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Star-like theorems and convex-like theorems in an annulus.

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1. Introduction.

It is the purpose of this paper to obtain some sufficient conditions for p-valency of $f(z)$, meromorphic and single-valued in an annulus.

It is well known that if $f(z)=z+a_2z^2+\cdots$, regular for $|z|<r$, satisfies the relation $\Re[zf'(z)/f(z)]>0$ for $|z|<r$, then $f(z)$ is univalent in $|z|<r$ [5]. This theorem has been generalized considerably by S. Ozaki [1,2] and the author [3]. We shall give in § 2 generalizations of the above theorems to the case of meromorphic functions defined in an annulus.

In § 3, we shall study analytic functions convex in one direction in a ring domain, which will be defined precisely later on, extending the results given by the author [4].

2. Star-like theorems.

THEOREM 1. *Let $f(z)$ be meromorphic and single-valued in $r\leqq|z|\leqq R$ and let $n(0)$ and $n(\infty)$ be the number of zeros and poles of $f(z)$ in $r\leqq|z|\leqq R$, respectively. Further let*

$$\frac{1}{2\pi}\int_{|z|=R}\Re\frac{zf'(z)}{f(z)}\,d\theta=p\,,\qquad \frac{1}{2\pi}\int_{|z|=r}\Re\frac{zf'(z)}{f(z)}\,d\theta=q\,.$$

(i) *If $\Re[zf'(z)/f(z)]>0$ on $|z|=R$ and $|z|=r$, then $f(z)$ is at most $p+n(\infty)$ $(=q+n(0))$-valent and at least* $\mathrm{Max}\,[n(0)-p,1]$*-valent in $r\leqq|z|\leqq R$.*

(ii) *If $\Re[zf'(z)/f(z)]<0$ on $|z|=R$ and $|z|=r$, then $f(z)$ is at most $n(0)-p(=n(\infty)-q)$-valent and at least* $\mathrm{Max}\,[n(\infty)+p,1]$*-valent in $r\leqq|z|\leqq R$.*

PROOF. By Lemma 1 which will be stated later, $f(z) \neq 0$ and $\neq \infty$ on $|z|=R$ and on $|z|=r$.

Now let $n(w)$ be the number of roots of the equation $f(z)=w$ in $r \leqq |z| \leqq R$ and let C_1 and C_2 be the images of $|z|=r$ and $|z|=R$ under $f(z)$, respectively. Put $C_1+C_2=C$. Then we make use of the following geometrical principle [6]:

$n(w)$ can change its value only when w arrives at a value assumed by $f(z)$ on $|z|=r$ and $|z|=R$, and hence, if w moves along a continuous curve without crossing the line C, then $n(w)$ is invariant. When w crosses the line C, then the saltus of $n(w)$ is an integer.

In the case (i), C_1 and C_2 surround the origin positively and star-likely with respect to the origin p times and q times respectively, and hence every straight-line starting from the origin and ending at ∞ must cut these curves in exactly $p+q$ points.

Suppose that $n(w_0) > p+n(\infty)$ for a certain point w_0.

By the geometrical principle the linear segment $\overline{0w_0}$ must cut C at least $n(w_0)-n(0)$ times and the half straight-line $\overline{w_0\infty}$ must cut C at least $n(w_0)-n(\infty)$ times. Consequently the half straight-line starting from the origin and passing through w_0 must cut C at least $2n(w_0)-n(0)-n(\infty)\big(>2p+2n(\infty)-n(0)-n(\infty)=p+q\big)$ times, which is a contradiction. Hence $n(w_0) \leqq p+n(\infty)$, which shows that $f(z)$ is at most $p+n(\infty)$-valent.

Next, suppose that $n(w_1) < n(0)-p$ for certain point w_1. By the same consideration as done above, the half straight-line starting from the origin and passing through w_1 must cut C at least $n(0)-n(w_1)+n(\infty)-n(w_1)\big(>n(0)+n(\infty)+2p-2n(0)=p+q\big)$ times, which is again a contradiction. And hence $f(z)$ is at least $n(0)-p$-valent in $r \leqq |z| \leqq R$.

Analogously we can complete the proof of the case (ii), which may be omitted here. Q. E. D.

