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An elementary proof of the fundamental theorem of normed fields.

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In the theory of normed rings founded by I. Gelfand[1], the most fundamental is the following theorem of Mazur-Galfand: " A normed field over the complex number field is, in the sense of isomorphism, nothing but the complex number field itself", the proof of which was depending on the notion of analyticity, Cauchy's integral formula and Liouville's theorem, etc[2].

Quite recently, Prof. E. Artin in his lecture note 'Algebraic numbers and algebraic functions I (1950–51) Princeton' gives a proof of this theorem as one of the bases of his theory, replacing contour integral by its approximating sum.

The aim of this short note is also to give an elementary proof of the above theorem, using no function-theoretical methods but the notion of continuity. Moreover the proof does not assume the completeness of the normed fields, though it is easy to see, by way of completion, that the assumption of completeness does not harm the generality of this theorem.

Let K be a normed field, that is to say, a field in the sence of algebra and at the same time a linear space, over the complex number field C, in which is given the norm $\| \ \|$ satisfying

$$x \in K \rightarrow \|x\| \geqq 0,$$

$$x \neq 0 \rightleftarrows \|x\| > 0,$$

$$x, y \in K \rightarrow \|x+y\| \leqq \|x\| + \|y\|, \quad \|xy\| \leqq \|x\| \cdot \|y\|,$$

$$\lambda \in C,\ x \in K \rightarrow \|\lambda x\| = |\lambda| \cdot \|x\|.$$

Then the unit e of K, being $\neq 0$, has positive norm, and consequently, $\|e\| \geqq 1$.

1) I. Gelfand: "Normierte Ringe", Recueil Math. T. 9 (51) No. 1 (1941).
2) ibid.

Throughout this note, the condition of completeness is not assumed. Thus, with respect to the norm, K becomes a metric space in which, as a function of x, $\|x\|$ is continuous. Also the product xy is a continuous function of x and y,

LEMMA 1. *There exists a neighborhood V of e where $\|x^{-1}\| \leqq 1 + \|e\|$ holds.*

PROOF. If $\|x-e\| < 1$ $(\leqq \|e\|)$ then $x \neq 0$ and x^{-1} exists, which shows in view of the following identity

$$\big(e-(e-x)\big)\sum_{j=0}^{n}(e-x)^j = e-(e-x)^{n+1}$$

that, if $\|x-e\| < 1$, then, as $n \to \infty$,

$$\Big\|\sum_{j=0}^{n}(e-x)^j - x^{-1}\Big\| \leqq \|x^{-1}\| \cdot \|e-x\|^{n+1} \to 0\,. \tag{1}$$

Now, if $\|x-e\| < 2^{-1}$, then, again from the above identity

$$\begin{aligned}\Big\|x^{-1}-\sum_{j=0}^{n}(e-x)^j\Big\| &\geqq \|x^{-1}\|-\|e\|-\sum_{j=1}^{n}\|e-x\|^j \\ &\geqq \|x^{-1}\|-\|e\|-\sum_{j=1}^{n}2^{-j} \geqq \|x^{-1}\|-\|e\|-1\end{aligned}$$

from which and from (1) follows $0 \geqq \|x^{-1}\|-(\|e\|+1)$. This completes the proof.

LEMMA 2. *x^{-1} is a continuous function defined on $K-\{0\}$.*

PROOF. Let $\{x_n\}$ be any sequence of points in K converging to e. Then, from a certain number on, $x_n \in V$ holds, which shows

$$\|x_n^{-1}-e\| \leqq \|x_n^{-1}\| \cdot \|e-x_n\| \leqq (\|e\|+1)\,\|e-x_n\| \to 0\,,$$

that is,

$$x_n^{-1} \to e \qquad \text{as} \qquad x_n \to e\,. \tag{2}$$

Now, if $x_n \to x \neq 0$, then by the continuity of product,

$$x_n\, x^{-1} \to x\, x^{-1} = e$$

which shows by (2)

$$x\, x_n^{-1} = (x_n\, x^{-1})^{-1} \to e\,.$$

But again by the continuity of product

$$x_n^{-1} = x^{-1} \cdot x \cdot x_n^{-1} \to x^{-1}\,,$$

and the result follows.

This gives as an immediate consequence the following

LEMMA 3. $\| x^{-1} \|$ *is continuous at every point* $x \neq 0$.

THEOREM OF MAZUR-GELFAND. $K=\{\lambda e \mid \lambda \in C\}$.

PROOF. Suppose $K \neq \{\lambda e \mid \lambda \in C\}=C^*$.

