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On the dimension of homogeneous spaces.

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Let $\mathfrak{G}$ be a topological group and $\mathfrak{H}$ a closed subgroup of $\mathfrak{G}$. We shall prove in this paper that the equality

(1) $$\dim \mathfrak{G}=\dim \mathfrak{H}+\dim \mathfrak{G}/\mathfrak{H}$$

holds, when $\mathfrak{G}$ is a locally compact group satisfying the second axiom of countability. It is known that the equality (I) does not hold in general [8], but it holds if $\mathfrak{G}$ is a Lie group or a compact group, etc, [1, 2]. The equality (I) for the locally compact group would be easily dedueed, if it could be shown that every locally compact finite dimensional group $\mathfrak{G}$ admits a local cross-section over a closed subgroup $\mathfrak{H}$. But the author could not decide whether this is the case.

This paper consists of two parts. The part I is of a preliminary character; we shall provethere some lemmas used in the part II. In the part II we shall first reduce the general case to the case where $\mathfrak{G}$ is a locally compact finite dimensional connected group, and the prove the equality (I) for this case. We need for the proof the fact that every locally compact finite dimensional connected group satisfying the second axiom of countability is a projective limit group of Lie groups [4]. Perhaps a proof of our theorem without using this fact would be desirable.

In the following G denotes invariantly a locally compact group satisfying the second axiom of countability, and H, H', etc. closed subgroups of G.

Then the spaces and the homogeneous spaces of such groups G, G/H, etc. are metric separable and we shall assume that also other spaces considered are all metric separable, so that we can make free use of the dimension theory.

I. Some lemmas.

Let us begin with proving the following

LEMMA 1. *If* $G \supset H \supset H'$, *then*

$$\dim G/H' \leqq \dim H/H' + \dim G/H. \tag{1}$$

PROOF. If the dimension of H/H' is infinite, then (1) holds formally and the lemma is trivial. When then dimension of H/H' is finite, then we may assume moreover that $\dim G/H' > \dim G/H$. Now let us denote by $\overline{U}$ a compact neighborhood of the coset of H' in the homogeneous space G/H', and by p the natural projection of the homogeneous space G/H' onto the homogeneous space G/H by the inclusion of cosets. If we restrict the natural projection p to $\overline{U}$ then p is a closed mapping, and the dimension of the inverse image of each point of $p(\overline{U})$ does not exceed $\dim H/H'$. So we gave by Hurewicz's theorem [9],

$$\dim G/H' = \dim \overline{U} \leqq \dim H/H' + \dim p(\overline{U}) = \dim H/H' + \dim G/H.$$

As a special case of lemma 1 we have

LEMMA 1'.

$$\dim G \leqq \dim H + \dim G/H.$$

In the sequel we shall denote by G^* the component of the identity of the group G. Then, by lemma 1', it is easy to see that we have $\dim G = \dim G^*$ since we have obviously $\dim G/G^* = 0$.

LEMMA 2. *If there is an open mapping* p *of a* 0-*dimensional locally compact space* X *onto a space* Y, *then we have*

$$\dim Y = 0.$$

PROOF. Take anarbitrary point y in Y and an arbitrary open neighborhood U of y. Furthermore let us take a point x belonging to $p^{-1}(y)$, then there exists a compact open neighborhood $\overline{V}$ of x such that

$$p^{-1}(U) \supset \overline{V} \ni x$$

because X is a locally compact 0-dimensional space. Since p is an open mapping, $p(\overline{V})$ is a compact open neighborhood of y which is included in the neighborhood U. In other words $\dim Y = 0$.

COROLLARY 1. *If* $\dim G = 0$, *then* $\dim G/H = 0$

COROLLARY 2. *If* $G \supset H \supset G^*$, *then*

$$\dim G/H = 0 .$$

PROOF. The dimension of G/G^* is zero and there exists the natural projection which is an open mapping of G/G^* onto G/H by the inclusion of cosets into cosets. By lemma 2 $\dim G/H = 0$.

