Journal of the Mathematical Society of Japan Vol. 2, Nos. 3-4, March, 1951.

On Invariant Differential Forms on Group Varieties

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(Received Apr. 17, 1950)

In this note we shall discuss the invariant differential foims on group varieties[1] and prove that for any group variety, there corresponds to it a Lie ring composed of invariant dervations of the (abstract) field of functions defined on that group variety. We shall also discuss some of its properties, which are the analogues of the case of usual Lie groups.

§ 1. Differential Forms on an Algebraic Variety.

Let V^n be a variety in S^N, $\mathfrak{D}_1(V)$ the totality of functions defined on $V\times V$ which induce on $\varDelta_V$ the constant 0. $\mathfrak{D}_1(V)$ is a module over the field of constants Ω[2]. Let $\theta \varepsilon \mathfrak{D}_1(V)$ and let k be a field of definition for θ, P a generic point of V over k and $H_1(X),\ldots\ldots,H_n(X)$ a uniformizing set of linear forms of V at P. We shall denote by $\varLambda_j^{2N-n+1}$ the linear variety in $S^N\times S^N$ defined by $H_i(X-X')=0$ $(i=1,\ldots\ldots,\hat{j},\ldots\ldots,n)$ (here $\hat{j}$ means to omit j). Then by F-VI_1 th. 1[3], $V\times V\cap\varLambda_j$ has a unique proper component W_j^{n+1} containing $\varDelta_V$, W_j has the multiplicity 1 in this intersection and $\varDelta_V$ is simple on W_j. If, therefore, the function θ_{W_j} induced by θ on W_j is not the conatant 0, $(\theta)\cdot W_j$ is defined and we have

$$v_{\varDelta_V}(\theta_{W_j})=\text{coeff. of } \varDelta_V \text{ in } (\theta)\cdot W_j\geqq 1.$$

Proposition 1. *Let $H_i'(X)$ $(i=1,\ldots\ldots,n)$ be another uniformizing set of linear forms of V at P and W_j' be defined from $H_i'(X)$ as W_j were from $H_i(X)$. If for some j $(1\leqq j\leqq n)$, θ_{W_j} is not the constant* 0 *and $v_{\varDelta_V}(\theta_{W_j})=1$, then the same is true for some $\theta_{W_l'}$ $(1\leqq l\leqq n)$.*

Proof. Let $P=(x)$, and $Q=(x')$ be a generic point of V over $k(x)$, then $P\times Q$ is a generic point of $V\times V$ over k. As θ is in the specialization ring of $\varDelta_V$ in $k(x, x')$,

$$\theta(x, x')=\frac{f(x, x')}{g(x, x')}$$

where $f(X, X')$, $g(X, X')\in k[X, X']$ and $g(x, x)\neq 0$.
Since we are concerned with the components containing $\varDelta_V$, it does not matter whether we consider the function θ or f. If we consider the function F on $S^N\times S^N$ defined by $F(\bar{x}, \bar{x}')=f(\bar{x}, \bar{x}')$ where $(\bar{x}),(\bar{x}')$ are in-

dependent generic points of S^N over k, F induces f on $V\times V$ and $F_{W_j}=f_{W_j}$ on W_j. Therefore $(F)\cdot W_j$ is defined and by F-VIII$_2$ th.4 we have

$$\text{coeff. of } \Delta_V \text{ in } (F)\cdot W_j=v_{\Delta V}(F_{W_j})=v_{\Delta V}(f_{W_j})=1.$$

This means that $(F)_0$ has a unique component containing Δ_V, and this component contains Δ_V as a simple subvariety and is transversal to W_j along Δ_V. Therefore if $\Delta_x F_k(X)=\sum_\mu \frac{\partial f}{\partial X_\mu}(X_\mu-x_\mu)=0$ $(k=1,\ldots\ldots,N-n)$ ($F_k(x)$ being in the ideal defining V in S^N) are the set of equations of the tangent linear variety of V at P, the linear forms

$$\Delta_{x,x}f(X,X')=\sum_\mu \frac{\partial f}{\partial X_\mu}(X_\mu-x_\mu)+\sum\frac{\partial f}{\partial X'_\mu}(X'_\nu-x_\nu),\ \Delta_x F_k(X),$$

$$\Delta_x F_k(X')\quad (k=1,\ldots\ldots,N-n) \text{ and } H_i(X-X')\quad (i=1,\ldots,\hat{j},\ldots,n)$$

are linearly independent. (Here $\frac{\partial f}{\partial X}$, $\frac{\partial f}{\partial X'}$ are taken at $X=x$, $X'=x$.) But as $H_i(X-X')$ are linear combinations of $\Delta_x F_k(X)$, $\Delta_x F_k(X')$ and $H'_i(X-X')$ $(i=1,\ldots,n)$, for a suitable l, $\Delta_{x,x}f(X,X')$, $\Delta_x F_k(X')$, $\Delta_x F_k(X')$, $H'_i(x-X')$ $(i\neq l)$ are linearly independent. From this we can arrive at the assertion of the proposition by reasoning in the inverse direction.