COROLLARY 1. *Let $f(z)$ be meromorphic and single-valued in $r \leqq |z| \leqq R$ and let p be a positive integer.*

(i) *If $p+1 > \Re[zf'(z)/f(z)] > 0$ on $|z|=R$ and $\Re[zf'(z)/f(z)] > 0$ on $|z|=r$, then $f(z)$ is at most $p+n(\infty)$-valent and at least* $\mathrm{Max}\,[n(0)-p, 1]$-*valent in $r \leqq |z| \leqq R$.*

(ii) *If $-p-1 < \Re[zf'(z)/f(z)] < 0$ on $|z|=R$ and $\Re[zf'(z)/f(z)] < 0$ on $|z|=r$, then $f(z)$ is at most $p+n(0)$-valent and at least* $\mathrm{Max}\,[n(\infty)-p, 1]$-*valent in $r \leqq |z| \leqq R$.*

COROLLARY 2. *Let $f(z)$ be meromorphic and single-valued in $r\leqq|z|\leqq R$ and let p be a positive integer. If, on $|z|=r$ and $|z|=R$,*

$$\left|\Re\frac{zf'(z)}{f(z)}-\frac{p+1}{2}\right|<\frac{p+1}{2}\quad\text{or}\quad\left|\Re\frac{zf'(z)}{f(z)}+\frac{p+1}{2}\right|<\frac{p+1}{2},$$

then $f(z)$ is at most $p+n(\infty)$-valent and at least $\operatorname{Max}[n(0)-p,1]$*-valent, or at most $p+n(0)$-valent and at least* $\operatorname{Max}[n(\infty)-p,1]$*-valent in $r\leqq|z|\leqq R$, respectively.*

LEMMA 1. *Let $f(z)$ be meromorphic and single-valued in $r\leqq|z|\leqq R$. If $\Re[zf'(z)/f(z)]>0$ or <0 in $r\leqq|z|\leqq R$, then $f(z)$ is regular and has no zeros in $r\leqq|z|\leqq R$.*

Proof of this lemma is analogous to the case where $r=0$, which is known as Kobori's lemma [5], and may be omitted.

Using this Lemma with Theorem 1 or Corollary 2 we have

COROLLARY 3. *Let $f(z)$ be meromorphic and single-valued in $r\leqq|z|\leqq R$. If $f(z)$ satisfies one of the following conditions:*

(i) $\Re[zf'(z)/f(z)]>0$ *for* $r\leqq|z|\leqq R$ *and* $2>\Re[zf'(z)/f(z)]$ *on* $|z|=R$ *or on* $|z|=r$,

(ii) $\Re[zf'(z)/f(z)]<0$ *for* $r\leqq|z|\leqq R$ *and* $-2<\Re[zf'(z)/f(z)]$ *on* $|z|=R$ *or on* $|z|=r$,

(iii) $|\Re[zf'(z)/f(z)]\pm1|<1$ *for* $r\leqq|z|\leqq R$,

then $f(z)$ is regular and univalent in $r\leqq|z|\leqq R$.

THEOREM 2. *Let $f(z)$ be meromorphic and single-valued in $r\leqq|z|\leqq R$. Further let $f(z)$ be expanded as follows*

$$f(z)=\sum_{-\infty}^{\infty}a_nz^n\quad\text{on } |z|=r\quad\text{and}\quad f(z)=\sum_{-\infty}^{\infty}b_nz^n\quad\text{on } |z|=R.$$

If, for an arbitrary integer k such that $2|k|\geqq p+1>|k|>0$,

$$|a_k|>\sum_{n=1}^{\infty}\left(1+\frac{n}{p+1-|k|}\right)(|a_{n+k}|r^n+|a_{k-n}|r^{-n}),\tag{2.1}$$

$$|b_k|>\sum_{n=1}^{\infty}\left(1+\frac{n}{p+1-|k|}\right)(|b_{n+k}|R^n+|b_{k-n}|R^{-n}),\tag{2.2}$$

then $f(z)$ is at most $p+n(\infty)$-valent and at least $\operatorname{Max}[n(0)-p,1]$*-valent in $r\leqq|z|\leqq R$ where $n(\infty)$ and $n(0)$ denote the number of poles and zeros of $f(z)$ in $r<|z|<R$, respectively. If, for an arbitrary integer k such that $p+1\geqq2|k|>0$,*

$$|a_k| > \sum_{n=1}^{\infty}\left(1+\frac{n}{|k|}\right)(|a_{n+k}|r^n+|a_{k-n}|r^{-n}), \tag{2.3}$$

$$|b_k| > \sum_{n=1}^{\infty}\left(1+\frac{n}{|k|}\right)(|b_{n+k}|R^n+|b_{k-n}|R^{-n}), \tag{2.4}$$

then $f(z)$ is at most $p+n(0)$-valent and at least $\mathrm{Max}\,[n(\infty)-p, 1]$*-valent in $r \leqq |z| \leqq R$.*

