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On n-dimensional homogeneous spaces of Lie groups of dimension greater than $n(n-1)/2$.

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0. Introduction

The purpose of this note is to determine the Lie groups of dimension greater than $n(n-1)/2$ which can be treated as groups of isometries on an n-dimensional Riemannian space and study the differential-geometrical and topological structure of the space.

In this regard, K. Yano [5] has recently proved the following interesting theorem.

THEOREM A. *A necessary and sufficient condition that an n-dimensional Riemannian space for $n>4$, $n\neq 8$ admit a group of motions of order $n(n-1)/2+1$ is that the space be the product space of a straight line and an $(n-1)$-dimensional Riemannian space of constant curvature or that the space be of negative constant curvature.*

In this theorem the cases $n=4$ and $n=8$ are exceptional. For $n=4$, S. Ishihara [1] has solved the problem completely by determining all 4-dimensional homogeneous Riemannian spaces, but it was open for $n=8$.

On the other hand, to prove Theorem A, K. Yano used essentially the following theorem due to D. Montgomery and H. Samelson [3].

THEOREM B. *The rotation group $R(n)$ in an n-dimensional vector space, for $n\neq 4$, $n\neq 8$, contains no proper closed subgroup whose dimension is greater than $(n-1)(n-2)/2$. If H is a subgroup whose dimension is equal to $(n-1)(n-2)/2$, then H is the subgroup which leaves fixed one and only one direction.*

As to the case $n=8$, it has already been known that $R(8)$ contains the universal covering group of $R(7)$[1]. This implies that the

1) Prof. S. Murakami has kindly informed me this fact and others concerned. I should like to express my hearty thanks to him.

Lie algebra of $R(7)$ admits an irreducible representation in an 8-dimensional vector space. Using this fact it will be proved that if $n=8$ the possible exceptional space in Theorem A is locally flat and homeomorphic to a Euclidean space.

After a preliminary section 1, we shall study in § 2 the case where the group is of dimension $n(n+1)/2$ and prepare some theorems and lemmas concerning the rotation group, the Lorentz group and the homogeneous space of the group in question. In § 3, applying the results of § 2 we shall treat of the case where the group is of dimension less than $n(n+1)/2$. We shall give an algebraic treatment of Theorem A by determining the Lie algebra of the group. The last section is concerned with the case $n=8$.

1. Preliminaries

Let G be a connected Lie group of dimension r and H a compact subgroup of dimension $r-n$ $(0<n\leqq r)$. Since H is compact, on the Lie algebra $\mathfrak{g}$ of G there exists a positive-definite bilinear form B invariant under $\mathrm{ad}(H)$. Then the subset

$$\mathfrak{m}=\{X\ ;\ X\in\mathfrak{g},\quad B(X,U)=0\quad \text{for all}\quad U\in\mathfrak{h}\}$$

is a subspace of $\mathfrak{g}$ such that $\mathfrak{g}=\mathfrak{m}+\mathfrak{h}$ (direct sum of vector spaces) and $\mathrm{ad}(h)\mathfrak{m}\subset\mathfrak{m}$ for all h in H, $\mathfrak{h}$ being the subalgebra of $\mathfrak{g}$ corresponding to the identity component of H.

The group G is said to be *effective* on the homogeneous space G/H as a transformation group of the homogeneous space G/H if every element of G, except the identity, moves at least one point on G/H. This is the case if H does not contain any non-trivial normal subgroup of G. Now we shall say G is *almost effective* if $\mathfrak{h}$ contains no non-trivial ideal of $\mathfrak{g}$, or equivalently if the representation $\mathfrak{h}\rightarrow\mathrm{ad}(\mathfrak{h})$ in $\mathfrak{m}$ is faithful. Of course, if G is effective, then it is also almost effective. Throughout this note we assume that G is almost effective on G/H.

2. The case $\dim G\geqq n(n+1)/2$

In this section we assume $\dim G=r\geqq n(n+1)/2$.

