1)\big) &= \dim H^0\big(\mathscr{A}_x(m_1 - m_0)\big) \dim W_0' - \dim \tilde{W}_1' \\ \dim H^0\big(\mathscr{E}(W_0')(m_2)\big) &= \dim H^0\big(\mathscr{A}_x(m_2 - m_0)\big) \dim W_0' \\ &\qquad - \dim H^0\big(\mathscr{A}_x(m_2 - m_1)\big) \dim \tilde{W}_1' + \dim \tilde{W}_2'. \end{aligned}$$

Since the family $\{\mathscr{E}(W_0')\}$ is bounded, we can take by using Lemma 4.5 a positive integer $m_0 \gg 0$ and a positive number $\epsilon > 0$ such that

$$\frac{h^0(\mathscr{E}(W_0')(m_0))}{P(m_0)} < \frac{a_0(\mathscr{E}(W_0'))}{a_0(P)} - \epsilon$$

for any W_0' such that

$$\frac{\chi(\mathscr{E}(W_0')(m))}{\chi(\mathscr{E}(W_0')(n))} < \frac{P(m)}{P(n)}$$

for $n \gg m \gg 0$. Here we write

$$\chi\bigl(\mathscr{E}(W_0')(n)\bigr) = \sum_{i=0}^{d} a_i(\mathscr{E}(W_0'))\binom{n+d-i}{d-i}, \quad P(n) = \sum_{i=0}^{d} a_i(P)\binom{n+d-i}{d-i}$$

with $a_i(\mathscr{E}(W_0'))$ and $a_i(P)$ integers. Since

$$\lim_{m_1\to\infty} \frac{h^0(\mathscr{E}(W_0')(m_1))}{P(m_1)} = \frac{a_0(\mathscr{E}(W_0'))}{a_0(P)},$$

we can take $m_1 \gg m_0$ such that

$$\frac{h^0(\mathscr{E}(W_0')(m_1))}{P(m_1)} > \frac{a_0(\mathscr{E}(W_0'))}{a_0(P)} - \frac{\epsilon}{2}.$$

Since

$$\begin{aligned}&\lim_{N\to\infty} \frac{(h^0(\mathscr{A}_x(m_2-m_1))+N)h^0(\mathscr{E}(W_0')(m_1)) - h^0(\mathscr{E}(W_0')(m_2))}{(h^0(\mathscr{A}_x(m_2-m_1))+N)P(m_1) - P(m_2)}\\ &= \frac{h^0(\mathscr{E}(W_0')(m_1))}{P(m_1)},\end{aligned}$$

we can take $N \gg m_2$ such that

$$\begin{aligned}&\frac{(h^0(\mathscr{A}_x(m_2-m_1))+N)h^0(\mathscr{E}(W_0')(m_1)) - h^0(\mathscr{E}(W_0')(m_2))}{(h^0(\mathscr{A}_x(m_2-m_1))+N)P(m_1) - P(m_2)}\\ &\quad > \frac{h^0(\mathscr{E}(W_0')(m_1))}{P(m_1)} - \frac{\epsilon}{2}.\end{aligned}$$

Then we have

$$
\begin{aligned}
&\frac{h^0(\mathscr{E}(W_0')(m_0))}{P(m_0)} \\
&\quad < \frac{a_0(\mathscr{E}(W_0'))}{a_0(P)} - \epsilon \\
&\quad < \frac{h^0(\mathscr{E}(W_0')(m_1))}{P(m_1)} + \frac{\epsilon}{2} - \epsilon \\
&\quad < \frac{(h^0(\mathscr{A}_x(m_2-m_1)) + N)h^0(\mathscr{E}(W_0')(m_1)) - h^0(\mathscr{E}(W_0')(m_2))}{(h^0(\mathscr{A}_x(m_2-m_1)) + N)P(m_1) - P(m_2)} + \frac{\epsilon}{2} + \frac{\epsilon}{2} - \epsilon \\
&\quad = \frac{(h^0(\mathscr{A}_x(m_2-m_1)) + N)h^0(\mathscr{E}(W_0')(m_1)) - h^0(\mathscr{E}(W_0')(m_2))}{(h^0(\mathscr{A}_x(m_2-m_1)) + N)P(m_1) - P(m_2)}
\end{aligned}
$$

for any W_0' such that

$$\frac{\chi(\mathscr{E}(W_0')(m))}{\chi(\mathscr{E}(W_0')(n))} < \frac{P(m)}{P(n)}$$

for $n \gg m \gg 0$. Take W_0' such that

$$\frac{\chi(\mathscr{E}(W_0')(m))}{\chi(\mathscr{E}(W_0')(n))} = \frac{P(m)}{P(n)}$$

for $n \gg m \gg 0$. Then we can see by Lemma 4.5 that

$$
\begin{aligned}
\frac{h^0(\mathscr{E}(W_0')(m_0))}{P(m_0)} &= \frac{a_0(\mathscr{E}(W_0'))}{a_0(P)} \\
&= \frac{(h^0(\mathscr{A}_x(m_2-m_1)) + N)h^0(\mathscr{E}(W_0')(m_1)) - h^0(\mathscr{E}(W_0')(m_2))}{(h^0(\mathscr{A}_x(m_2-m_1)) + N)P(m_1) - P(m_2)}.
\end{aligned}
$$