PROOF. By our hypothesis (2.1), we can prove

$$\left|\Re\frac{zf'(z)}{f(z)}-\frac{p+1}{2}\right| < \frac{p+1}{2} \quad \text{on} \quad |z|=r \tag{2.5}$$

as follows

$$\begin{aligned}
p+1-|\Re[2zf'(z)/f(z)]-(p+1)| &\geqq p+1-|[2zf'(z)/f(z)]-(p+1)| \\
&= p+1-\left|\frac{\sum_{-\infty}^{\infty}\{2n-(p+1)\}a_n z^{n-k}}{a_k+\sum_{n\neq k}^{\infty}_{-\infty} a_n z^{n-k}}\right| \\
&= p+1-\left|\{2k-(p+1)\}+\frac{\sum_{-\infty}^{\infty} 2(n-k)\,a_n z^{n-k}}{a_k+\sum_{n\neq k}^{\infty}_{-\infty} a_n z^{n-k}}\right| \\
&> p+1-\left|2k-(p+1)\right|-\frac{2\sum_{n=1}^{\infty} n(|a_{k+n}|r^n+|a_{k-n}|r^{-n})}{|a_k|-\sum_{n\neq k}^{\infty}_{-\infty}(|a_{k+n}|r^n+|a_{k-n}|r^{-n})} \\
&= 2(p+1-|k|)\left\{|a_k|-\sum_{n=1}^{\infty}\left(1+\frac{n}{p+1-|k|}\right)(|a_{n+k}|r^n+|a_{k-n}|r^{-n})\right\} \\
&\quad : \left\{|a_k|-\sum_{n=1}^{\infty}(|a_{k+n}|r^n+|a_{k-n}|r^{-n})\right\} > 0 \qquad \text{by (2.1).}
\end{aligned}$$

Quite similarly we can prove (2.5) on $|z|=R$ by using (2.2). Consequently, by Corollary 2 $f(z)$ is at most $p+n(\infty)$-valent and at least $\mathrm{Max}\,[n(0)-p, 1]$-valent in $r \leqq |z| \leqq R$.

If we use (2.3) and (2.4) we have

$$\left|\Re\frac{zf'(z)}{f(z)}+\frac{p+1}{2}\right| < \frac{p+1}{2} \quad \text{on } |z|=r \text{ and } |z|=R \tag{2.6}$$

and hence $f(z)$ is $p+n(0)$-valent and at least $\mathrm{Max}\,[n(\infty)-p, 1]$-valent in $r \leqq |z| \leqq R$ by Corollary 2.

COROLLARY 4. *The function $f(z)=\sum_{-\infty}^{\infty} a_n z^n$, regular for $r \leqq |z| \leqq R$,*

is at most p-valent for $r \leqq |z| \leqq R$ *if we have one of the following conditions.*

(i) *For an arbitrary integer* k *such that* $2|k| \geqq p+1 > |k| > 0$,

$$(2.7)\qquad \begin{aligned} |a_k| &> \sum_{n=1}^{\infty}\left(1+\frac{n}{p+1-|k|}\right)(|a_{n+k}|r^n+|a_{k-n}|r^{-n}), \\ |a_k| &> \sum_{n=1}^{\infty}\left(1+\frac{n}{p+1-|k|}\right)(|a_{n+k}|R^n+|a_{k-n}|R^{-n}). \end{aligned}$$

(ii) $f(z) \neq 0$ *for* $r < |z| < R$ *and for an arbitrary integer* k *such that* $p+1 \geqq 2|k| > 0$,

$$(2.8)\qquad \begin{aligned} |a_k| &> \sum_{n=1}^{\infty}\left(1+\frac{n}{|k|}\right)(|a_{n+k}|r^n+|a_{k-n}|r^{-n}), \\ |a_k| &> \sum_{n=1}^{\infty}\left(1+\frac{n}{|k|}\right)(|a_{n+k}|R^n+|a_{k-n}|R^{-n}). \end{aligned}$$

