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Theorems of Bertini on Linear Systems

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As the fundamental theorems of the classical algebraic geometry we have these of Bertini:

I. *The general section U_{r-1} of an algebraic variety U_r by a linear system without fixed components is irreducible, provided that the linear system is not composed of an algebraic pencil.*

II. *The general section U_{r-1} of U_r by a linear system can not have any singular points outside the singlar points of U_r and outside the base points of the linear system.*

The first proposition was proved purely algebraically first by Zariski,[1] when the basic field k of U_r is of characteristic $p=0$. Matsusaka[2] remarked that this holds even when $p>0$ under an additional condition.

Zariski[3] has also given an adequate formulation to the second proposition for the case $p>0$, as it cannot be maintained in the above formulation in this case.

In this paper we shall study how the above formulation will not be maintained when $p>0$, and will give a sufficient condition that it should be maintained. Thereby we shall give also a new proof the first proposition. Further we shall add a new elementary proof of the second proposition in the classical case.[4]

1. Let U_r be an r-dimensional irreducible algebraic variety immersed in an N-dimensional projective space S^N and defined over a field k of characteristic $p \geqq 0$. We denote by $(\xi_0, \xi_1, \dots, \xi_N)$ the homogeneous coordinates of the generic point of U/k. And we assume that the linear system on U

$$\lambda_0 f_0(\xi) + \lambda_1 f_1(\xi) + \dots + \lambda_m f_m(\xi) \tag{1}$$

has no fixed components.

1) See Zariski [1].
2) See Matsusaka [5].
3) See Zariski [2].
4) We shall use the same terminalogies in Weil's book [3].

We now consider the algebraic correspondence W in doubly projective space $S_N \times S_m$, attaching to the linear system (1) as follows:

$$\eta_0 f_i(\xi) - \eta_i f_0(\xi) = 0 \quad (1 \leqq i \leqq m) \tag{2}$$

Let P, Q be the pair of corresponding points on U and W, and P not to belong to the base variety of the linear system. As W is rational over U, if P is simple on U, then Q is simple on W, and vice versa.[5)]

The geometrical projection V of W in the second factor S^m of $S^N \times S^m$ is an algebraic variety defined over k, for $(\eta_i) = (f_i(\xi))$ is regular with respect to k, as (ξ) is so.

Let now C_λ be the generic element of the linear system (1) on U, then it is readily be seen from our assumption the linear system to be without fixed components that

$$C_\lambda = \mathrm{proj}_U[(S_N \times H) \cdot W],$$

where H is the generic hyperplane in S_m

$$\lambda_0 Y_0 + \lambda_1 Y_1 + \dots + \lambda_m Y_m = 0.$$

2. Let the inhomogeneous coordinates of the generic point P of U be $(x_1, \dots, x_N)$ such that

$$x_i = \xi_i / \xi_0 \qquad 1 \leqq i \leqq N,$$

then we can assume, since P does not belong to the base variety,

$$f_0(x) = f_0(1, x_1, \dots, x_N) \neq 0.$$

If we now define

$$y_i = f_i(x)/f_0(x) \qquad (1 \leqq i \leqq m), \tag{3}$$

then $(x_1, \dots, x_N, y_1, \dots, y_m)$ is the inhomogeneous coordinates of the generic point of W in the affine space L^{N+m}.

Let us assume that the dimension of V/k, *i.e.* of the field $k(y) = k(y_1, \dots, y_m)$ over k is not less than 2. Since $k(y)$ is regular over k,

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5) See Weil's book [3]. Theor. 15, p. 108.

we can conclude, by the fundamental lemma of Zariski,[6] that if it is not $k(y)\subset\{k(x)\}^p$, $K(\sum_{i=1}^m\lambda_i y_i)$ is algebraically closed in $K(x,y)=K(x)$, where $K=k(\lambda_2,\ldots,\lambda_m)$ and the λ_s are independent indeterminates. Therefore

$$(S^N\times H)\cdot W=m\Gamma \tag{4}$$

and Γ is defined over $\overline{K(\lambda_0)}$ (algebraic closure of $K(\lambda_0)=k(\lambda_0,\lambda_1,\ldots,\lambda_m)$), namely Γ is absolutely irreducible.