Hence we have the inequality

$$
\begin{aligned}
&h^0\big(\mathscr{E}(W_0')(m_0)\big) \\
&\quad \leq \frac{(h^0(\mathscr{A}_x(m_2-m_1)) + N)h^0(\mathscr{E}(W_0')(m_1)) - h^0(\mathscr{E}(W_0')(m_2))}{(h^0(\mathscr{A}_x(m_2-m_1)) + N)P(m_1) - P(m_2)} P(m_0) \qquad (2)
\end{aligned}
$$

for any $\mathscr{E}(W_0')$. Moreover, the equality holds in (2) if and only if $\chi(\mathscr{E}(W_0')(n))/a_0(\mathscr{E}(W_0')) = P(n)/a_0(P)$ as polynomials in n. From the inequality (2), we obtain the inequality

$$(r_2 + Nr_1)\dim W_0' - Nr_0 \dim \tilde{W}_1' - r_0 \dim \tilde{W}_2' \geq 0$$

by using $\dim W_0' \leq h^0(\mathscr{E}(W_0')(m_0))$. Since $\dim W_1' \leq \dim \tilde{W}_1'$ and $\dim W_2' \leq \dim \tilde{W}_2'$, we have

$$\alpha_0 \dim W_0' + \alpha_1 \dim W_1' + \alpha_2 \dim W_2' \geq 0. \tag{3}$$

Thus x becomes a geometric point of $Z^{ss}(\chi)$.

In the inequality (3), the equality holds if and only if $\dim \tilde{W}_1' = \dim W_1'$, $\dim \tilde{W}_2' = \dim W_2'$, $h^0(\mathscr{E}(W_0')) = \dim W_0'$ and $\chi(\mathscr{E}(W_0')(n))/a_0(\mathscr{E}(W_0')) = P(n)/a_0(P)$ as polynomials in n. So, if x is a geometric point of Y^s, we have

$$(r_2 + Nr_1)\dim W_0' - Nr_0 \dim W_1' - r_0 \dim W_2' > 0.$$

for any (W_0', W_1', W_2') with $(0,0,0) \neq (W_0', W_1', W_2') \subsetneq ((W_0)_x, (W_1)_x, (W_2)_x)$, which means that x becomes a geometric point of $Z^s(\chi)$. □

By [**5**] and [**9**], there exists a GIT quotient $\phi : Y \cap Z^{ss}(\chi) \to (Y \cap Z^{ss}(\chi))//G$.

Lemma 4.7. *$\phi^{-1}(\phi(Y^{ss})) = Y^{ss}$.*

Proof. It is sufficient to show that $\phi^{-1}(\phi(Y^{ss})) \subset Y^{ss}$. Take any k-valued geometric point x of $\phi^{-1}(\phi(Y^{ss}))$. Let s be the induced k-valued geometric point of S. Since $\phi(x)$ is a geometric point of $\phi(Y^{ss})$, there exists a k-valued geometric point y of Y^{ss} such that $\phi(x) = \phi(y)$.

Let $\mathscr{E}$ be the Y^{ss}-flat $\mathscr{A}_{Y^{ss}}$-module corresponding to $\tilde{E}_{Y^{ss}}$ as in the proof of Lemma 4.6. Then there is a Jordan-Hölder filtration

$$0 = F^{(0)} \subset F^{(1)} \subset \cdots \subset F^{(l)} = \mathscr{E} \otimes k(y)$$

of $\mathscr{E} \otimes k(y)$. For each i with $1 \leq i \leq l$, we define $K_1^{(i)}$, $K_2^{(i)}$ by exact sequences

$$0 \longrightarrow K_1^{(i)} \longrightarrow H^0\big(F^{(i)}(m_0)\big) \otimes \mathscr{A}(-m_0) \longrightarrow F^{(i)} \longrightarrow 0$$
$$0 \longrightarrow K_2^{(i)} \longrightarrow H^0\big(K_1^{(i)}(m_1)\big) \otimes \mathscr{A}(-m_1) \longrightarrow K_1^{(i)} \longrightarrow 0.$$

Then y corresponds to the representation of quiver given by

$$H^0\big(K_2^{(l)}(m_2)\big) \longrightarrow H^0\big(K_1^{(l)}(m_1)\big) \otimes H^0(\mathscr{A}_s(m_2 - m_1))$$

$$H^0\big(K_1^{(l)}(m_1)\big) \longrightarrow H^0\big(F^{(l)}(m_0)\big) \otimes H^0(\mathscr{A}_s(m_1 - m_0))$$

and the Jordan-Hölder filtration of $\mathscr{E} \otimes k(y)$ corresponds to the filtration of the quiver representation given by

$$\begin{aligned} &0 \subset H^0\big(K_2^{(1)}(m_2)\big) \subset \cdots \subset H^0\big(K_2^{(l)}(m_2)\big) \\ &0 \subset H^0\big(K_1^{(1)}(m_1)\big) \subset \cdots \subset H^0\big(K_1^{(l)}(m_1)\big) \\ &0 \subset H^0\big(F^{(1)}(m_0)\big) \subset \cdots \subset H^0\big(F^{(l)}(m_0)\big). \end{aligned}$$