REMARK. Obviously it is sufficient to use only one inequality

$$(2.8)\qquad |a_k| \geqq \sum_{n=1}^{\infty}\left(1+\frac{n}{p+1-|k|}\right)(|a_{n+k}|R^n+|a_{k-n}|r^{-n})$$

instead of the two inequalities (2.7) and also

$$(2.10)\qquad |a_k| \geqq \sum_{n=1}^{\infty}\left(1+\frac{n}{|k|}\right)(|a_{n+k}|R^n+|a_{k-n}|r^{-n})$$

instead of (2.8). The assumption $f(z) \neq 0$ in (ii) is then unnecessary since we get (2.6) for $r \leqq |z| \leqq R$ and we can use Lemma 1 if we have (2.10). In this form Corollary 4 is reduced to S. Ozaki's result [2].

3. Convex-like theorems.

DEFINITION 1. Let $F(z)$ be meromorphic and single-valued in $r \leqq |z| \leqq R$. Further let C_1 and C_2 be the image curves of $|z|=r$ and $|z|=R$, respectively. Put $C_1+C_2=C$.

(i) If a certain straight-line passing through the origin cuts C in $2(p+1)$ and not more than $2(p+1)$ points, then $F(z)$ is said to be star-like of order p in one direction (in the direction of the straight-line). Let the class of such functions be denoted by $\mathfrak{S}_1(p)$.

(ii) If every straight-line parallel to a direction cut C in not more than $2(p+1)$ points and there exists at least one such straight-line which cuts C in $2(p+1)$ points, then $F(z)$ is said to be convex of order p in one direction. Let the class of such functions be denoted by $\mathfrak{K}_1(p)$.

LEMMA 2. *If $zF'(z)\in\mathfrak{S}_1(p)$, then* (i) *$F(z)$ is convex of order at most p in one direction,* (ii) *$F(z)$ is at most $p+n(\infty)+1$-valent and at least* $\mathrm{Max}\,[n(\infty)-p-1, 1]$*-valent in $r\leqq|z|\leqq R$, where $n(\infty)$ denote the number of poles of $F(z)$ in $r\leqq|z|\leqq R$.*

PROOF. We can prove this lemma analogously to the one given in [4, p. 199]. But a few remarks must be added.

Now by the hypothesis, $\Im\{zF'(z)\}$ may be considered to change its sign $2(p+1)$ times as z moves along the circumferences of the circles $|z|=r$ and $|z|=R$. In view of the identity

$$\Im\{zF'(z)\}=-\partial\Re F(z)/\partial\theta \qquad \text{for } z=re^{i\theta} \text{ and } z=Re^{i\theta},$$

$\Re F(z)$ then increases and decreases alternatively $p+1$ times respectively, whence it follows that $F(z)$ is convex of order at most p in the direction of the imaginary axis.

As for the proof of (ii) it is quite analogous to [4. p. 199] and will be omitted.

LEMMA 3. *Let $\varphi(z)$ be meromorphic and single-valued in $r\leqq|z|\leqq R$ and continuous and $\neq 0$ on $|z|=r$ and $|z|=R$. If there holds the relation*

$$\int_{|z|=r}\left|\Re\frac{z\varphi'(z)}{\varphi(z)}\right|d\theta+\int_{|z|=R}\left|\Re\frac{z\varphi'(z)}{\varphi(z)}\right|d\theta<2\pi(p+2), \tag{3.1}$$

p: positive integer,

then $\varphi(z)$ is star-like of order at most p in one direction in $r\leqq|z|\leqq R$.

The proof of this lemma is also analogous to the one given in [4, p. 200] and will be omitted.

By using Lemmas 1 and 2 we easily obtain the following:

THEOREM 3. *Let $F(z)$ be meromorphic and single-valued in $r\leqq|z|\leqq R$, continuous and $F'(z)\neq 0$ on $|z|=R$ and $|z|=r$. If there holds the relation*

$$\int_{|z|=r}\left|1+\Re\frac{zF''(z)}{F'(z)}\right|d\theta+\int_{|z|=R}\left|1+\Re\frac{zF''(z)}{F'(z)}\right|d\theta \tag{3.2}$$
$$<2\pi\big(k-n(\infty)+1\big), \qquad k: \text{positive integer},$$

then $F(z)$ is convex of order at most $k-n(\infty)-1$ in one direction in $r\leqq|z|\leqq R$. Further $F(z)$ is at most k-valent and at least $\mathrm{Max}\,[2n(\infty)-k, 1]$*-valent in $r\leqq|z|\leqq R$.*

LEMMA 4. *If $F(z)$ is meromorphic and*

$$\Re\left[e^{i\alpha}\left(1+\frac{zF''(z)}{F'(z)}\right)\right]>K \qquad (\alpha, K: \textit{real constants})$$

in a domain D, then $F(z)$ is regular in D (except the origin), and $F'(z)\neq 0$ in D (except the origin).