From Prop. 1 we see that

$\mathfrak{D}_2(V)=\{\theta|\in\mathfrak{D}_1(V),\ v_{\Delta V}(\theta_{W_j})\geqq 2$ whenever θ_{W_j} is not the constant $0.\}$

is a submodule of $\mathfrak{D}_1(V)$ defined independently of the choice of $H_i(x)$.

Next we prove that $\mathfrak{D}_1$, $\mathfrak{D}_2$ are birationally invariant. Let V^n and U^n be two varieties respectively in S^N and S^M, T^n be a birational correspendence between V and U. Then the transform T' of $T\times T$ by the transformation of the product $S^N\times S^M\times S^N\times S^M$ which interchanges the second and the third factors is a birational correspondence between $V\times V$ and $U\times U$, and it is biregular along Δ_V.

Proposition 2. *Let θ be a function on $V\times V$ and k a field of definition for V, U, T and θ. Let $P\times Q$ and $R\times S$ be corresponding generic points of $V\times V$ and $U\times U$ by T' over k. Then the formula $\theta'(R\times S)=\theta(P\times Q)$ defines a function on $U\times U$ and if $\theta\in\mathfrak{D}_1(V)$ or $\theta\in\mathfrak{D}_2(V)$, we have respectiqely $\theta'\in\mathfrak{D}_1(U)$ or $\theta'\in\mathfrak{D}_2(U)$.*

Proof. Only the assertion about $\mathfrak{D}_2$ is not evident. To prove this we assume $\theta'\notin\mathfrak{D}_2(U)$. Then for some j, we have $v_{\Delta_V}(\theta'_{Y_j})=1$ where Y_j are constructed for U as W_j were for V before. As T' is biregular along

Δ_U, there is a subvariety Y of $V\times V$ corresponding to Y_j, and we have

$$1=v_{\Delta_V}(\theta'_{Y_j})=\text{coeff. of } \Delta_U \text{ in } (\theta')\cdot Y_j=\text{coeff. of } \Delta_V \text{ in } (\theta)\cdot Y.$$

This means that that there is a unique component of (θ) containing Δ_V and it is transversal to Y along Δ_V.

Now we put $P=(x)$, $Q=(x')$, $\theta(x, x')=\dfrac{f(x, x')}{g(x, x')}$ as in the proof of Prop. 1, and find that the linear forms

$$\Delta_{x,x}f(X, X'),\ \Delta_x F_k(X),\ \Delta_x F_k(X') \text{ and } \Delta_{x,x}\Phi_i\ (X, X')$$
$$(i=1,\dots, n-1\ ;\ k=1,\dots, N-n)$$

are linearly independent, where $\Phi_i(X, X')$ belong to the ideal defining Y in $S^N\times S^N$ and $\Delta_{x,x}\Phi_i\ (X, X')$ form, together with $\Delta_x F_k(X)$ and $\Delta_x F_k(X')$, the equations of tangent linear variety of Y at $P\times P$. But Y contains Δ_V so we have $\Phi_i(x, x)=0$ and therefore $\Phi_i(X, X)=\sum h_l(X)G_l(X)$ ($h_l(X)\in k[X]$ and $G_l(X)$ belong to the ideal defining V), and hence

$$\frac{\partial\Phi}{\partial x_\mu}(x, x)+\frac{\partial\Phi}{\partial x'_\mu}(x, x)=\sum h_l(x)\frac{\partial G_l}{\partial x_\mu}(x).$$

This shows that we can choose Φ_i so that $\Delta_{x,x}\Phi_i(X, X')$ have the form $H_i(X-X')$. If we take these $H_i(X)$ $(i=1,\dots, n-1)$ and a suitable $H_n(X)$ for a set of linear forms to define W_j, we have $v_{\Delta V}(\theta_{Wj})=1$, that is a contradiction.

In the above proof, we saw that the tangent linear variety of Y^{n+1} at $P\times P$ can be defined by the system of equations

$$\Delta_x F_k(X)=0,\ \Delta_x F_k(X')=0,\ H_i(X-X')=0\ \ (i=1,\dots,n-1\ ;\ k=1,\dots, N-n).$$

Evidently this remark holds true for any subvariety Z^{n+r} of $V\times V$ containing Δ_V, $(1\leqq r\leqq n-1)$, that is to say, the tangent linear variety of Z at $P\times P$ can be defined by the system of equations of the form

$$\Delta_x F_k(X)=0,\quad \Delta_x F_k(X')=0,\ H_i(X-X')=0.\quad (i=1,\dots n-r,\ k=1,\dots,N-n).$$

