Journal of the Mathematical Society of Japan Vol. 7, No. 2, April, 1955

Function of U-class and its applications.

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(Received July 23, 1954)

1. Function of U-class.

Let $w=f(z)$ be regular and $|f(z)|<1$ in $|z|<1$, then by Fatou's theorem, $\lim_{z\to e^{i\theta}} f(z)=f(e^{i\theta})$ exists almost everywhere on $|z|=1$, when $z\to e^{i\theta}$ from the inside of any Stolz domain, whose vertex is at $e^{i\theta}$. If $|f(e^{i\theta})|=1$ almost everywhere, we say with Seidel[1] that $f(z)$ belongs to U-class and denote $f(z)\in U$. If $(f(z)-a)/\rho\in U$, we write $f(z)\in U_\rho(a)$. Functions of U-class play an important rôle in several problems. In this paper, we shall show some applications of them. In this paper, "capacity" means "logarithmic capacity" and $\gamma(E)$ denotes the capacity of E.

LEMMA 1.[2] *(Extension of Löwner's theorem). Let $w=f(z)$ be regular and $|f(z)|<1$ in $|z|<1, f(0)=0$. Let E be the set of $e^{i\theta}$ on $|z|=1$, such that $|f(e^{i\theta})|=1$ and E^* be the set $f(e^{i\theta})$ $(e^{i\theta}\in E)$ on $|w|=1$. Then E and E^* are measurable and $mE^*\geqq mE$.*

LEMMA 2. *If $f(z)\in U$, then $f(z)$ takes any value of $|w|<1$ at least once, except a set of capacity zero.*

PROOF. Let E be the set of a $(|a|<1)$, such that $f(z)\neq a$ in $|z|<1$ and suppose that $\gamma(E)>0$, then by taking a suitable closed sub-set, we may assume that E is a closed set, contained entirely in $|w|<1$. Let D be the domain, which is bounded by E and $|w|=1$. We solve the Dirichlet problem for D, with the boundary value 1 on E and 0 on $|w|=1$, and let $u(w)$ be its solution, then since $\gamma(E)>0$, E contains a regular point of Dirichlet problem, so that $u(w)\not\equiv 0$. If we put $u(f(z))=v(z)$, then $v(z)$ is a bounded harmonic function in $|z|<1$.

1) W. Seidel: On the distribution of values of bounded analytic functions. Trans. Amer. Math. Soc. 36 (1934).

2) M. Tsuji: On an extension of Löwner's theorm. Proc. Imp. Acad. 18 (1942). The special case, where $f(z)$ is schlicht in $|z|<1$, is proved by Y. Kawakami: On an extension of Löwner's lemma. Jap. Journ. Math. 17 (1941).

Since $f(z) \in U$, $v(e^{i\theta})=0$ almost everywhere, so that $v(z)\equiv 0$, or $u(w)\equiv 0$, which is absurd. Hence $\gamma(E)=0$.

THEOREM 1. *Let $f(z) \in U$ and F be the Riemann surface, generated by $w=f(z)$ on the w-plane.*

(i) *Let F_ρ be a connected piece of F, which lies above a disc $K: |w-a_0|<\rho$, which lies in $|w|<1$. If we map F_ρ conformally on $|\zeta|<1$ by $w=\varphi(\zeta)$, then $\varphi(\zeta) \in U_\rho(a_0)$*[3].

(ii) *Let a be any point of K and be covered $n(a)$-times by F_ρ and $n_0=\sup\limits_a n(a)$. Then F_ρ covers any point of K n_0-times, except a set of capacity zero. If $n_0<\infty$, then F_ρ covers any point of K n_0-times.*

(iii) *If $f(z)$ is of the form: $f_0(z)=\varepsilon\prod\limits_{\nu=1}^{n}\dfrac{z-z_\nu}{1-\bar{z}_\nu z}$ $(|z_\nu|<1, |\varepsilon|=1)$, then F covers any point of $|w|<1$ n-times. If $f(z)$ is not of the form $f_0(z)$, then F covers any point of $|w|<1$ infinitely often, except a set of capacity zero.*[4]

PROOF of (i). Let $\varDelta_0$ be the image of F_ρ in $|z|<1$, then $\varDelta_0$ is simply connected, so that F_ρ is simply connected.