Then there would exist $x \in K-C^*$. Obviously

$$a, b \in C, \quad a \neq 0 \rightarrow a \cdot x+b \cdot e \in K-C^* . \tag{3}$$

Since $x-\lambda e \neq 0$ for every $\lambda \in C$, $\varphi(\lambda)=(x-\lambda e)^{-1}$ and $\| \varphi(\lambda) \|$ are, by Lemma 2 and 3, continuous on C. Furthermore

$$\| \varphi(\lambda) \| \leqq |\lambda^{-1}| \cdot \| (\lambda^{-1} x-e)^{-1} \| \rightarrow 0 \quad \text{as} \quad |\lambda| \rightarrow \infty ,$$

since $\lambda^{-1} x-e \in V$ for sufficiently large $|\lambda|$.

From this fact and the continuity of $\| \varphi(\lambda) \|$ would follow the existence of a number $\lambda_0 \in C$ such that $\|\varphi(\lambda_0)\|=\sup_{\lambda \in C} \| \varphi(\lambda) \|$.

In view of the property (3), it may be assumed that $\lambda_0=0$ and $\| \varphi(\lambda_0) \|=1$, so that

$$\lambda \in C \rightarrow \| \varphi(\lambda) \|=\| (x-\lambda e)^{-1} \| \leqq \| x^{-1} \|=1 . \tag{4}$$

Now it will be shown that (4) implies

$$\| (x-2^{-1}e)^{-1} \|=1 . \tag{5}$$

Once this established, it would be possible to replace x by $x-2^{-1}e$ in view of (3) and, by repetition of this process, would hold

$$1=\|(x-2^{-1} \cdot n \cdot e)^{-1} \| \quad (n=1, 2, \cdots)$$

which would lead to a contradiction, since

$$\| (x-2^{-1} \cdot n \cdot e)^{-1} \| \rightarrow 0 \qquad (n \rightarrow \infty) .$$

It remains only to prove that (4) implies (5).

If $\| \varphi(2^{-1}) \| < 1$, then $\| \varphi(2^{-1}) \|=1-2\delta$ where $\delta > 0$. By the continuity of $\|\varphi(\lambda)\|$, there would exist a positive number η such that

$$|\lambda-2^{-1}| \leqq \eta \rightarrow \| \varphi(\lambda) \| < 1-\delta . \tag{6}$$

Let $\xi_1, \cdots, \xi_n$ be the n^{th} roots of 1. Then it is easy to see

$$\sum_{j=0}^{n} (x-2^{-1} \xi_j \cdot e)^{-1}=n \cdot x^{n-1}(x^n-2^{-n}e)^{-1}$$

$$=n\cdot x^{-1}\Big(e+x^{-1}(2^{-1}x^{-1})^n(e-(2^{-1}x^{-1})^n)^{-1}\Big)$$

or

$$n^{-1}\sum_{j=0}^{n}(x-2^{-1}\xi_j\cdot e)^{-1}=x^{-1}+x^{-1}(2^{-1}x^{-1})^n\Big(e-(2^{-1}x^{-1})^n\Big)^{-1}$$

from which

(7) $$n^{-1}\sum_{j=0}^{n}\|\varphi(2^{-1}\xi_j)\|\geqq\|x^{-1}\|-\|x^{-1}\|\cdot\|2^{-1}x^{-1}\|^n\cdot\|\Big(e-(2^{-1}x^{-1})\Big)^{n-1}\|.$$

Let n' be the number of $2^{-1}\xi_j$ satisfying $|2^{-1}\xi_j-2^{-1}|\leqq\eta$. Then from (6), (4) and (7), follows

(8) $$n^{-1}\Big(n'(1-\delta)+(n-n')\Big)>1-2^{-n}\|(e-(2^{-1}x^{-1})^n)^{-1}\|,$$

in which the right hand-side of (8) tends to 1 since $e-(2^{-1}x^{-1})^n\in V$ for sufficiently large n, while the left hand-side, $1-(n'/n)\delta$, tends to $1-\dfrac{l}{2\pi 2^{-1}}\delta$ where $l/(2\pi 2^{-1})$ is the ratio of the length l of the arc defined by $|\lambda-2^{-1}|\leqq\eta$ and $|\lambda|=2^{-1}$ to the total length of the circle $|\lambda|=2^{-1}$.

This obviously would give

$$1-\frac{l}{\pi}\delta\geqq 1$$

which is a contradiction. (March 3rd, 1952)

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