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A metamathematical theorem on the theory of ordinal numbers.

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The purpose of this paper is to show that the consistency proof of a theory of ordinal numbers in the weakened form considered in G. Gentzen's logical system LK (cf. Gentzen [1]) can be reduced to that of a weakened theory of ordinal numbers $<\omega^{\omega}$, this latter theory being considered in a logical system which is obtained in extending slightly the system LK by the use of the symbol Min: if $\mathfrak{A}(a)$ is a formula and x is any bound variable not contained in $\mathfrak{A}(a)$, the figure $\mathrm{Min}\,(x)\,\mathfrak{A}(x)$ is a 'term', a figure for a particular object. (We follow the terminology of Gentzen [1].) What these theories mean, will be described below by sets of axioms $1.1,\cdots,1.16$ and $2.1,\cdots,2.19$ respectively. Thus, we shall prove that any set of axioms indicated in $1.1,\cdots,1.16$, containing no special object other than $0, \omega$ and no function other than $*'$ ($*$ indicates an argument-place), is consistent, in assuming that any set of axioms indicated in $2.1,\cdots,2.19$ can not lead to a contradiction.

To perform this, we shall establish a metatheorem called Representation Theorem, which is meaningful in the weakened theory of ordinal numbers.

Each of $1.12,\cdots,1.16$, $2.16,\cdots,2.19$ stands for a finite number of arbitrary axioms of the indicated form, and $[z]$ stands for a row of symbols of the form $\forall z_1\cdots\forall z_k$; properly we should write $\mathfrak{A}(x, z_1,\cdots, z_k)$ or $\mathfrak{A}(x, y, z_1,\cdots, z_k)$ for $\mathfrak{A}(x)$ or $\mathfrak{A}(x, y)$ respectively, but it seems improbable that any confusion should occur from our simplified expression.

Some metamathematical lemmas, e.g. the one formulated immediately below, will be useful in our consideration, but it seems unimportant to give all such lemmas used, which are merely explicit and rather long formulation of mathematicians' common sense.

LEMMA. Let $\mathfrak{M}_i (i=0, 1)$ be two formulas obtained exactly in the same way from formulas $\mathfrak{A}_i(a_1,\cdots,a_k)$, $\mathfrak{C}_1,\cdots,\mathfrak{C}_n$ by means of logical operations ($\neg$, $\wedge$, $\vee$, $\forall$ or $\exists$) and, possibly, of substitution among variables (for instance, $\mathfrak{M}_0$ is $\mathfrak{C}_1 \vee \forall x\, \mathfrak{A}_0(x, b, a_2,\cdots, a_k)$ and $\mathfrak{M}_1$ is $\mathfrak{C}_1 \vee \forall x\, \mathfrak{A}_1(x, b, a_2, \cdots a_k)$). Then the sequence (cf. Gentzen [1])

$\forall x_1 \cdots \forall x_k \big(\mathfrak{A}_0(x_1,\cdots, x_k) \leftrightarrow \mathfrak{A}_1(x_1,\cdots, x_k)\big) \rightarrow \mathfrak{M}_0 \leftrightarrow \mathfrak{M}_1$ is provable.

$\mathfrak{A} \leftrightarrow \mathfrak{B}$ means $(\mathfrak{A} \vdash \mathfrak{B}) \wedge (\mathfrak{B} \vdash \mathfrak{A})$, and $\mathfrak{A} \vdash \mathfrak{B}$ means $\neg \mathfrak{A} \vee \mathfrak{B}$; the symbols $\leftrightarrow$ and $\vdash$ have 'weaker adhering power' than the other logical symbols: e. g. $\mathfrak{A} \wedge \mathfrak{B} \vdash \neg \mathfrak{C}$ means $(\mathfrak{A} \wedge \mathfrak{B}) \vdash (\neg \mathfrak{C})$, i. e. $\neg(\mathfrak{A} \wedge \mathfrak{B}) \vee \neg \mathfrak{C}$. Throughout this paper, the last index in a series of figures $\mathfrak{F}_1,\cdots, \mathfrak{F}_k$ may be 0, in which case the series is void.

COROLLARY. Let $m+1$ formulas $\mathfrak{M}_i (i=0, 1,\cdots, m)$ be as before, and let $\mathfrak{B}_j (j=1,\cdots, m)$ be the formula

$$\forall x_1 \cdots \forall x_k \left(\mathfrak{A}_0(x_1, \cdots, x_k) \leftrightarrow \mathfrak{A}_j(x_1,\cdots, x_k)\right).$$

Then, if $m+1$ sequences

$$\Gamma \rightarrow \mathfrak{B}_1,\cdots, \mathfrak{B}_m$$

$$\Gamma \rightarrow \mathfrak{M}_j \qquad (j=1, \cdots, m)$$

are provable, the sequence

$$\Gamma \rightarrow \mathfrak{M}_0$$

is also provable.

1. (THEORY OF ORDINAL NUMBERS)

1.1 $\forall x\, (x=x)$ $(=$ is a predicate$)$

1.2 $0 < \omega$ (0 and ω are special objects; $<$ is a predicate.)

1.3 $\forall x\, \forall y\, (x < y \vee x=y \vee y < x)$

1.4 $\forall x\, \forall y\, \neg (x=y \wedge x < y)$

1.5 $\forall x\, \forall y\, \neg (x < y \wedge y < x)$

1.6 $\forall x\, \forall y\, \forall z\, (x < y \wedge y < z \vdash x < z)$

1.7 $\forall x\, (0 < x \vee 0=x)$

1.8 $\forall x\, \forall y\, (x < y \vdash x'=y \vee x' < y)$ ($*'$ is a function)

1.9 $\forall x\, (x < x')$

1.10 $\forall x\, \forall y\, (x'=y' \vdash x=y)$

1.11 $\forall x\, (x < \omega \vdash x' < \omega)$

1.12 $[z]\, \forall x\, \forall y\, \{x=y \vdash (\mathfrak{A}(x) \leftrightarrow \mathfrak{A}(y))\}$

1.13 $[z]\,\forall x\{\mathfrak{A}(0)\wedge\forall y(\mathfrak{A}(y)\vdash\mathfrak{A}(y'))\wedge x<\omega\vdash\mathfrak{A}(x)\}$

1.14 $[z]\,\forall x\{\mathfrak{A}(0)\wedge\forall y\{\forall u(u<y\vdash\mathfrak{A}(u))\vdash\mathfrak{A}(y)\}\vdash\mathfrak{A}(x)\}$

1.15 $[z]\,\forall u[\forall x\,\forall y\,\forall s(\mathfrak{A}(x,s)\wedge\mathfrak{A}(y,s)\vdash x=y)$
$\vdash \mathrm{E}x\,\forall y\{\mathrm{E}s(\mathfrak{A}(y,s)\wedge s<u)\vdash y<x\}]$

1.16 $\forall u\,\mathrm{E}v[z]\,[\forall x\,\forall y\,\forall s(\mathfrak{A}(x,s)\wedge\mathfrak{A}(y,s)\vdash x=y)$
$\vdash \mathrm{E}x\{x<v\wedge\forall y\,\neg\,(\mathfrak{A}(x,y)\wedge y<u)\}]$

2. (THEORY OF ORDINAL NUMBERS $<\omega^{\omega}$)

2.1 $\forall x(x=x)$

2.2 $0<\omega$

2.3 $\forall x\,\forall y(x<y\vee x=y\vee y<x)$

2.4 $\forall x\,\forall y\,\neg\,(x=y\wedge x<y)$

2.5 $\forall x\,\forall y\,\neg\,(x<y\wedge y<x)$

2.6 $\forall x\,\forall y\,\forall z(x<y\wedge y<z\vdash x<z)$

2.7 $\forall x(0\leqq x)$
$a\leqq b$ means $a<b\vee a=b$; $a<b$ and $a\leqq b$ are often written $b>a$ and $b\geqq a$ respectively.

