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Concave modulars.

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We have defined and discussed modulars on semi-ordered linear space in a book[1]. Let R be a semi-ordered linear space and universally continuous, that is, for every system of positive elements $a_\lambda \in R$ $(\lambda \in \Lambda)$ there exists $\bigcap_{\lambda \in \Lambda} a_\lambda$. A functional $m(x)$ $(x \in R)$ is called a *modular* on R, if 1) $0 \leqq m(x) \leqq +\infty$ for every $x \in R$, 2) $m(\xi a)=0$ for every $\xi \geqq 0$ implies $a=0$, 3) for any $a \in R$ we can find $\alpha > 0$ such that $m(\alpha a) < +\infty$, 4) for each $x \in R$, $m(\xi x)$ is a convex function of ξ: $m\left(\frac{\alpha+\beta}{2}x\right) \leqq \frac{1}{2}\{m(\alpha x)+m(\beta x)\}$, 5) $|x| \leqq |y|$ implies $m(x) \leqq m(y)$, 6) $x \frown y=0$ implies $m(x+y)=m(x)+m(y)$, 7) $0 \leqq x_\lambda \uparrow_{\lambda \in \Lambda} x_0$ implies $m(x_0)=\sup_{\lambda \in \Lambda} m(x_\lambda)$.

In this paper we shall consider a functional $m(x)$ $(x \in R)$ which satisfies instead of 4) the condition: $m(\xi x)$ is a concave function of $\xi \geqq 0$, i.e., we define a *concave modular* $m(x)$ $(x \in R)$ by the postulates: 1) $0 \leqq m(x) < +\infty$, 2) $m(x)=0$ implies $x=0$, 3) $|x| \leqq |y|$ implies $m(x) \leqq m(y)$, 4) $x \frown y=0$ implies $m(x+y)=m(x)+m(y)$, 5) $m(\xi x)$ is a concave function of $\xi \geqq 0$:

$$m\left(\frac{\lambda+\mu}{2}x\right) \geqq \frac{1}{2}\{m(\lambda x)+m(\mu x)\} \qquad \text{for } \lambda, \mu \geqq 0,$$

6) $\lim_{\xi \to 0} m(\xi x)=0$, 7) $0 \leqq x_\nu \uparrow_{\nu=1}^{\infty}$, $\sup_{\nu \geqq 1} m(x_\nu) < +\infty$ implies the existence of an element x_0 for which $x_\nu \uparrow_{\nu=1}^{\infty} x_0$ and $m(x_0)=\lim_{\nu \to \infty} m(x_\nu)$.

Concerning the concave modulars $m(x)$ on R, we can prove

$$m(x+y) \leqq m(x)+m(y) \qquad \text{for every } x, y \in R.$$

Thus, every concave modular $m(x)$ on R is a quasi-norm by which R is a Fréchet space.

For a concave modular $m(x)$ on R, we can prove easily

$$\frac{m(\xi x)}{\xi} \leqq \frac{m(\eta x)}{\eta} \qquad \text{for } \xi > \eta > 0,$$

and hence there exists the limit

$$m_1(x) = \lim_{\xi \to \infty} \frac{m(\xi x)}{\xi} \qquad \text{for every } x \in R.$$

A concave modular $m(x)$ is said to be of the *first kind*, if $m_1(x)=0$ implies $x=0$, and of the *second kind*, if $m_1(x)=0$ for every $x \in R$. With this definition, R may be devided in two normal manifolds F and S such that $m(x)$ is of the first kind in F and of the second kind in S.

If $m(x)$ is of the first kind in R, then $m_1(x)$ is a norm on R and

$$m_1(x+y) = m_1(x) + m_1(y) \qquad \text{for } x, y \geqq 0.$$

By this norm $m_1(x)$, R is complete, and hence a so-called generalized L_1-space, if and only if $\sup_{m_1(x) \leq 1} m(x) < +\infty$.

If $m(x)$ is of the second kind on R, and R has no discrete element, then there is no bounded linear functional on R except for the identical zero 0.

Finally we shall consider the case where R is a discrete space with a basis $a_\lambda \geqq 0$ $(\lambda \in \Lambda)$, that is, every positive element $x \in R$ may be represented uniquely in the form $x = \bigcup_{\lambda \in \Lambda} \alpha_\lambda a_\lambda$. In this case, the conjugate space of R is a generalized (m)-space, if and only if we can find $\alpha_\lambda > 0$ $(\lambda \in \Lambda)$ such that

$$\inf_{\lambda \in \Lambda} m(\alpha_\lambda a_\lambda) > 0, \quad \lim_{\xi \to 0} \sup_{\lambda \in \Lambda} m(\xi \alpha_\lambda a_\lambda) = 0.$$

As applications, we consider the following modulars in the space of measurable functions:

$$m(\varphi) = \int_0^1 |\varphi(t)|^{p(t)} dt, \qquad 0 < p(t) < 1 \text{ for } 0 < t < 1,$$

$$m(\varphi) = \int_0^1 \frac{|\varphi(t)|}{1+\varphi(t)} dt,$$

and in the space of number sequences $(\xi_1, \xi_2, \cdots)$

$$m(\xi_1, \xi_2, \cdots) = \sum_{\nu=1}^{\infty} |\xi_\nu|^{p_\nu}, \qquad 0 < p_\nu < 1 \ (\nu = 1, 2, \cdots),$$

$$m(\xi_1, \xi_2, \cdots) = \sum_{\nu=1}^{\infty} \frac{1}{2^\nu} \frac{|\xi_\nu|}{1 + |\xi_\nu|}.$$

§ 1. Concave modulars.

Let R be a continuous semi-ordered linear space. A functional $m(x)$ $(x \in R)$ on R is said to be a *concave modular* on R, if

1) $0 \leqq m(x) < +\infty$ for every $x \in R$,
2) $m(x) = 0$ implies $x = 0$,
3) $|x| \leqq |y|$ implies $m(x) \leqq m(y)$,
4) $x \frown y = 0$ implies $m(x+y) = m(x) + m(y)$,
5) $m(\xi x)$ is a concave function of $\xi \geqq 0$:

$$m\left(\frac{\lambda + \mu}{2} x\right) \geqq \frac{m(\lambda x) + m(\mu x)}{2} \qquad \text{for } \lambda, \ \mu \geqq 0,$$

6) $\lim_{\xi \to 0} m(\xi x) = 0$,
7) $0 \leqq x_\nu \uparrow_{\nu=1}^{\infty}$, $\sup_{\nu \geqq 1} m(x_\nu) < +\infty$ implies the existence of x_0 such that

$$x_\nu \uparrow_{\nu=1} x_0, \quad m(x_0) = \lim_{\nu \to \infty} m(x_\nu).$$

On account of the postulate 3), $m(\xi x)$ is a non-decreasing function of $\xi \geqq 0$, and $m(0) = 0$ by 4). Thus we can conclude easily from 5), 6) that $m(\xi x)$ is a continuous concave function of $\xi \geqq 0$ and

(1) $m((\lambda\xi + \mu\eta)x) \geqq \lambda\, m(\xi x) + \mu m(\eta x)$ for $\lambda + \mu = 1$; $\lambda, \mu, \xi, \eta \geqq 0$.