2.1. Determination of the space $[\mathfrak{m}, \mathfrak{m}]$. We shall first prove

LEMMA 1. *G is of dimension $n(n+1)/2$ and $\mathfrak{h}$ is isomorphic to the Lie algebra $\mathfrak{r}(n)$ of the rotation group $R(n)$ in the vector space $\mathfrak{m}$ for any n.*

PROOF. Since H is compact and $\dim \mathfrak{m}=n$, $\mathrm{ad}(\mathfrak{h})$ in $\mathfrak{m}$ is a subalgebra of $\mathfrak{r}(n)$ in the vector space $\mathfrak{m}$. G being almost effective on G/H, the representation $\mathfrak{h}\rightarrow\mathrm{ad}(\mathfrak{h})$ in $\mathfrak{m}$ is faithful, so that $\mathfrak{h}$ is isomorphic to $\mathrm{ad}(\mathfrak{h})$ in $\mathfrak{m}$. Therefore we have

$$\dim \mathrm{ad}(\mathfrak{h})=\dim \mathfrak{h}=r-n\geqq \frac{1}{2} n(n-1)=\dim \mathfrak{r}(n).$$

On the other hand, $\mathrm{ad}(\mathfrak{h})$ in $\mathfrak{m}$ being a subalgebra of $\mathfrak{r}(n)$, we have $\dim \mathrm{ad}(\mathfrak{h})\leqq \dim \mathfrak{r}(n)=n(n-1)/2$. Therefore we have $\dim \mathrm{ad}(\mathfrak{h}) =n(n-1)/2$ and $\mathrm{ad}(\mathfrak{h})=\mathfrak{r}(n)$ in $\mathfrak{m}$. Hence we have $\dim G=n(n+1)/2$.

From Lemma 1 it follows that in case $n=1$ G is 1-dimensional and H is a finite group and the structures of G and H are known. We shall accordingly assume $n\geqq 2$ in the rest of this note.

LEMMA 2. *If $n\neq 3$, $n\neq 4$, we have either $[\mathfrak{m}, \mathfrak{m}]=\mathfrak{h}$ or $[\mathfrak{m}, \mathfrak{m}]=(0)$, where $[\mathfrak{m}, \mathfrak{m}]$ is the subspace spanned by all elements of the form $[X, Y]$, $X, Y\in\mathfrak{m}$.*

To prove this, we need a trivial lemma.

LEMMA 3. *Let $\mathfrak{g}$ be a vector space of a semi-simple representation of a group. If $\mathfrak{g}=\mathfrak{g}_1+\mathfrak{g}_2$ is a decomposition of $\mathfrak{g}$ as a direct sum of irreducible subspaces and $\dim \mathfrak{g}_1\neq\dim \mathfrak{g}_2$, then there exists no proper non-trivial invariant subspace except $\mathfrak{g}_1$ and $\mathfrak{g}_2$.*

PROOF OF LEMMA 2. The subspace $[\mathfrak{m}, \mathfrak{m}]$ is invariant under $\mathrm{ad}(\mathfrak{h})$ in $\mathfrak{g}$. In fact, for any $X, Y\in\mathfrak{m}$ and $U\in\mathfrak{h}$, we have $[U, X]\in\mathfrak{m}$ and $[U, Y]\in\mathfrak{m}$, and the Jacobi identity shows that

$$[U, [X, Y]]=[[U, X], Y]+[X, [U, Y]].$$

Therefore $[U, [X, Y]]\in[\mathfrak{m}, \mathfrak{m}]$, which proves that $[\mathfrak{m}, \mathfrak{m}]$ is invariant under $\mathrm{ad}(\mathfrak{h})$.

On the other hand, by Lemma 1 $\mathrm{ad}(\mathfrak{h})$ in $\mathfrak{m}$ coincides with $\mathfrak{r}(n)$ in $\mathfrak{m}$, and accordingly $\mathrm{ad}(\mathfrak{h})$ in $\mathfrak{m}$ is irreducible. $\mathfrak{r}(n)$ being simple for $n\neq 4$, $\mathrm{ad}(\mathfrak{h})$ in $\mathfrak{h}$ is also irreducible. Therefore the decomposition $\mathfrak{g}=\mathfrak{m}+\mathfrak{h}$ is an irreducible one of $\mathfrak{g}$ under the representation $\mathfrak{h}\rightarrow\mathrm{ad}(\mathfrak{h})$ in $\mathfrak{g}$. Furthermore we have $\dim \mathfrak{m}\neq\dim \mathfrak{h}$ for $n\neq 3$.