Applying this to the case of $r=n-1$, we have the following

Proposition 3. *If $Z_1^{2n-1},\dots\dots, Z_n^{2n-1}$, are n subvarieties of $V^n\times V^n$ and Δ_V is a proper component of $Z_1\cap\dots\dots\cap Z_n$ with multiplicity* 1, *then we can find a uniformizing set of linear forms $H_1(X),\dots\dots,H_n(X)$ of V at a generic point P of V over a field of definition k of V_1, $Z_1,\dots\dots,Z_n$, such that the tangent linear varicty of Z_j at $P\times P$ are defined by the system of equations*

$$\Delta_x F_k(X)=0,\ \Delta_x F_k(X')=0 \quad \text{and} \quad H_j(X-X')=0 \quad (k=1,\dots\dots,N-n),$$

where $F_k(X)$ are in the ideal defining V in S^N.

If an abstract variety V is given, we can define $\mathfrak{D}_1(V)$, $\mathfrak{D}_2(V)$ independently of its representative, by Prop. 2. $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are Ω-modules and $\mathfrak{D}_2$ is a submodule of $\mathfrak{D}_1$, so we can construct the factor module $\mathfrak{D}(V)=\mathfrak{D}_1(V)/\mathfrak{D}_2(V)$. We can, as in Weil's book[4], define in $\mathfrak{D}(V)$ multiplication by the element of $\Omega(V)$, and make $\mathfrak{D}(V)$ a $\Omega(V)$-module. This module is called the module of differential forms of the first degree on V, and its element ω is called the differential form of the first degree on V. If θ is in the class ω, we say that θ defines ω and write $\omega=\{\theta\}$, if one of the functions of class ω is defined over K, we say ω is defined over K.

Let φ be a function defined on V over k, then we define a differential form $d\varphi$ on V defined over k, by the formula

$$d\varphi=\{\varphi_\partial\},\quad \varphi_\partial(P\times Q)=\varphi(Q)-\varphi(P),$$

where P and Q are independent generic points of V over k. This differential form is called the differential of the function φ.

Proposition 4. *Let V be a variety in S^N, k a field of definition for V, and $P=(x)$ a generic point of V over k. Let $H_i(X)$ $(i=1,\dots,n)$ be a uniformizing set of liner forms with coefficients in k at P. Consider the functions u_i defined by $u_i(x)=H_i(x)$.* Then

(1) *du_i $(i=1,\dots,n)$ are linearly independent over $\Omega(V)$,*

(2) *any $\omega\in\mathfrak{D}$ is a linear combination of du_i with coefficients in $\Omega(V)$, and especially with coefficients in $\Omega_k(V)$ if ω is defined over k.*

Proof. (1) Notations being as above, assume that

$$\theta(x,\ x')=\varphi_1(x)H_1(x-x')+\cdots+\varphi_n(x)H_n(x-x')\ \in\mathfrak{D}_2$$

with φ_i not all 0. We can here assume $\varphi_i(x)$ to be polynomials. If $\varphi_1(x)\neq 0$, we have $\theta_{W1}=\varphi_1(x)\cdot H_1(x-x')\neq 0$, and considering the function F defined in $S^N\times S^N$ by $F(\bar{x},\bar{x}')=\varphi_1(\bar{x})H_1(\bar{x}-\bar{x}')$, we have $v_{\Delta V}(\theta_{W1})=$coeff. of Δ_V in $(F)\cdot W_1=1$, which is a contradiction.

(2) Let $\omega=\{\theta\}$, we can here also assume that $\theta(X,X')$ is a polynomial in X,X'. As $\theta(x,\ x)=0$ we have as in the above proof of Prop. 1 and Prop. 2,

$$\Delta_{x,x}\theta(X,X')=\sum a_iH_i(X-X')+\sum b_k\Delta_xF_k(X)+\sum c_k\Delta_xF_k(X')$$

$$(a_i,\ b_k,\ c_k\in\Omega(V)),$$

and therefore if we put $\psi(X,\ X')=\theta(X,X')-\sum a_i H_i(X-X')$, $\Delta_{x,x'}\psi(X,\ X')$

is linearly dependent on $\Delta_x F_k(X)$, $\Delta_x F_k(X')$. This implies that W_j and the component of $(\psi)_0$ containing Δ_V are not transversal. Therefore $\psi \in \mathfrak{D}_2$, and we have $\omega = \sum a_i\, du_i$.

The last assertion is now immediate.

Proposition 5. *Notations being as above, the fuction induced on* Δ_V *by* $\left[\dfrac{\theta(x, x')}{H_j(x-x')}\right]_{W_j}$ *shall be denoted by* $\chi_j(x, x)$ (*this is not* 0 *when* $v_{\Delta V}(\theta_{W_j}) = 1$, *and* 0 *otherwise*). *Then the functions* a_j *such that* $\omega = \sum_j a_j du_j$ *are given by* $a_j(x) = \chi_j(x, x)$.