Putting $\eta = 0$ in (1), we obtain

$$\frac{m(\lambda x)}{\lambda} \geqq \frac{m(\mu x)}{\mu} \qquad \text{for } 0 < \lambda < \mu. \tag{2}$$

As $m(\xi x)$ is a continuous concave function of $\xi \geqq 0$, we have

$$\frac{m((\lambda + \xi)x) - m(\xi x)}{\lambda} \leqq \frac{m((\mu + \eta)x) - m(\eta x)}{\mu}$$

for $\xi > \eta \geqq 0$, $\lambda, \mu > 0$. Especially, putting $\eta = 0$, $\lambda = \mu > 0$, we obtain

$$\frac{m((\lambda+\xi)x)-m(\xi x)}{\lambda} \leqq \frac{m(\lambda x)}{\lambda}.$$

Thus we have

(3) $\quad m((\lambda+\mu)x) \leqq m(\lambda x)+m(\mu x) \qquad$ for $\lambda, \mu \geqq 0$.

THEOREM 1.1 *$[p_\nu]\downarrow_{\nu=1}^{\infty} 0$ implies $\lim_{\nu\to\infty} m([p_\nu]x)=0$ for every $x \in R$.*

PROOF. If $[p_\nu]\downarrow_{\nu=1}^{\infty} 0$, then we have $1-[p_\nu]\uparrow_{\nu=1}^{\infty} 1$, and hence by 7)

$$\lim_{\nu\to\infty} m((1-[p_\nu])a)=m(a) \qquad \text{for every } a \geqq 0.$$

As $m(x)=m(|x|)$ by 3) and $m((1-[p_\nu])a)+m([p_\nu]a)=m(a)$ by 4), we conclude hence $\lim_{\nu\to\infty} m([p_\nu]x)=0$ for every $x \in R$.

In the sequel, we assume that a concave modular $m(x)$ is defined on R.

THEOREM 1.2. *R is superuniversally continuous and totally continuous.*

PROOF. For an orthogonal system a_λ $(\lambda \in \Lambda)$ and a positive element a, we have by 3) and 4)

$$\sum_{\nu=1}^{\kappa} m([a_{\lambda_\nu}]a)=m(\sum_{\nu=1}^{\kappa}[a_{\lambda_\nu}]a) \leqq m(a)$$

for every finite number of elements $\lambda_\nu \in \Lambda$ $(\nu=1,2,\cdots,\kappa)$, and hence we have $m([a_\lambda]a)=0$ for every $\lambda \in \Lambda$ up to at most countable $\lambda \in \Lambda$. Thus we have $[a_\lambda]a=0$ except for at most countable $\lambda \in \Lambda$. Therefore R is superuniversally continuous by MSLS[1] Theorem 13.2.

If $[p] \geqq [p_{\nu,\mu}]\downarrow_{\mu=1}^{\infty} 0$ $(\nu=1,2,\cdots)$, then we have by Theorem 1.1

$$\lim_{\mu\to\infty} m([p_{\nu,\mu}]p)=0 \qquad (\nu=1,2,\cdots).$$

Thus we can find $\mu_{\nu,\rho}\uparrow_{\rho=1}^{\infty}+\infty$ $(\nu=1,2,\cdots)$ such that

$$\sum_{\nu=1}^{\infty} m([p_{\nu,\mu_{\nu,\rho}}]p) \leqq \frac{1}{\rho} \qquad (\rho=1,2,\cdots).$$

For such $\mu_{\nu,\rho}$, putting $[p_\rho]=\bigcup_{\nu=1}^{\infty}[p_{\nu,\mu_{\nu,\rho}}]$, we see easily by 3), 4)

$$m([p_\rho]p) \leqq \sum_{\nu=1}^{\infty} m([p_{\nu, \mu_{\nu}, \rho}]p) \leqq \frac{1}{\rho}$$

and $[p_\rho] \downarrow_{\rho=1}^{\infty}$. Putting $[p_0] = \bigcap_{\rho=1}^{\infty} [p_\rho]$, we have obviously by 3)

$$m([p_0]p) \leqq m([p_\rho]p) \leqq \frac{1}{\rho} \qquad \text{for every } \rho = 1, 2, \cdots,$$

and hence $m([p_0]p)=0$. This relation yields by 2) $[p_0]p=0$, and consequently $[p_0]=0$, because $[p_0] \leqq [p]$. Therefore R is totally continuous by MSLS Theorem 14.1.

THEOREM 1.3. *R is totally unbounded.*

PROOF. If $a = \sum_{\nu=1}^{\infty} a_\nu$, $a_\nu \frown a_\mu = 0$ for $\nu \neq \mu$, then we have by 4), 7)

$$\sum_{\nu=1}^{\infty} m(a_\nu) = m(a) < +\infty.$$

Thus there exists a sequence $1 \leqq \alpha_\nu \uparrow_{\nu=1}^{\infty} + \infty$ such that $\sum_{\nu=1}^{\infty} \alpha_\nu m(a_\nu) < +\infty$. As $m(\alpha_\nu a_\nu) \leqq \alpha_\nu m(a_\nu)$ by (2), we obtain by 4)

$$m(\sum_{\nu=1}^{\kappa} \alpha_\nu a_\nu) = \sum_{\nu=1}^{\kappa} m(\alpha_\nu a_\nu) \leqq \sum_{\nu=1}^{\infty} \alpha_\nu m(a_\nu)$$

for every $\kappa = 1, 2, \cdots$. Therefore $\sum_{\nu=1}^{\infty} \alpha_\nu a_\nu$ is convergent by 7).

Recalling MSLS Theorem 19.7, we obtain by Theorems 1.2 and 1.3

THEOREM 1.4. *Every bounded linear functional on R is universally continuous.*

§ 2. Spectral theory.

As $m([p]a)=0$ implies by 2) $[p]a=0$, and $[p_\nu] \uparrow_{\nu=1}^{\infty} [p]$ implies by (7) $\lim_{\nu \to \infty} m([p_\nu]a) = m([p]a)$, we see by MSLS Theorem 36.1 that, putting

$$\omega(\xi, a, \mathfrak{p}) = \lim_{[p] \to \mathfrak{p}} \frac{m(\xi[p]a)}{m([p]a)} \qquad (\xi \geqq 0, \mathfrak{p} \in U_{[a]}),$$

we obtain a continuos function $\omega(\xi, a, \mathfrak{p})$ on $U_{[a]}$, and

$$m(\xi[p]a)=\int_{[p]}\omega(\xi, a, \mathfrak{p})m(d\mathfrak{p}a)$$

$\omega(\xi, a, \mathfrak{p})$ is called the *spectrum* by m. $\omega(\xi, a, \mathfrak{p})$ is obviously a non-decreasing concave function of $\xi \geqq 0$. Recalling (2), (3) in §1, we see easily by definition

(1) $\omega(1, a, \mathfrak{p})=1,$

(2) $\frac{1}{\xi}\omega(\xi, a, \mathfrak{p}) \leqq \frac{1}{\eta}\omega(\eta, a, \mathfrak{p})$ for $\xi \geqq \eta > 0$,

(3) $\omega(\xi+\eta, a, \mathfrak{p}) \leqq \omega(\xi, a, \mathfrak{p})+\omega(\eta, a, \mathfrak{p})$.