We may assume that $\varDelta_0$ has boundary points on $|z|=1$ and let e_0 be the set of such boundary points. We map F_ρ on $|\zeta|<1$ conformally by $w=\varphi(\zeta)$, then $\lim\limits_{r\to 1}\varphi(re^{i\varphi})=\varphi(e^{i\varphi})$ exists almost everywhere. Let e_1 be the set of $e^{i\varphi}$, such that $|\varphi(e^{i\varphi})-a_0|<\rho$. If $\zeta=re^{i\varphi}\to e^{i\varphi}$, then $w\to\varphi(e^{i\varphi})$ along a curve L. Let L correspond to a curve $\varLambda$ in $\varDelta_0$, which ends at a point $e^{i\theta}\in e_0$. Then if $z\to e^{i\theta}$ on $\varLambda$, $w=f(z)\to\varphi(e^{i\varphi})$. Since $f(z)$ is bounded, $\lim\limits_{r\to 1} f(re^{i\theta})=\varphi(e^{i\varphi})$ by Hardy's theorem. Since $f(z)\in U$, the set of such $e^{i\theta}$ is of measure zero. Hence by Lemma 1, e_1 is a null set, so that $\varphi(\zeta)\in U_\rho(a_0)$.

To prove (ii), we shall prove a lemma.

LEMMA 3. *Let K_0 be a disc contained in K. If every point of K_0 is covered n-times by F_ρ $(1\leqq n<\infty)$, then every point of K is covered n-times by F_ρ.*

PROOF. Let D be the domain, which contains K_0 and every point of which is covered n-times by F_ρ. Suppose that D does not coincide

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3) K. Noshiro: Contributions to the theory of the singularities of analytic functions. Jap. Journ. Math. 19 (1944-48).

4) O. Frostman: Potentiel d'équilibre et capacité des ensembles. Lund. (1935).

with K and let Γ be the part of the boundary of D, which lies in K and $w_0 \in \Gamma$.

Then w_0 is covered at most n-times by F_ρ. We shall prove that w_0 is covered at most $(n-1)$-times by F_ρ. Suppose that w_0 is covered n-times by F_ρ, then the part of F_ρ, which lies above a small disc K_1 about w_0 contains n discs: $F_1, \cdots, F_n$ consisting of inner points, where the part of the Riemann surface of $(w-w_0)^{\frac{1}{k}}$ is considered as k discs. If there is no other connected piece of F_ρ above K_1, then K_1 is covered n-times by F_ρ, so that w_0 belongs to D, which is absurd. Hence there is another connected piece F_0 of F_ρ above K_1 other than $F_1, \cdots, F_n$.

By Lemma 2 and part (i), F_0 covers any points of K_1 at least once, except a set of capacity zero, but F_0 does not cover $D_0 = D.K_1$, which is of positive capacity, which is absurd. Hence every point of Γ is covered at most $(n-1)$-times by F_ρ. Next we shall prove that $\gamma(\Gamma)=0$. Suppose that $\gamma(\Gamma)>0$. Let Γ_k be the sub-set of Γ, which is covered k-times by F_ρ, then for some k, $\gamma(\Gamma_k)>0$. Since by Lemma 2 and the part (i), F_ρ covers any point of K at least once, except a set of capacity zero, $\gamma(\Gamma_0)=0$, so that $1 \leqq k \leqq n-1$. By taking a suitable closed sub-set, we may assume that Γ_k is a closed set, contained entirely in K. Then there exists a point $w_0 \in \Gamma_k$, such that $\gamma(\Gamma_k \cdot K_1)>0$, for any small disc K_1 about w_0.

Since $w_0 \in \Gamma_k$, w_0 is covered k-times by F_ρ, there exists k discs $F_1, \cdots, F_k$ above K_1 consisting of inner points.

Since $1 \leqq k \leqq n-1$, there is another connected piece F_0 above K_1, other than $F_1, \cdots, F_k$, then similarly as before, F_0 covers any point of K_1 at least once, except a set of capacity zero, but since $\Gamma_k \cdot K_1$ is covered k-times in $F_1, \cdots F_k$, F_0 does not cover $\Gamma_k \cdot K_1$, which is of positive capacity, which is absurd. Hence $\gamma(\Gamma)=0$.