We put $E^{(i)} := F^{(i)}/F^{(i-1)}$ and $\overline{\mathscr{E}} := \bigoplus_{i=1}^l E^{(i)}$. For $i = 1, \ldots, l$, we define $\bar{K}_1^{(i)}$, $\bar{K}_2^{(i)}$ by the exact sequences

$$\begin{aligned} &0 \longrightarrow \bar{K}_1^{(i)} \longrightarrow H^0\big(E^{(i)}(m_0)\big) \otimes \mathscr{A}(-m_0) \longrightarrow E^{(i)} \longrightarrow 0 \\ &0 \longrightarrow \bar{K}_2^{(i)} \longrightarrow H^0\big(\bar{K}_1^{(i)}(m_1)\big) \otimes \mathscr{A}(-m_1) \longrightarrow \bar{K}_1^{(i)} \longrightarrow 0. \end{aligned}$$

We can see from the proof of Lemma 4.6 that the quiver representation y_i given by

$$\begin{aligned} &H^0\big(\bar{K}_2^{(i)}(m_2)\big) \longrightarrow H^0\big(\bar{K}_1^{(i)}(m_1)\big) \otimes H^0(\mathscr{A}_s(m_2 - m_1)) \\ &H^0\big(\bar{K}_1^{(i)}(m_1)\big) \longrightarrow H^0\big(E^{(i)}(m_0)\big) \otimes H^0(\mathscr{A}_s(m_1 - m_0)) \end{aligned}$$

is stable with respect to the weight $(\alpha_0, \alpha_1, \alpha_2)$. The direct sum $y_1 \oplus \cdots \oplus y_l$ corresponds to a point y' of Y_s^{ss} given by the exact sequence

$$\begin{aligned} H^0\bigg(\bigoplus_{i=1}^l \bar{K}_2^{(i)}(m_2)\bigg) \otimes \mathscr{A}(-m_2) \longrightarrow H^0\bigg(\bigoplus_{i=1}^l \bar{K}_1^{(i)}(m_1)\bigg) \otimes \mathscr{A}(-m_1) \\ \longrightarrow H^0\bigg(\bigoplus_{i=1}^l E^{(i)}(m_0)\bigg) \otimes \mathscr{A}(-m_0) \longrightarrow \bigoplus_{i=1}^l E^{(i)} \longrightarrow 0. \end{aligned}$$

Then we can see that the quiver representations determined by y and y' are S-equivalent. So we have $\phi(x) = \phi(y) = \phi(y')$. Note that $G_s y'$ is a closed orbit in $(Y \cap Z^{ss}(\chi))_s$ by [**5**, Proposition 3.2]. Thus the closure of the G_s-orbit of x must contain y'. Then, by Proposition 4.3, x becomes a geometric point of Y_s^{ss}. □

Proof of Theorem 4.4. If we put

$$\overline{M_{\mathscr{D}}^{P,\mathscr{L}}} := \phi(Y^{ss}),$$

then we can see by Lemma 4.7 that $\overline{M_{\mathscr{D}}^{P,\mathscr{L}}}$ is an open subset of $(Y \cap Z^{ss}(\chi))//G$. We can see by a similar argument to that of [**8**, Proposition 7.3], that there is a canonical morphism $\Phi : \overline{\mathscr{M}_{\mathscr{D}}^{P,\mathscr{L}}} \to \overline{M_{\mathscr{D}}^{P,\mathscr{L}}}$. For two geometric points $x_1, x_2 \in Y^{ss}$ over a geometric point s of S, $\phi(x_1) = \phi(x_2)$ if and only if the corresponding representations of quiver are S-equivalent ([**5**]), that is, the corresponding objects of $\mathscr{D}_s$ are S-equivalent. Thus for any algebraically closed field k over S, $\Phi(k) : \overline{\mathscr{M}_{\mathscr{D}}^{P,\mathscr{L}}}(k) \to \overline{M_{\mathscr{D}}^{P,\mathscr{L}}}(k)$ is bijective. We can see by a standard argument that $\overline{M_{\mathscr{D}}^{P,\mathscr{L}}}$ has the universal property of the coarse moduli scheme. If we put $M_{\mathscr{D}}^{P,\mathscr{L}} := Y^s/G$, then $M_{\mathscr{D}}^{P,\mathscr{L}}$ becomes an open subset of $\overline{M_{\mathscr{D}}^{P,\mathscr{L}}}$ and we can easily see that $M_{\mathscr{D}}^{P,\mathscr{L}}$ is a coarse moduli scheme of $\mathscr{M}_{\mathscr{D}}^{P,\mathscr{L}}$. So we have proved Theorem 4.4. □

Theorem 4.8. *Assume that S is of finite type over a universally Japanese ring Ξ. Then the moduli scheme $\overline{M_{\mathscr{D}}^{P,\mathscr{L}}}$ is projective over S.*

For the proof of Theorem 4.8, the following lemma is essential.