As $\omega(\xi, a, \mathfrak{p})$ is a non-decreasing concave function of $\xi \geqq 0$ and $[p_\nu]\downarrow_{\nu=1}^{\infty} 0$ implies by Theorem 1.1 $\lim_{\nu\to\infty} m([p_\nu]a)=0$, we see easily that we can find an open set $A \subset U_{[a]}$ such that A is dense in $U_{[a]}$, and $\omega(\xi, a, \mathfrak{p})$ is a finite continuous function of $\xi \geqq 0$ for every $\mathfrak{p} \in A$. For two positive elements $a, b \in R$, we can find an open set B being dense in $U_{[a]}$ such that the relative spectrum $\left(\frac{b}{a}, \mathfrak{p}\right)$ is finite and continuous in B. For an arbitrary $\mathfrak{p}_0 \in AB$, if $\left(\frac{b}{a}, \mathfrak{p}_0\right) < \lambda$, then we can find a projector $[p]$ such that $\mathfrak{p}_0 \in U_{[p]} \subset AB$ and $\left(\frac{b}{a}, \mathfrak{p}\right) < \lambda$ for every $\mathfrak{p} \in U_{[p]}$. This relation yields $[p]b \leqq \lambda[p]a$, and hence we obtain by the postulate 3) $m([p]b) \leqq m(\lambda[p]a)$. From this relation we can conclude

$$\lim_{[p]\to\mathfrak{p}_0}\frac{m([p]b)}{m([p]a)} \leqq \omega(\lambda, a, \mathfrak{p}_0).$$

As $\omega(\xi, a, \mathfrak{p}_0)$ is a continuous function of $\xi \geqq 0$, and $\lambda > \left(\frac{b}{a}, \mathfrak{p}\right)$ may be arbitrary, we obtain hence

$$\lim_{[p]\to\mathfrak{p}_0}\frac{m([p]b)}{m([p]a)} \leqq \omega\left(\left(\frac{b}{a}, \mathfrak{p}\right), a, \mathfrak{p}_0\right).$$

We can prove likewise

$$\lim_{[p]\to\mathfrak{p}_0}\frac{m([p]b)}{m([p]a)} \geqq \omega\left(\left(\frac{b}{a}, \mathfrak{p}\right), a, \mathfrak{p}_0\right)$$

considering an arbitrary positive number $\lambda < \left(\frac{b}{a}, \mathfrak{p}_0\right)$. Therefore we have

$$\lim_{[p]\to\mathfrak{p}_0} \frac{m([p]b)}{m([p]a)} = \omega\left(\left(\frac{b}{a}, \mathfrak{p}\right), a, \mathfrak{p}\right) \qquad \text{for every } \mathfrak{p} \in AB.$$

As AB also is dense in $U_{[a]}$, we conclude hence by MSLS Theorem 36.1

$$(4) \qquad m([a]b) = \int_{[a]} \omega\left(\left(\frac{b}{a}, \mathfrak{p}\right), a, \mathfrak{p}\right) m(d\mathfrak{p}a)$$

for every two positive elements $a, b \in R$.

For two arbitrary elements $a, b \in R$, putting $c = |a| + |b|$, we have by the formulas (3) and (4)

$$\begin{aligned} m(a+b) &\leqq m(|a|+|b|) \\ &= \int_{[c]} \omega\left(\left(\frac{|a|+|b|}{c}, \mathfrak{p}\right), c, \mathfrak{p}\right) m(d\mathfrak{p}c) \\ &\leqq \int_{[c]} \omega\left(\left(\frac{a}{c}, \mathfrak{p}\right), c, \mathfrak{p}\right) m(d\mathfrak{p}c) + \int_{[c]} \omega\left(\left(\frac{b}{a}, \mathfrak{p}\right), c, \mathfrak{p}\right) m(d\mathfrak{p}c) \\ &= m(a) + m(b), \end{aligned}$$

that is,

$$(5) \qquad m(a+b) \leqq m(a) + m(b).$$

Thus we see that $m(x)$ $(x \in R)$ is a quasi-norm on R.

As $\omega(\xi, a, \mathfrak{p})$ is a non-decreasing concave function of $\xi \geqq 0$, we have

$$\omega(\lambda\xi + \mu\eta, a, \mathfrak{p}) \geqq \lambda\omega(\xi, a, \mathfrak{p}) + \mu\omega(\eta, a, \mathfrak{p})$$

for $\lambda + \mu = 1$; $\lambda, \mu, \xi, \eta \geqq 0$. Thus for two positive elements $a, b \in R$, putting $c = a + b$, we have by (4) for $\lambda + \mu = 1$; $\lambda, \mu \geqq 0$

$$\begin{aligned} m(\lambda a + \mu b) &= \int_{[c]} \omega\left(\left(\frac{\lambda a + \mu b}{c}, \mathfrak{p}\right), c, \mathfrak{p}\right) m(d\mathfrak{p}c) \\ &\geqq \lambda\, m(a) + \mu\, m(b), \end{aligned}$$

that is, for $a, b \geqq 0, \lambda+\mu=1, \lambda, \mu \geqq 0$

$$(6) \qquad m(\lambda a+\mu b) \geqq \lambda\, m(a)+\mu\, m(b).$$

THEOREM 2.1. *$\lim_{\mu,\nu\to\infty} m(a_\mu-a_\nu)=0$ implies $\lim_{\nu\to\infty} m(a_\nu-a)=0$ for some $a \in R$, that is, $m(x)\ (x \in R)$ is complete as a quasi-norm.*

PROOF. If $\lim_{\mu,\nu\to\infty} m(a_\mu-a_\nu)=0$, then we can find a subsequence μ_ν $(\nu=1, 2, \cdots)$ such that

$$m(a_{\mu_\nu}-a_{\mu_{\nu+1}}) \leqq \frac{1}{2^\nu} \qquad (\nu=1, 2, \cdots).$$

Then we see easily by (5) and the postulate 7) that $\sum_{\nu=1}^{\infty} | a_{\mu_\nu}-a_{\mu_{\nu+1}} |$ is convergent, and for every $\rho=1, 2, \cdots$

$$m(\sum_{\nu=\rho}^{\infty} | a_{\mu_\nu}-a_{\mu_{\nu+1}} |) \leqq \sum_{\nu=\rho}^{\infty} \frac{1}{2^\nu} = \frac{1}{2^{\rho-1}}.$$