This lemma is due to S. Ozaki [2].

COROLLARY 5. *Let $F(z)$ be meromorphic and single-valued in $r\leqq|z|\leqq R$. If there holds the relation*

$$\left|1+\Re\frac{zF''(z)}{F'(z)}\right|<\frac{k-n(\infty)+1}{2} \qquad \textit{on } |z|=r \textit{ and } |z|=R, \tag{3.3}$$

then we have the same conclusion as Theorem 3.

COROLLARY 6. *Let $F(z)$ be meromorphic and single-valued in $r\leqq|z|\leqq R$. If*

$$\left|1+\Re\frac{zF''(z)}{F'(z)}\right|<\frac{p+1}{2} \qquad \textit{for } r\leqq|z|\leqq R,$$

then $F(z)$ is regular and convex of order at most $p-1$ in one direction and at most p-valent in $r\leqq|z|\leqq R$.

COROLLARY 7. *Let $F(z)$ be meromorphic and single-valued in $r\leqq|z|\leqq R$. Further let $F(z)$ be expanded as follows*

$$F(z)=\sum_{-\infty}^{\infty}a_n z^n \quad \textit{on } |z|=r \textit{ and} \quad F(z)=\sum_{-\infty}^{\infty}b_n z^n \quad \textit{on } |z|=R.$$

If, for an arbitrary constant such that $p+1>2|k|$,

$$\begin{aligned} |a_k| &> \sum_{n=1}^{\infty}\left(1+\frac{2n}{p+1-2|k|}\right)\left(\left|1+\frac{n}{k}\right|\left|a_{k+n}\right|r^n+\left|1-\frac{n}{k}\right|\left|a_{k-n}\right|r^{-n}\right) \\ |b_k| &> \sum_{n=1}^{\infty}\left(1+\frac{2n}{p+1-2|k|}\right)\left(\left|1+\frac{n}{k}\right|\left|b_{k+n}\right|R^n+\left|1-\frac{n}{k}\right|\left|b_{k-n}\right|R^{-n}\right) \end{aligned} \tag{3.4}$$

then $F(z)$ is convex of order at most $p-1$ in one direction in $r\leqq|z|\leqq R$. Further $F(z)$ is at most $p+n(\infty)$-valent and at least $\mathrm{Max}\,[n(\infty)-p, 1]$*-valent in $r\leqq|z|\leqq R$, where $n(\infty)$ denotes the number of poles of $F(z)$ in $r\leqq|z|\leqq R$.*

PROOF. By our hypothesis (3.4), we can prove

$$\left|1+\Re\frac{zF''(z)}{F'(z)}\right|<\frac{p+1}{2} \quad \text{on } |z|=r \text{ and } |z|=R \tag{3.5}$$

as follows:

$$\frac{p+1}{2}-\left|1+\Re\frac{zF''(z)}{F'(z)}\right|\geqq\frac{p+1}{2}-\left|k+\frac{\sum_{-\infty}^{\infty}n(n-k)a_nz^{n-k}}{ka_k+\sum_{n\neq k}^{\infty}na_n\,z^{n-k}}\right|\geqq\frac{p+1}{2}-|k|$$

$$-\frac{\sum_{n=1}^{\infty}|n(n+k)a_{k+n}|\cdot r^n+\sum_{n=1}^{\infty}|-n(k-n)a_{k-n}|\cdot r^{-n}}{|ka_k|-\sum_{n=1}^{\infty}|(k+n)a_{k+n}|\cdot r^n-\sum_{n=1}^{\infty}|(k-n)a_{k-n}|\cdot r^{-n}}>0$$

on $|z|=r$.

Quite similarly we obtain (3.5) on $|z|=R$. Consequently, by Corollary 5 we have our conclusion.

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