Let $w_0 \in \Gamma$ and $z=z_i(w)$ $(i=1, 2, \cdots, n)$ be n branches of the inverse function $z=z(w)$ of $w=f(z)$ and consider

$$\prod_{i=1}^{n} (z-z_i(w)) = z^n + a_1(w)z^{n-1} + \cdots + a_n(w) = 0,$$

then $a_i(w)$ is one-valued, regular and bounded in a neighbourhood of w_0 and since $\gamma(\Gamma)=0$, $a_i(w)$ is regular at w_0, so that w_0 is covered n-times by F_ρ, which is absurd. Hence D coincides with K, so that every point of K is covered n-times by F_ρ.

PROOF of (ii) and (iii).

(ii) Let $n < n_0$ and E_n be the set of a, such that $n(a)=n$. We shall prove that $\gamma(E_n)=0$. Suppose that $\gamma(E_n)>0$, then we may assume that E_n is a closed set, contained entirely in K. Then there exists a point $w_0 \in E_n$, such that $\gamma(E_n \cdot K_1)>0$, for any small disc K_1 about w_0. Since $w_0 \in E_n$, w_0 is covered n-times by F_ρ, so that there exists n discs $F_1, \cdots, F_n$ above K_1 consisting of inner points. Since $n<n_0$, there is a point a, such that $n(a)>n$, hence by Lemma 3, there is another connected piece F_0 above K_1, other than $F_1, \cdots, F_n$. Then as before, F_0 covers any point of K_1 at least once, except a set of capacity zero, but since $E_n \cdot K_1$ is covered n-times in $F_1, \cdots, F_n$, F_0 does not cover $E_n \cdot K_1$, which is of positive capacity, which is absurd. Hence $\gamma(E_n)=0$, $n<n_0$. Hence F_ρ covers any point of K n_0-times, except a set of capacity zero. Suppose that $n_0<\infty$ and let E be the set of a, such that $n(a)<n_0$, then E is a closed set of capacity zero, so that from the proof of Lemma 3, F_ρ covers any point of K n_0-times.

(iii) We take K: $|w| \leqq \rho < 1$ and we choose F_ρ, such that $F_\rho \subset F_{\rho'}$, if $\rho<\rho'$ and let $n_0=n_0(\rho)$, $\lim_{\rho \to 1} n_0(\rho)=\bar{n}_0$. If $\bar{n}_0<\infty$, then since $\lim_{\rho\to 1} F_\rho = F$, F consists of $\bar{n}_0$ sheets and by (ii) F covers any point of $|w|<1$ $\bar{n}_0$-times. By (ii), F_ρ consists of n sheets $F_\rho^{(i)}$ $(i=1,\cdots,n)$ $(n \leqq \bar{n}_0)$. Let v_ρ be the sum of orders of branch points in F_ρ and $\rho^{(i)}$ be the Euler's characteristic of $F_\rho^{(i)}$, then $\rho^{(i)} \geqq -1$. If we consider the image of F_ρ in $|z|<1$, then we see that F_ρ is simply connected, hence by Hurwitz's relation, we have

$$-1=\sum_{i=1}^{n} \rho^{(i)}+v_\rho \geqq -n+v_\rho \geqq -\bar{n}_0+v_\rho, \quad v_\rho \leqq \bar{n}_0-1.$$

Hence there is only a finite number of branch points in F, so that $f(z)$ is regular on $|z|=1$. Since $|f(z)|=1$ on $|z|=1$, we see, by the principle of inversion, that $f(z)$ is a rational function of the form $f_0(z)$. Hence if $f(z)$ is not of the form $f_0(z)$, then $\bar{n}_0=\infty$, so that F covers any point of $|w|<1$ infinitely often, except a set of capacity zero.

2. Open Riemann surface with null boundary.

Let F be an open Riemann surface with null boundary, spread over the z-plane. If F consists of a finite number of sheets, we shall call it a quasi-closed surface.

THEOREM 2. *Let F_ρ be a connected piece of F, which lies above a disc K: $|z-a_0|<\rho$.*

(i) *If we map the universal covering surface of F_ρ conformally on $|\zeta|<1$ by $z=\varphi(\zeta)$, then $\varphi(\zeta)\in U_\rho(a_0)$.*

(ii) *Let a be any point of K and be covered $n(a)$-times by F_ρ and $n_0=\sup_a n(a)$. Then F_ρ covers any point of K n_0-times, except a set of capasity zero.*[5]

(iii) *If F is not quasi-closed, then F covers any point z infinitely often, except a set of capacity zero.*[6]

PROOF. (i) If F_ρ is compact, (i) follows easily from Fatou's theorem, so that we assume that F_ρ is non-compact. We map the universal covering surface of F conformally on $|x|<1$ by $z=\psi(x)$, then by a theorem,[6] proved by the author, the ideal boundary of F is mapped on a null set on $|x|=1$.