Thus, putting $a=a_{\mu_1}+\sum_{\nu=1}^{\infty}(a_{\mu_{\nu+1}}-a_{\mu_\nu})$, we have

$$\overline{\lim_{\nu\to\infty}}\, m(a-a_{\mu_\nu}) \leqq \overline{\lim_{\rho\to\infty}}\, m(\sum_{\nu=\rho}^{\infty} | a_{\mu_\nu}-a_{\mu_{\nu+1}} |)=0,$$

and hence we conclude further by (4) and the assumption

$$\lim_{\nu\to\infty} m(a-a_\nu)=0.$$

A linear functional φ on R is said to be *modular bounded* by m, if we can find a positive number ε such that

$$\sup_{m(x) \leqq \varepsilon} | \varphi(x) | < +\infty.$$

With this definition we have

THEOREM 2.2. *A linear functional φ on R is modular bounded, if and only if φ is bounded.*

PROOF. If φ is modular bounded, then we have by definition

$$\sup_{m(x) \leqq \varepsilon} | \varphi(x) | < +\infty \qquad \text{for some } \varepsilon > 0.$$

For any $a \geqq 0$, we can find by 6) $\lambda > 0$ such that $m(\lambda a) < \varepsilon$, and we have obviously

$$\sup_{0 \leqq x \leqq a} |\varphi(x)| = \frac{1}{\lambda} \sup_{0 \leqq x \leqq \lambda a} |\varphi(x)| \leqq \frac{1}{\lambda} \sup_{m(x) \leqq \varepsilon} |\varphi(x)| < +\infty .$$

Thus φ is bounded.

If φ is positive but not modular bounded, then we can find $a_\nu \geqq 0$ $(\nu = 1, 2, \cdots)$ such that $m(a_\nu) \leqq \frac{1}{2^\nu}$, $\varphi(a_\nu) \geqq 2^\nu$ for every $\nu = 1, 2, \cdots$. Then we see easily by (4) and 7) that $\sum_{\nu=1}^{\infty} a_\nu$ is convergent, but we have

$$\varphi(\sum_{\nu=1}^{\infty} a_\nu) \geqq \varphi(a_\nu) \geqq 2^\nu \qquad (\nu = 1, 2, \cdots),$$

contradicting $\varphi(\sum_{\nu=1}^{\infty} a_\nu) < +\infty$. Thus, if φ is bounded, then φ is modular bounded.

THEOREM 2.3. *If $\lim_{\lambda \to +\infty} \frac{1}{\lambda} \omega(\lambda, a, \mathfrak{p}_0) = 0$ and $\mathfrak{p}_0$ is not an isolated point, then for every bounded linear functional φ on R we have*

$$\lim_{[p] \to \mathfrak{p}_0} \frac{\varphi([p]a)}{m([p]a)} = 0 .$$

PROOF. If $\lim_{\lambda \to +\infty} \frac{1}{\lambda} \omega(\lambda, a, \mathfrak{p}_0) = 0$, then we can find a sequence of positive numbers λ_ν $(\nu = 1, 2, \cdots)$ such that

$$\frac{1}{\lambda_\nu} \omega(\lambda_\nu, a, \mathfrak{p}_0) < \frac{1}{2^\nu} \qquad (\nu = 1, 2, \cdots).$$

If $\mathfrak{p}_0$ is not an isolated point, then we can find a sequence of projectors $[p_\nu] \downarrow_{\nu=1}^{\infty} 0$ such that $\mathfrak{p}_0 \in U_{[p_\nu]}$ for every $\nu = 1, 2, \cdots$, and

$$\frac{1}{\lambda_\nu} \omega(\lambda_\nu, a, \mathfrak{p}) < \frac{1}{2^\nu} \qquad \text{for every } \mathfrak{p} \in U_{[p_\nu]},$$

because R is superuniversally continuous by Theorem 1.2. For a positive linear functional φ on R, if

$$\overline{\lim_{[p]\to\mathfrak{p}_0}} \frac{\varphi([p]a)}{m([p]a)} > \varepsilon > 0\,,$$

then we can find $[q_\nu]\downarrow_{\nu=1}^{\infty}0$ such that $[q_\nu]\leqq[p_\nu]$, $m([q_\nu]a)\leqq\frac{1}{\lambda_\nu}$, and

$$\frac{\varphi([q_\nu]a)}{m([q_\nu]a)} > \varepsilon \qquad (\nu=1,2,\cdots).$$

Then, putting

$$\alpha_\nu = \frac{1}{m([q_\nu]a)} \qquad (\nu=1,2,\cdots),$$

we have $\alpha_\nu \geqq \lambda_\nu$, and hence

$$\frac{1}{\alpha_\nu}\omega(\alpha_\nu, a, \mathfrak{p}) \leqq \frac{1}{2^\nu} \qquad \text{for every } \mathfrak{p}\in U_{[q_\nu]}\,,$$

because $\frac{1}{\alpha_\nu}\omega(\alpha_\nu, a, \mathfrak{p}) \leqq \frac{1}{\lambda_\nu}\omega(\lambda_\nu, a, \mathfrak{p})$ by (2). Thus we have by (4)

$$m(\alpha_\nu[q_\nu]a) = \int_{[q_\nu]} \omega(\varphi_\nu, a, \mathfrak{p})m(d\mathfrak{p}\,a)$$

$$\leqq \frac{\alpha_\nu}{2^\nu}\int_{[q_\nu]} m(d\mathfrak{p}\,a) = \frac{\alpha_\nu}{2^\nu}m([q_\nu]a) = \frac{1}{2^\nu},$$

and hence we see by (5) and 7) that $\sum_{\nu=1}^{\infty}\alpha_\nu[q_\nu]a$ is convergent. But we have

$$\varphi(\sum_{\nu=1}^{\infty}\alpha_\nu[q_\nu]a) \geqq \sum_{\nu=1}^{\infty}\alpha_\nu\,\varepsilon\, m([q_\nu]a) = +\infty\,,$$

contradicting $\varphi(\sum_{\nu=1}^{\infty}\alpha_\nu[q_\nu]a) < +\infty$. Therefore we obtain

$$\lim_{[p]\to\mathfrak{p}_0} \frac{\varphi([p]a)}{m([p]a)} = 0\,.$$

§ 3. The first kind concave modulars.

By virtue of the formula §1 (2), we can put

$$m_1(x) = \lim_{\xi\to+\infty} \frac{m(\xi x)}{\xi}\,,$$

and we obtain a functional $m_1(x)$ on R. This functional $m_1(x)$ will be called the *limit modular* of a concave modular $m(x)$.