LEMMA 3. *Let* $\dim G = n\,(<\infty)$. *Then a necessary and sufficient condition for* $\dim H = n$ *is the inclusion* $H \supset G^*$.

PROOF. Sufficiency is obvious from the following relations

$$n \geqq \dim H \geqq \dim G^* = n .$$

To prove necessity, assume $\dim H = n$. Let H^* be the component of the identity of H. As we have

$$\dim H \geqq \dim (H \frown G^*) \geqq \dim H^* = \dim H ,$$

we have

$$n = \dim H = \dim (H \frown G^*) .$$

But the dimension of a proper subgroup of a finite dimensional connected group is exactly smaller than the dimension of the original group as shown by Montgomery [5]. Thus we obtain $H \frown G^* = G^*$.

From corollary 2 for lemma 2 and lemma 3 follows immediately

LEMMA 5. *If* $\dim G = \dim H = n\ (<\infty)$, *then*

$$\dim G/H = 0 .$$

DEFINITON. We say a topological group G acts on a space X when the following conditions are satisfied:

(1) each elements of G is a homeomorphism of the space X onto itself,

(2) for every $x \in X$ and $g_1, g_2 \in G$

$$(g_1 \cdot g_2)(x) = g_1\big(g_2(x)\big) ,$$

(3) $g(x)$ is a continuous mapping of the product space $G \times X$ onto the space X.

It is easily seen from the definition that the identity of G is the identity mapping of the space X.

LEMMA 5. *When a group G acts on a space X and $G(x)$ is the orbit of the group G for $x \in X$, we have*

$$\dim G(x) = \dim G/G_x$$

where G_x is the subgroup of G consisting of all elements which fix the element x.

PROOF. For an arbitrary element gG_x in the space G/G_x regarded as the left coset space, we set $T(gG_x)=g(x)$. Then we obtain a continuous mapping T of the space, G/G_x onto the space $G(x)$. It is clear that T is an one-to-one mapping. Next we can cover the space G/G_x with a countable set of compact neighborhoods, because the space G/G_x is a locally compact space satisfying the second axiom of countability. Let us denote by C_i $(i=1, 2,\cdots)$ these countable compact neighborhoods. Then we have

$$T(G/G_x)=T(\bigcup_{i=1}^{\infty} C_i)=\bigcup_{i=1}^{\infty} T(C_i).$$

As the continuous mapping T is a homeomorphism on each C_i, the following relation holds true

$$\dim T(C_i)=\dim C_j \,(=\dim G/G_x).$$

On the other hand each $T(C_i)$ $(i=1, 2,\cdots)$ is closed in $G(x)$, consequently using the sum theorem of the dimension theory, we have

$$\dim G(x)=\dim \bigcup_{i=1}^{\infty} T(C_i)=\dim T(C_j)=\dim G/G_x.$$

REMARK. This lemma holds also true if the dimension of G/G_x is infinite.

LEMMA 6. *When a group G acts on a space X and the orbit $G^*(x)$ of G^* for $x\in X$ is closed in X, then we have*

$$\dim G(x)=\dim G^*(x).$$

PROOF. By lemma 5 we have

$$\dim G(x)=\dim G/G_x.$$

Furthermore, as the orbit $G^*(x)$ of G^* for x is closed in X, we have

$$G^*(x)=G^*G_x(x)=\overline{G^*G_x}(x).$$

On the other hand, the group $\overline{G^*G_x}$ acts on the space X in the same manner as G acts on X, consequently making use of lemma 5, we have

$$\dim G^*(x)=\dim \overline{G_*G_x}(x)=\dim \overline{G^*G_x}/G_x$$

Next, the group $\overline{G^*G_x}$ is a subgroup of G and the group G_x is a subgroup of $\overline{G^*G_x}$. Therefore we can apply lemma 1, and obtain