2.8 $\forall x\,\forall y(x<y\vdash x'\leqq y)$

2.9 $\forall x(x<x')$

2.10 $\forall x\,\forall y(x'=y'\vdash x=y)$

2.11 $\forall x(x<\omega\vdash x'<\omega)$

2.12 $\forall x(x+0=x)$ $\quad(*+*$ is a function)

2.13 $\forall x\,\forall y\,\forall z\,(y<z\vdash x+y<x+z)$

2.14 $\forall x\,\forall y\,\mathrm{E}z(x\leqq y\vdash x+z=y)$

2.15 $\forall x\,\mathrm{E}y\{x<y\wedge\forall u\,\forall v(u<y\wedge v<y\vdash u+v<y)\}$

2.16 $[z]\,\forall x\,\forall y\{x=y\vdash(\mathfrak{A}(x)\dashv\vdash\mathfrak{A}(y))\}$

2.17 $[z]\,\forall x\{\mathfrak{A}(0)\wedge\forall y(\mathfrak{A}(y)\vdash\mathfrak{A}(y'))\wedge x<\omega\vdash\mathfrak{A}(x)\}$

2.18 $[z]\,\forall x\{\mathfrak{A}(x)\vdash x\geqq\mathrm{Min}\,(y)\mathfrak{A}(y)\}$

2.19 $[z]\,\{\mathrm{E}x\,\mathfrak{A}(x)\vdash\mathfrak{A}\,(\mathrm{Min}\,(y)\,\mathfrak{A}(y))\}$

The series of axioms $2.1,\cdots,2.19$ will be denoted by Γ_0; but these axioms will not be written explicitly in the left sides of sequences, that is, $\Gamma\to\Delta$ means $\Gamma_0,\Gamma\to\Delta$. Also a formula $\mathfrak{A}$ will be said to be 'provable' when $\Gamma_0\to\mathfrak{A}$ is provable.

In order to show formulas to be provable, we shall write the proofs of formulas $\mathfrak{A}$ in the usual mathematical language instead of showing the formal proof-figures to the corresponding sequences $\Gamma_0 \to \mathfrak{A}$, which would cost too much space; thus we procede as if we were constructing, in a naïve stand point, a usual mathematical system from the axioms Γ_0 with mathematical 'meaning'.

According to 2.18 and 2.19, we have transfinite induction of the form 1.14, whence we see easily that the following axioms are provable.

3.1 $\forall x \forall y (x+y \geqq y)$

3.2 $\forall x \forall y (x+y' = (x+y)')$

3.3 $\forall x \forall y \forall z \{0 < y \wedge \forall u (u < y \vdash x+u < z) \vdash x+y \leqq z\}$

3.4 $\forall x \forall y \forall z ((x+y)+z = x+(y+z))$

3.5 $\forall x \forall y \forall z (x < y \vdash x+z \leqq y+z)$

3.6 $\forall x (0+x = x)$

3.7 $\forall x \forall y (x < \omega \wedge y < \omega \vdash x+y < \omega)$

Now we define a special object ω^n for each concretely given natural number n:

4.1 $\omega^1 = \omega$

4.2 $\omega^{n+1} = \mathrm{Min}\,(z)\{\omega^n < z \wedge \forall x \forall y (x < z \wedge y < z \vdash x+y < z)\}$

$(n = 1, 2, 3, \cdots)$

A 'definition' of such type should be understood as follows: ω^{n+1} is an 'abbreviation' of the right hand side of 4.2; that is, a formula $\mathfrak{A}(\omega^{n+1})$ is a formula of the form

$$\mathfrak{A}\,(\mathrm{Min}\,(z)\{\omega^n < z \wedge \forall x \forall y (x < z \wedge y < z \vdash x+y < z)\}),$$

in which x, y, z stand for any bound variables admissible in the construction of formulas.

From 2.15, 2.18, 2.19 and 3.7 follow

5.1 $\omega^n < \omega^{n+1}$

5.2 $\forall x \forall y (x < \omega^n \wedge y < \omega^n \vdash x+y < \omega^n)$

5.3 $\forall x (x < \omega^n \vdash x + \omega^n = \omega^n)$.

Now we introduce new functions $*-*$ and $\overset{n}{\mathrm{ess}}(*)$ ('n-th essential part' or 'n-th principal part' of $*$) defined as follows

6.1 $a-b=\mathrm{Min}\,(z)\ (b+z=a)$

6.2 $\overset{n}{\mathrm{ess}}\,(a)=\mathrm{Min}\,(z)\ (z+\omega^n>a)$

Clearly

7.1 $\forall x\,\forall y\{x\geqq y\vdash y+(x-y)=x\}$

7.2 $\forall x\,(\overset{n}{\mathrm{ess}}\,(x)+\omega^n>x)$

7.3 $\forall x\,(x\geqq\overset{n}{\mathrm{ess}}\,(x))$

7.4 $\forall x\,\forall y\,\forall z\,(x=y+z\vdash z=x-y)$

7.5 $\forall x\,\forall y\,\forall z\,(x>y\wedge y\geqq z\vdash x-z>y-z)$

Using this function $\overset{n}{\mathrm{ess}}\,(*)$, we define the following metamathematical concept.

A function $f(*)$ is called 'periodic of order n' $(n=1,2,3,\cdots)$, when

$$\forall x\,\{f(x)=\overset{n}{\mathrm{ess}}\,(x)+f(x-\overset{n}{\mathrm{ess}}\,(x))\}$$
$$\wedge\,\forall x(x<\omega^n\vdash f(x)<\omega^n)\wedge\forall x\,\forall y\,(x<y\vdash f(x)\leqq f(y))$$

is provable.

After some simple calculations, we can show that our new functions $\overset{n}{\mathrm{ess}}\,(*)$ and $*-*$ have the following properties.

8.1 $\forall x\,\forall y\,(x>y\vdash\overset{n}{\mathrm{ess}}\,(x)\geqq\overset{n}{\mathrm{ess}}\,(y))$.

8.2 $\forall x\,(\omega^n>x-\overset{n}{\mathrm{ess}}\,(x))$;

this follows from the provable formula $\overset{n}{\mathrm{ess}}\,(x)+(x-\overset{n}{\mathrm{ess}}\,(x))=x<\overset{n}{\mathrm{ess}}(x)+\omega^n$, in which (and in similar expressions) the bound variable stands for an arbitrary free variable not contained in the given figure.

8.3 $\{\omega^n>x-y\wedge x\geqq y\vdash y\geqq\overset{n}{\mathrm{ess}}\,(x)\}$,

because we have

$$x\geqq y\ \rightarrow\ y+(x-y)=x\,,$$

and so from 5.3

$$x\geqq y,\qquad \omega^n>x-y\ \rightarrow\ y+\omega^n=y+\{(x-y)+\omega^n\}=x+\omega^n>x$$

whence

$$x\geqq y,\qquad \omega^n>x-y\ \rightarrow\ y\geqq\overset{n}{\mathrm{ess}}\,(x)\,.$$

8.4 $\overset{n+1}{\text{ess}}\,(x+\omega^n)=\overset{n+1}{\text{ess}}\,(x)$

because, from $\omega^{n+1}>x-\overset{n+1}{\text{ess}}\,(x)$ and 5.3 we obtain

$$\omega^{n+1}>(x-\overset{n+1}{\text{ess}}\,(x))+\omega^n\,,$$

so that $\overset{n+1}{\text{ess}}\,(x)+\omega^{n+1}>\overset{n+1}{\text{ess}}\,(x)+\{(x-\overset{n+1}{\text{ess}}\,(x))+\omega^n\}=x+\omega^n$,

and $\overset{n+1}{\text{ess}}\,(x)\geqq\overset{n+1}{\text{ess}}\,(x+\omega^n)$.

8.5 $\overset{n+1}{\text{ess}}\,(\overset{n}{\text{ess}}\,(x))=\overset{n+1}{\text{ess}}\,(x)$,

because from 8.4 follows

$$\overset{n+1}{\text{ess}}\,(\overset{n}{\text{ess}}\,(x))=\overset{n+1}{\text{ess}}\,(\overset{n}{\text{ess}}\,(x)+\omega^n)\geqq\overset{n+1}{\text{ess}}\,(x)\,.$$

8.6 $a\geqq b\ \rightarrow\ (a-b)+c=(a+c)-b$,

because $a\geqq b\ \rightarrow\ b+(a-b)+c=a+c$.