THEOREM 3.1. *For the limit modular $m_1(x)$, $\{x: m_1(x)=0\}$ is a normal manifold of R.*

PROOF. Putting $S=\{x: m_1(x)=0\}$, we see easily by definition that S is a semi-normal manifold of R. If

$$[p_\nu]\uparrow_{\nu=1}^{\infty}[a],\quad [p_\nu]a\in S\quad (\nu=1,2,\cdots),$$

then we have $a\in S$. Because, for any $\varepsilon>0$, recalling Theorem 1.1, we can find by assumption ν such that

$$m(([a]-[p_\nu])a)<\varepsilon,$$

and for such ν we can find further by assumption $\xi>1$ such that

$$\frac{m(\xi[p_\nu]a)}{\xi}<\varepsilon.$$

Then we have by the postulate 4) and the formula (2) in § 1

$$\frac{m(\xi a)}{\xi}=\frac{m(\xi([a]-[p_\nu])a)}{\xi}+\frac{m(\xi[p_\nu]a)}{\xi}<2\varepsilon.$$

Therefore we obtain $a\in S$. If $0\leqq a_\nu\uparrow_{\nu=1}^{\infty}a$, $a_\nu\in S$ $(\nu=1,2,\cdots)$, then we have by MSLS Theorems 6.2 and 6.19

$$[a_\nu]\uparrow_{\nu=1}^{\infty}[a],\ [(\mu a_\nu-a)^+]\uparrow_{\mu=1}^{\infty}[a_\nu],$$

$$0\leqq[(\mu a_\nu-a)^+]a\leqq\mu a_\nu\in S.$$

Thus we have $[a_\nu]a\in S$ for every $\nu=1,2,\cdots$, and hence $a\in S$, as proved just above. As R is superuniversally continuous by Theorem 1.2, we see easily hence that $0\leqq a_\lambda\uparrow_{\lambda\in\Lambda}a$, $a_\lambda\in S$ $(\lambda\in\Lambda)$ implies $a\in S$. Therefore S is by MSLS Theorem 4.9 a normal manifold of R.

A concave modular $m(x)$ on R is said to be of the ***first kind***, if, $m_1(x)\neq 0$ for every $x\neq 0$, and $m(x)$ is said to be of the *second kind*, if $m_1(x)=0$ for every $x\in R$. With this definition, we have obviously by Theorem 3.1.

THEOREM 3.2. *For a concave modular $m(x)$ on R, we can devide R uniquely in two orthogonal normal manifolds F and S, such that*

$m(x)$ is of the first kind in F and of the second kind in S.

Now we suppose that $m(x)$ is the first kind concave modular on R. The limit modular $m_1(x)$ satisfies obviously by definition

$$m_1(\lambda x)=\lim_{\xi\to+\infty}\frac{m(\xi\lambda x)}{\xi}=|\lambda|\,m_1(x)$$

and $x\frown y=0$ implies $m_1(x+y)=m_1(x)+m_1(y)$. Therefore the limit modular $m_1(x)$ is a *linear modular* on R. (c.f., MSLS § 41).

THEOREM 3.3. *If a concave modular $m(x)$ on R is of the first kind, then the limit modular $m_1(x)$ of $m(x)$ is a linear modular on R. This linear modular $m_1(x)$ is monotone complete, if and only if*

$$\sup_{m_1(x)\leq 1} m(x)<+\infty\,.$$

PROOF. We suppose firstly $\alpha=\sup_{m_1(x)\leq 1} m(x)<+\infty$. If

$$0\leq a_\nu\uparrow_{\nu=1}^{\infty},\ \sup_{\nu\geq 1} m_1(a_\nu)=\beta<+\infty\,,$$

then we have obviously

$$m_1\Big(\frac{1}{\beta}a_\nu\Big)=\frac{1}{\beta}m_1(a_\nu)\leq 1 \qquad \text{for every } \nu=1,2,\cdots,$$

and hence by assumption $\sup_{\nu\geq 1} m\Big(\frac{1}{\beta}a_\nu\Big)\leq\alpha$. Thus we can find by the postulate 7) in § 1 $a\in R$ such that $a_\nu\uparrow_{\nu=1}^{\infty}a$. Therefore $m_1(x)$ is monotone complete by definition.

If $\sup_{m_1(x)\leq 1} m(x)=+\infty$, then we can find $a_\nu\geq 0\ (\nu=1,2,\cdots)$ such that

$$m_1(a_\nu)\leq 1,\ m(a_\nu)\geq 2^\nu \qquad (\nu=1,2,\cdots).$$

For such $a_\nu\in R\ (\nu=1,2,\cdots)$, as $m_1(x)$ is a linear modular, we have by MSLS Theorem 36.9 for every $\kappa=1,2,\cdots$

$$m_1\Big(\sum_{\nu=1}^{\kappa}\frac{1}{2^\nu}a_\nu\Big)=\sum_{\nu=1}^{\kappa}\frac{1}{2^\nu}m_1(a_\nu)\leq 1\,,$$

but $\sum_{\nu=1}^{\infty}\frac{1}{2^\nu}a_\nu$ is not convergent, because we have by § 2 (6)

$$m\Big(\sum_{\nu=1}^{\kappa}\frac{1}{2^\nu}a_\nu\Big)\geq\sum_{\nu=1}^{\kappa}\frac{1}{2^\nu}m(a_\nu)\geq\kappa \qquad \text{for every } \kappa=1,2,\cdots.$$

Therefore $m_1(x)$ is not monotone complete.

§ 4. The second kind concave modulars.

In this §, we shall consider the second kind concave modulars.

THEOREM 4.1. *If a concave modular $m(x)$ is of the second kind, then for any $a \in R$ we can find an open set A of the proper space of R such that A is dense in $U_{[a]}$ and*

$$\lim_{\xi \to +\infty} \frac{1}{\xi} \omega(\xi, a, \mathfrak{p}) = 0 \qquad \text{for every } \mathfrak{p} \in A.$$

PROOF. For any $\varepsilon > 0$,

$$\left\{\mathfrak{p} : \lim_{\xi \to +\infty} \frac{1}{\xi} \omega(\xi, a, \mathfrak{p}) \geqq \varepsilon\right\} = \prod_{\nu=1}^{\infty} \left\{\mathfrak{p} : \frac{1}{\nu} \omega(\xi, a, \mathfrak{p}) \geqq \varepsilon\right\}$$

is obviously a closed set. Furthermore this closed set is nowhere dense. Because, if there is a projector $[p]$ such that $0 \neq [p] \leqq [a]$ and

$$\lim_{\xi \to +\infty} \frac{1}{\xi} \omega(\xi, a, \mathfrak{p}) \geqq \varepsilon \qquad \text{for every } \mathfrak{p} \in U_{[p]},$$

then we have for every $\xi > 0$

$$\frac{m(\xi [p] a)}{\xi} = \int_{[p]} \frac{1}{\xi} \omega(\xi, a, \mathfrak{p}) m(d\mathfrak{p}\, a) \geqq \varepsilon\, m([p] a),$$

contradicting $m_1([p] a) = 0$. As

$$\left\{\mathfrak{p} : \lim_{\xi \to +\infty} \frac{1}{\xi} \omega(\xi, a, \mathfrak{p}) \neq 0\right\} = \sum_{\nu=1}^{\infty} \left\{\mathfrak{p} : \lim_{\xi \to +\infty} \frac{1}{\xi} \omega(\xi, a, \mathfrak{p}) \geqq \frac{1}{\nu}\right\},$$

we see by MSLS Theorem 14.5 that

$$\left\{\mathfrak{p} : \lim_{\xi \to +\infty} \frac{1}{\xi} \omega(\xi, a, \mathfrak{p}) \neq 0\right\}$$

is nowhere dense, because R is totally continuous and superuniversally continuous by Theorem 1.2.