3. Let $\mathfrak{P}=(F_1(\xi),\ldots,F_\mu(\xi))$ be the defining ideal of U in S^N and $P(\xi)$ a point not belonging to the base variety with $f_0(\xi)\neq 0$. Then the affine model W may be defined locally at P by

$$(\ldots,F_i(X),\ldots;\ldots,Y_j f_0(X)-f_j(X),\ldots) \tag{5}$$
$$1\leq i\leq\mu,\ 1\leq j\leq m$$

Lemma. *Let $P'(x')$ be a simple point on U not belonging to the base variety of the linear system with $f_0(x')\neq 0$, and $Q'(x',y')$ be the point on W which coresponds to the point P'. Then the hyperplane $S^N\times H'$ passing throvgh P'*

$$\sum\lambda_j(Y_i-y'_i)$$

is not transversal at Q' to W, if and only if the equations

$$\sum_{i=1}^m\lambda_i D_j(y_i)=0,\quad y_i=f_i(x)/f_0(x) \tag{6}$$

are consistent at P' for all derivations D_j of $k(x)$.

Proof. It is clear that the hyperplane $S^N\times H'$ is not transversal at Q' to W, if and only if the rank of the matrix

$$\begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_N} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial F_{N-r}}{\partial x_1} & \cdots & \frac{\partial F_{N-r}}{\partial x_N} & 0 & \cdots & 0 \end{pmatrix}$$

6) See for the cae $p=0$ Zariski's poper [1], Lem.5. Also see for the case $p>0$ Matsusaka [5], Theor. 2, 4. Also cf. Igusa [7].

$$\begin{pmatrix} y_1\frac{\partial f_0}{\partial x_1}-\frac{\partial f_1}{\partial x_1} \cdots\cdots y_1\frac{\partial f_0}{\partial x_N}-\frac{\partial f_1}{\partial x_N} & f_0 \cdots\cdots 0 \\ \cdots\cdots & \cdots\cdots \\ y_m\frac{\partial f_0}{\partial x_1}-\frac{\partial f_m}{\partial x_1} \cdots\cdots y_m\frac{\partial f_0}{\partial x_N}-\frac{\partial f_m}{\partial x_N} & 0 \cdots\cdots f_0 \\ 0 \cdots\cdots 0 & \lambda_1 \cdots\cdots \lambda_m \end{pmatrix}$$

at Q' is not $N+m+1-r$ for any choice of $F_1,\dots, F_{N-r}$ among polynomials belonging to the ideal $\mathfrak{P}$. Since $f_0(x')\neq 0$, this is equivalent to the condition, as we can see by easy calculation, that the rank of the matrix

$$A=\begin{pmatrix} \frac{\partial F_1}{\partial x_1} \cdots\cdots \frac{\partial F_1}{\partial x_N} \\ \cdots\cdots \\ \frac{\partial F_{N-r}}{\partial x_1} \cdots\cdots \frac{\partial F_{N-r}}{\partial x_1} \\ \sum\lambda_i\frac{\partial}{\partial x_1}\left(\frac{f_i}{f_0}\right) \cdots\cdots \sum\lambda_i\frac{\partial}{\partial x_N}\left(\frac{f_i}{f_0}\right) \end{pmatrix} \tag{7}$$

at P' is not $N-r+1$. While P' is smiple on U, we can choose $F_1,\dots, F_{N-r}$ such that the rank of the first $N-r$ rows of A is $N-r$. Therefore any derivation D_j of $k(x)$ over k as a solution-vector of equations

$$\sum_{l=1}^{N}\frac{\partial F_i}{\partial x_l}D_jx_l=0, \qquad i=1,\dots, N-r$$

satisfies also

$$\sum_{l=1}^{N}\left(\sum\lambda_i\frac{\partial}{\partial x_l}\left(\frac{f_i}{f_0}\right)\right)D_jx_l=0$$

at P', namely at P' for any D_j which is finite at P'

$$\sum\lambda_iD_j(f_i/f_0)=0, \quad q.e.d.$$

4. Let $P(x)$ be a generic point of U, $Q(x,y)$ the point on W which corresponds to P, and $\lambda_1,\dots,\lambda_m$ independent indeterminates with respect to $k(x)$. If we set λ_0 such that

$$-\lambda_0=\sum\lambda_i y_i,$$

then the hyperplane

$$\lambda_0+\lambda_1 Y_1+\dots+\lambda_m Y_m=0$$

passes through Q. In order the hyperplane not to be transversal to W at Q, as we see from the preceeding lemma, it must be

$$\sum\lambda_i D(y_i)=0$$

for any derivation D of $k(x)$. Whereas $D(y_i)\in k(x)$, and λ_i are linearly independent over $k(x)$, hence it must be

$$D(y_1)=\dots=D(y_m)=0$$

for any derivation D. Therefore the field must satisfy

$$k(y_1,\dots,y_m)\subset\{k(x)\}^p,$$

where p is the characteristic of k. Thus we get the

Proposition. *If there is at least one y_i which is not the p^{th} power of an element of $k(x)$, then the hyperplane in L^{N+m}*

$$\lambda_0+\lambda_1 Y_1+\dots+\lambda_m Y_m=0$$

whose coefficients λ are independent indeterminates, is transversal to W.