By this, we can prove as Theorem 1, that $\varphi(\zeta)\in U_\rho(a_0)$.

(ii) Suppose that a disc K_0 contained entirely in K be covered exactly n-times by F_ρ $(1\leqq n<\infty)$ and let D be the domain, which contains K_0 and every point of which is covered n-times by F_ρ, then as before, we can prove that the part Γ of the boundary of D in K is of capacity zero, so that F_ρ covers any point of K n-times, except a set of capacity zero. In this case $n_0=n$.

Next suppose that there exists no such a disc K_0 and let E_n $(n=0, 1, 2, \cdots)$ be the set of a, such that $n(a)=n$. Then we can prove as before, that $\gamma(E_n)=0$, if $n<n_0$. Hence F_ρ covers any point of K, n_0-times, except a set of capacity zero.

(iii) We put $n_0=n_0(\rho)$ and $\lim\limits_{\rho\to\infty} n_0(\rho)=\bar{n}_0$, If $\bar{n}_0<\infty$, then since $\lim\limits_{\rho\to\infty} F_\rho=F$, F consists of $\bar{n}_0$ sheets, so that F is quasi-closed, hence if F is not quasi-closed, then $\bar{n}_0=\infty$, so that F covers any point z infinitely often, except a set of capacity zero.

5) Y. Nagai: On the behaviour of the boundary of Riemann surfaces. II. Proc. Jap. Acad. 26 (1950).

6) M. Tsuji: Some metrical theorems on Fuchsian groups. Kodai Math. Seminar Reports. (1950).

From Theorem 2, we have easily

THEOREM 3. *The projection of direct transcendental singularities of F on the z-plane is of capacity zero.*

3. Implicit function $y(x)$ defined by an integral relation $G(x, y)=0$.

Let $G(x,y)$ be an integral function of two variables x and y and $y(x)$ be an analytic function defined by $G(x,y)=0$ and F be its Riemann surface spread over the x-plane. If $G(x,y)$ is of the form:

$$G(x,y)=A_0(x)y^n+A_1(x)y^{n-1}+\cdots+A_n(x),$$

where $A_i(x)$ are integral functions of x, then $y(x)$ is an algebroid function and F consists of n sheets. We shall prove

THEOREM 4. *Let F_ρ be a connected piece of F, which lies above a disc K: $|x-a_0|<\rho$.*

(i) *If we map the universal covering surface of F_ρ conformally on $|\zeta|<1$ by $x=\varphi(\zeta)$, then $\varphi(\zeta)\in U_\rho(a_0)$.*

(ii) *Let a be any point of K and be covered $n(a)$-times by F_ρ and $n_0=\sup\limits_a n(a)$. Then F_ρ covers any point of K n_0-times, except a set of capacity zero. If $n_0<\infty$, then F_ρ covers any point of K n_0-times.*

(iii) *If $y(x)$ is not an algebroid function, F covers any point x infinitely often, except a set of capacity zero.*[7]

PROOF. (i) As Julia[8] proved, if x tends to an accessible boundary point of F, then $\lim y(x)=\infty$. Let E be the set of $e^{i\theta}$ on $|\zeta|=1$, such that $|\varphi(e^{i\theta})-a_0|<\rho$, then if ζ tends to $e^{i\theta}$ from the inside of any Stolz domain, whose vertex is at $e^{i\theta}$, then $x=\varphi(\zeta)$ tends to an accessible boundary point of F, so that $\lim y(\varphi(\zeta))=\infty$, hence by Lusin-Priwaloff's theorem, E is a null set, hence $\varphi(\zeta)\in U_\rho(a_0)$.

(ii) Let K_0 be a disc contained in K and suppose that every point of K_0 is covered n-times by F_ρ $(1\leqq n<\infty)$ and let D be the domain, which contains K_0 and every point of which is covered n-times by F_ρ,

7) M. Tsuji: Theory of meromorphic functions in a neighbourhood of a closed set of capacity zero. Jap. Journ. Math. 19 (1944).

8) G. Julia: Sur le domaine d'existence d'une fonction implicite définie par une relation entière $G(x,y)=0$. Bull. Soc. Math. France (1926).

then similarly as before, we can prove that if D does not coincide with K, the part Γ of the boundary of D in K is of capacity zero.