THEOREM 4.2. *If a concave modular $m(x)$ is of the second kind,* *and R has no discrete element, then there is no bounded linear functional on R up to* 0.

PROOF. Let φ be a positive linear functional on R. As φ is by Theorem 1.4 universally continuous, the characteristic set of φ is open by MSLS Theorem 22.5. Thus, if $\varphi \neq 0$, then we can find a positive element $a \neq 0$, such that $\varphi(x)=0$, $x \geqq 0$, implies $[a]x=0$. For such a, we can find by Theorem 4.1 a positive element $p \neq 0$, such that $[p] \leqq [a]$ and

$$\lim_{\xi\to+\infty} \frac{1}{\xi}\omega(\xi, a, \mathfrak{p})=0 \qquad \text{for every } \mathfrak{p} \in U_{[p]}.$$

Then, for every $\mathfrak{p} \in U_{[p]}$ we have by Theorem 2.3

$$\lim_{[p]\to\mathfrak{p}} \frac{\varphi([p]a)}{m([p]a)}=0,$$

because R has no discrete element, and hence the proper space of R has no isolated point. Thus, for any $\varepsilon > 0$, corresponding to every $\mathfrak{p} \in U_{[p]}$, we can find a normal manifold $P_{\mathfrak{p}}$ such that $U_{[P_{\mathfrak{p}}]} \ni \mathfrak{p}$ and

$$\varphi([p]a) \leqq \varepsilon\, m([p]a) \qquad \text{for } \mathfrak{p} \in U_{[P_{\mathfrak{p}}]} \subset U_{[p]}.$$

As $U_{[p]}$ is compact, we can find a finite number of points $\mathfrak{p}_\nu \in U_{[p]}$ $(\nu=1, 2, \cdots, \kappa)$ such that

$$U_{[p]}=\sum_{\nu=1}^{\kappa} U_{[P_{\mathfrak{p}_\nu}]}.$$

For such $[P_{\mathfrak{p}_\nu}]$ $(\nu=1, 2, \cdots, \kappa)$ we can find obviously projection operators $[P_\nu]$ $(\nu=1, 2, \cdots, \kappa)$ such that

$$[p]=\sum_{\nu=1}^{\kappa}[P_\nu],\quad [P_\nu] \leqq [P_{\mathfrak{p}_\nu}],\quad [P_\nu][P_\mu]=0 \text{ for } \nu \neq \mu.$$

Then we have

$$\varphi([p]a)=\sum_{\nu=1}^{\kappa} \varphi([P_\nu]a) \leqq \sum_{\nu=1}^{\kappa} \varepsilon m([P_\nu]a)=\varepsilon m([p]a).$$

As $\varepsilon > 0$ may be arbitrary, we obtain hence $\varphi([p]a]=0$, contradicting $[a][p]a=[p]a \neq 0$. Therefore we have $\varphi=0$.

§ 5. Discrete spaces.

Let R be now a discrete space, and $a_\lambda \geqq 0$ $(\lambda \in \Lambda)$ a *discrete basis* of R, that is, every positive element $a \in R$ may be represented uniquely as $a = \bigcup_{\lambda \in \Lambda} \alpha_\lambda a_\lambda$. For a concave modular $m(x)$ on R, we make use of the notation

$$\omega_\lambda = \lim_{\xi \to +\infty} m(\xi a_\lambda) \qquad (\lambda \in \Lambda).$$

THEOREM 5.1. *For any bounded linear functional φ on R we can find $\varepsilon > 0$ such that $\omega_\lambda \leqq \varepsilon$ implies $\varphi(a_\lambda) = 0$.*

PROOF. For a positive linear functional φ on R, if there is a sequence of elements $\lambda_\nu \in \Lambda$ $(\nu = 1, 2, \cdots)$ such that

$$\alpha_{\lambda_\nu} = \varphi(a_{\lambda_\nu}) \neq 0, \qquad \omega_{\lambda_\nu} \leqq \frac{1}{2^\nu} \qquad (\nu = 1, 2, \cdots),$$

then we have

$$m\left(\frac{1}{\alpha_{\lambda_\nu}} a_{\lambda_\nu}\right) \leqq \omega_{\lambda_\nu} \leqq \frac{1}{2^\nu} \qquad (\nu = 1, 2, \cdots),$$

and hence by the formula § 2 (4)

$$m\left(\sum_{\nu=1}^{\kappa} \frac{1}{\alpha_{\lambda_\nu}} a_{\lambda_\nu}\right) \leqq \sum_{\nu=1}^{\kappa} \frac{1}{2^\nu} \leqq 1.$$

Thus $\sum_{\nu=1}^{\infty} \frac{1}{\alpha_{\lambda_\nu}} a_{\lambda_\nu}$ is convergent by the postulate 7) in § 1, but we have

$$\varphi\left(\sum_{\nu=1}^{\infty} \frac{1}{\alpha_{\lambda_\nu}} a_{\lambda_\nu}\right) \geqq \sum_{\nu=1}^{\infty} \frac{1}{\alpha_{\lambda_\nu}} \varphi(a_{\lambda_\nu}) = +\infty.$$

Therefore we obtain our assertion.

Next we shall consider the case where $\inf_{\lambda \in \Lambda} \omega_\lambda > 0$. In this case, we can find $\varepsilon > 0$ such that

$$\omega_\lambda > \varepsilon \qquad \text{for every } \lambda \in \Lambda.$$

Then we can find $\alpha_\lambda > 0$ $(\lambda \in \Lambda)$ such that

$$m(\alpha_\lambda a_\lambda) \geqq \varepsilon \qquad \text{for every } \lambda \in \Lambda .$$

For an arbitrary positive element $x \in R$, we can find uniquely $\xi_\lambda \geqq 0$ $(\lambda \in \Lambda)$ such that

$$x = \bigcup_{\lambda \in \Lambda} \xi_\lambda a_\lambda , \quad m(x) = \sum_{\lambda \in \Lambda} m(\xi_\lambda a_\lambda) < +\infty .$$

Here we have naturally $\xi_\lambda = 0$ except for at most countable $\lambda \in \Lambda$. Furthermore we have $m(\xi_\lambda a_\lambda) \leqq \varepsilon$ except for a finite number of $\lambda \in \Lambda$. For every $\lambda \in \Lambda$ subject to $m(\xi_\lambda a_\lambda) \leqq \varepsilon$, we have obviously $\xi_\lambda \leqq \alpha_\lambda$, and hence by the formula § 1 (2)

$$\frac{\xi_\lambda}{\alpha_\lambda} \varepsilon \leqq m(\xi_\lambda a_\lambda) .$$