Lemma 4.9. *Let R be a discrete valuation ring over S with quotient field K and residue field k. Assume that E is an object of $\mathscr{D}_K$ which is $\mathscr{L}$-semistable. Then there is an object $\tilde{E} \in \mathscr{D}_R$ such that $\tilde{E}_K \cong E$ and $\tilde{E}_k$ is $\mathscr{L}$-semistable.*

Proof. The above E corresponds to a coherent $\mathscr{A}_K$-module $\mathscr{E}$ and it suffices to show that there exists an R-flat coherent $\mathscr{A}_R$-module $\tilde{\mathscr{E}}$ such that $\tilde{\mathscr{E}} \otimes_R K \cong \mathscr{E}$ and $\tilde{\mathscr{E}} \otimes k$ satisfies the semistability condition given by the inequality in Remark 3.11. For a sufficiently large integer N, we have $H^i(\mathscr{E}(N)) = 0$ for $i > 0$ and $\mathscr{E}(N)$ is generated by global sections. Then there is a surjection $\mathscr{A}_K(-N)^{\oplus r} \to \mathscr{E}$ which determies a K-valued point η of the Quot-scheme $\mathrm{Quot}^P_{\mathscr{A}(-N)^{\oplus r}}$ for some numerical polynomial P, where $r = \dim H^0(\mathscr{E}(N))$. Let $\mathscr{F} \subset \mathscr{A}(-N)^{\oplus r}$ be the universal subsheaf and Y be the maximal closed subscheme of $\mathrm{Quot}^P_{\mathscr{A}(-N)^{\oplus r}}$ such that $\mathscr{A} \otimes \mathscr{F}_Y \to \mathscr{A}(-N)^{\oplus r}_Y$ factors through $\mathscr{F}_Y$. Then η is a K-valued point of Y and extends to an R-valued point ξ of Y because Y is proper over S. ξ corresponds to an R-flat quotient coherent $\mathscr{A}_R$-module $\mathscr{E}'$ of $\mathscr{A}(-N)^{\oplus r}_R$ and we have $\mathscr{E}' \otimes_R K \cong \mathscr{E}$. From the proof similar to that of Langton's theorem ([**3**, Theorem 2.B.1]), we can obtain an R-flat coherent $\mathscr{A}_R$-module $\tilde{\mathscr{E}}$ by taking succesive elementary transforms of $\mathscr{E}'$ along $\boldsymbol{P}(V_1) \times \operatorname{Spec} k$ such that $\tilde{\mathscr{E}} \otimes_R K \cong \mathscr{E}' \otimes_R K \cong \mathscr{E}$ and $\tilde{\mathscr{E}} \otimes k$ is semistable as $\mathscr{A} \otimes k$-module. □

Now we prove Theorem 4.8. By construction, the moduli scheme $\overline{M_{\mathscr{D}}^{P,\mathscr{L}}}$ is quasi-projective over S. So it is sufficient to show that $\overline{M_{\mathscr{D}}^{P,\mathscr{L}}}$ is proper over S. Let R be a discrete valuation ring over S with quotient field K and let $\varphi : \operatorname{Spec} K \to \overline{M_{\mathscr{D}}^{P,\mathscr{L}}}$ be a morphism over S. Then there is a finite extension field K' of K such that the composite $\psi : \operatorname{Spec} K' \to \operatorname{Spec} K \xrightarrow{\varphi} \overline{M_{\mathscr{D}}^{P,\mathscr{L}}}$ is given by an $\mathscr{L}$-semistable object E'. We can take a discrete valuation ring R' with quotient field K' such that $K \cap R' = R$. Let k' be the residue field of R'. By Lemma 4.9, there exists an object E of $\mathscr{D}_{R'}$ such that $E_{K'} \cong E'$ and $E_{k'}$ is $\mathscr{L}$-semistable. Then E gives a morphism $\overline{\psi} : \operatorname{Spec} R' \to \overline{M_{\mathscr{D}}^{P,\mathscr{L}}}$ which is an extension of ψ. We can easily see that $\overline{\psi}$ factors through $\operatorname{Spec} R$. Thus $\overline{M_{\mathscr{D}}^{P,\mathscr{L}}}$ is proper over S by the valuative criterion of properness. □

5. Examples.

In this section, we give several examples of moduli spaces of stable objects determined by a strict ample sequence.

Example 5.1. Let $f : X \to S$ be a flat projective morphism of noetherian schemes and let $\mathscr{O}_X(1)$ be an S-very ample line bundle on X such that $H^i(\mathscr{O}_{X_s}(m)) = 0$ for $i > 0$, $s \in S$ and $m > 0$. Consider the fibered triangulated category $\mathscr{D}_{X/S}$ defined by $(\mathscr{D}_{X/S})_U = D^b(\mathrm{Coh}(X_U/U))$ for $U \in (\mathrm{Sch}/S)$. Then $\mathscr{L} = \{\mathscr{O}_X(-n)\}_{n \geq 0}$ becomes a strict ample sequence in $\mathscr{D}_{X/S}$.