Proof. Put $\psi(x, x') = \theta(x, x') - \sum \chi_j(x, x)\, H_j(x-x')$, then it can easily be seen that $\psi_{W_j} = 0$ or $v_{\Delta V}(\psi_{W_j}) > 1$, and therefore $\psi \in \mathfrak{D}_2$. This implies $\omega = \sum_j \chi_j(x, x)\, du_j$.

By definition, $d(\varphi + \psi) = d\varphi + d\psi$, and $d(\varphi\psi) = \psi d\varphi + \varphi d\psi$, for two functions φ, ψ on V. So, if we put $d\varphi = \sum z_i\, du_i$ we can define n derivations of $\Omega(V)$ over Ω by $D_i \varphi = z_i$[5]. As $D_i u_j = \delta_{ij}$, D_i are linearly independent over $\Omega(V)$, and therefore form a $\Omega(V)$-base of the module $\Delta(V)$ of derivations of $\Omega(V)$ over Ω. Therefore we can consider a differential form of the first degree ω as a linear mapping of $\Delta(V)$ into $\Omega(V)$, by the relation: $\omega * D = \sum \varphi_i\, Du_i$ if $\omega = \sum \varphi_i\, du_i$. And it is easy to see that $\mathfrak{D}(V)$ and $\Delta(V)$ are dual modules with respect to this product. By the last assertion of Prop. 3, this duality holds when we restrict the field of definition of differentials, derivations and functions to any field K which is at the same time a field of definition for V[5)6)]. Thus our definition of differential forms of the first degree agree with that given by Weil in F-IX$_2$. We can therefore, speak of whether a differential form is finite or not at a point of V, and also speak of differential forms of higher degrees. Cf. also Koizumi's paper[7].

Concerning the Prop. 6 in Koizumi's paper, we have the following

Proposition 6. *Let* V^n *be a variety and* Z^r *be its simple subvariety. If a differential form of the first degree* ω *on* V *is finite along* Z, *there is a function* θ *in* $\mathfrak{D}_1(V)$ *such that* $\{\theta\} = \omega$ *and* $\theta_{Z\times Z}$ *defines the differential form which is induced on* Z *by* ω (*in Koizumi's sense*). *Moreover if* $\{\varphi\} = 0$ *on* V *and if* φ *is defined along* Δ_Z, *then we have* $\varphi_{Z\times Z} \in \mathfrak{D}_2(Z)$.

Proof. We can assume that V is in an affine space S^N, Let k be a field of definition for V, Z and ω, and $P = (x)$ and $P' = (x')$ be two independent generic points of V over k, Q a generic point of Z over k. If $\omega = \sum_i z_i\, du_i$, where u_i are uniformizing parameters of V at Q, then our first assertion is satisfied by

$$\theta(x,x')=-\sum z_i(x)\{u_i(x)-u_i(x')\}.$$

Next we suppose $\{\varphi\}=0$, we choose a uniformizing set of linear forms $H_i(x)$, $(i=1,\ldots\ldots,r)$ of Z at Q. As Q is simple on V, we can choose a uniformizing set of linear forms $H_1(x),\ldots\ldots,H_r(x)$, $H_{r+1}(x),\ldots\ldots,H_n(x)$ of V at Q, in such a way that $H_1(x),\ldots\ldots,H_r(x)$ appear among them, and $H_{r+1}(x),\ldots\ldots H_n(x)$ appear among the equations for tangent linear variety of Z at Q. As above we define the varieties $W_j(j=1,\ldots\ldots,n)$ for $V\times V$ and $W_j'(j=1,\ldots\ldots,r)$ for $Z\times Z$.

We write φ' instead of $\varphi_{Z\times Z}$. We have for some j $(1\leqq j\leqq r)$

$$1=\text{coeff. of } \Delta_Z \text{ in } \{(\varphi')\cdot W_j'\}_{Z\times Z}.$$

But the right hand side is equal to the coefficient of Δ_Z in $\{(\overline{\varphi})\cdot W_j\}_{Z\times V}$ where $\overline{\varphi}$ is the function on $Z\times V$ induced by φ. Consider W_j and $Z\times V$. These are transversal to each other at Q on $V\times V$, and therefore $W_j\cap(Z\times V)$ has a unique component W_j'' containing Δ_Z, and W_j'' has Δ_Z as a simple subvariety. Moreover W_j'' and W_j' have the same tangent linear variety at Q in $S^N\times S^N$. If, therefore, we denote by A the component of $(\overline{\varphi})_0$ containing Δ_Z, we have $i[A\cdot W_j'', \Delta_Z\,;\ Z\times V]=1$, and by F–VI$_3$ th. 9, $i[A\cdot W_j, \Delta_Z\,;\ V\times V]=1$. From this we deduce by F-VI$_2$ th. 5 $v_{\Delta V}(\varphi_{Wj})=1$, which is a contradiction.