Therefore we conclude $\sum_{\lambda \in \Lambda} \frac{\xi_\lambda}{\alpha_\lambda} < +\infty$. Thus putting

$$\varphi(x) = \sum_{\lambda \in \Lambda} \frac{\xi_\lambda}{\alpha_\lambda} \qquad \text{for every positive } x = \bigcup_{\lambda \in \Lambda} \xi_\lambda a_\lambda ,$$

we obtain a positive linear functional φ on R. This linear functional φ is complete in R, that is, $\varphi(x) = 0$, $x \geqq 0$, implies obviously $x = 0$. Thus we can state

THEOREM 5.2. *If* $\inf_{\lambda \in \Lambda} \omega_\lambda > 0$, *then R is regular.* (c.f. MSLS § 19)

For a system of positive numbers α_λ $(\lambda \in \Lambda)$ and $\varepsilon > 0$, if

$$m(\alpha_\lambda a_\lambda) \geqq \varepsilon \qquad \text{for every } \lambda \in \Lambda ,$$

then, putting

$$\varphi(x) = \sum_{\lambda \in \Lambda} \frac{\xi_\lambda}{\alpha_\lambda} \qquad \text{for every positive } x = \bigcup_{\lambda \in \Lambda} \xi_\lambda a_\lambda ,$$

we obtain a complete positive linear functional φ on R, as proved just above. Furthermore, if

$$\lim_{\xi \to 0} \sup_{\lambda \in \Lambda} m(\xi \alpha_\lambda a_\lambda) = 0 ,$$

then for every positive linear functional ψ on R we have $\sup_{\lambda \in \Lambda} \psi(\alpha_\lambda a_\lambda) < +\infty$. Because, if there is a sequence $\lambda_\nu \in \Lambda$ $(\nu = 1, 2, \cdots)$ such that

$\psi(\alpha_{\lambda_\nu} a_{\lambda_\nu}) \geqq \nu$, then we have by assumption

$$\lim_{\nu\to\infty} m\left(\frac{1}{\nu}\alpha_{\lambda_\nu} a_{\lambda_\nu}\right) \leqq \lim_{\nu\to\infty} \sup_{\lambda\in\Lambda} m\left(\frac{1}{\nu}\alpha_\lambda a_\lambda\right) = 0,$$

and hence we can find a subsequence λ_{ν_μ} ($\mu=1, 2, \cdots$) such that

$$\sum_{\mu=1}^{\infty} m\left(\frac{1}{\nu_\mu}\alpha_{\lambda_{\nu_\mu}} a_{\lambda_{\nu_\mu}}\right) < +\infty .$$

For such λ_{ν_μ}, we see easily by § 2 (5) and the postulate 7) in § 1 that $\sum_{\mu=1}^{\infty} \frac{1}{\nu_\mu} \alpha_{\lambda_{\nu_\mu}} a_{\lambda_{\nu_\mu}}$ is convergent, but we have

$$\psi\left(\sum_{\mu=1}^{\infty} \frac{1}{\nu_\mu}\alpha_{\lambda_{\nu_\mu}} a_{\lambda_{\nu_\mu}}\right) \geqq \sum_{\mu=1}^{\infty} \frac{1}{\nu_\mu}\psi(\alpha_{\lambda_{\nu_\mu}} a_{\lambda_{\nu_\mu}}) = +\infty ,$$

contradicting $\psi\left(\sum_{\mu=1}^{\infty} \frac{1}{\nu_\mu}\alpha_{\lambda_{\nu_\mu}} a_{\lambda_{\nu_\mu}}\right) < +\infty$. Thus we have $\sup_{\lambda\in\Lambda} \psi(\alpha_\lambda a_\lambda) < +\infty$, and hence, putting

$$\gamma = \sup_{\lambda\in\Lambda} \psi(\alpha_\lambda a_\lambda) ,$$

we have $\psi \leqq \gamma\varphi$. Therefore the conjugate space $\bar{R}$ of R is bounded, because $\bar{R} \ni \varphi$ by Theorem 1.4.

Conversely, if there is a positive linear functional φ on R, such that for any positive linear functional ψ on R we can find $\gamma > 0$ for which $\psi \leqq \gamma\varphi$, then we have naturally $\varphi(a_\lambda) > 0$ for every $\lambda \in \Lambda$, and hence, putting

$$\alpha_\lambda = \frac{1}{\varphi(a_\lambda)} \qquad (\lambda \in \Lambda) ,$$

we have

$$\inf_{\lambda\in\Lambda} m(\alpha_\lambda a_\lambda) > 0 .$$

Because, if $\inf_{\lambda\in\Lambda} m(\alpha_\lambda a_\lambda) = 0$, then we can find a sequence $\lambda_\nu \in \Lambda$ ($\nu=1, 2, \cdots$) such that $\sum_{\nu=1}^{\infty} m(\alpha_{\lambda_\nu} a_{\lambda_\nu}) < +\infty$, and hence $\sum_{\nu=1}^{\infty} \alpha_{\lambda_\nu} a_{\lambda_\nu}$ is convergent, but

$$\varphi\Big(\sum_{\nu=1}^{\infty}\alpha_{\lambda_\nu} a_{\lambda_\nu}\Big) \geqq \sum_{\nu=1}^{\infty}\alpha_{\lambda_\nu}\varphi(a_{\lambda_\nu}) = +\infty\,,$$

contradicting $\varphi\Big(\sum_{\nu=1}^{\infty}\alpha_{\lambda_\nu} a_{\lambda_\nu}\Big) < +\infty$. Furthermore we have

$$\lim_{\xi\to 0}\sup_{\lambda\in\Lambda} m(\xi\alpha_\lambda a_\lambda) = 0\,.$$

Because, if there is $\delta > 0$ such that

$$\lim_{\xi\to 0}\sup_{\lambda\in\Lambda} m(\xi\alpha_\lambda a_\lambda) > \delta\,,$$

then we can find a sequence $\lambda_\nu \in \Lambda$ $(\nu = 1, 2, \cdots)$ such that

$$m\Big(\frac{1}{\nu^3}\alpha_{\lambda_\nu} a_{\lambda_\nu}\Big) > \delta \qquad \text{for every } \nu = 1, 2, \cdots .$$