8.7 $a\geqq c+b\ \rightarrow\ (a-c)-b=a-(c+b)$,

because $a\geqq c+b\ \rightarrow\ c+\{b+((a-c)-b)\}=a$.

8.8 $\overset{n}{\text{ess}}\,(a+b)=\overset{n}{\text{ess}}\,(a)+\overset{n}{\text{ess}}\,(b)$.

In fact, if $b<\omega^n$, this is evidently equivalent with $\overset{n}{\text{ess}}\,(a+b)=\overset{n}{\text{ess}}\,(a)$ and so can be verified in the same way as 8.4.

Therefore we assume $b\geqq\omega^n$. Then

$$\overset{n}{\text{ess}}\,(a)+\overset{n}{\text{ess}}\,(b)=a+\overset{n}{\text{ess}}\,(b)$$

and $$a+\overset{n}{\text{ess}}\,(b)+\omega^n>a+b$$

so $$\overset{n}{\text{ess}}\,(a)+\overset{n}{\text{ess}}\,(b)\geqq\overset{n}{\text{ess}}\,(a+b)\,.$$

On the other hand

$$(\overset{n}{\text{ess}}\,(a+b)-a)+\omega^n>(a+b)-a=b$$

so $$\overset{n}{\text{ess}}\,(a+b)\geqq a+\overset{n}{\text{ess}}\,(b)$$

and $$\overset{n}{\text{ess}}\,(a+b)=\overset{n}{\text{ess}}\,(a)+\overset{n}{\text{ess}}\,(b)\,.$$

8 9 Any periodic function of order n is periodic of order $n+1$.

PROOF: Let $f(*)$ be a periodic function of order n. We have only to prove

$$\forall x\{f(x)=\overset{n+1}{\text{ess}}(x)+f(x-\overset{n+1}{\text{ess}}(x))\}.$$

As $\quad \overset{n+1}{\text{ess}}(x)+f(x-\overset{n+1}{\text{ess}}(x))$

$$=\overset{n+1}{\text{ess}}(x)+\overset{n}{\text{ess}}(x-\overset{n+1}{\text{ess}}\,x))+f((x-\overset{n+1}{\text{ess}}(x))-\overset{n}{\text{ess}}(x-\overset{n+1}{\text{ess}}(x)))$$

$$=\{\overset{n+1}{\text{ess}}(x)+\overset{n}{\text{ess}}(x-\overset{n+1}{\text{ess}}(x))\}+f(x-\{\overset{n+1}{\text{ess}}(x)+\overset{n}{\text{ess}}(x-\overset{n+1}{\text{ess}}(x))\}),$$

it is sufficient to prove that $\overset{n}{\text{ess}}(x)=\overset{n+1}{\text{ess}}(x)+\overset{n}{\text{ess}}(x-\overset{n+1}{\text{ess}}(x))$.

But $\quad (\overset{n}{\text{ess}}(x)-\overset{n+1}{\text{ess}}(x))+\omega^n=(\overset{n}{\text{ess}}(x)+\omega^n)-\overset{n+1}{\text{ess}}(x)>x-\overset{n+1}{\text{ess}}(x)$,

so $\quad \overset{n}{\text{ess}}(x)-\overset{n+1}{\text{ess}}(x)\geqq\overset{n}{\text{ess}}(x-\overset{n+1}{\text{ess}}(x))$

and $\quad \overset{n}{\text{ess}}(x)\geqq\overset{n+1}{\text{ess}}(x)+\overset{n}{\text{ess}}(x-\overset{n+1}{\text{ess}}(x))$.

On the other hand we have

$$\overset{n+1}{\text{ess}}(x)+\overset{n}{\text{ess}}(x-\overset{n+1}{\text{ess}}(x))+\omega^n>\overset{n+1}{\text{ess}}(x)+(x-\overset{n+1}{\text{ess}}(x))=x$$

and so $\quad \overset{n+1}{\text{ess}}(x)+\overset{n}{\text{ess}}(x-\overset{n+1}{\text{ess}}(x))\geqq\overset{n}{\text{ess}}(x)$.

Hence $\quad \overset{n}{\text{ess}}(x)=\overset{n+1}{\text{ess}}(x)+\overset{n}{\text{ess}}(x-\overset{n+1}{\text{ess}}(x))$. q. e. d.

8.10 $\overset{n}{\text{ess}}(x)+(x-\overset{n}{\text{ess}}(x))'=\{\overset{n}{\text{ess}}(x)+(x-\overset{n}{\text{ess}}(x))\}'=x'$

in other words, $*'$ is a periodic function of order n. $(n=1,2,3,\cdots)$

9.1 For any given function $f(*)$, we define new functions $f^{\square}(*)$, $f_{\square}(*)$ as follows:

$$f^{\square}(a)=\text{Min}\,(z)\ \ (f(z)>a)$$
$$f_{\square}(a)=\text{Min}\,(z)\ \ (f(z)\geqq a).$$

9.2 If $f(*)$ is a periodic function of order n, then $f^{\square}(*)$ and $f_{\square}(*)$ are periodic of order $n+1$.

PROOF: We have clearly

$$f(x+\omega^n)\geqq\overset{n}{\text{ess}}(x+\omega^n)=\overset{n}{\text{ess}}(x)+\omega^n>x.$$

So it is easily verified that $f(f^{\square}(a))>a$, $f(f_{\square}(a))\geqq a$.
Moreover $a+\omega^n\geqq f^{\square}(a)$, $a+\omega^n\geqq f_{\square}(a)$.
So if $a<\omega^{n+1}$, then $f^{\square}(a)<\omega^{n+1}$ and $f_{\square}(a)<\omega^{n+1}$.
So we have only to prove

$$f^{\square}(a)=\overset{n}{\text{ess}}(a)+f^{\square}(a-\overset{n}{\text{ess}}(a)),$$

$$f_{\square}(a)=\overset{n}{\mathrm{ess}}\,(a)+f_{\square}(a-\overset{n}{\mathrm{ess}}\,(a)).$$

We prove this only for $f^{\square}(*)$, as the proof for $f_{\square}(*)$ goes in the same way. We have

9.3 $f(\overset{n}{\mathrm{ess}}\,(a)+b)$

$$=\overset{n}{\mathrm{ess}}\,(\overset{n}{\mathrm{ess}}\,(a)+b)+f((\overset{n}{\mathrm{ess}}\,(a)+b)-\overset{n}{\mathrm{ess}}\,(\overset{n}{\mathrm{ess}}\,(a)+b))$$
$$=\overset{n}{\mathrm{ess}}\,(a)+\overset{n}{\mathrm{ess}}\,(b)+f((\overset{n}{\mathrm{ess}}\,(a)+b)-(\overset{n}{\mathrm{ess}}\,(a)+\overset{n}{\mathrm{ess}}\,(b)))$$
$$=\overset{n}{\mathrm{ess}}\,(a)+\overset{n}{\mathrm{ess}}\,(b)+f(b-\overset{n}{\mathrm{ess}}\,(b))$$
$$=\overset{n}{\mathrm{ess}}\,(a)+f(b),$$

so $\quad f(\overset{n}{\mathrm{ess}}\,(a)+f^{\square}(a-\overset{n}{\mathrm{ess}}\,(a)))$

$$=\overset{n}{\mathrm{ess}}\,(a)+f(f^{\square}(a-\overset{n}{\mathrm{ess}}\,(a)))$$
$$>\overset{n}{\mathrm{ess}}\,(a)+(a-\overset{n}{\mathrm{ess}}\,(a))=a,$$

therefore $\quad \overset{n}{\mathrm{ess}}\,(a)+f^{\square}(a-\overset{n}{\mathrm{ess}}\,(a))\geqq f^{\square}(a)$.

Conversely we have

$$f((\overset{n}{\mathrm{ess}}\,(a)+b)-\overset{n}{\mathrm{ess}}\,(a))=f(b)=f(\overset{n}{\mathrm{ess}}\,(a)+b)-\overset{n}{\mathrm{ess}}\,(a),$$

thus $f(b-\overset{n}{\mathrm{ess}}\,(a))=f(b)-\overset{n}{\mathrm{ess}}\,(a)$ provided that $b\geqq\overset{n}{\mathrm{ess}}\,(a)$.