If ω and θ are differential forms on V, we can define the product $\omega\cdot\theta$ and $d\omega$ in the usual way[8]. These induce $\omega'\cdot\theta'$ and $d\omega'$ respectively on a simple subvariety Z of V, where ω', θ' are forms induced on Z by ω, θ respectively.

Let Λ be a function defined on V with values in U[9] such that its values exhaust U. If k is a field of definition for V, U and Λ, and if x is a generic point of V over k and $y=\Lambda\cdot x$, we have $k(y)\subset k(x)$. Therefore a differential form $\omega'=\sum z_{i_1\ldots\ldots i_p}\,du_{i_1}\ldots\ldots du_{i_p}$ on U can be considered as one on V, by thinking $z_{i_1\ldots\ldots i_p}$ as functions on V and du_i as differentials of functions on V. We denote, following Chevalley[8], the differential form thus obtained by $\omega=\delta\Lambda\cdot\omega'$.

§ 2. Invariant Differential Forms on a Group Variety.

Let G^n be a group variety and a a point of G. The left translation T_a is an everywhere biregular birational correspondence between G and G itself, defined over a field of definition K for G over which a is rational.[10]

If ω is a differential form on G, we write ω^a instead of $\delta T_a \cdot \omega$.

If $\omega^a = \omega$ for any $a \in G$, we say that ω is (left) invariant. It is clear that invariant differential forms of given degree form a Ω-module.

In the case of a form $\omega = \{\theta\}$ of the first degree, ω^a is given by $\omega^a = \{\theta^a\}$ where θ^a is defined by $\theta^a(x,x') = \theta(ax, ax')$[11].

If therefore, $\theta^a = \theta$ for any $a \in G$, $\omega = \{\theta\}$ is invariant.

Theorem 1. *If G^n is a group variety, then there are n invariant differential forms of the first degree on G, linearly independent over Ω. Moreover they form a $\Omega(G)$-base of $\mathfrak{D}(G)$.*

Proof. Let k be a field of definition for G, and x, y be independent generic points of G over k and $z = x^{-1}y$. Let G_0 be a representative of G in S^N in which the neutral element e has a representative with coordinates (0) and is a simple point, and let z_ν ($\nu = 1, \ldots\ldots, N$) be the coordinates of the representative of z in G_0. Then the functions φ_ν on $G \times G$ defined by $\varphi_\nu(x, y) = z_\nu$ give invariant differential forms on G. As z is generic on G over k, we can find among z_ν a set of uniformizing parameters at e. We can assume that $z_1, \ldots\ldots, z_n$ are such. Then $z_\nu \to 0$ ($\nu = 1, \ldots\ldots, N$) is a proper specialization of (z_ν) over $z_1, \ldots\ldots, z_n \to 0$. On the other hand if we denote by Λ the locus of $x \times y \times z$ over k in $G \times G \times G$, we have $\Lambda \cdot (G \times G \times e) = \Delta_G \times e$. These imply that Δ_G is a proper component of $(\varphi_1)_0 \cap \ldots\ldots \cap (\varphi_n)_0$. We shall prove that Δ_G is contained in this intersection with multiplicity 1.

To do this, let $U = G \times G \times G$, and $V = G \times G \times D^n$, where D^n is a product of n projective lines. We consider the loci Λ', Δ' of $x \times y \times x^{-1}y \times x \times y \times (z)$ and $x \times x \times e \times x \times x \times (0)$ respectively in $U \times V$ over k, and the locus Λ'' of $x \times y \times (z_1, \ldots\ldots, z_n)$ in V over k. We can apply F-VII$_5$ th. 8 to calculate $i[\Lambda'' \cdot (G \times G \times (0), \Delta_G \times (0)\,;\, \Lambda]$. The formula in that theorem becomes

$$\begin{aligned}&[\Lambda' : \Lambda''] \cdot i[\Lambda'' \cdot (G \times G \times (0)), \Delta_G \times (0)\,;\, V] \\ &\qquad = [\Delta' : \Delta_G \times (0)] \cdot i[\Lambda' \cdot (G \times G \times G \times G \times G \times (0), \Delta'\,;\, U \times U].\end{aligned}$$

On the right hand side of this equality, the first factor is clearly 1, the second is calculated by F-VII$_3$ theor. 9 and gives the value 1. Therefore $i[\Lambda'' \cdot (G \times G \times (0)), \Delta_G \times (0)\,;\, V] = 1$. From this we deduce, by successive use of F-VII$_6$ theor. 16, that Δ_G is contained in $(\varphi_1)_0 \cap \ldots\ldots \cap (\varphi_n)_0$ with multiplicity 1.