By this proposittion and the general theory of intersections we see the multiplicity of the intersection $(L^N\times H)\cdot W$ to be one, and thus together with the result at the end of the section 2

$$(S^N\times H)\cdot W=\Gamma, \tag{8}$$

if $\dim V\geqq 2$. And then $Q(x,y)$ is regular over $k(\lambda)=k(\lambda_0\dots\lambda_m)$, so it is $P(x)$ over $k(\lambda)$. Therefore

$$C=\mathrm{proj}_U\{(S\ \times H)\cdot W\} \tag{9}$$

is absolutely irreducible over $k(\lambda)$. Hence we get the Bertini's theorem (the first proposition in the introduction).

Theorem. *If a linear system without fixed components is not composed of an algebraic pencil, the general section of the linear system is absolutely irreducible, provided that it is not $k(y)\supset\{k(x)\}^p$.*

5. We assume througout hereafter in this paper not to be $k(y) \subset \{k(x)\}^p$.

Let P' be a point on the general section C of the linear system, simple on U not belonging to the base variety of the linear system, Q' the point on W which corresponds to P', Γ the subvariety on W which corresponds to C. We know already that Q' is simple on W and Γ is absolutely irreducible. Futher we can see that if and only if P' is simple on C, then Q' is simple on Γ.[7]

If the hyperplane $S^N \times H$ is transversal to W at Q', then Q' is simple on the intersetion $\Gamma = (S^N \times H) \cdot W$. Therefore by the lemma in section 3, we see that, if P'(or Q') is singular on C(or Γ), then it must be

$$\sum \lambda_i D(y_i) = 0$$

at P' for any derivation D.

As to the converse let us suppose that P'(or Q') is simple on C(or Γ). Since the intersection $\Gamma = (L \times H) \cdot W$, as we have seen in the above lemma, has multiplicity 1 and Q' is simple on Γ, the hyperplane $L^N \times H$ is transversal to W at Q'.[8] Therefore if it were for every derivation D of $k(x)$

$$\sum \lambda_i D(y_i) = 0$$

at P', as we can conclude from the same lemma, P' would not be simple on C. Hence we get the

Proposition. *Let $P'(x')$ be a simple point on U not belonging to the base variety of the linear system with $f_0(x') \neq 0$. In order that P' is singular on the general section of the linear system, it is neccessary and sufficient that*

$$\sum_{i=1}^m \lambda_i D(y_i) = 0, \qquad y_i = \frac{f_i(x)}{f_0(x)}$$

at P' for any derivation D of $k(x)$.

6. Does there exist such a point with the conditions in the preceeding propoition? The classical Bertini's theorem asserts that it does never

7) See Theor. 15 (p. 108) in Weil's book [3]. 8) Prop. 21 (p. 141) in Weil's book [3].

occur, if p is zero. But if $p>0$, as Zariski has pointed out, it does occur. Namely let us take a plane as U and the linear pencil

$$C:\ x^p+y^2=\lambda.$$

Then the point $P(\lambda^{\frac{1}{p}},0)$ is singnlar on C, though P is simple on U and is not the base point of the pencil.

Let us now consider in each affine model of U the locus of points $\bar{P}(\bar{x})$ for which the r equations for each fixed σ $(0\leq\sigma\leq m)$

$$\sum\lambda_i D_j(y_i^{(\sigma)})=0 \quad (1\leqq j\leqq r)$$

are solvable with respect to $\lambda_1,\ldots,\lambda_m$, where $D_1,\ldots,D_r$ are independent derivations of $k(x)$ and $\bar{y}_i^{(\sigma)}=f_i(\bar{x})/f_\sigma(\bar{x}),\ f_\sigma(\bar{x})\neq 0$.

If $m>r$, the variety U itself is such a locus, but if $m\leqq r$ the locus may be empty. In general the locus is clearly a bunch of varieties over $\bar{k}$, where $\bar{k}$ means the algebralc closure of k. These varieties shall be called the *resultant varieties* of the linear system.