Therefore

$$f(f^{\square}(a)-\overset{n}{\mathrm{ess}}\,(a))=f(f^{\square}(a))-\overset{n}{\mathrm{ess}}\,(a)>a-\overset{n}{\mathrm{ess}}\,(a)$$

so $\quad f^{\square}(a)-\overset{n}{\mathrm{ess}}\,(a)\geqq f^{\square}(a-\overset{n}{\mathrm{ess}}\,(a))$

and $\quad f^{\square}(a)\geqq\overset{n}{\mathrm{ess}}\,(a)+f^{\square}(a-\overset{n}{\mathrm{ess}}\,(a))$,

Hence $\quad f^{\square}(a)=\overset{n}{\mathrm{ess}}\,(a)+f^{\square}(a-\overset{n}{\mathrm{ess}}\,(a))$. q. e. d.

From the preceding results follows

9.4 $\quad \forall x\,\forall y\{f(y)>x \vdash\!\dashv y\geqq f^{\square}(x)\}$

$$\wedge\ \forall x\,\forall y\{f(y)\geqq x \vdash\!\dashv y\geqq f_{\square}(x)\}$$
$$\wedge\ \forall x\,\forall y\{f(y)<x \vdash\!\dashv y< f_{\square}(x)\}$$
$$\wedge\ \forall x\,\forall y\{f(y)\leqq x \vdash\!\dashv y< f^{\square}(x)\}$$

for an arbitrary periodic function $f(*)$.

9.5 If $f(*)$, $g(*)$ are periodic of order n, then $f(g(*))$ is periodic of order n,

because $f(g(x))=f(\overset{n}{\mathrm{ess}}(x)+g(x-\overset{n}{\mathrm{ess}}(x)))=\overset{n}{\mathrm{ess}}(x)+f(g(x-\overset{n}{\mathrm{ess}}(x)))$.

THEOREM 1 (REPRESENTATION THEOREM).

Let $\mathfrak{A}(a_1,\cdots,a_k)$ be an arbitrary formula consisting solely of $\forall$, E, $\wedge$, $\vee$, $\neg$, $*<*$, $*=*$, $*'$, special objects, bound variables and $a_1,\cdots,a_k$. Then there exists a set of formulas $\mathfrak{B}_1(a_1,\cdots,a_k),\cdots,\mathfrak{B}_n(a_1,\cdots,a_k)$ consisting solely of $\wedge$, $\vee$, $<$, $=$, special objects, periodic functions, and $a_1,\cdots,a_k$ ($*'$, $+$, $\neg$, $\forall$, E, Min and bound variables may be used to construct a periodic function or a special object, but should be used nowhere else) such that the following sequence (without free variable) is provable.

$$\Gamma_0 \to \forall x_1\cdots\forall x_k(\mathfrak{A}(x_1,\cdots,x_k)\leftrightarrow\mathfrak{B}_1(x_1,\cdots,x_k)),\cdots,$$
$$\forall x_1\cdots\forall x_k(\mathfrak{A}(x_1,\cdots,x_k)\leftrightarrow\mathfrak{B}_n(x_1,\cdots,x_k)).$$

PROOF: Let us call 'specialized symbols' the kind of symbols enumerated above as admissible in $\mathfrak{B}_1(a_1,\cdots,a_k),\cdots,\mathfrak{B}_n(a_1,\cdots,a_k)$. If $\mathfrak{A}(a_1,\cdots,a_k)$ has no $\forall$, E, then our assertion is evident, because $*'$ is periodic and $\neg(a=b)$, $\neg(a<b)$ are equivalent with $a<b\vee a>b$, $a=b\vee a>b$ respectively.

So we prove the theorem by induction on the number of $\forall$, E, in $\mathfrak{A}(a_1,\cdots,a_k)$. Clearly we may assume that $\mathfrak{A}(a_1,\cdots,a_k)$ is of the form $\mathrm{E}x\,\mathfrak{A}'(x,a_1,\cdots,a_k)$ or $\forall x\,\mathfrak{A}'(x,a_1,\cdots,a_k)$. Since $\forall x\,\mathfrak{A}'(x,a_1,\cdots,a_k)$ is equivalent with $\neg\mathrm{E}x\neg\mathfrak{A}'(x,a_1,\cdots,a_k)$, we treat only the case where $\mathfrak{A}(a_1\cdots a_k)$, is of the form $\mathrm{E}x\,\mathfrak{A}'(x,a_1,\cdots,a_k)$.

By the hypothesis of induction, there exist $\mathfrak{B}_1(a_0,a_1,\cdots,a_k),\cdots,\mathfrak{B}_n(a_0,a_1,\cdots,a_k)$ consisting of specialized symbols, such that the following sequence is provable:

$$\to\forall x_0\cdots\forall x_k(\mathfrak{A}'(x_0,\cdots,x_k)\leftrightarrow\mathfrak{B}_1(x_0,\cdots,x_k)),\cdots,$$
$$\forall x_0\cdots\forall x_k(\mathfrak{A}'(x_0,\cdots,x_k)\leftrightarrow\mathfrak{B}_n(x_0,\cdots,x_k)).$$

So we can assume that $\mathfrak{A}(a_1,\cdots,a_k)$ is of the form $\mathrm{E}x\,\mathfrak{B}(x,a_1,\cdots,a_k)$ where $\mathfrak{B}(a_0,a_1,\cdots,a_k)$ consists of specialized symbols and, possibly, a_0.

According to 8.9 and 9.5 $\mathfrak{B}(x,a_1,\cdots,a_k)$ is obtained in combining figures of the form

$$f(x)<g(x),\qquad f(x)=g(x),\qquad f(x)<t,\qquad f(x)=t,$$
$$g(x)>t,\qquad t_1<t_2,\qquad t_1=t_2$$

by $\wedge$ and $\vee$, where $f(*)$ and $g(*)$ are arbitrary periodic functions and t, t_1, t_2 are terms of the form $p(a)$, where $p(*)$ is a periodic function and a is one of the free variables $a_1, \cdots, a_k$ or a special object.

Then we can transform, in virtue of 9.4,

$$f(x) < t \quad \text{to} \quad x < f_{\square}(t)$$
$$f(x) > t \quad \text{to} \quad x \geqq f^{\square}(t)$$
$$f(x) = t \quad \text{to} \quad x \geqq f_{\square}(t) \wedge x < f^{\square}(t).$$

So $\mathfrak{B}(x, a_1, \cdots, a_k)$ may be obtained in combining figures

$$f(x) < g(x), \quad f(x) = g(x), \quad x < t, \quad x > t,$$
$$x = t, \quad t_1 < t_2, \quad t_1 = t_2$$

by $\wedge$ and $\vee$. ($f(x) < g(x)$, etc. merely indicate the *forms* of figures considered.)

Now, the combination of figures by $\wedge$ and $\vee$ can be brought to a form in which the combination by $\vee$ is executed after the combination by $\wedge$ is accomplished. We shall call such normal form, the '$\vee - \wedge$ normal form', in opposision to another normal form, the '$\wedge - \vee$ normal form', in which the combination by $\wedge$, $\vee$ are executed in the inverse order.