Let $x_0 \in \Gamma$ and $y_i(x)\,(i=1,2,\cdots,n)$ be n branches of $y(x)$ in D and suppose that $y_i(x)\,(i=1,2,\cdots,k)\,(k \leqq n)$ are not meromorphic at x_0 and consider

$$\prod_{i=1}^{k}\left(\frac{1}{y}-\frac{1}{y_i(x)}\right)=\frac{1}{y^k}+\frac{a_1(x)}{y^{k-1}}+\cdots+a_k(x)=0,$$

Then since $1/y(x)$ tends to zero, when x tends to an accessible boundary point of F, $a_i(x)$ is one-valued, regular and bounded in a neighbourhood of x_0, and since $\gamma(\Gamma)=0$, $a_i(x)$ is regular at x_0, so that x_0 is covered n-times by F_ρ, which is absurd. Hence D coincides with K, so that F_ρ covers any pont of K n-times. From this we can prove the remaning part of the theorem similarly as Theorem 1.

From Theorem 4, we have

THEOREM 5. *The projection of direct transcendental singularities of F on the x-plane is of capacity zero.*[7]

4. Cluster set of a meromorphic function.

Let Δ be a domain on the z-plane and Γ be its boundary and z_0 be a non-isolated boundary point. We denote the part of Δ, contained in $|z-z_0|<r$ by Δ_r, and that of Γ in $|z-z_0|\leqq r$ by Γ_r. Let $w=f(z)$ be one-valued and meromporphic in Δ and W_r be the set of values taken by $w=f(z)$ in Δ_r and $\overline{W_r}$ be its closure, then

$$\lim_{r\to 0}\overline{W_r}=H_\Delta(z_0) \tag{1}$$

is called the cluster set of $f(z)$ in Δ at z_0.

Let e be a set of capacity zero on Γ, such that $z_0 \in e$ and e_r be the part of e lying in $|z-z|\leqq r$. Let

$$V_r(\Gamma-e)=\sum_{\zeta\in\Gamma_r-e_r} H_\Delta(\zeta), \text{ added for all } \zeta\in\Gamma_r-e_r, \tag{2}$$

and $\overline{V_r}(\Gamma-e)$ be its closure, then

$$\lim_{r\to 0} V_r(\Gamma-e)=H_{\Gamma-e}(z_0) \tag{3}$$

is called the cluster set of $f(z)$ on $\Gamma-e$ at z_0, Evidently, $H_\Delta(z_0)$ and $H_{\Gamma-e}(z_0)$ are closed sets and $H_{\Gamma-e}(z_0) \subset H_\Delta(z_0)$.

In the former paper[9] I have proved:

THEOREM 6. *Every boundary point of $H_\Delta(z)$ belongs to $H_{\Gamma-e}(z_0)$.*

When e consists of only one point z_0, the theorem is proved by Iversen.[10]

First we shall prove a lemma.

LEMMA 4. *Let Δ be a bounded domain and Γ be its boundary and e be a set of capacity zero on Γ and $z_0 \in e$ be a non-isolated boundary p int, which is a regular point of Dirichlet problem for Δ. Let $w=f(z)$ be one-valued, regular and bounded in Δ.*

If $\overline{\lim\limits_{\zeta \to z_0}} \overline{\lim\limits_{z \to \zeta \in \Gamma - e}} |f(z)| \leq M$, where $z \to \zeta \in \Gamma - e$, from the inside of Δ, then $\overline{\lim\limits_{z \to z_0}} |f(z)| \leq M$, where $z \to z_0$ from the inside of Δ.

PROOF. We may assume that $|f(z)| \leq 1$ in Δ. For any small $\varepsilon > 0$, we choose ρ, such that

$$\overline{\lim_{z \to \zeta \in \Gamma_\rho \to e_\rho}} |f(z)| \leq M + \varepsilon.$$

We solve the Dirichlet problem for Δ, with the boundary value $M+\varepsilon$ on Γ_ρ and 1 on $\Gamma - \Gamma_\rho$ and let $u(z)$ be its solution. Since z_0 is a regular point of Dirichlet problem, $\lim\limits_{z \to z_0} u(z) = M+\varepsilon$, when $z \to z_0$ from the inside of Δ. $u(z)$ takes the given boundary value, except a set of capacity zero and since $|f(z)|$ is a continuous bounded subharmonic function and $|f(z)| \leq u(z)$ on Γ, except a set of capacity zero, we have $|f(z)| \leq u(z)$ in Δ, so that $\overline{\lim\limits_{z \to z_0}} |f(z)| \leq M+\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have $\overline{\lim\limits_{z \to z_0}} |f(z)| \leq M$.