Let $\bar{P}(\bar{x})$ ee the generic point of a resultant variety Φ_0 (we assume here $\sigma=0$) and $r-s$ the rank of the matrix

$$(D_j(\bar{y}_j)).$$

If there exist such points $\bar{P}'(\bar{x}')$ on Φ_0 that the rank of the matrix

$$(D_j(\bar{y}'_i))$$

at P' will be less than $r-s$, the locus of these P' is a bunch of subvarieties of Φ_0. These subvarieties Φ_1 shall be called *critical varieties*. Further if there exist such points $\bar{P}''(\bar{x}'')$ on Φ_1 that the rank of the matrix

$$(D_j(\bar{y}''))$$

is less than that of the matrix $(D_j(\bar{y}'))$, where $\bar{P}'(\bar{x}')$ is generic for Φ_1 over $\bar{k}$, then the locus of $\bar{P}''$ shall be called critical varieties of higher order, and so on.

We now consider the system of equations

$$\begin{cases} \lambda_0+\sum\lambda_i y'_i=0 & \\ \sum\lambda_i D_j(y'_i)=0 & (1\leqq j\leqq r) \\ f_0(x')y'_i-f_i(x')=0 & (1\leqq i\leqq m) \end{cases} \tag{10}$$

defines an algebraic correspondence Δ_l between the m-dimensional projective space $\Lambda(\lambda_0,\ldots,\lambda_m)$ and the resultant (or critical) variety Φ_l, where (x') means the generic point of Φ_l.

Then it is readily to be seen that the Bertini's theorem on movable singularities will be maintained if $\mathrm{proj}_\Lambda \Delta_l$ can not cover the the whole space Λ for any Φ_l.

7. Theorem. *Let (x) be the generic point of an affine model of U on which the critical (or resultant) variety Φ_l lies, and $P'(x')$ be the generic point of Φ_l with $f_0(x')\neq 0$ and $y'_i=f_i(x')/f_0(x')$.*

If $\bar{k}(x')$ is separably generated over $\bar{k}(y')$ for any Φ_l, then the Bertini's theorem holds in the classical formulation.

Proof We assume here $\sigma=0$. Let ρ be the dimension of $\Phi=\Phi_l$ over $\bar{k}$, namely $\dim \bar{k}(x')=\rho$, and s be the dimension of $\bar{k}(x')$ over $\bar{k}(y')$. Since $\bar{k}(x')$ is separably geneated over $\bar{k}(y')$, the dimension of the derivation-module of $\bar{k}(x')$ over $\bar{k}(y')$ is equal to s. If we denote by D' derivations of $\bar{k}(x')$ over $\bar{k}$, there are ρ linearly independent derivations $D'_1,\ldots,D'_\rho$. The derivations $\bar{D}$ of $\bar{k}'(x')$ over $\bar{k}(y')$ may be written in the form

$$\bar{D}=\mu_1 D'_1+\ldots+\mu_\rho D'_\rho.$$

Since $\bar{D}y'_i=0$ for any i, it must be

$$\mu_1 D'_1 y'_i+\ldots+\mu_\rho D'_\rho y'_i=0,\quad 1\leqq i\leqq m.$$

Hence the rank of the matrix

$$(D'_j(y'_i))\quad 1\leqq i\leqq m,\ 1\leqq j\leqq \rho$$

is $\rho-s$, as it is by hypothesis $\dim \{\bar{D}\}=s$. Therefore we can take $D'_1,\ldots,D'_{\rho-s}$ linearly independent on $\bar{k}(y')$, snch that $\rho-s$ derivations D''_i of $\bar{k}(y')$ induced by $D'_i(1\leqq i\leqq \rho-s)$ on $\bar{k}(y')$ form a complete system of the derivation-module of $\bar{k}(y')$ over $\bar{k}$. For, it is $\dim\ [\bar{k}(y'):\bar{k}]=\rho-s$ and $\bar{k}(y')$ is of course regular over $\bar{k}$.

Let us now consider the correspondence Δ_l in the preceeding section:

$$\begin{cases} \lambda_0+\sum\lambda_i y'_i=0 \\ \sum\lambda_i D_j y'_i=0 \quad (0\leqq j\leqq r)\end{cases} \tag{10'}$$

If (λ) corresponds to $P'(x')$, it must be satisfied

$$\begin{cases} \lambda_0+\sum\lambda_i y'_i=0 \\ \sum\lambda_i D'_j y'_i=0 \quad (1\leqq j\leqq \rho-s) \end{cases} \tag{11}$$

for any derivation D'_j of $\bar{k}(x')$.[9] But we can consider this system $\{D'_1,\ldots, D'_{\rho-s}\}$ as above mentioned as a complete system of linearly independent derivations $\{D''_1,\ldots, D''_{\rho-s}\}$ of $\bar{k}(y')$ over $\bar{k}$. Therefore (λ) must satisfy

$$\begin{cases} \lambda_0+\sum\lambda_i y'_i=0 \\ \sum\lambda_i D''_j y'_i=0 \quad (1\leqq j\leqq \rho-s) \end{cases} \tag{12}$$

As all coefficicients of these equations are rational in $k(y')$ and as these equations are evidently all linearly independent, the dimension of $K(\lambda_1,\ldots,\lambda_m)$ over $K=k(y',\lambda_0)$ is not greater than $m-\rho+s-1$. And the dimension of $k(y')$ over k is $\rho-s$.