Since dim $\mathfrak{m}=n$, we have dim $[\mathfrak{m},\mathfrak{m}]\leqq n(n-1)/2$ and $[\mathfrak{m},\mathfrak{m}]$ is a proper subspace invariant under ad($\mathfrak{h}$) in $\mathfrak{g}$. By Lemma 3, if it is not trivial, we have either $[\mathfrak{m},\mathfrak{m}]=\mathfrak{h}$ or $[\mathfrak{m},\mathfrak{m}]=\mathfrak{m}$.

Now, we shall prove that the case $[\mathfrak{m},\mathfrak{m}]=\mathfrak{m}$ cannot occur. In order to do this, suppose that $[\mathfrak{m},\mathfrak{m}]=\mathfrak{m}$ and denote by $\mathfrak{r}$ the radical of $\mathfrak{g}$. Then, being an ideal, in particular $\mathfrak{r}$ is invariant under ad($\mathfrak{h}$) in $\mathfrak{g}$. $\mathfrak{h}$ being semi-simple, $\dim\mathfrak{r}\leqq\dim\mathfrak{g}-\dim\mathfrak{h}=n$, whence $\mathfrak{r}$ must be $\mathfrak{m}$ or (0). By our assumption $[\mathfrak{m},\mathfrak{m}]=\mathfrak{m}$, $\mathfrak{m}$ cannot be solvable. Therefore $\mathfrak{r}=(0)$ and $\mathfrak{g}$ is semi-simple. In our case $\mathfrak{m}$ being an ideal in $\mathfrak{g}$, there exists a supplementary ideal $\mathfrak{h}'$ such that $\mathfrak{g}$ is the direct sum of $\mathfrak{m}$ and $\mathfrak{h}'$. Since $\mathfrak{h}'$ is invariant under ad($\mathfrak{h}$) and $\dim\mathfrak{h}'=\dim\mathfrak{h}$, we must have $\mathfrak{h}'=\mathfrak{h}$ by Lemma 3. On the other hand, from $[\mathfrak{h},\mathfrak{m}]=\mathfrak{m}$, $\mathfrak{h}$ is not an ideal, which leads to a contradiction.

We have thus proved that either $[\mathfrak{m},\mathfrak{m}]=(0)$ or $[\mathfrak{m},\mathfrak{m}]=\mathfrak{h}$, which is the statement of Lemma 2.

LEMMA 4. *If* $[\mathfrak{m},\mathfrak{m}]=\mathfrak{h}$ *and* $n\neq 3$, $n\neq 4$, *then* $\mathfrak{g}$ *is simple and semi-simple.*

PROOF. Let $\mathfrak{a}$ be an ideal of $\mathfrak{g}$. Then $\mathfrak{a}$ is invariant under ad($\mathfrak{h}$). On the other hand, $\mathfrak{h}$ and $\mathfrak{m}$ are not ideals because we have $[\mathfrak{h},\mathfrak{m}]=\mathfrak{m}$ and $[\mathfrak{m},\mathfrak{m}]=\mathfrak{h}$. Therefore $\mathfrak{a}$ must be either $\mathfrak{g}$ or (0) by Lemma 3. Moreover, since $n\geqq 2$, we have $\dim\mathfrak{g}=n(n+1)/2>1$, so that the simple Lie algebra $\mathfrak{g}$ is semi-simple.

