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A remark on the prolongation of Riemann surfaces of finite genus.

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Let F be an abstract Riemann surface. If there exists no one-valued, regular analytic and non-constant function on F such that its Dirichlet integral taken over F is finite, we shall say that F is a *surface of class* $N_{\mathfrak{D}}$ (F has "einen hebbaren Rand" in Sario's terminology[1]).

If F is of finite genus p, we can map F conformally onto a part $\bar{F}$ of a closed Riemann surface F^* of the same genus[2]. Then, Nevanlinna stated the following conjecture[3]:

THEOREM. *The prolongation of a Riemann surface F of finite genus p onto a closed Riemann surface F^* is unique, if and only if F is a surface of class* $N_{\mathfrak{D}}$.

The "uniqueness" means: if F is mapped conformally onto a part $\bar{F}$ of F^* and a part $\bar{F}_1$ of F_1^* respectively, then the analytic function which maps $\bar{F}$ onto $\bar{F}_1$ maps necessarily F^* onto F_1^*.

This conjecture was proved by Ahlfors and Beurling[4] for the case $p=0$: *A plane region Ω is of class* $N_{\mathfrak{D}}$ *if and only if every univalent* (*schlicht*) *function in Ω is linear.* In this note we shall show that the conjecture for an arbitrary p can be easily proved by means of this Ahlfors-Beurling's theorem.

Let E be a bounded closed set of points on the complex z-plane. If any one-valued regular analytic function in a neighbourhood $U-E$ of E with finite Dirichlet integral taken over $U-E$ is regular also on E, we shall say, for convenience' sake, that E is a *null-set of class* $N_{\mathfrak{D}}$[5].

We cut F along a non-decomposing system of p analytic loop cuts on F having no points in common with each others, and map the resulting surface of planar character (schlichtartig) conformally onto a domain D on the z-plane, which is bounded by $2p$ closed analytic curves C_i, C_i' ($i=1, \cdots, p$) and a bounded closed set of points E, so that C_i and C_i' correspond to one and the same loop cut on F and E corresponds to the ideal boundary of F. Let $D^*=D+E$ be the domain bounded by C_i, C_i' $(i=1, \cdots, p)$. Since there exist analytic correspondences between C_i and C_i', D^* can be regarded as a closed Riemann surface F^* of genus p, while we identify the corresponding points on C_i and C_i'. F is conformally equivalent to the part $\bar{F}=F^*-E$ of F^*.

First we shall prove:

LEMMA. *F is a surface of class $N_{\mathfrak{D}}$ if and only if the set E is a null-set of class $N_{\mathfrak{D}}$.*

The sufficiency of this condition and its necessity for the case $p=0$ were proved by Sario.[6]

PROOF. *Sufficiency.* Suppose that E is of class $N_{\mathfrak{D}}$. Let f be a one-valued regular analytic function on F with finite Dirichlet integral taken over F. Then, considered as a function of $z \in D$, $f=f(z)$ is regular also on E. Hence, as a function on F^*, f is everywhere regular, so that $f \equiv$ const., q. e. d.

Necessity. Suppose that E is not of class $N_{\mathfrak{D}}$. Then there exists a function $\varphi(z)$ one-valued and regular in a neighbourhood $U-E$ of E, which is not everywhere regular on E and whose Dirichlet integral taken over $U-E$ is finite. If E is of positive areal measure, we can choose, as $\varphi(z)$, the function which maps the complement of E onto the corresponding Koebe's minimal slit-domain, whose slits have the areal measure zero as is well-known.

First, suppose that E is totally disconnected. Let E_0 be the closed subset of E consisting of all singular points of $\varphi(z)$ on E. If E is of areal measure zero, the Dirichlet integral of $\varphi(z)$ taken over $U-E_0$ is also finite. The same holds also for the case of positive areal measure by the mentioned choice of $\varphi(z)$. Then, $\varphi(z)$ can have neither poles nor isolated essential singularities, so that E_0 is a totally disconnected perfect set. We divide E_0 into $2p+1$ disjoint closed subsets E_k $(k=1, \cdots, 2p+1)$ and take a neighbourhood U_k-E_k of E_k for each k, such that $U_k \subset U$, $U_k U_j=0$ $(k \neq j)$. We put $\varphi_k(z) \equiv \varphi(z)$ for $z \in U_k-E_k$.