Proof. Definition 3.1 (1), (2), (3) are easy to verify. Let us prove Definition 3.1 (4). Take any $U \in (\mathrm{Sch}/S)$ and any object $E^\bullet \in (\mathscr{D}_{X/S})_U$. We may assume that $E^\bullet$ is given by a complex

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow E^{l_1} \xrightarrow{d^{l_1}} E^{l_1+1} \xrightarrow{d^{l_1+1}} \cdots \xrightarrow{d^{l_2-1}} E^{l_2} \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots,$$

where each E^i is a coherent sheaf on X_U flat over U. By flattening stratification theorem, there is a stratification $U = \coprod_{j=1}^m Y_j$ of U by subschemes Y_j such that each $\operatorname{coker}(d^i)_{Y_j} = \operatorname{coker}(d^i_{Y_j})$ is flat over Y_j for any i and j. Then we can see that $\operatorname{im}(d^i_{Y_j})$ and $\ker(d^i_{Y_j})$ are flat over Y_j for any i and j. For any point $s \in U$, the sequence

$$0 \longrightarrow \operatorname{im}\left(d^{i-1}_{Y_j}\right) \otimes k(s) \longrightarrow E^i \otimes k(s) \longrightarrow \operatorname{coker}\left(d^i_{Y_j}\right) \otimes k(s) \longrightarrow 0$$

is exact because $\operatorname{coker}(d_{Y_j})$ is flat over Y_j. Then the homomorphism $\operatorname{im}(d^{i-1}_{Y_j}) \otimes k(s) \longrightarrow \ker(d^i_{Y_j}) \otimes k(s)$ is injective for any $s \in Y_j$. Thus the cohomology sheaf $\mathscr{H}^i(E^\bullet_{Y_j}) := \ker(d^i_{Y_j})/\operatorname{im}(d^{i-1}_{Y_j})$ is flat over Y_j for any i and j. We can take a positive integer n_0 such that for any $n \geq n_0$, $R^p(f_{Y_j})_*(E^i_{Y_j}(n)) = 0$, $R^p(f_{Y_j})_*(\operatorname{im}(d^i_{Y_j})(n)) = 0$ and $R^p(f_{Y_j})_*(\ker d^i_{Y_j}(n)) = 0$ for any $p > 0$ and any i, j. Then we have $R^p(f_{Y_j})_*(\mathscr{H}^i(E^\bullet_{Y_j}(n))) = 0$ for any $p > 0$, any i, j and $n \geq n_0$. From the spectral sequence $R^p(f_{Y_j})_*(\mathscr{H}^q(E^\bullet_{Y_J}(n))) \Rightarrow R^{p+q}(f_{Y_j})_*(E^\bullet_{Y_j}(n))$, we have an isomorphism $R^i(f_{Y_j})_*(E^\bullet_{Y_j}(n)) \cong (f_{Y_j})_*(\mathscr{H}^i(E^\bullet_{Y_j})(n))$ for any i, j and $n \geq n_0$. So we can see that $\boldsymbol{R}(f_{Y_j})_*(E^\bullet_{Y_j}(n))$ is quasi-isomorphic to the complex

$$\begin{aligned}\cdots \longrightarrow 0 \longrightarrow (f_{Y_j})_*\big(E^{l_1}_{Y_j}(n)\big) \longrightarrow (f_{Y_j})_*\big(E^{l_1+1}_{Y_j}(n)\big)\\ \longrightarrow \cdots \longrightarrow (f_{Y_j})_*\big(E^{l_2}_{Y_j}(n)\big) \longrightarrow 0 \longrightarrow \cdots\end{aligned}$$

for any i, j and $n \geq n_0$. Note that there are canonical isomorphisms

$$\begin{aligned}\boldsymbol{H}^i\big(E^\bullet_s(n)\big) &\cong R^i(f_{Y_j})_*\big(E^\bullet_{Y_j}(n)\big) \otimes k(s) \cong (f_{Y_j})_*\big(\mathscr{H}^i(E^\bullet_{Y_j})(n)\big) \otimes k(s)\\ &\cong H^0\big(X_s, \mathscr{H}^i(E^\bullet_s)(n)\big).\end{aligned}$$

for any i, j, any $s \in Y_j$ and $n \geq n_0$. If we take n_0 sufficiently larger, we may assume that the homomorphism

$$(f_{Y_j})^*(f_{Y_j})_*\big(\mathscr{H}^i(E^\bullet_{Y_j}(n))\big) \longrightarrow \mathscr{H}^i\big(E^\bullet_{Y_j}\big)(n)$$

is surjective for any $n \geq n_0$ and any i, j. Thus there exists a positive integer $N_0 \gg n$ such that

$$(f_{Y_j})_*(\mathscr{O}_{X_{Y_j}}(N-n)) \otimes (f_{Y_j})_*\big(\mathscr{H}^i(E^\bullet_{Y_j}(n))\big) \longrightarrow (f_{Y_j})_*\big(\mathscr{H}^i(E^\bullet_{Y_j})(N)\big)$$

is surjective for any $N \geq N_0$ and any i, j. So we obtain a commutative diagram

$$\begin{array}{ccc}
\begin{array}{c}(f_{Y_j})_*(\mathscr{O}_{X_{Y_j}}(N-n)) \otimes k(s)\\ \otimes (f_{Y_j})_*\big(\mathscr{H}^i(\mathscr{E}^\bullet_{Y_j}(n))\big) \otimes k(s)\end{array} & \longrightarrow & (f_{Y_j})_*\big(\mathscr{H}^i(E^\bullet_{Y_j}(N)\big) \otimes k(s)\\
\downarrow\cong & & \downarrow\cong\\
H^0(\mathscr{O}_{X_s}(N-n)) \otimes \boldsymbol{H}^i\big(E^\bullet_s(n)\big) & \longrightarrow & \boldsymbol{H}^i\big(E^\bullet_s(N)\big)\\
\downarrow\cong & & \downarrow\cong\\
\begin{array}{c}\operatorname{Hom}(\mathscr{O}_{X_s}(-N), \mathscr{O}_{X_s}(-n))\\ \otimes \operatorname{Ext}^i\big(\mathscr{O}_{X_s}(-n), E^\bullet_s\big)\end{array} & \longrightarrow & \operatorname{Ext}^i\big(\mathscr{O}_{X_s}(-N), E^\bullet_s\big).
\end{array}$$