REMARK. Since in both cases of Lemma 2 we have $[\mathfrak{m},\mathfrak{m}]\subset\mathfrak{h}$, we can define an involutive automorphism σ of $\mathfrak{g}$:

$$X^\sigma=-X \quad\text{for}\quad X\in\mathfrak{m}, \qquad U^\sigma=U \quad\text{for}\quad U\in\mathfrak{h}.$$

If this is the case the homogeneous space G/H is called to be a *locally symmetric homogeneous space.*

2.2. Determination of the Lie algebra $\mathfrak{g}$.

Since the bilinear form B is positive-definite, on $\mathfrak{m}\times\mathfrak{m}$ we may take a base $\{X_1,\dots,X_n\}$ of $\mathfrak{m}$ such that $B(X_i,X_j)=\delta_{ij}$ $(1\leqq i,j\leqq n)$. Since ad($\mathfrak{h}$)$=\mathfrak{r}(n)$ in $\mathfrak{m}$, we can find a base $\{X_{ij}\}$ $(1\leqq i<j\leqq n)$ of $\mathfrak{h}$ such that

$$[X_{ij},\ X_k]=\delta_{ik}X_j-\delta_{jk}X_i \qquad (1\leqq i<j\leqq n,\quad 1\leqq k\leqq n).$$

Then we can easily see that for any i,j,k,

$$[X_{ij},\ X_{kl}]=\delta_{ik}X_{jl}-\delta_{jk}X_{il}-\delta_{il}X_{jk}+\delta_{jl}X_{ik},$$

where for convenience we put $X_{ii}=0$, and $X_{ij}=-X_{ji}$ for $i>j$ if necessary. We denote by $B_{\mathfrak{m}}$ and $B_{\mathfrak{h}}$ the restrictions of B to $\mathfrak{m}\times\mathfrak{m}$ and $\mathfrak{h}\times\mathfrak{h}$ respectively. They are clearly invariant under ad($\mathfrak{h}$) and positive-definite. If $n\neq 4$, since $\mathfrak{h}$ is simple, from the beginning we may assume that $-2(n-2)B_{\mathfrak{h}}$ is identical with the fundamental bilinear form of $\mathfrak{h}$ itself which is invariant under ad($\mathfrak{h}$) and negative-definite. Then it is easily seen that

$$B(X_{ij},\ X_{kl})=\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}.$$

We consider the case $[\mathfrak{m},\mathfrak{m}]=\mathfrak{h}$. In this case $\mathfrak{g}$ is simple and semi-simple by Lemma 4. Let ϕ be the fundamental bilinear form of $\mathfrak{g}$, then ϕ is non-degenerate. Since it is invariant under every automorphism of $\mathfrak{g}$, we have in particular $\phi(X, U)=\phi(X^{\sigma}, U^{\sigma})=\phi(-X, U)$ and hence $\phi(X, U)=0$ for any $X\in\mathfrak{m}$ and $U\in\mathfrak{h}$. Let $\phi_{\mathfrak{m}}$ and $\phi_{\mathfrak{h}}$ be the restrictions of ϕ to $\mathfrak{m}\times\mathfrak{m}$ and $\mathfrak{h}\times\mathfrak{h}$ respectively. Then they are invariant under ad($\mathfrak{h}$) and non-degenerate. Furthermore since $\mathfrak{m}$ and $\mathfrak{h}$ are irreducible under ad($\mathfrak{h}$) we have $aB(X, Y)=\phi_{\mathfrak{m}}(X, Y)$ and $bB(U, V)=\phi_{\mathfrak{h}}(U, V)$ for $X, Y\in\mathfrak{m}$, $U, V\in\mathfrak{h}$, where a and b are non-zero real numbers.

LEMMA 5. *If we put* $X_i^*=\sqrt{|c|}\,X_i$ $(c=b/a)$, *then*

$$[X_i^*,\ X_j^*]=\mathrm{sgn}(c)X_{ij}\quad \textit{for all}\quad 1\leqq i<j\leqq n.$$

PROOF. Using the fact that ϕ is invariant under ad($\mathfrak{g}$) and $[X_i, X_j]\in\mathfrak{h}$, we have

$$\begin{aligned}
B([X_i,\ X_j],\ X_{kl}) &= \frac{1}{b}\ \phi([X_i,\ X_j],\ X_{kl}) && (\text{in } \mathfrak{h})\\
&= \frac{1}{b}\ \phi(X_j,\ [X_{kl},\ X_i]) && (\text{in } \mathfrak{g})\\
&= \frac{a}{b}\ B(X_j,\ \delta_{ik}X_l-\delta_{il}X_k) && (\text{in } \mathfrak{m})\\
&= \frac{1}{c}\ (\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})
\end{aligned}$$

for any $1 \leqq i < j \leqq n$, $1 \leqq k < l \leqq n$. Accordingly we have

$$B([X_i^*, X_j^*], X_{kl}) = \mathrm{sgn}(c)(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) = \begin{cases} \mathrm{sgn}(c), & \text{if } i=k,\ j=l \\ 0, & \text{otherwise.} \end{cases}$$