Now we consider the representative $G_0 \times G_0$ of $G \times G$ and its point $x \times x$. (For simplicity's sake we use the same symbol for a point on G and its representative in G_0, which will cause no confusion.) We take a uniformizing set of linear forms $H_1(x), \ldots\ldots, H_n(x)$ of G_0 at x as in Prop. 3, and put $u_i(x) = H_i(x)$ and define W_j as in Prop. 1. It is easy to see that $v_{\Delta G}$ $([\varphi_i]_{W_j}) = 1$ only for $i=j$. It follows from Prop. 5. that $\{\varphi_i\} = a_i \ du_i$, which shows that $\{\varphi_i\}$ form a $\Omega(G)$-base of $\mathfrak{D}(G)$.

Corollary. *n invariant differential form in Th.* 1 *can be found among* $\{\varphi_\nu\}$. *where* $\varphi_\nu(x,y)$ *is the* ν*-th coordinate of* $x^{-1}y$ *in the representative* G_0.

Let K be an (eventually abstract) field containing a field k, and let a derivation D of K over k and an automorphism σ of K over k be given. We put $Dz = z'$ for $z \in K$, then the operation D^σ defined by $D^\sigma z^\sigma = z'^\sigma$ is a derivation of K over k.

Let G be a group variety and a a point of G, and let D be a derivation of $\Omega(G)$ over Ω. As $\varphi \to \varphi^a$ is an automorphism of $\Omega(G)$ over Ω, we have as above a derivation D^a. D^a is defined by

$$D^a \psi = (D\psi^{a^{-1}})^a \text{ for } \psi \in \Omega(G).$$

If $D^a = D$ for any $a \in G$, we say that D is an invariant derivation. The totality of invariant derivations of G forms a Ω-module $\mathfrak{L}$.

$$\text{If } \omega = \sum \varphi_i \ du_i, \text{ we have } \omega^a = \sum \varphi_i^a (du_i)^a = \sum \varphi_i^a \ d(u_i^a) \text{ and}$$
$$\omega^a * D^a = \sum \varphi_i^a \ D^a u_i^a = (\sum \varphi_i D u_i)^a = (\omega * D)^a.$$

Therefore if ω and D are both invariant, we have

$$\omega * D = constant.$$

Theorem 2. *The Ω-module $\mathfrak{M}$ of invariant differential forms of the first degree on a group variety G and the Ω-module $\mathfrak{L}$ of invariant derivations of $\Omega(G)$ are dual with each other with respect to the product $\omega * D$. Their common rank over Ω is equal to the dimension n of G.*

Proof. That $\omega * D$ is bilinear is evident.

By Th. 1 there are n forms $\omega_1, \ldots\ldots, \omega_n$ in $\mathfrak{M}$ which form a $\Omega(G)$-base of $\mathfrak{D}(G)$. Therefore $\omega * D = 0$ for all $\omega \in \mathfrak{M}$ implies $D = 0$. If we consider the derivations D_j such that $\omega_i * D_j = \delta_{ij}$, we have $\omega_i * D_j^a = \omega_i^a * D_j^a = (\omega_i * D_j)^a = \delta_{ij}$, so that $D_j^a = D_j$; D_j are invariant. An arbitrary $\omega \in \mathfrak{M}$ is expressed in the form $\omega = \sum \varphi_i \ \omega_i$ where $\varphi_i \in \Omega(G)$, then $\omega * D_j = \varphi_j$ and φ_j must be a constant. This shows that $\mathfrak{M}$ has rank n over Ω, and that $\omega * D=0$ for all $D \in \mathfrak{L}$ implies $\omega=0$.

Let X^{n-1} be a subvariety of G^n, and let $\omega=\sum z_{i1\cdots\cdots ip}\, du_{i1}\cdots\cdots du_{ip}$ be an expression of a differential form ω by the uniformizing parameters along X, van der Waerden defined the order $v_X(\omega)$ of ω on X as $v_X(\omega)=\min v_X\ (z_{i1\cdots\cdots ip})$. This is independent of the choice of the uniformizing parameters.

If ω is an invariant differential form on G and $v_X(\omega)\neq 0$ for a X^{n-1}, then $v_{Xa}(\omega)\neq 0$ for any $a \in G$ (X_a is the transform of X by T_a), which is impossible. This shows on one hand:

Proposition 7. *Invariant differential forms are everywhere finite.*

On the other hand, let D_i be a base of $\mathfrak{L}$ defined over k, and ω_i a base of $\mathfrak{M}$ dual to D_i, and let a be a point of G. Then ω_i are expressed by uniformizing parameters u_i at a as $\omega_i=\sum z_{ij}\, du_j$, and therefore

$$\delta_{ij}=\omega_i * D_j=\sum_k z_{ik}\, D_j u_k.$$

As $\omega_1\cdots\cdots\omega_n=det\ |z_{ij}|\cdot du_1\cdots\cdots du_n$ is an invariant differential form on G, we have $det\ |z_{ij}|\neq 0$ at a, therefore $D_k u_j$ are defined and finite at a. Tnus we have

Paoposition 8. *An invariant derivation D is, so to speak, everywhere finite; that is to say, by D, a function on G defined and finite at a point of G goes over to a function of the same kind*[12].