PROOF of THEOREM 6.

Suppose that there exists a boundary point w_0 of $H_\Delta(z_0)$, which

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9) M. Tsuji: On the cluster set of a meromorphic function. Proc. Imp. Acad. **19** (1943).

10) F. Iversen: Sur quelques propriétés des fonctions monogênes au voisinage d'un point singulier. Öfv. af Finska Vet. Soc. Förh. **58** (1916). K. Kunugui: Sur un théorême de MM. Seidel-Beurling. Proc. Imp. Acad. 15 (1939).

does not belong to $H_{\Gamma-e}(z_0)$ and we assume that $w_0=0$. Then we take r and ρ small, such that

$$\overline{V_r}(\Gamma-e) \text{ lies outside of } |w|=\rho>0. \tag{1}$$

Since $w_0=0$ is a boundary point of $H_\Delta(z_0)$, there exists w_1 ($|w_1|<\rho/2$), which does not belong to $H_\Delta(z_0)$. Since $H_\Delta(z_0)$ is a closed set, $\dfrac{1}{f(z)-w_1}$ is bounded in a neighbourhood of z_0.

(i) First suppose that z_0 is a regular point of Dirichlet problem. Then by Lemma 4 and (1), since $w=0$ belongs to $H_\Delta(z_0)$,

$$\frac{1}{|w_1|}\leqq\overline{\lim_{z\to z_0}}\frac{1}{|f(z)-w_1|}\leqq\overline{\lim_{\zeta\to z_0}}\ \overline{\lim_{\zeta\in\Gamma-e}}\frac{1}{|f(z)-w_1|}\leqq\overline{\lim_{\zeta\to z_0}}\ \overline{\lim_{\zeta\in\Gamma-e}}\frac{1}{|f(z)|-|w_1|}$$
$$\leqq\frac{1}{\rho-|w_1|},$$

so that $|w_1|\geqq\rho/2$, which is absurd. Hence the theorem is proved in this case.

(ii) Next suppose that z_0 is an irregular point of Dirichlet problem. Then in any small neighbourhood of z_0, there is a Jordan curve C in Δ, surrounding z_0. We assume that C lies in $|z-z_0|<r$ and there is no zero points of $f(z)$ on it, then by taking r and ρ small, we may assume that

$$\overline{V_r}(\Gamma-e) \text{ lies outside of } |w|=2\rho \text{ and } |f(z)|>2\rho \text{ on } C. \tag{2}$$