If E contains a continuum γ, we take $2p+1$ disjoint sub-continua E_k $(k=1, \cdots, 2p+1)$ of γ and $2p+1$ domains U_k containing E_k as above. In this case, let $\varphi_k(z)$ be an arbitrary function, which is one-valued and regular in U_k-E_k but not everywhere regular in U_k, and whose

Dirichlet integral taken over $U_k - E_k$ is finite. The existence of such functions is obvious.

By the well-known smoothing process, we construct a one-valued regular harmonic function u_k on $F^* - E_k$, such that $u_k(z) - \Re\varphi_k(z)$ is harmonic throughout U_k.[7] The Dirichlet integral of u_k taken over $F^* - E_k$ is finite. Let v_k be a conjugate harmonic function of u_k. Then $u_k + iv_k$ is one-valued and regular in $U_k - E_k$.

Let $\alpha_1, \cdots, \alpha_{2p}$ be a base of loop cuts on F^* described in $F^* - E = \bar{F}$. v_k has $2p$ moduli of periodicity $(a_1^{(k)}, \cdots, a_{2p}^{(k)})$ along these loop cuts. Then, we can find $2p+1$ not all vanishing real numbers $c_1, \cdots, c_{2p+1}$ such that

$$\sum_{k=1}^{2p+1} c_k a_i^{(k)} = 0 \qquad (i = 1, \cdots, 2p)$$

hold. Then $f = \sum c_k(u_k + iv_k)$ is a one-valued, regular and non-constant function on $F^* - \sum E_k \supset F^* - E = \bar{F}$, whose Dirichlet integral taken over $\bar{F}$ is finite. Hence, F is not of class $N_{\mathfrak{D}}$, q. e. d.

REMARK. As is seen from the above proof, the Lemma remains valid, if we replace the surface and the null-set of class $N_{\mathfrak{D}}$ by those of class $N_{\mathfrak{B}}$ defined similarly with respect to the family $\mathfrak{B}$ of one-valued, regular and bounded functions.

PROOF OF THE THEOREM,

Sufficiency. Suppose that F is of class $N_{\mathfrak{D}}$, and that F is mapped conformally onto $\bar{F} = F^* - E$ and $\bar{F}_1 = F_1^* - E_1$ respectively. Let D, D^*, D_1 and D_1^* be the corresponding domains on the z-plane. Then, by the conformal mapping $\bar{F} \to F \to \bar{F}_1$, the domain D is mapped onto D_1. Since $E = D^* - D$ is of class $N_{\mathfrak{D}}$ by the lemma, D^* is necessarily mapped onto D_1^* by this mapping, so that F^* is mapped onto F_1^*, q. e. d.

Necessity. Suppose that F is not of class $N_{\mathfrak{D}}$, so that, by the lemma, the corresponding set $E = D^* - D$ on the z-plane is not of class $N_{\mathfrak{D}}$. Then, again by the lemma (for $p=0$), the complement Ω of E is not of class $N_{\mathfrak{D}}$. Hence, by Ahlfors-Beurling's theorem, there exists a univalent function $\varphi(z)$ in Ω, which has a pole in Ω and is not everywhere regular on E. Let D_1 be the image of D by $\varphi(z)$, and D_1^*, $\bar{F}_1$ and F_1^* be the corresponding domain and Riemann surfaces. $\varphi(z)$ provides a conformal mapping of $\bar{F}$ onto $\bar{F}_1$. But since

$\varphi(z)$ can not be analytically prolonged onto D^*, it does not map F^* onto F_1^*. Thus, the prolongation of F is not unique, q. e. d.

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References.

1) L. Sario: Über Riemannsche Flächen mit hebbarem Rand, Ann. Acad. Sci. Fennicae A. I. Nr. 50 (1948).

2) S. Bochner: Fortsetzung Riemannscher Flächen, Math. Ann. 98 (1928). L. Sario: loc. cit. Also our argument in this note contains a proof of the possibility of such mappings.

3) R. Nevanlinna: Eindeutigkeitsfragen in der Theorie der konformen Abbildung. 10. Congr. Math. Scand. Copenhagen 1946.

4) L. Ahlfors and A. Beurling: Conformal invariants and function-theoretic null-sets, Acta Math. 83 (1950), Theorem 6.

5) Ahlfors-Beurling: loc. cit. The equivalency of our definition with theirs is shown by Theorem 5 there or by the Lemma in this note.

6) L. Sario: loc. cit. Ahlfors-Beurling: loc. cit., Theorem 5.

7) C. Neumann: Abelsche Integrale, 2. Aufl. 1884. W. F. Osgood: Lehrbuch der Funktionentheorie II, 2, Kap. 5.