We consider now the equations (12) as the algebraic correspondence Δ' between the projective space Λ and the image Φ' of Φ by the correspondence W. As (y') is generic for Φ' and

$$(\rho-s)+(m-\rho+s-1)=m-1,$$

we can readily deduce by the principle of "Konstantentenzählung," that the subvarietiy of Φ' is empty which corresponds to a generic point of Λ by the correspondence Δ'.

Thus $\mathrm{proj}_\Lambda\Delta$ can not cover the whole space Λ. Hence by the remark at the end of the preceeding section we get the theorem.

8. The special case, in which the linear system is the general linear function of (x):

$$\lambda_0+\lambda_1x_1+\ldots+\lambda_Nx_N$$

is very important. That the Bertini's theorem holds in this case without any other conditions even when $p>0$, has been proved very elegantly by Y. Nakai.[10] But it follows also from our general theorem.

In fact in this case the resultant variety is U itself. Further the

9) cf. S. Koizumi [8], Prop. 6, 7, p. 277.

10) See Y. Nakai [6].

matrix $(D_j(y_i))$ becomes

$$(D_j(x_i))=\begin{pmatrix} 1\ 0 \ldots\ldots\ldots 0 & * \ldots\ldots\ldots * \\ 0\ 1 \ldots\ldots\ldots 0 & * \ldots\ldots\ldots * \\ \ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \\ 0\ 0 \ldots\ldots\ldots 1 & * \ldots\ldots\ldots * \end{pmatrix},$$

and the rank of this matrix is r at every point on U. Hence there does not exist any critical subvariety. Moreover $k(x)$ is separably generated over $k(y)=k(x)$. Therefore by the preceeding theorem the Bertini's theorem holds in this case.

9. We will give here a new elementary proof of the Bertini's theorem in the classical case.

We reduce it also as usual[11] to the case of linear pencil $\lambda_0 f_0+\lambda_1 f_1$.

Then the locus of singular points of the sections is clearly contained in the locus defined by equations

$$f_1 D(f_0)-f_0 D(f_1)=0$$

for every derivation D of $k(x)$ over k, and this locus is clearly a bunch of subvarieties Ψ_j.

Let $\bar{P}(\bar{x})$ be a generic point of a Ψ. If Ψ does not lie on the singular varieties of U nor the base variety of the pencil, $\bar{P}$ is a simple point not belonging to the base variety of the pencil and we can assume without loss of generality $f_0(\bar{x})\neq 0$, and then

$$\mu=\frac{\lambda_0}{\lambda_1}=-\frac{f_1(\bar{x})}{f_0(\bar{x})}$$

must be *algebraic* over the field k.

In fact if we derivate the relation

$$\mu f_0(x)+f_1(x)=0$$

then we have

$$(D\mu)\cdot f_0(x)+\mu D f_0(x)+D f_1(x)=0$$

11) See v. d. Waerden's book [4].

While, as $\bar{P}$ lies on Ψ, by putting $x=\bar{x}$

$$\mu Df_0(\bar{x})+Df_1(\bar{x})=0$$

therefore, as $f_0(\bar{x})\neq 0$, it is on Ψ $D\mu=0$ for any derivation D of $k(x)$ over k, consequently $\bar{D}\mu=0$ for any derivation $\bar{D}$ of $k(\bar{x})$ over k. Hence μ must be algebraic over k, as the characteristic of k is zero.

While there exists Ψ_j of only finite number, therefore the above μ also of finite number. Except for these finite μ, the above section has no singularities outside the singular points of U and outside the base variety of the pencil, q. e. d.

The idea of this apparently algebraic proof is essentially analytic, so we can successfully apply this to the proof of

Oka's lemma. Let R be a finite open region of n-dimensional complex space, $\sum$ an analytic vasiety in R, $f(z_1,\ldots,z_n)$ $(0\leq i\leq m)$ a set of analytic functions of complex variables $z_1,\ldots,z_n$. Then the set of points $(\lambda_0,\lambda_1,\ldots,\lambda_m)$ for which the section of $\sum$ by the hypersurface $\lambda_0 f_0+\ldots+\lambda_m f_m=0$ may have singularities outside the singular points of $\sum$ and outside the base points of the linesr system is only of the first category.

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