for any i, j, any $s \in Y_j$ and $N \geq N_0$. Hence

$$\mathrm{Hom}(\mathscr{O}_{X_s}(-N), \mathscr{O}_{X_s}(-n)) \otimes \mathrm{Ext}^i\left(\mathscr{O}_{X_s}(-n), E^\bullet_s\right) \longrightarrow \mathrm{Ext}^i\left(\mathscr{O}_{X_s}(-N), E^\bullet_s\right)$$

is surjective for any $s \in U$, any i and $N \geq N_0$ and we have proved Definition 3.1 (4).

Now we prove Definition 3.1 (5). Assume that an object $E \in (\mathscr{D}_{X/S})_U$ and integers i, n_0 are given such that $\mathrm{Ext}^i(\mathscr{O}_{X_s}(-n), E^\bullet_s) = 0$ for any $s \in U$ and $n \geq n_0$. Replacing n_0 by a sufficiently large integer, we have

$$\mathrm{Ext}^i\left(\mathscr{O}_{X_s}(-n), E^\bullet_s\right) \cong \boldsymbol{H}^i\big(E^\bullet_s(n)\big) \cong H^0\big(X_s, \mathscr{H}^i(E^\bullet_s)(n)\big) = 0$$

for any $s \in U$ and any $n \geq n_0$. Then we have $\mathscr{H}^i(E^\bullet_s) = 0$. If $E^\bullet$ is given by

$$E^{l_1} \xrightarrow{d^{l_1}} E^{l_1+1} \xrightarrow{d^{l_1+1}} \cdots \xrightarrow{d^{l_2-1}} E^{l_2},$$

such that each E^j is flat over U, then the induced homomorphism $\mathrm{coker}(d^{i-1}) \otimes k(s) \to E^{i+1} \otimes k(s)$ is injective for any $s \in U$. Then $\mathrm{coker}(d^i)$ is flat over U and $\mathrm{coker}(d^{i-1}) \to E^{i+1}$ is injective. Let $F^\bullet$ be the complex given by

$$\cdots \longrightarrow 0 \longrightarrow \mathrm{coker}(d^i) \longrightarrow E^{i+2} \xrightarrow{d^{i+2}} \cdots \xrightarrow{d^{l_2-1}} E^{l_2} \longrightarrow 0 \longrightarrow \cdots.$$

Then there is a canonical morphism $u : E^\bullet \to F^\bullet$. Note that

$$R^j \mathrm{Hom}_f(\mathscr{O}_{X_U}(-n), E^\bullet) = R^j(f_U)_*(E^\bullet(n)) \cong (f_U)_*(\mathscr{H}^j(E^\bullet)(n))$$

for $n \gg 0$. So u induces isomorphisms

$$\begin{aligned} R^j \mathrm{Hom}_f(\mathscr{O}_{X_U}(-n), E^\bullet) &\xrightarrow{\sim} (f_U)_*(\mathscr{H}^j(E^\bullet)(n)) \\ &\xrightarrow{\sim} (f_U)_*(\mathscr{H}^j(F^\bullet)(n)) \xrightarrow{\sim} R^j \mathrm{Hom}_f(\mathscr{O}_{X_U}(-n), F^\bullet) \end{aligned}$$

for $j > i$ and $n \gg 0$. By definition we have $R^j \mathrm{Hom}_f(\mathscr{O}_{X_U}(-n), F^\bullet) = (f_U)_*(\mathscr{H}^j(F^\bullet(n))) = 0$ for $j \leq i$ and $n \gg 0$. Thus we have proved Definition 3.1 (5).

Finally, let us prove Definition 3.1 (6). Let $E^\bullet$ and $F^\bullet$ be objects of $(\mathscr{D}_{X/S})_U$. Assume that $R^j(f_U)_*(E^\bullet(n)) = 0$ for $j \geq 0$ and $n \gg 0$ and that $R^j(f_U)_*(F^\bullet(n)) = 0$ for $j < 0$ and $n \gg 0$. Since $R^j(f_U)_*(E^\bullet(n)) \cong (f_U)_*(\mathscr{H}^j(E^\bullet)(n))$ for $n \gg 0$, we have $\mathscr{H}^j(E^\bullet) = 0$ for $j \geq 0$. Then $E^\bullet$ is quasi-isomorphic to the complex given by

$$\cdots \longrightarrow 0 \longrightarrow E^{l_1} \xrightarrow{d_E^{l_1}} E^{l_1+1} \longrightarrow \cdots \longrightarrow E^{-2} \longrightarrow \ker(d_E^{-1}) \longrightarrow 0 \longrightarrow \cdots .$$