For such $\lambda_\nu \in \Lambda$ $(\nu = 1, 2, \cdots)$, if

$$\sum_{\nu=1}^{\infty} m(\xi_\nu\,\alpha_{\lambda_\nu} \alpha_{\lambda_\nu}) < +\infty\,,\ \xi_\nu \geqq 0 \qquad (\nu = 1, 2\cdots)\,,$$

then we have $\xi_\nu \leqq \frac{1}{\nu^3}$ except for a finite number of $\nu = 1, 2, \cdots$, and hence $\sum_{\nu=1}^{\infty}\nu\xi_\nu < +\infty$. Thus, putting

$$\psi(x) = \sum_{\nu=1}^{\infty}\frac{\nu}{\alpha_{\lambda_\nu}}\,\xi_{\lambda_\nu} \qquad \text{for every positive } x = \bigcup_{\lambda\in\Lambda}\xi_\lambda a_\lambda\,,$$

we obtain a positive linear functional ψ on R for which $\psi(\alpha_{\lambda_\nu} a_{\lambda_\nu}) = \nu$ $(\nu = 1, 2, \cdots)$, contradicting $\psi \leqq \gamma\,\varphi$ for some positive number γ. Therefore we can state

THEOREM 5.3. *The conjugate space $\bar{R}$ of R is bounded as a semi-ordered linear space, if and only if we can find $\alpha_\lambda > 0$ $(\lambda \in \Lambda)$ such that*

$$\inf_{\lambda\in\Lambda} m(\alpha_\lambda a_\lambda) > 0, \qquad \lim_{\xi\to 0}\sup_{\lambda\in\Lambda} m(\xi\alpha_\lambda a_\lambda) = 0\,,$$

and then, putting

$$\varphi(x)=\sum_{\lambda\in\Lambda}\frac{1}{\alpha_\lambda}\xi_\lambda \qquad \textit{for every positive } x=\bigcup_{\lambda\in\Lambda}\xi_\lambda a_\lambda,$$

we obtain a positive linear functional $\varphi\in\bar{R}$ *by which* $\bar{R}$ *is bounded.*

§ 6. Examples.

Let $p(t)$ be a measurable function on the interval (0, 1) such that $0<p(t)<1$. Denoting by $L_{p(t)}$ the totality of measurable functions $\varphi(t)$ for which

$$\int_0^1 |\varphi(t)|^{p(t)}dt<+\infty,$$

we see easily that $L_{p(t)}$ is a continuous semi-ordered linear space, defining $\varphi\geqq\psi$ to mean $\varphi(t)\geqq\psi(t)$ in (0, 1) up to a zero measure set. Putting

$$m(\varphi)=\int_0^1 |\varphi(t)|^{p(t)}dt \qquad \text{for } \varphi\in L_{p(t)},$$

we see easily that $m(\varphi)$ is a concave modular of the second kind on $L_{p(t)}$. Therefore we conclude by Theorem 4.2 that there is no bounded functional on $L_{p(t)}$ except for 0. This result was proved first by M. M. Day and G. Sirvint independently in the special case where $p(t)$ is a constant.[2)]

The totality of measurable functions on the interval (0, 1) is denoted by (S). (S) is obviously a continuous semi-ordered linear space in the usual sense. Putting

$$m(\varphi)=\int_0^1 \frac{|\varphi(t)|}{1+|\varphi(t)|}dt \qquad \text{for every } \varphi\in(S),$$

we obtain a concave modular of the second kind. Thus we see by Theorem 4.2 that there is no bounded linear functional on (S) except for 0.[3)]

For a sequence of positive numbers $p_\nu<1$ $(\nu=1,2,\cdots)$, denoting by $l(p_1,p_2,\cdots)$ the totality of $x=(\xi_1,\xi_2,\cdots)$ for which

$$\sum_{\nu=1}^{\infty} |\xi_\nu|^{p_\nu} < +\infty ,$$

we obtain a continuous semi-ordered linear space $l(p_1, p_2, \cdots)$ which is obviously a discrete space. Putting

$$m(x) = \sum_{\nu=1}^{\infty} |\xi_\nu|^{p_\nu} \qquad \text{for } x = (\xi_1, \xi_2, \cdots) \in l(p_1, p_2, \cdots),$$

we obtain a concave modular $m(x)$ on $l(p_1, p_2, \cdots)$. Every bounded linear functional φ on $l(p_1, p_2, \cdots)$ is continuous by Theorem 1.4, and hence represented uniquely in the form

$$\varphi(x) = \sum_{\nu=1}^{\infty} \alpha_\nu \xi_\nu \qquad \text{for } x = (\xi_1, \xi_2, \cdots) \in l(p_1, p_2, \cdots).$$

As $\sum_{\nu=1}^{\infty} |\xi_\nu|^{p_\nu} < +\infty$ implies $\sum_{\nu=1}^{\infty} |\xi_\nu| < +\infty$, putting

$$\varphi_0(x) = \sum_{\nu=1}^{\infty} \xi_\nu \qquad \text{for } x = (\xi_1, \xi_2, \cdots) \in l(p_1, p_2, \cdots),$$

we obtain a positive linear functional φ_0 on $l(p_1, p_2, \cdots)$.

If $\varliminf_{\nu\to\infty} p_\nu > 0$, then we can find $\varepsilon > 0$ such that $p_\nu > \varepsilon$ for every $\nu = 1, 2, \cdots$, and hence

$$\lim_{\xi\to 0} \sup_{\nu\geq 1} |\xi|^{p_\nu} \leq \lim_{\xi\to 0} |\xi|^{\varepsilon} = 0 .$$

Thus we conclude by Theorem 5.3 that every bounded linear functional φ on $l(p_1, p_2, \cdots)$ is represented in the form

$$\varphi(x) = \sum_{\nu=1}^{\infty} \alpha_\nu \xi_\nu, \quad \sup_{\nu\geq 1} |\alpha_\nu| < +\infty ,$$

for $x = (\xi_1, \xi_2, \cdots) \in l(p_1, p_2, \cdots)$.

The totality of sequences $(\xi_1, \xi_1, \cdots)$ is denoted by (s). (s) is obviously a continuous semi-ordered linear space. Putting

$$m(x) = \sum_{\nu=1}^{\infty} \frac{|\xi_\nu|}{2^\nu(1+|\xi_\nu|)} \qquad \text{for } x = (\xi_1, \xi_2, \cdots),$$

we obtain a concave modular $m(x)$ on (s). For this concave modular $m(x)$ we have obviously

$$\omega_\nu = \lim_{\xi \to +\infty} \frac{|\xi|}{2^\nu(1+|\xi|)} = \frac{1}{2^\nu} \qquad (\nu=1, 2, \cdots).$$

Thus we see by Theorem 5.1 that every bounded linear functional φ on (s) may be represented in the form

$$\varphi(x) = \sum_{\nu=1}^{\kappa} \alpha_\nu \xi_\nu \qquad \text{for } x=(\xi_1, \xi_2, \cdots)$$

for a finite number of real numbers $\alpha_1, \alpha_2, \cdots, \alpha_\kappa$.

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References.

1) H. Nakano: Modulared semi-ordered linear spaces, Tokyo Math. Book Series I (1950). This book will be denoted by MSLS in this paper.
2) M. M. Day: The space L^p with $0 < p < 1$, Bull. Amer. Math. Soc., 46 (1940), 816-823. G. Sirvint: Espace des fonctionells linéaires, Com. Rend. URSS., 26 (1940), 123-126.
3) C.f. 2).