$$\dim G/G_x \leqq \dim \overline{G^*G_x}/G_x + \dim G/\overline{G^*G_x}\,.$$

On the other hand we have $\dim G/\overline{G^*G_x}$ by corollary 2 for lemma 2. Therefore we have

$$\dim G/G_x \leqq \dim \overline{G^*G_x}/G_x\,.$$

At the same time

$$\dim G/G_x \geqq \dim \overline{G^*G_x}/G_x\,,$$

holds true, because the spce $\overline{G^*G_x}/G_*$ is a subspace of the space G/G_x. In other words

$$\dim G/G_x = \dim \overline{G^*G_x}/G_x$$

holds, whence follows our conclusion.

THEOREM 1. *Dim G/H is equal to the dimension of a component of the homogeneous space G/H.*

PROOF- As in the proof of lemma 6, we have

$$\dim G/H = \dim \overline{G^*H}/H\,.$$

So it is sufficient for the proof to verify that the space $\overline{G^*H}/H$ coincides with the component K of the coset of H in thespace G/H. The space $\overline{G^*H}/H$ is obviously included in K. Let p be the natural projection of the space G onto the space G/H, P the natural one of the space G onto the space $G/\overline{G^*H}$, respectively. If the component K includes the set $\overline{G^*H}/H$ as a proper subset, so $P\big(p^{-1}(K)\big)$ includes more than one points, and theset $p' \circ p \circ p^{-1}(K) = p'(K) = P \circ p^{-1}(K)$ is connected as $P = p' \circ p$ in the diagramm.

$$\begin{array}{ccc} G & \xrightarrow{\quad P \quad} & G/\overline{G^*H} \\ & {\scriptstyle p}\searrow \quad \nearrow{\scriptstyle p'} & \\ & G/H & \end{array}$$

On the other hand, we have clearly $\dim G/\overline{G^*H} = 0$. Thus we have arrived at a contradiction. Therefore the component K is the space $\overline{G^*H}/H$.

LEMMA 7.

$$\dim G^*H/H = \dim G^*/G^* \frown H .$$

PROOF. We can consider that the group G^* acts on the homogeneous space G/H in the same manner as the group G acts on the homogeneous space G/H. Then the orbit of G^* for H in G/H is the space G^*H/H and the group $G^* \frown H$ is exactly the subgroup of G^* consisting of all elements which fix the point H. By lemma 5 we have

$$\dim G^*H/H = \dim G^*/G^* \frown H .$$

II. Reduction and the final proof.

LEMMA 8. *If the property (I) holds true for all finite dimensional connected groups and their subgroups, then the property (I) holds also true in case where the orbit of G^* for H is closed in the space G/H.*

PROOF. When the dimension of the group G is infinite, then by lemma 1′ the property (I) holds true formally. So we shall assume $\dim G < \infty$. Then we can consider that the group G acts on the space G/H. The point H of this space G/H will be denoted by $\bar{e}$. By lemma 6 we have

$$\dim G/H = \dim G(\bar{e}) = \dim G^*(\bar{e}) ,$$

and by lemma 7

$$\dim G^*(\bar{e}) = \dim G^*H/H = \dim G^*/G^* \frown H .$$

On the otherhand, we have

$$\dim G = \dim G^* ,$$

and

$$\dim H = \dim (G^* \frown H) .$$

Therefore to prove the relation $\dim G = \dim H + \dim G/H$ is equivalent to prove the relation

$$\dim G^* = \dim (G^* \frown H) + \dim G^*/G^* \frown H .$$

Thus the proof of (I) in our case is reduced to the case where G is finite dimensional and connected.