On the other hand, we have $\mathscr{H}^j(F^\bullet) = 0$ for $j < 0$, because $R^j(f_U)_*(F^\bullet(n)) \cong (f_U)_*(\mathscr{H}^j(F^\bullet)(n))$ for $n \gg 0$. Then $F^\bullet$ is quasi-isomorphic to the complex given by

$$\cdots \longrightarrow 0 \longrightarrow \operatorname{coker} d_F^{-1} \longrightarrow F^1 \xrightarrow{d_F^1} \cdots \longrightarrow F^{m_2} \longrightarrow 0 \longrightarrow \cdots .$$

We can take a complex

$$\cdots \longrightarrow 0 \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$$

such that each I^j is an injective sheaf on X_U and that $I^\bullet$ is quasi-isomorphic to $F^\bullet$. Then we have $\operatorname{Hom}_{(\mathscr{D}_{X/S})_U}(E^\bullet, F^\bullet) \cong H^0(\operatorname{Hom}^\bullet(E^\bullet, I^\bullet)) = 0$. So we have proved Definition 3.1 (6). □

For an object $E \in (\mathscr{D}_{X/S})_U$, $\operatorname{Ext}^i(\mathscr{O}_{X_s}(-n), E_s) = 0$ for $n \gg 0$, $i \neq 0$ and $s \in U$ if and only if $E^\bullet$ is quasi-isomorphic to a coherent sheaf on X_U flat over U. Hence, for a numerical polynomial P, the moduli space $M^{P,\mathscr{L}}_{\mathscr{D}_{X/S}}$ (resp. $\overline{M^{P,\mathscr{L}}_{\mathscr{D}_{X/S}}}$) is just the usual moduli space of $\mathscr{O}_X(1)$-stable sheaves (resp. moduli space of S-equivalence classes of $\mathscr{O}_X(1)$-semistable sheaves) on X over S.

Example 5.2. Let X, S, $\mathscr{O}_X(1)$ and $\mathscr{D}_{X/S}$ be as in Example 5.1. Take a vector bundle G on X. Replacing $\mathscr{O}_X(1)$ by some multiple, $\mathscr{L}_G = \{G \otimes \mathscr{O}_X(-n)\}_{n \geq 0}$ also becomes a strict ample sequence in $\mathscr{D}_{X/S}$ and the moduli space $M^{P,\mathscr{L}_G}_{\mathscr{D}_{X/S}}$ (resp. $\overline{M^{P,\mathscr{L}_G}_{\mathscr{D}_{X/S}}}$) is the moduli space of G-twisted $\mathscr{O}_X(1)$-stable sheaves (resp. moduli space of S-equivalence classes of G-twisted $\mathscr{O}_X(1)$-semistable sheaves) on X over S.

Example 5.3. Let X, Y be projective schemes over an algebraically closed field k and let $\mathscr{O}_X(1)$ be a very ample line bundle on X such that $H^i(X, \mathscr{O}_X(m)) = 0$ for $i > 0$ and $m > 0$. Assume that a Fourier-Mukai transform

$$\Phi : D^b_c(X) \xrightarrow{\sim} D^b_c(Y)$$
$$E \mapsto \boldsymbol{R}(p_Y)_*(p_X^*(E) \otimes \mathscr{P})$$

with the kernel $\mathscr{P} \in D^b_c(X \times Y)$ is given. Then Φ extends to an equivalence of fibered triangulated categories

$$\Phi : \mathscr{D}_{X/k} \xrightarrow{\sim} \mathscr{D}_{Y/k}.$$

Since $\mathscr{L} = \{\mathscr{O}_X(-n)\}_{n\geq 0}$ is a strict ample sequence in $\mathscr{D}_{X/k}$, $\mathscr{L}^\Phi = \{\Phi(\mathscr{O}_X(-n))\}_{n\geq 0}$ is a strict ample sequence in $\mathscr{D}_{Y/k}$. Moreover Φ determines an isomorphism

$$\Phi : M^{P,\mathscr{L}}_{\mathscr{D}_{X/k}} \xrightarrow{\sim} M^{P,\mathscr{L}^\Phi}_{\mathscr{D}_{Y/k}}$$

of the moduli space of stable sheaves on X to the moduli space of stable objects in $D^b_c(Y)$.

Example 5.4. Let G be a finite group and X be a projective variety over $\boldsymbol{C}$ on which G acts. Take a G-linearized very ample line bundle $\mathscr{O}_X(1)$ on X such that $H^i(X, \mathscr{O}_X(m)) = 0$ for $i > 0$ and $m > 0$. Let $\rho_0, \rho_1, \dots, \rho_s$ be the irreducible representations of G. Consider the fibered triangulated category $\mathscr{D}^G_{X/\boldsymbol{C}}$ defined by $(\mathscr{D}^G_{X/\boldsymbol{C}})_U = D^G(\mathrm{Coh}(X_U/U))$, for $U \in (\mathrm{Sch}/\boldsymbol{C})$, where $D^G(\mathrm{Coh}(X_U/U))$ is the full subcategory of the derived category of bounded complexes of G-equivariant coherent sheaves on X_U consisting of the objects of finite Tor-dimension over U. For positive integers $r_0, r_1, \dots, r_s$, $\mathscr{L}^G_{(r_0,\dots,r_s)} = \{\mathscr{O}_X(-n) \otimes (\rho_0^{\oplus r_0} \oplus \cdots \oplus \rho_s^{\oplus r_s})\}_{n\geq 0}$ becomes a strict ample sequence in $\mathscr{D}^G_{X/\boldsymbol{C}}$. The moduli space $M^{P,\mathscr{L}^G_{(r_0,\dots,r_s)}}_{\mathscr{D}^G_{X/\boldsymbol{C}}}$ is just the moduli space of G-equivariant sheaves $\mathscr{E}$ on X satisfying the stability condition: $\mathscr{E}$ is of pure dimension $d = \deg P$ and for any G-equivariant subsheaf $0 \neq \mathscr{F} \subsetneq \mathscr{E}$, the inequality