We consider the image of $|w|<\rho$ on the z-plane, which lies in C. It consists of at most a countable number of connected domains $\{\Delta_i\}_{i=1,2\cdots}$. We shall prove that there is one Δ_0 among $\{\Delta_i\}$, which contains z_0 on its boundary. If otherwise, then since $w=0$ belongs to $H_\Delta(z_0)$ and $w=0$ is a boundary point of $H_\Delta(z_0)$, there are infinitely many $\{\Delta_\nu\}_{\nu=1,2\cdots}$ among $\{\Delta_i\}$, such that the boundary Γ_ν of Δ_ν has common points with Γ and contains $z_\nu\to z_0$, such that $f(z_\nu)\to 0$. Then we shall prove that Δ_ν converges to z_0. For, if otherwise, Γ_ν has a common point ζ_ν with a certain Jordan curve C' in Δ, surrounding z_0, which is contained inside of C. Let ζ be one of limit points of ζ_ν, then $f(z)$ is meromorphic at ζ and in any small neighbourhood of ζ, there are infinitely many niveau curves $|f(z)|=\text{const.}=\rho$, which is absurd. Hence Δ_ν converges to z_0. By (2), the common part e_ν of Γ_ν with Γ belongs to e, so that it is of capacity zero. If we map the universal covering surface of Δ_ν conformally on $|\zeta|<1$ by $z=\varphi_\nu(\zeta)$, then e_ν is mapped on a null set on $|\zeta|=1$, so that if we put $w=f(\varphi_\nu(\zeta))=F_\nu(\zeta)$, then $\frac{F_\nu(\zeta)}{\rho}$ belongs to U-class, hence $F_\nu(\zeta)$ takes any value of $|w|<\rho$ at least once, except a set of capacity zero. Since there are infinitely many Δ_ν converging to z_0, $f(z)$ takes any value of $|w|<\rho$ infinitely often, in any small neighbourhood of z_0, except a set of capacity zero, hence $|w|<\rho$ belongs to $H_\Delta(z_0)$, which is absurd. Hence there is one Δ_0 among $\{\Delta_i\}$, which contains z_0 on its boundary. Since $w=0$ is a boundary point of $H_\Delta(z_0)$, we see from the above proof, that there is only a finite number of such Δ_0, hence one fixed Δ_0 contains infinitely many $z_\nu\to z_0$, such that $f(z_\nu)\to 0$. If we consider the images of $|w|<\frac{\rho}{n}$ $(n=1, 2,\cdots)$ in Δ_0, we see that there exists a curve L in Δ_0, which ends at z_0, such that $f(z)\to 0$, when $z\to z_0$ on L. We take off L from Δ_0 and put $\tilde{\Delta}_0=\Delta_0-L$, then z_0 is a regular point of Dirichlet problem for $\tilde{\Delta}_0$. Let w_1 $\left(|w_1|<\frac{\rho}{2}\right)$ lie outside of $H_\Delta(z_0)$. If we apply Lemma 4 to $\frac{1}{f(z)-w_1}$ for $\tilde{\Delta}_0$, then $\overline{\lim_{z\to z_0}}\frac{1}{|f(z)-w_1|}\leqq\frac{1}{|w_1|}$, or $\underline{\lim_{z\to z_0}}\ |f(z)-w_1|\geqq|w_1|$, hence $|w-w_1|<|w_1|$ does not belong to $H_\Delta(z_0)$. Similarly we see that $0<|w|\leqq|w_1|$ does not belong to $H_\Delta(z_0)$, which is absurd, since $H_\Delta(z_0)$ contains a continuum, which connects z_0 to $H_{\Gamma-e}(z)$. Hence the theorem is proved in this case.

From Theorem 6, we see that the same result as Lemma 4 holds, if z_0 is an irregular point of Dirichlet problem. Hence

THEOREM 7. *The same result holds as Lemma* 4, *for any non-isolated boundary point* z_0.

By Theorem 6, $H_\Delta(z_0)-H_{\Gamma-e}(z_0)$ is an open set, if it is not empty, so that it consists of at most a countable number of connected domains (components).

THEOREM 8.[9] *Let D be one of components of $H_{\Delta}(z_0)-H_{\Gamma-e}(z_0)$. Then in any small neighbourhood of z_0, $f(z)$ takes any value of D infinitely often, except a set of capacity zero.*

PROOF. Let E_n $(n=0,1,2,\cdots)$ be the set of points of D, which are taken n-times by $w=f(z)$ in a neighbourhood U: $|z-z_0|<r$ of z_0, and suppose that $\gamma(E_n)>0$, then by taking a suitable closed sub-set, we may assume that E_n is a closed set. Hence by taking r small, we may assume that $f(z)$ does not take the values $\in E_n$ in U. There exists a point $w_0 \in E_n$, such that $\gamma(E_n \cdot K)>0$, for any small disc K about w_0. We assume that $w_0=0$. We can choose r, such that $|z-z|=r$ does not contain points of e and zero points of $f(z)$, then by taking r and ρ small, we assume that

$$V_\nu(\Gamma-e) \text{ lies outside of } |w|=2\rho \text{ and } |f(z)|>2\rho \text{ on } |z-z_0|=r. \tag{1}$$

We consider the images of $|w|<\rho$ on the z-plane, then there is one domain Δ_0 among the images, which lies in $|z-z_0|<r$. By (1), if the boundary Γ_0 of Δ_0 has common points with Γ, then the common part e_0 is a sub-set of e, so that it is of capacity zero. By mapping the universal covering surface of Δ_0 conformally on $|\zeta|<1$, we see as before, that $f(z)$ takes in Δ_0 any value of $K:|w|<\rho$ at least once, except a set of capacity zero, but $f(z)$ does not take values $\in E_n \cdot K$, which is of positive capacity, which is absurd. Hence $\gamma(E_n)=0$ $(n=0, 1,2,\cdots)$, so that in U, $f(z)$ takes any value of D infinitely often, except a set of capacity zero.

REMARK. If e consists of only one point z_0, then $f(z)$ takes any value of D infinitely often, with two possible exceptions in any neighbourhood of z_0[11].

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