If we transform $\mathfrak{B}(x, a_1, \cdots, a_k)$ to the $\vee - \wedge$ normal form, then we have clearly a provable formula

$$\mathsf{E}x\, \mathfrak{B}(x, a_1, \cdots, a_k) \mapsto \mathsf{E}x\, \mathfrak{C}_1(x, a_1, \cdots, a_k) \vee \cdots \vee \mathsf{E}x\, \mathfrak{C}_n(x, a_1, \cdots, a_k),$$

where $\mathfrak{C}_i(x, a_1, \cdots, a_k)$ has no $\vee$, that is, a combination of

$$f(x) < g(x), \quad f(x) = g(x), \quad x < t, \quad x > t,$$
$$x = t, \quad t_1 < t_2, \quad t_1 = t_2$$

by $\wedge$ alone. Then we can reduce the proof of the theorem to the case where $\mathfrak{A}(a_1, \cdots, a_k)$ is of the form $\mathsf{E}x\, \mathfrak{C}(x, a_1, \cdots, a_k)$ in which $\mathfrak{C}(x, a_1, \cdots, a_k)$ is of the type described above. After that, we can transform $\mathfrak{C}(x, a_1, \cdots, a_k)$ to $\mathfrak{C}'(x, a_1, \cdots, a_k) \wedge \mathfrak{C}''$ where $\mathfrak{C}'(x, a_1, \cdots, a_k)$ is a combination of

$$f(x) < g(x), \quad f(x) = g(x), \quad x < t, \quad x = t, \quad x > t$$

by $\wedge$, and $\mathfrak{C}''$ is a combination of $t_1 < t_2$, $t_1 = t_2$ by $\wedge$. By this reduction we can assume, without loss of generality, that $\mathfrak{C}(x, a_1, \cdots, a_k)$ is a combination of

$$f(x) < g(x), \quad f(x) = g(x), \quad x < t, \quad x = t, \quad x > t \qquad \text{by } \wedge .$$

Moreover, if $\mathfrak{C}(x, a_1, \cdots, a_k)$ contains a figure of the type $x = t$, say $x = t_j$, then $\mathsf{E}x\, \mathfrak{C}(x, a_1, \cdots, a_k)$ is equivalent to a formula obtained in combining

$$f(t_j) < g(t_j), \qquad f(t_j) = g(t_j), \qquad t_j < t, \qquad t_j = t, \qquad t_j > t$$

by $\wedge$. Therefore the proof is completed if $x = t$ appears in $\mathfrak{C}(x, a_1, \cdots a_k)$.

So we may assume that $\mathfrak{C}(x, a_1, \cdots, a_k)$ is a combination of

$$f(x) < g(x), \qquad f(x) = g(x), \qquad x < t, \qquad x > t \qquad \text{by } \wedge.$$

If no figures of the type $x < t$ or $x > t$ appear in $\mathfrak{C}(x, a_1, \cdots, a_k)$, $\mathsf{E}x\, \mathfrak{C}(x, a_1 \cdots, a_k)$ is of the form $\mathsf{E}x\, \mathfrak{C}(x)$ and we have easily

$$\rightarrow \quad \mathsf{E}x\, \mathfrak{C}(x) \vdash\!\dashv 0 = 0, \quad \mathsf{E}x\, \mathfrak{C}(x) \vdash\!\dashv 0 > 0$$

and all is proved.

Therefore we may assume that $\mathsf{E}x\, \mathfrak{C}(x, a_1, \cdots, a_k)$ is of the form

$$\mathsf{E}x\{x < t_1 \wedge \cdots \wedge x > t_n \wedge x > t^1 \wedge \cdots \wedge x > t^m \wedge \mathfrak{D}(x)\},$$

where $\mathfrak{D}(x)$ is a combination of $f(x) < g(x)$ and $f(x) = g(x)$ by $\wedge$ and either m or n is $\geqq 1$.

Let $i_1, \cdots, i_n$ be any permutation of $1, \cdots, n$; and let $j_1, \cdots, j_m$ be any permutation of $1, \cdots, m$. Then we have the sequence

$$\begin{aligned}
\rightarrow \mathsf{E}x\, \mathfrak{C}(x, a_1, \cdots, a_k) \vdash\!\dashv \\
&[\{t_1 \leqq \cdots \leqq t_n \wedge t^1 \geqq \cdots \geqq t^m \wedge \mathsf{E}x\, \mathfrak{C}(x, a_1, \cdots, a_k)\} \\
&\quad \vee \cdots \\
&\qquad \cdots \\
&\vee \{t_{i_1} \leqq \cdots \leqq t_{i_n} \wedge t^{j_1} \geqq \cdots \geqq t^{j_m} \wedge \mathsf{E}x\, \mathfrak{C}(x, a_1, \cdots, a_k)\} \\
&\qquad \cdots \\
&\quad \vee \cdots \qquad]
\end{aligned}$$

($t_1 \leqq t_2 \leqq \cdots \leqq t_{n-1} \leqq t_n$ means, as usual, $t_1 \leqq t_2 \wedge \cdots \wedge t_{n-1} \leqq t_n$). Hence we have only to consider the formula

$$t_1 \leqq \cdots \leqq t_n \wedge t^1 \geqq \cdots \geqq t^m \wedge \mathsf{E}x\, \mathfrak{C}(x, a_1, \cdots, a_k).$$

This is equivalent to

$$t_1 \leqq \cdots \leqq t_n \wedge t^1 \leqq \cdots \leqq t^m \wedge \mathsf{E}x\{x < t_1 \wedge x > t^1 \wedge \mathfrak{D}(x)\}\,.$$

Therefore we may restrict ourselves to the following three types:

$$\mathsf{E}x\{x > t \wedge \mathfrak{D}(x)\}$$
$$\mathsf{E}x\{x < t \wedge \mathfrak{D}(x)\}$$
$$\mathsf{E}x\{x < t_1 \wedge x > t_2 \wedge \mathfrak{D}(x)\}\,,$$

where $\mathfrak{D}(x)$ is of the form $f(x) < g(x) \wedge \cdots \wedge f'(x) = g'(x) \wedge \cdots$ and all $f(*)$, $g(*), \cdots$, $f'(*)$, $g'(*), \cdots$ are periodic of order n.

10.1 The case $\mathsf{E}x\{x > t \wedge \mathfrak{D}(x)\}$.

First, let us assume $\mathfrak{D}(a)$ i. e.

$$f(a) < g(a) \wedge \cdots \wedge f'(a) = g'(a) \wedge \cdots\,.$$

This implies

$$\overset{n}{\mathrm{ess}}\,(a) + f(a - \overset{n}{\mathrm{ess}}\,(a)) < \overset{n}{\mathrm{ess}}\,(a) + g(a - \overset{n}{\mathrm{ess}}\,(a)) \qquad \text{etc.}$$

and $$\overset{n}{\mathrm{ess}}\,(a) + f'(a - \overset{n}{\mathrm{ess}}\,(a)) = \overset{n}{\mathrm{ess}}\,(a) + g'(a - \overset{n}{\mathrm{ess}}\,(a)) \qquad \text{etc.}$$

so $$f(a - \overset{n}{\mathrm{ess}}\,(a)) < g(a - \overset{n}{\mathrm{ess}}\,(a)) \qquad \text{etc.}$$

and $$f'(a - \overset{n}{\mathrm{ess}}\,(a)) = g'(a - \overset{n}{\mathrm{ess}}\,(a)) \qquad \text{etc.}$$

Hence we have

$$\begin{aligned} f(t + \omega^n + a) &= \overset{n}{\mathrm{ess}}\,(t + \omega^n + a) + f((t + \omega^n + a) - \overset{n}{\mathrm{ess}}\,(t + \omega^n + a)) \\ &= \overset{n}{\mathrm{ess}}\,(t + \omega^n + a) + f(a - \overset{n}{\mathrm{ess}}(a)) \\ &< \overset{n}{\mathrm{ess}}\,(t + \omega^n + a) + g(a - \overset{n}{\mathrm{ess}}\,(a)) = g(t + \omega^n + a)\,, \end{aligned}$$

therefore $f(t + \omega^n + a) < g(t + \omega^n + a)$ etc.

After a similar calculation as above, we have

$$f'(t + \omega^n + a) = g'(t + \omega^n + a) \qquad \text{etc.}$$

Therefore $\mathfrak{D}(t + \omega^n + a)$. Henceforth follows

$$\mathfrak{D}(a) \rightarrow t + \omega^n + a > t \wedge \mathfrak{D}(t + \omega^n + a)$$

so $$\mathfrak{D}(a) \rightarrow \mathsf{E}x\{x > t \wedge \mathfrak{D}(x)\}$$

and $$\mathsf{E}x\, \mathfrak{D}(x) \rightarrow \mathsf{E}(x)\{x > t \wedge \mathfrak{D}(x)\}\,.$$

So we have

$$\rightarrow \forall y\,[\mathsf{E}x\{x > \breve{t} \wedge \mathfrak{D}(x)\} \mapsto 0=0]\,,$$
$$\forall y\,[\mathsf{E}x\{x > \breve{t} \wedge \mathfrak{D}(x)\} \mapsto 0>0]\,,$$

where $\breve{t}$ is obtained from t by substituting y for the free variable in t if such a variable exists, or else $\breve{t}$ is t itself.

