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A generalization of Riesz-Fischer's theorem.

By Ichiro AMEMIYA

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L. Kantorovitch [1] has proved, generalizing Riesz-Fischer's theorem to an abstract form, that a normed semi-ordered linear space is complete if the norm satisfies the following two conditions:

(1) The norm is monotone complete, i. e., every non-decreasing and norm-bounded sequence of positive elements has the least upper bound.

(2) The norm is continuous, i. e., every non-increasing sequence of elements which is order-convergent to 0 also converges to 0 by the norm.

H. Nakano [2] has shown that the condition (2) can be weakened to the condition of semi-continuity, i. e.,

(2′) For every non-decreasing sequence of positive elements, $a_\nu \uparrow_{\nu=1}^{\infty}$, such that $a=\bigcup_{\nu=1}^{\infty} a_\nu$ exists, we have

$$\|a\|=\sup_{\nu\geq 1}\|a_\nu\|.$$

Here we shall show that even this condition (2′) is superfluous, i. e. only the condition (1) suffices for the completeness of the normed semi-ordered linear space.

First we shall prove the following

LEMMA. *If a normed semi-ordered linear space R has a monotone complete norm, then there exists a positive real number $\alpha\leqq 1$ such that $0\leqq a_\nu \uparrow_{\nu=1}^{\infty} a$ implies always*

$$\sup_{\nu\geq 1}\|a_\nu\|\geqq\alpha\|a\|.$$

PROOF. If we can not find such a number α, then there exists a double sequence of positive elements $a_{\mu,\nu}$ and a sequence a_μ $(\mu=1, 2,\cdots)$, such that we have

$$0\leqq a_{\mu,\nu}\uparrow_{\nu=1}^{\infty} a_\mu, \qquad \|a_\mu\|\geqq\mu \quad \text{and} \quad \sup_{\nu\geq 1}\|a_{\mu,\nu}\|\leqq\frac{1}{2^\mu}.$$

Now putting

$$b_\nu = a_{1,\nu} + a_{2,\nu} + \cdots + a_{\nu,\nu}$$

for every $\nu = 1, 2, \cdots$, we obtain a norm bounded non-decreasing sequence and hence there exists the least upper bound $b = \bigcup_{\nu=1}^{\infty} b_\nu$, because we have

$$\|b_\nu\| \leqq \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^\nu} < 1 .$$

Then we have $b \geqq a_\mu$ ($\mu = 1, 2, \cdots$) and hence $\|b\| \geqq \mu$, a contradiction.

By virtue of this lemma, we can see easily that for any sequence of elements a_ν ($\nu = 1, 2, \cdots$) such that $\|a_\nu\| \leqq \frac{1}{2^\nu}$, the series $a_1 + a_2 + \cdots$ is convergent to the same sum both in order and in norm. Hence we can select from any Cauchy sequence a subsequence a_ν ($\nu = 1, 2, \cdots$) such that $\nu \leqq \mu, \rho$ implies $\|a_\mu - a_\rho\| \leqq \frac{1}{2^\nu}$; then $a = a_1 + (a_2 - a_1) + (a_3 - a_2) + \cdots$ is the limit of a_ν by norm-topology. Thus we have:

THEOREM. *If the norm of a normed semi-ordered linear space R is monotone complete, then R is complete.*

REMARK. This theorem is also valid in the case that the topology of R is sequential and compatible with the order, i. e., as a basis of the neighbourboods of 0 we can take V_ν ($\nu = 1, 2, \cdots$) such that

$$x \in V_\nu \quad \text{and} \quad |x| \geqq |y| \quad \text{imply} \quad y \in V_\nu .$$

Here the monotone completeness of the topology is defined similarly as in (1); we have only to replace the word "norm-bounded" by "topologically bounded".

Department of Mathematics,
Hokkaido University

References

[1] L. Kantorovitch, Linear halbgeordnete Räume, Recueil Math., 44 (1937), pp. 121-165.
[2] H. Nakano, Modulared semi-ordered linear spaces, 1950, Tokyo Math. Book Series vol. 1, Theorem 30. 17.