LEMMA 9. *If the property (I) holds true for all finite dimensional connected groups and their subgroups, then the property (I) holds true for any group and any subgroup.*

PROOF. We may assume that the dimension of G is finite. Now as the group H/H^* is a 0-dimensional group, there exists a closed and open subgroup H' of the group H such that the homogeneous space H/H' is discrete and the group H'/H^* is a compact group. From the discreteness of the space H/H', we have

$$\dim G/H=\dim G/H' .$$

On the other hand we have $\dim H=\dim H'$, so that we can replace H by H' in the proof of (I).

We shall show that for H' the orbit of G^* in G/H' is closed. By theorem 1, we have

$$\dim G/H'=\dim \overline{G^*H'}/H' .$$

On the other hand, let p' be the projection of the group G onto the space G/H^*, then there exists a compact set C in the group G such that the following relation holds

$$p'(C)=H'/H^* ,$$

because the group H'/H^* is compact. This shows $H'=CH^*$. Then the set G^*C is closed in G, for the set C is compact and the group G^* is closed [1]. Furthermore it holds true that

$$G^*C=G^*CG^*\supset G^*CH^*=G^*H'\supset G^*C .$$

Therefore we have

$$\overline{G^*H'}=G^*H' .$$

In other words the space G^*H'/H' is closed in G/H'. Our lemma follows then from the preceding lemma.

LEMMA 10. *Let G be an n-dimensional connected group and H a subgroup of G, then the properted (I) holds true.*

PROOF. As the group G is a generalized Lie group, there exists a neighborhood U of the identity having the following properties:

$$U=Z\cdot L$$

where Z is a compact 0-dimensional central group and L is a local Lie group with $\dim L = \dim G$. Furthermore as any closed subgroup of a generalized Lie group is also a generalized Lie group, there exists a neighborhood V of the identity in the group G such that the following properties are satisfies:

$$U \supset V, \quad V \frown H = Z' \cdot L', \quad Z' \subset Z, \quad L' \subset L,$$

where Z' is a compact 0-dimensional central group and L' is a local Liegroup with $\dim L' = \dim H$.

Now $\dim G/Z'$ is clearly equal to the dimension of a neighborhood of the identity of the factor group G/Z', so we have

$$\dim G/Z' = \dim Z/Z' + \dim L = \dim L = \dim G .$$

Furthermore the group G/Z' is an n-dimensional group containing the local Lie group L_1 which is isomorphic to the local Lie group L', therefore there exists a compact local cross-section set M at the identity such that the following equality holds

$$\dim G/Z' = \dim (L_1 \cdot M) = \dim (L' \times M) .$$

On the other hand L' may be regarded as an n-dimensional cell, therefore by the Hurewicz's theorem we have

$$\begin{aligned} \dim (L' \times M) &= \dim L' + \dim M \\ &= \dim H + \dim G/Z'L' , \end{aligned}$$

where $\dim G/Z'L'$ means the dimension at the coset $Z'L'$ of the local coset space $G/Z'L'$. In other words

$$\dim G = \dim H + \dim G/H . \qquad \text{Q. E. D.}$$

From lemma 9 and 10 follows immediately

THEOREM 2.

$$\dim G = \dim H + \dim G/H .$$

By lemma 5 we can generalize this theorem to the following theorem.

THEOREM 3. *Let G be a group acting on a space X, $G(x)$ the orbit of G for $x \in X$, and G_x the subgroup of G consisting of the elements which fix the point x. Then we have*

$$\dim G = \dim G_x + \dim G(x) .$$

Finally we have also as a generalization of lemma 6

THEOREM 4. *If we use the same notations as in lemma then we have*

$$\dim G(x) = \dim G^*(x) .$$

PROOF. In case where the dimension of the group G is finite, the following relations holds true

$$\begin{aligned}\dim G^*(x) = \dim G^*/G^* \frown G_x &= \dim G^* - \dim (G^* \frown G_x) \\ &= \dim G - \dim G_x \\ &= \dim G/G_x \\ &= \dim G(x) .\end{aligned}$$

If $\dim G$ is infinite, we can prove in the same way as in the proof of lemma 9.

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