Therefore we have $[X_i^*, X_j^*] = \mathrm{sgn}(c) X_{ij}$.

Since we have considered the structure of the Lie algebra $\mathfrak{g}$ in the case $n \neq 3$, $n \neq 4$, we shall study it in the cases $n=3$ and $n=4$.

We shall begin with the case $n=4$. All 4-dimensional homogeneous Riemannian spaces have already been studied by S. Ishihara [1], so that we shall not enter into detail. As is well known, $\mathfrak{h} \cong \mathfrak{r}(4)$ is the direct sum of the special unitary algebra $\mathfrak{su}(2)$ in 2-complex-variables by itself. Taking account of this fact it is easily seen that the space $[\mathfrak{m}, \mathfrak{m}]$ is contained in $\mathfrak{h}$ and therefore the homogeneous space G/H is a locally symmetric homogeneous space. K. Nomizu [4] has proved that if G/H is a locally symmetric homogeneous space and $\mathrm{ad}(\mathfrak{h})$ in $\mathfrak{m}$ is irreducible then we have either $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{h}$ or $[\mathfrak{m}, \mathfrak{m}] = (0)$. We have thus

LEMMA 2′. *If* $n=4$, *we have either* $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{h}$ *or* $[\mathfrak{m}, \mathfrak{m}] = (0)$.

Modifying the proof of Lemma 4, we have immediately

LEMMA 4′. *If* $n=4$ *and* $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{h}$, *then* $\mathfrak{g}$ *is simple and semi-simple.*

Furthermore using the Jacobi identity and performing a simple calculation we have

LEMMA 5′. *If* $n=4$ *and* $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{h}$, *there exists a non-zero real number* c *such that* $[X_i, X_j] = cX_{ij}$ *for* $1 \leqq i < j \leqq 4$.

We shall next consider the case $n=3$. In this case it is easily seen that $[X_i, X_j]$'s are as follows:

$$[X_2, X_3] = aX_1 + bX_{23}, \quad [X_3, X_1] = aX_2 + bX_{31}, \quad [X_1, X_2] = aX_3 + bX_{12},$$

where a and b are real numbers. If we denote by $\mathfrak{m}'$ the vector space spanned by the elements

$$X_1' = 2X_1 - aX_{23}, \qquad X_2' = 2X_2 - aX_{31}, \qquad X_3' = 2X_3 - aX_{12},$$

then $\mathfrak{m}'$ is invariant under $\mathrm{ad}(\mathfrak{h})$ and $[X_i', X_j'] = cX_{ij}$ and $\mathfrak{g} = \mathfrak{m}' + \mathfrak{h}$ where we put $c = a^2 + 4b$. It follows that we may take $\mathfrak{m}'$ for $\mathfrak{m}$. Then the same situations occur as in the case $n \neq 3$, and the corresponding lemmas hold good.

Stating the structure of $\mathfrak{g}$ corresponding to the sign of the number c, we have the followings:

(I) The case $c>0$.

The fundamental bilinear form ϕ is definite and G is compact; therefore G/H is compact. If we put $X_i^*=X_{0i}=-X_{i0}$, then $\mathfrak{g}$ has the following structure:

$$[X_{ij}, X_{kl}]=\delta_{ik}X_{jl}-\delta_{jk}X_{il}-\delta_{il}X_{jk}+\delta_{jl}X_{ik} \qquad (0\leqq i, j, k, l\leqq n),$$

that is, $\mathfrak{g}$ is isomorphic to the Lie algebra $\mathfrak{r}(n+1)$ of the rotation group $R(n+1)$.

(II) The case $c<0$.