§ 3. Lie Ring of a Group Variety.

In the following the characteristic of the universal domain is assumed not to be equal to 2.

In $\mathfrak{L}$ we can define a commutator product which makes $\mathfrak{L}$ a Lie ring. In fact, if $D_1, D_2 \in \mathfrak{L}$, $[D_1 D_2]=D_1 D_2-D_2 D_1$ is evidently a derivation of $\Omega(G)$ over Ω and it is also invariant, so that $[D_1, D_2] \in \mathfrak{L}$. Let c_{ijk} be the structure constants of Lie ring $\mathfrak{L}$: $[D_i D_j]=\sum c_{ijk}\, D_k$, then it is shown as usual[13] that

$$d\omega_k=-\frac{1}{2}\sum_{i,j} c_{ijk}\ \omega_i\cdot\omega_j$$

Now let G^n be a gioup variety and H^r its group subvatriety, both defined over k. As H is simple on G and $\omega \in \mathfrak{M}_G$ is everywhere finite, ω induces a differential form ω' on H, which is evidently invariant.

Let x, x' be two independent generic points of G over k and y, y' such of H, and let $\varphi_\nu(x, x')$ be, as in Th. 1 ν-th coordinate of $x^{-1}x'$ in some representative G_0 of G in which H has a representative H_0. Then $\varphi_\nu(y, y')$ are defined and are coordinates of $y^{-1}y'$. Therefore, by Th. 1 Coroll. r of φ's define r differential forms which form a Ω-base of $\mathfrak{M}_H$. This means that whole $\mathfrak{M}_H$ is induced by $\mathfrak{M}_G$ on H, and therefore we can choose a base $\omega_1, \ldots\ldots, \omega_r, \omega_{r+1}, \ldots\ldots, \omega_n$ of $\mathfrak{M}_G$ so that $\omega_1', \ldots\ldots, \omega_r'$ form a base of $\mathfrak{M}_H$ and $\omega'_{r+1} = \ldots\ldots = \omega_n' = 0$. Then the formula

$$d\omega_k = -\frac{1}{2}\sum_{i,j} c_{ijk}\ \omega_i \cdot \omega_j$$

implies

$$d\omega_k' = -\frac{1}{2}\sum_{i,j} c_{ijk}\ \omega_i' \cdot \omega_j'.$$

and therefore $c_{ijk}=0$ for $i, j \leqq r$, $k > r$.

If we consider the base $D_1, \ldots\ldots, D_n$ of $\mathfrak{L}_G$ dual to $\omega_1, \ldots\ldots, \omega_n$ we see $D_1, \ldots\ldots, D_r$ form a base of a Lie subring of $\mathfrak{L}_G$ which is isomorphic to $\mathfrak{L}_H$.

The relation between $\mathfrak{L}_G$ and $\mathfrak{L}_H$ can further be explained as follows: We can suppose that D_i and ω_i are defined over k. Let us denote by $\sum$ the specialization ring of y in $k(x)$, and by $\mathfrak{p}$ the ideal of $\sum$ composed of elements which have specialization 0 over $x \to y$. Let $v \in \mathfrak{p}$, we have $dv = \sum_{i=1}^{n} (D_i v) \cdot \omega_i$ and by Prop. 8[12] $D_i v \in \sum$. But dv induces 0 on H, so that $D_i v$ $(i=1, \ldots\ldots, r)$ have specialization 0 over $x \to y$, therefore $D_i\ v \in \mathfrak{p}$ for $i=1, \ldots\ldots, r$.

Since $D_i \sum \subset \sum$ and $D_i \mathfrak{p} \subset \mathfrak{p}$ for $1 \leqq i \leqq r$, D_i define r derivations of $\sum/\mathfrak{p} \cong k(y)$ over k, which form precisely the dual base of $\mathfrak{L}_H$ to $\omega_1', \ldots\ldots \omega_r'$.

We resume the result in the following

Theorem 3. *If H is a group subvariety of a group variety G, then $\mathfrak{L}_H$ is isomorphic with a Lie subring of $\mathfrak{L}_G$, and this isomorphism is given by that the elements of the subring in question can be considered, in a natural way, as derivations of $\Omega(H)$ over Ω.*

Next we consider homomorphisms of group varieties.

Let Λ be a homomorphism of a group variety G^n onto a group variety H^r. Let a be a point of G, and k a field of definition for G, H and Λ, over which a is rational. Let x be a generic point of G over k, then $y=$

$\Lambda\cdot x$ is a generic point of H over k and $b=\Lambda\cdot a$ is rational over k. It is easily seen that if ω' is a differential form on H and $\omega=\delta\Lambda\cdot\omega'$, then $\omega^a=\delta\Lambda\cdot\omega'^b$, so invariant ω' goes to an invariant ω by $\delta\Lambda$.