10.2 The case $\mathsf{E}x\{x < t \wedge \mathfrak{D}(x)\}$.

Put $s_0 = \mathrm{Min}\,(z)\,\mathfrak{D}(z)$; s_0 is a special object. Then

$$\mathsf{E}x\,\mathfrak{D}(x) \rightarrow \mathsf{E}x\{x < t \wedge \mathfrak{D}(x)\} \mapsto s_0 < t$$

so
$$\rightarrow \forall y\,[\mathsf{E}x\{x < \breve{t} \wedge \mathfrak{D}(x)\} \mapsto s_0 < \breve{t}\}]\,,$$
$$\forall y\,[\mathsf{E}x\{x < \breve{t} \wedge \mathfrak{D}(x)\} \mapsto 0 > 0]\,,$$

where $\breve{t}$ has the same meaning as in 10.1.

10.3 The case $\mathsf{E}x\{x < t_1 \wedge x > t_2 \wedge \mathfrak{D}(x)\}$.

We introduce a new function $h(*)$ by

$$h(a) = \mathrm{Min}\,(z)[\{z > a \wedge \mathfrak{D}(z)\} \vee \{\neg\,\mathsf{E}y\,\mathfrak{D}(y) \wedge z=a\}].$$

If $\neg\,\mathsf{E}y\,\mathfrak{D}(y)$, then $h(*)$ is periodic of order 1.

Next we shall assume $\mathsf{E}y\,\mathfrak{D}(y)$. We see easily by the same consideration as in 10.1

$$\forall x\,\mathsf{E}y(y > x \wedge \mathfrak{D}(y))$$

so
$$\forall x\{h(x) > x \wedge \mathfrak{D}(h(x))\}\,.$$

So
$$f(\overset{n+1}{\mathrm{ess}}(x) + h(x - \overset{n+1}{\mathrm{ess}}(x))) = \overset{n+1}{\mathrm{ess}}(x) + f(h(x - \overset{n+1}{\mathrm{ess}}(x)))$$
$$< \overset{n+1}{\mathrm{ess}}(x) + g(h(x - \overset{n+1}{\mathrm{ess}}(x))) = g(\overset{n+1}{\mathrm{ess}}(x) + h(x - \overset{n+1}{\mathrm{ess}}(x)))\,.$$

Therefore

$$f(\overset{n+1}{\mathrm{ess}}(x) + h(x - \overset{n+1}{\mathrm{ess}}(x))) < g(\overset{n+1}{\mathrm{ess}}(x) + h(x - \overset{n+1}{\mathrm{ess}}(x))) \qquad \text{etc.}$$

In the similar way we have

$$f'(\overset{n+1}{\mathrm{ess}}(x) + h(x - \overset{n+1}{\mathrm{ess}}(x))) = g'(\overset{n+1}{\mathrm{ess}}(x) + h(x - \overset{n+1}{\mathrm{ess}}(x))) \qquad \text{etc.}$$

Hence
$$\mathfrak{D}(\overset{n+1}{\mathrm{ess}}(x) + h(x - \overset{n+1}{\mathrm{ess}}(x)))\,.$$

Therefore
$$\overset{n+1}{\mathrm{ess}}(x) + h(x - \overset{n}{\mathrm{ess}}(x)) \geqq h(x)\,.$$

On the other hand

$$f(h(x) - \overset{n+1}{\mathrm{ess}}(x)) = f(h(x)) - \overset{n+1}{\mathrm{ess}}(x)$$

$$<g(h(x))-\overset{n+1}{\mathrm{ess}}(x)=g(h(x)-\overset{n+1}{\mathrm{ess}}(x)).$$

So $$f(h(x)-\overset{n+1}{\mathrm{ess}}(x))<g(h(x)-\overset{n+1}{\mathrm{ess}}(x)) \qquad \text{etc.}$$

In the similar way we have

$$f'(h(x)-\overset{n+1}{\mathrm{ess}}(x))=g'(h(x)-\overset{n+1}{\mathrm{ess}}(x)) \qquad \text{etc.}$$

Therefore $\mathfrak{D}(h(x)-\overset{n+1}{\mathrm{ess}}(x))$,

so $$h(x)-\overset{n+1}{\mathrm{ess}}(x)\geqq h(x-\overset{n+1}{\mathrm{ess}}(x))$$

and $$h(x)\geqq \overset{n+1}{\mathrm{ess}}(x)+h(x-\overset{n+1}{\mathrm{ess}}(x)).$$

This gives, together with $\overset{n+1}{\mathrm{ess}}(x)+h(x-\overset{n+1}{\mathrm{ess}}(x))\geqq h(x)$,

$$h(x)=\overset{n+1}{\mathrm{ess}}(x)+h(x-\overset{n+1}{\mathrm{ess}}(x)).$$

Moreover, as $\mathsf{E}y\,\mathfrak{D}(y)$, we may assume $\mathfrak{D}(a)$; so, by the same consideration as in 10.1, we have

$$\mathfrak{D}(a-\overset{n}{\mathrm{ess}}(a)) \qquad \text{and} \qquad \mathfrak{D}(t+\omega^n+(a-\overset{n}{\mathrm{ess}}(a))).$$

Hence $$h(x)\leqq x+\omega^n+(a-\overset{n}{\mathrm{ess}}(a)), \qquad \text{so}$$
$$\forall x\,(x<\omega^{n+1}\vdash h(x)<\omega^{n+1}).$$

Thus $h(*)$ is periodic of order $n+1$.

Now it is easily verified that

$$\mathsf{E}x\,\mathfrak{D}(x)\to \mathsf{E}x\,\{x<t_1\wedge x>t_2\wedge \mathfrak{D}(x)\}\vdash t_1>h(t_2)$$

and $$\neg\,\mathsf{E}x\,\mathfrak{D}(x)\to \mathsf{E}x\{x<t_1\wedge x>t_2\wedge \mathfrak{D}(x)\}\vdash 0>0.$$

Hence

$$\to \forall y_1\,\forall y_2\,[\mathsf{E}x\{x<\tilde{t}_1\wedge x>\tilde{t}_2\wedge \mathfrak{D}(x)\}\vdash \tilde{t}_1>h(\tilde{t}_2)],$$
$$\forall y_1\,\forall y_2\,[\mathsf{E}x\{x<\tilde{t}_1\wedge x>\tilde{t}_2\wedge \mathfrak{D}(x)\}\vdash 0>0],$$

where $\tilde{t}_1$ or $\tilde{t}_2$ is obtained from t_1 or t_2 by substituting y_1 or y_2 for the variable (if contained) in t_1 or t_2 respectively. Thus the theorem is completely proved.

Now we shall prove the consistency of 1.1, $\cdots$, 1.16. Clearly we have only to prove the following:

11. If $\mathfrak{A}(*,*)$ contains only $*<*$ and $*=*$ as predicates, $*'$ as function and 0, ω as special objects and does not contain Min, then 1.15 and 1.16 are provable.

First we prove $\Gamma'_0 \rightarrow 1.15$. Let $[z]$ be $\forall z_1 \cdots \forall z_k$ and let $\mathfrak{A}(x, s)$ be of the form $\mathfrak{A}(x, s, z_1, \cdots, z_k)$. According to theorem 1 there exist $\mathfrak{C}_1(*, \cdots, *), \cdots, \mathfrak{C}_n(*, \cdots, *)$ consisting solely of $\wedge$, $\vee$, $<$, $=$, special objects and periodic functions satisfying the following sequence:

$$
\begin{aligned}
\rightarrow\ & \forall x\, \forall z\, \forall z_1 \cdots \forall z_k \{\mathfrak{A}(x, z, z_1, \cdots, z_k) \vdash\!\dashv \mathfrak{C}_1(x, z, z_1, \cdots, z_k)\}\,, \\
& \cdots, \\
& \forall x\, \forall z\, \forall z_1 \cdots \forall z_k \{\mathfrak{A}(x, z, z_1, \cdots, z_k) \vdash\!\dashv \mathfrak{C}_n(x, z, z_1, \cdots, z_k)\}\,.
\end{aligned}
$$

Therefore we may assume that $\mathfrak{A}(*, \cdots, *)$ consists of the symbols enumerated above.