$$\frac{\mathrm{Hom}_G\left(\rho_0^{\oplus r_0} \oplus \cdots \oplus \rho_s^{\oplus r_s}, H^0(X, \mathscr{F} \otimes \mathscr{O}_X(n))\right)}{a_0(\mathscr{F})} < \frac{\mathrm{Hom}_G\left(\rho_0^{\oplus r_0} \oplus \cdots \oplus \rho_s^{\oplus r_s}, H^0(X, \mathscr{E} \otimes \mathscr{O}_X(n))\right)}{a_0(\mathscr{E})}$$

holds for $n \gg 0$, where we define

$$\chi(\mathscr{E}(m)) = \sum_{i=0}^{d} a_i(\mathscr{E}) \binom{m+d-i}{d-i} \quad \text{and} \quad \chi(\mathscr{F}(m)) = \sum_{i=0}^{d} a_i(\mathscr{F}) \binom{m+d-i}{d-i}$$

and so on.

EXAMPLE 5.5. Let X be a projective variety over $\boldsymbol{C}$ and let $\mathscr{O}_X(1)$ be a very ample line bundle on X such that $H^i(X, \mathscr{O}_X(m)) = 0$ for $i > 0$ and $m > 0$. For a torsion class $\alpha \in H^2(X, \mathscr{O}_X^{\times})$, consider the fibered triangulated category $\mathscr{D}^{\alpha}_{X/\boldsymbol{C}}$ over $(\mathrm{Sch}/\boldsymbol{C})$ defined by $(\mathscr{D}^{\alpha}_{X/\boldsymbol{C}})_U := D^b(\mathrm{Coh}(X_U/U), \alpha_U)$, where $D^b(\mathrm{Coh}(X_U/U), \alpha_U)$ is the derived category of bounded complexes of coherent α_U-twisted sheaves on $X \times U$ of finite Tor-dimension over U and α_U is the image of α in $H^2(X_U, \mathscr{O}_{X_U}^{\times})$. For a locally free α-twisted sheaf G of finite rank on X, $\mathscr{L}_G^{\alpha} = \{G \otimes \mathscr{O}_X(-n)\}_{n \geq 0}$ becomes a strict ample sequence in $\mathscr{D}^{\alpha}_{X/\boldsymbol{C}}$, after replacing $\mathscr{O}_X(1)$ by some multiple. The moduli space $M^{P, \mathscr{L}_G^{\alpha}}_{\mathscr{D}^{\alpha}_{X/\boldsymbol{C}}}$ is just the moduli space of G-twisted stable α-twisted sheaves on X in the sense of [**10**].

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References

[1] T. Bridgeland, Stability conditions on triangulated categories. Ann. Math. (2), **166** (2007), 317–345.

[2] A. Bondal and D. Orlov, Reconstruction of a variety from the derived category and the groups of autoequivalences, Comp. Math. **125** (2001), 327–344.

[3] D. Huybrechts and M. Lehn, The geometry of moduli spaces of sheaves, Aspects of Mathematics, **E31**, Friedr. Vieweg & Sohn, Braunschweig, 1997.

[4] M. Inaba, Toward a definition of moduli of complexes of coherent sheaves on a projective scheme, J. Math. Kyoto Univ., **42** (2002), 317–329.

[5] A. D. King, Moduli of representations of finite-dimensional algebras, Quart. J. Math. Oxford Ser. (2), **45** (1994), no. 180, 515–530.

[6] A. Langer, Semistable sheaves in positive characteristic, Ann. of Math. (2), **159** (2004), 251–276.

[7] M. Lieblich, Moduli of complexes on a proper morphism, J. Algebraic. Geom., **15** (2006), 175–206.

[8] M. Maruyama, Construction of moduli spaces of stable sheaves via Simpson's idea, Moduli of vector bundles (Sanda, 1994; Kyoto, 1994), 147–187, Lecture Notes in Pure and Appl. Math., **179**, Dekker, New York, 1996.

[9] C. S. Seshadri, Geometric reductivity over arbitrary base, Adv. in Math., **26** (1977), 225–274.

[10] K. Yoshioka, Moduli spaces of twisted sheaves on a projective variety, Moduli spaces and arithmetic geometry, 1–30, Adv. Stud. Pure Math., **45**, Math. Soc. Japan, Tokyo, 2006.

[11] K. Yoshioka, Stability and the Fourier-Mukai transform I, Math. Z., **245** (2003), 657–665.

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