We have $k(x)\supset k(y)$, and if the order of inseparability $[k(x):k(y)]_i$ is equal to 1, $k(x)$ is separably generated over $k(y)$, and if ω' is not 0 on H, $\omega=\delta\Lambda\cdot\omega'\neq 0$ on G. Such a homorphism shall, provisionally, be called a separable homomorphism. If Λ is such, and if $\omega',\dots\dots,\omega_r'$ is a base of $\mathfrak{M}_H$, then $\omega_1=\delta\Lambda\cdot\omega'_1,\dots\dots,\omega_r=\delta\Lambda\cdot\omega_r'$ are independent on G, and we can find a bese $\omega_1,\dots\dots\omega_r,\ \omega_{r+1},\dots\dots,\omega_n$ of $\mathfrak{M}_G$ such that $\omega_i=\delta\Lambda\cdot\omega_i'$ $(i=1,\dots,r)$.

Now consider the dual base $D_1,\dots\dots,D_n$ of $\omega_1,\dots\dots\omega_n$. If $v\in k(y)$, dv (considered as a differential form on H) is a linear combination of $\omega_1',\dots\dots,\omega_r'$ with coefficients in $k(y)$, so that when considered as a differential form on G, it is a linear combination of $\omega_1,\dots\dots,\omega_r$ with coefficients in $k(y)$. Therefore $D_i\,k(y)\subset k(y)$ $1\leqq i\leqq r$, $D_ik(y)=0$ $r<i$, and $D_1,\dots\dots,D_r$ define r derivations $D_1',\dots\dots,D_r'$ of $k(y)$ over k. $D_1',\dots\dots,D_r'$ are invariant and form a base of $\mathfrak{L}_H$. We have thus defined a linear mapping $\lambda:\lambda(D_i)=D_i'$ $(i\leqq r)$, $\lambda(D_i)=0$ $(r<i)$ of $\mathfrak{L}_G$ onto $\mathfrak{L}_H$. λ is a homomorphism of $\mathfrak{L}_G$ onto $\mathfrak{L}_H$, because $\lambda(D_i)$ is the contraction of D_i to $k(y)$.

Now let C be the component containing e of the kernel of Λ, we can see by Prop. 6 that $\omega_1,\dots\dots,\omega_r$ induce 0 on C. But C is a group subvariety of G of dimension $n-r$[14]. Therefore the Lie subring of $\mathfrak{L}_G$ generated by $D_{r+1},\dots\dots,D_n$ (which is nothing but the kernel of λ) is the Lie subring corresponding to C.

Theorem 4. *If Λ is a separable homomorphism of a group variety G onto a group variety H, then there exists a homomorphism λ of $\mathfrak{L}_G$ onto $\mathfrak{L}_H$, and the kernel of λ is the Lie ring of the component of e of the kernel of Λ.*

In conclusion I express my heartful thanks to Prof. Akizuki for his kind interest taken in this work.

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Notes

1) Cf. A. Weil: Variétés Abeliennes et Courbes Algébriques, in the following this shall be cited as A.

2) We denote by $\Omega(V)$ the abstract field of all functions defined on V, (each function having a field of definition in the sense of F-I_1.[3]) The subfield of $\Omega(V)$ composed of functions equal to constants is denoted by Ω, this is isomorphic to the universal domain. We denote by

$\Omega_K(V)$ the field of functions on V defined over K, this is isomorphic to $K(P)$, where P is a generic point of V over K, and at times they are identified.

3) F-II_1. th. 1 means A. Weil: Foundaions of Algebraic Geometry, Chap. VI § 1. th. 1. This abbreviation is used throughout this note.

4) A. Weil: Courbes algébriques et Variétés qui s'en déduisent, I[re] partie II.

5) As $\Omega(V)$ is the compositum of $\Omega_K(V)$ and Ω, a derivation of $\Omega_K(V)$ over K is extended to that of $\Omega(V)$ over Ω. Conversely, as $\Omega(V)$ is finitely generated over Ω, a derivation D of $\Omega(V)$ over Ω can be considered as an extension of some derivation of $\Omega_K(V)$ over K. Here K is a field suitably chosen, in the sense of F-I_1. In this case we say, D is defined over K.

6) Of course in this case both are considered as $\Omega_K(V)$-modules.

7) S. Koizumi: Journ. Math. Soc. of Japan, vol. 1.

8) Cf. A. n° 1.

10) As in A. K is said to be a field of definition for a group variety G if K is a field of definition both for variety G, and for the law of composition in G.

11) Here x and x' are independent generic points of G over K.

12) If we identify $\Omega_K(G)$ and $K(x)$ as is stated in note (2), we can state this Prop. as follows: If D is an invariant derivation on G defined over K, D leaves the specialization ring of a in $K(x)$ invariant.

13) Cf. C. Chevalley: Loc. cit.

14) This can be seen by a slight modification of A. th. 11, in the case of onto homomorphism.