Now we bring $\mathfrak{A}(x, z)$ (precisely: $\mathfrak{A}(x, z, z, \cdots, z_k)$) to the $\vee - \wedge$ normal form

$$\mathfrak{B}_1(x, z) \vee \cdots \vee \mathfrak{B}_L(x, z)\,.$$

Then $\mathfrak{B}_i(x, z)$ has no logical symbol other than $\wedge$. Clearly we have

$$
\begin{aligned}
& \Gamma'_0,\ \forall x\, \forall y\, \forall z(\mathfrak{A}(x, z) \wedge \mathfrak{A}(y, z) \vdash x = y) \\
& \qquad \rightarrow \forall x\, \forall y\, \forall z(\mathfrak{B}_i(x, z) \wedge \mathfrak{B}_i(y, z) \vdash x = y)\,.
\end{aligned}
$$

So, if 1.15 is true for the case when $\mathfrak{A}(*, *)$ is $\mathfrak{B}_i(*, *)$, i. e. if

$$
\begin{aligned}
& \Gamma'_0,\ \forall x\, \forall y\, \forall z(\mathfrak{B}_i(x, z) \wedge \mathfrak{B}_i(y, z) \vdash x = y) \\
& \qquad \rightarrow \exists x\, \forall y\, \forall z\{\mathfrak{B}_i(y, z) \wedge z < b \vdash y < x\} \quad \text{for each } i,
\end{aligned}
$$

then, together with

$$
\begin{aligned}
& \forall y\, \forall z(\mathfrak{B}_1(y, z) \wedge z < b \vdash y < c_1), \qquad c_1 < d, \\
& \cdots, \\
& \forall y\, \forall z(\mathfrak{B}_L(y, z) \wedge z < b \vdash y < c_L), \qquad c_L < d \\
& \qquad \rightarrow \forall y\, \forall z(\mathfrak{A}(y, z) \wedge z < b \vdash y < d),
\end{aligned}
$$

we see easily that 1.15 is true in general.

Therefore we have only to prove 1.15 for the case when $\mathfrak{A}(x, z, a_1, \cdots, a_k)$ is one of the $\mathfrak{B}_i(x, z, a_1, \cdots, a_k)$, which may be written as

$$x < f(t_1) \wedge \cdots \wedge x = g(t_2) \wedge \cdots \wedge x > h(t_3) \wedge \cdots \wedge \mathfrak{D}(x) \wedge \mathfrak{C}\,.$$

Here each of $t_1, \cdots, t_2, \cdots, t_3, \cdots$ is either one of $z, a_1, \cdots, a_k$ or a special object; $f(*), \cdots,\ g(*), \cdots,\ h(*), \cdots$ are periodic functions; $\mathfrak{D}(x)$ has neither special object nor variable other than x; and $\mathfrak{C}$ has no x.

If $\mathfrak{B}_i(x, z)$ contains a figure of the form $x < f(t)$ or $x = g(t)$, we have easily $\mathsf{E}x\, \forall y\, \forall z\{\mathfrak{B}_i(y, z) \wedge z < a \vdash y < x\}$, and 1.15 is proved.

If $\mathfrak{B}_i(x, z)$ does not contain any figure of above forms, then $\mathfrak{B}_i(x, z, a_1, \cdots, a_k)$ will be of the form

$$x > h(t) \wedge \cdots \wedge \mathfrak{D}(x) \wedge \mathfrak{C}.$$

Here $\mathfrak{D}(x)$ is of the form $f(x) < g(x) \wedge \cdots \wedge f'(x) = g'(x) \cdots$, and $f(*)$, $g(*), \cdots, f'(*), g'(*), \cdots$ are periodic of order n. If $b > h(t) \wedge \cdots \wedge \mathfrak{D}(b) \wedge \mathfrak{C}$ holds, then

$$b + \omega^n + (b - \overset{n}{\mathrm{ess}}\,(b)) > h(t) \wedge \cdots \wedge \mathfrak{D}(b + \omega^n + (b - \overset{n}{\mathrm{ess}}\,(b))) \wedge \mathfrak{C},$$

and there are two x's for one z which satisfy $\mathfrak{B}_i(x, z)$. So

$$\Gamma_0,\ \forall x\, \forall y\, \forall z(\mathfrak{B}_i(x, z) \wedge \mathfrak{B}_i(y, z) \vdash x = y) \rightarrow \forall x\, \forall y\, \neg\, \mathfrak{B}_i(x, y)$$

and 1.15 is proved.

Now we shall consider $\Gamma_0 \rightarrow 1.16$.

In the same way as above, we may assume that $\mathfrak{A}(x, z)$ consists solely of $\wedge$, $\vee$, $<$, $=$, periodic functions, special objects and bound variables. Suppose that all the periodic functions in $\mathfrak{A}(*, \cdots, *)$ are of order n. We are going to prove

$$\begin{aligned} \rightarrow [z]\{ &\forall x\, \forall y\, \forall s\, (\mathfrak{A}(x, s) \wedge \mathfrak{A}(y, s) \vdash x = y) \\ &\vdash \mathsf{E}x\, \forall y(x < a + \omega^{n+2} \wedge \neg\, (\mathfrak{A}(x, y) \wedge y < a))\} \end{aligned}$$

where a is a free variable not contained in $\mathfrak{A}(x, z)$. It is sufficient to prove, for each i $(i = 1, \cdots, L)$, the sequence

11.1 $\Gamma_0,\ \forall x\, \forall y\, \forall z(\mathfrak{A}(x, z) \wedge \mathfrak{A}(y, z) \vdash x = y)$

$$\rightarrow \mathsf{E}y\, \forall x\, \forall z\{y < a + \omega^{n+2} \wedge \neg\, (y < x < a + \omega^{n+2} \wedge \mathfrak{B}_i(x, z) \wedge z < a)\}.$$

where $\mathfrak{B}_1(b, c) \vee \cdots \vee \mathfrak{B}_L(b, c)$ is a $\vee - \wedge$ normal form of $\mathfrak{A}(b, c)$ (b and c are free variables not contained in $\mathfrak{A}(x, z)$). It is to be noted that

11.2 $\forall x\, \forall y\, \forall z(\mathfrak{A}(x, z) \wedge \mathfrak{A}(y, z) \vdash x = y)$

$$\rightarrow \forall x\, \forall y\, \forall z(\mathfrak{B}_i(x, z) \wedge \mathfrak{B}_i(y, z) \vdash x = y)$$

is provable. As $\mathfrak{B}_i(x, z)$ has no logical symbol other than $\wedge$, we may and shall assume that it is a combination of figures of the forms

$$x < t, \quad x = t, \quad x > t, \quad f(x) < g(x), \quad f(x) = g(x), \quad \mathfrak{D}(z)$$

by $\wedge$, where $f(*)$, $g(*)$ are periodic functions of order n and neither t nor $\mathfrak{D}(z)$ contains x; consequently, if t contains z, then t should be of the form $h(z)$, $h(*)$ being a periodic function of order $n+1$.

If $\mathfrak{B}_i(x, z)$ contains a figure of the form $x < h(z)$ or of the form $x = h(z)$, then

$$\Gamma_0 \to \neg\, (a + \omega^{n+1} < b \wedge \mathfrak{B}_i(b, c) \wedge c < a)$$

can be proved easily, and 11.1 follows at once. If $\mathfrak{B}_i(x, z)$ does not contain such a figure, then as 11.2 is provable, we have only to prove the sequence

11.3 $\Gamma_0,\ t_1 \leqq \cdots \leqq t_N < a + \omega^{n+2} \leqq t_{N+1} < \cdots \leqq t_M,$

$$\forall x\, \forall y\, \forall z(\mathfrak{B}_i(x, z) \wedge \mathfrak{B}_i(y, z) \vdash x = y)$$

$$\to \forall x\, \forall z\, \neg\, (t_N < x < a + \omega^{n+2} \wedge \mathfrak{B}_i(x, z) \wedge z < a)$$

for each permutation $t_1, \cdots, t_M$ of all t without z and for each N, $0 \leqq N \leqq M$.