The fundamental bilinear form ϕ is not definite and G is not compact; therefore G/H is not compact. If we put $X_i^*=X_{0i}=-X_{i0}$ then $\mathfrak{g}$ has the following structure:

$$[X_{0i}, X_{0j}]=-X_{ij} \qquad (1\leqq i<j\leqq n),$$

$$[X_{ij}, X_{kl}]=\delta_{ik}X_{jl}-\delta_{jk}X_{il}-\delta_{il}X_{jk}+\delta_{jl}X_{ik} \quad (1\leqq i, j\leqq n, 0\leqq k, l\leqq n),$$

that is, $\mathfrak{g}$ is isomorphic to the Lie algebra $\mathfrak{l}(n+1)$ of the Lorentz group $L(n+1)$.

(III) The case $[\mathfrak{m}, \mathfrak{m}]=(0)$.

Obviously $\mathfrak{g}$ is isomorphic to the Lie algebra $\mathfrak{m}(n)$ of the group $M(n)$ of all proper motions in an n-dimensional Euclidean space.

Gathering all the results obtained in this section, we can state the following lemma.

LEMMA 6. *Let G be a connected Lie group of dimension $r=n(n+1)/2$ and H a compact subgroup of dimension $r-n$. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebra of G and H respectively. We assume that G is almost effective on the homogeneous space G/H as a transformation group. Then $\mathfrak{g}$ is isomorphic to one of the following Lie algebras:*

(I) *The Lie algebra $\mathfrak{r}(n+1)$ of the rotation group $R(n+1)$ in an $(n+1)$-dimensional vector space,*

(II) *The Lie algebra $\mathfrak{l}(n+1)$ of the Lorentz group $L(n+1)$ in an $(n+1)$-dimensional vector space,*

(III) *The Lie algebra $\mathfrak{m}(n)$ of the group $M(n)$ of motions in an n-dimensional Euclidean space.*

The subalgebra $\mathfrak{h}$ is isomorphic to $\mathfrak{r}(n)$ and there exists an auto-

morphism of $\mathfrak{g}$ *which maps* $\mathfrak{h}$ *onto the standard subalgebra* $\mathfrak{r}(n)$ *of* $\mathfrak{g}$.

2.3. On the group *G*. We can now state Theorem *B* in § 0 as follows.

THEOREM 1. *The rotation group* $R(n+1)$ *in an* $(n+1)$*-dimensional vector space for* $n \neq 3$ *contains no proper closed subgroup whose dimension is greater than* dim $R(n)$. *If* H *is a closed subgroup of* $R(n+1)$ *whose dimension is equal to* dim $R(n)$ *and* $n \neq 1, 3, 7$, *then* H *is the subgroup* $R(n)$ *which leaves fixed one and only one direction.*

PROOF. Since the Lie algebra $\mathfrak{r}(n+1)$ of $R(n+1)$ is simple for $n \neq 3$, it contains no non-trivial proper ideal. If H is a closed subgroup of $R(n+1)$ of dimension not less than dim $R(n)$, then the homogeneous space $R(n+1)/H$ is of dimension $m \leqq n$ and $R(n+1)$ is almost effective on $R(n+1)/H$ as a transformation group. Since we have $\dim R(n+1) \geqq m(m+1)/2$ and $\dim H = \dim R(n+1) - m$, we must have $\dim R(n+1) = m(m+1)/2$ by Lemma 1 and $m = n$. From Lemma 6 it follows that $\mathfrak{h}$ is mapped onto a standard subalgebra $\mathfrak{r}(n)$ by an automorphism of $\mathfrak{r}(n+1)$. If $n \neq 1, 3, 7$ this is induced by an automorphism of $R(n+1)$[2]. Since it is known that the automorphisms of $R(n+1)$ are conjugations by orthogonal matrices, H is conjugate with the standard subgroup $R(n)$ in the orthogonal group $O(n+1)$. This proves the assertion of the theorem for $n \neq 1, 3, 7$.