Proof is obvious when $\mathfrak{B}_i(x, z)$ contains $x < t_j$ or $x = t_j$ for some $j \leqq N$; in the remaining case we have, as in the last part of the proof of $\Gamma_0 \to 1.15$, the provable sequence

$$\Gamma_0,\ t_1 \leqq \cdots \leqq t_N < a + \omega^{n+2} \leqq t_{N+1} \leqq \cdots \leqq t_M$$

$$\to \{t_N < b < a + \omega^{n+2} \wedge \mathfrak{B}_i(b, c) \vdash \mathfrak{B}_i(b + \omega^{n+1} + (b - \overset{n+1}{\mathrm{ess}}(b)),\ c)\}$$

$$\wedge\, b < b + \omega^n + (b - \overset{n+1}{\mathrm{ess}}(b))$$

and so

$$\Gamma_0,\ t_1 \leqq \cdots \leqq t_N < a + \omega^{n+2} \leqq t_{N+1} \leqq \cdots \leqq t_M,$$

$$\forall x\, \forall y\, \forall z(\mathfrak{B}_i(x, z) \wedge \mathfrak{B}_i(y, z) \vdash x = y)$$

$$\to \neg\, (t_N < b < a + \omega^{n+2} \wedge \mathfrak{B}_i(b, z))$$

from which 11.3 follows at once.

Thus 11 and so the consistency of $1.1, \cdots, 1.16$ is established.

Appendix

In this appendix we prove a metatheorem called Representation Theorem in the theory of arithmetics. As the theory of arithmetics we consider the following axioms:

12 (THEORY OF ARITHMETICS)

12.1 $\forall x(x=x)$

12.2 $\forall x\, \forall y(x<y \vee x=y \vee y<x)$

12.3 $\forall x\, \forall y \urcorner (x=y \wedge x<y)$

12.4 $\forall x\, \forall y \urcorner (x<y \wedge y<x)$

12.5 $\forall x\, \forall y\, \forall z(x<y \wedge y<z \vdash x<z)$

12.6 $\forall x(1 \leqq x)$

12.7 $\forall x\, \forall y(x<y \vdash x' \leqq y)$

12.8 $\forall x(x<x')$

12.9 $\forall x\, \forall y(x'=y' \vdash x=y)$

12.10 $\forall x\, \exists y(1<x \vdash y'=x)$

12.11 $[z]\, \forall x\, \forall y\{x=y \vdash (\mathfrak{A}(x) \vdash\dashv \mathfrak{A}(y))\}$

(12.11 stands for a finite number of axioms of this kind.)

The series of axioms 12.1,$\cdots$, 12.11 will be denoted by Γ_a; but in this appendix these axioms will not be written explicitly in the left sides of sequences, that is, $\Gamma \rightarrow \Delta$ means $\Gamma_a, \Gamma \rightarrow \Delta$. Also a formula $\mathfrak{A}$ will be said to be provable when $\Gamma_a \rightarrow \mathfrak{A}$ is provable.

THEOREM 2 (REPRESENTATION THEOREM).

Let $\mathfrak{A}(a_1, \cdots, a_k)$ be an arbitrary formula consisting solely of $\forall$, $\exists$, $\wedge$, $\vee$, $\urcorner$, $*<*$, $*=*$, $*'$, 1, bound variables and $a_1, \cdots, a_k$. Then there exists suitable formula $\mathfrak{B}(a_1, \cdots, a_k)$ consisting just of $\wedge, \vee, <, =, 1, *'$ and $a_1, \cdots, a_k$, so that the following axiom is provable

$$\forall x_1 \cdots \forall x_k(\mathfrak{A}(x_1, \cdots, x_k) \vdash\dashv \mathfrak{B}(x_1, \cdots, x_k))$$

PROOF: From now on, we shall write for convenience's sake $x+n$ for $x'^{\cdots\prime}$, as usual, where n is the number of $'$'s in $'\cdots'$. In the same way as in the proof of theorem 1, we can assume that $\mathfrak{A}(a_1, \cdots, a_k)$ is of the form $\exists x\, \mathfrak{B}(x, a_1, \cdots, a_k)$ and $\mathfrak{B}(x, a_1, \cdots, a_k)$ is obtained in combining the figures of the forms

$$x'^{\cdots\prime} < x'^{\cdots\prime}, \quad x'^{\cdots\prime} = x'^{\cdots\prime}, \quad x'^{\cdots\prime} < t, \quad x'^{\cdots\prime} = t, \quad x'^{\cdots\prime} > t$$

by $\wedge$, where t is of the form $a'^{\cdots\prime}$, a being either 1 or one of the free variables $a_1, \cdots, a_k$.

Clearly we can remove the case where $\mathfrak{B}(x, a_1, \cdots, a_k)$ contains $x'^{\cdots\prime} < x'^{\cdots\prime}$ or $x'^{\cdots\prime} = x'^{\cdots\prime}$. Therefore we can assume that $\mathfrak{B}(x, a_1, \cdots, a_k)$

is a combination of the figures of the forms

$$x+l<t, \qquad x+m=t, \qquad x+n>t \qquad \text{by } \wedge .$$

Let m_0 be the greatest of the numbers l, m and n such that $x+l<t$, $x+m=t$ and $x+n>t$ are contained in $\mathfrak{B}(x, a_1\cdots, a_k)$.

In virtue of $a<b \mapsto a'<b'$ and $a=b \mapsto a'=b'$, $\mathsf{E}x\,\mathfrak{B}(x, a_1,\cdots, a_k)$ is equivalent with a formula of the following form

$$\mathsf{E}x(x+m_0<t_1 \wedge\cdots\wedge x+m_0=t_2 \wedge\cdots\wedge t_3<x+m_0 \wedge\cdots),$$

hence it is equivalent with

$$\mathsf{E}y(m_0<y \wedge y<t_1 \wedge\cdots\wedge y=t_2 \wedge\cdots\wedge t_3<y \wedge\cdots)$$

or $$\mathsf{E}y(y<t_1 \wedge\cdots\wedge y=t_2 \wedge\cdots\wedge t_3<y \wedge\cdots)$$

according as m_0 is positive or zero.

If this formula contains a figure of the form $y=t$, then $\mathsf{E}x\,\mathfrak{B}(x, a_1,\cdots, a_k)$ is equivalent to

$$m_0<t \wedge t<t_1 \wedge\cdots\wedge t=t_2 \wedge\cdots\wedge t_3<t \wedge\cdots,$$

in which $m_0<t$ should be omitted when m_0 is zero, and the theorem is proved.

Therefore we have only to prove the theorem where $\mathsf{E}x\,\mathfrak{B}(x, a_1,\cdots, a_k)$ is of the form

$$\mathsf{E}x(x<t_1 \wedge\cdots\wedge x<t_m \wedge t^1<x \wedge\cdots\wedge t^n<x).$$

In the same way as in the proof of theorem 1 we can reduce this case to the following three simple cases:

13.1 $\mathsf{E}x(x<t)$,

13.2 $\mathsf{E}x(t<x)$,

13.3 $\mathsf{E}x(t_1<x \wedge x<t_2)$,

where t_1, t_2 and t are of the form $a_i^{\prime\cdots\prime}$ or $1^{\prime\cdots\prime}$. Since we have clearly

$$\mathsf{E}x(x<t) \mapsto 1<t, \qquad \mathsf{E}x(t<x) \mapsto 1=1 \qquad \text{and}$$

$$\mathsf{E}x(t_1<x \wedge x<t_2) \mapsto t_1'<t_2,$$

the theorem is completely proved.

From theorem 2 follows immediately

THEOREM 3.

Let $\mathfrak{A}$ be an arbitrary formula consisting solely of $\forall$, E, $\wedge$, $\vee$, $\neg$, $*<*$, $*=*$, $*'$, 1 and bound variables. Then either $\Gamma_a \to \mathfrak{A}$ is provable or $\Gamma_a \to \neg\,\mathfrak{A}$ is provable.

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Reference

[1] G. Gentzen. Untersuchungen über das logische Schliessen I, II. Math. Zeitschr., 39 (1934).