As for the Lorentz group $L(n+1)$ we have the following

THEOREM 2. *Let* G *be a connected Lie group which is locally isomorphic to the Lorentz group* $L(n+1)$, $n \geqq 2$. *Then the maximal compact subgroup of* G *is locally isomorphic to* $R(n)$. *In particular,* $L(n+1)$ *in an* $(n+1)$*-dimensional vector space contains no compact subgroup whose dimension is greater than* dim $R(n)$. *If* H *is a compact subgroup of* $L(n+1)$ *whose dimension is equal to* dim $R(n)$, *then* H *is the subgroup* $R(n)$ *which leaves fixed one and only one direction.*

PROOF. Since the Lie algebra $\mathfrak{l}(n+1)$ of $L(n+1)$ is simple, for any closed subgroup H, G is almost effective on G/H as a transformation group. Therefore if $\mathfrak{h}$ is the Lie algebra of a compact subgroup of dimension $\geqq$ dim $R(n)$, the same argument as in Theorem 1 shows that $\mathfrak{h}$ is isomorphic to $\mathfrak{r}(n)$. We have thus proved the first part of

2) See the footnote 1).

Theorem 2. If $G=L(n+1)$, then its maximal compact subgroup is locally isomorphic to $R(n)$. On the other hand by a theorem of Iwasawa [2] maximal compact subgroups are connected and conjugate to each other; therefore H must be conjugate to the subgroup $R(n)$ which leaves fixed one and only one direction.

The Lorentz group $L(n+1)$, however, has non-compact subgroups whose dimension is $n(n-1)/2+1$. In fact, we have the following

LEMMA 8. *The notation being the same as in (II) of § 2.2, let* $\mathfrak{m}_1$, $\mathfrak{m}_2$, $\mathfrak{m}_2'$, *and* $\mathfrak{h}_0$ *be the subspaces of* $\mathfrak{l}(n+1)$ *spanned by* X_{01}; $X_{0i}-X_{1i}$, $2\leqq i\leqq n$; $X_{0i}+X_{1i}$, $2\leqq i\leqq n$; *and* X_{ij}, $2\leqq i<j\leqq n$ *respectively. Then the vector space* $\mathfrak{g}_0=\mathfrak{m}_1+\mathfrak{m}_2+\mathfrak{h}_0$ *is a subalgebra of* $\mathfrak{l}(n+1)$ *having the following structure:*

$$[\mathfrak{m}_1, \mathfrak{m}_1]=(0); \qquad [\mathfrak{m}_2, \mathfrak{m}_2]=(0); \qquad [\mathfrak{h}_0, \mathfrak{m}_1]=(0);$$

$$[\mathfrak{m}_1, \mathfrak{m}_2]=\mathfrak{m}_2 \quad \text{and} \quad [X, Y]=k(X)Y \quad \text{for all} \quad X\in\mathfrak{m}_1,\ Y\in\mathfrak{m}_2,$$

where $k(X)$ *is a non-trivial linear function of* $\mathfrak{m}_1$; $\mathfrak{h}_0$ *is isomorphic to* $\mathfrak{r}(n-1)$; $[\mathfrak{h}_0, \mathfrak{m}_2]=\mathfrak{m}_2$ *and* $\mathrm{ad}(\mathfrak{h}_0)$ *in* $\mathfrak{m}_2$ *coincides with* $\mathfrak{r}(n-1)$ *in* $\mathfrak{m}_2$. $\mathfrak{g}_0'=\mathfrak{m}_1+\mathfrak{m}_2'+\mathfrak{h}_0$ *is also a subalgebra of* $\mathfrak{l}(n+1)$ *isomorphic to* $\mathfrak{g}_0$.

This is a direct consequence of the structure of $\mathfrak{l}(n+1)$ and straightforward calculations. This will be used in § 3.3.

As for the group $M(n)$ of proper motions in an n-dimensional Euclidean space, we have the following theorem.

THEOREM 3. *Let G be a connected Lie group which is locally isomorphic to the group* $M(n)$, $n\geqq 2$. *Then the maximal compact subgroups of G are locally isomorphic to* $R(n)$. *In particular,* $M(n)$ *contains no compact subgroup of dimension greater than* $\dim R(n)$. *If H is a compact subgroup of* $M(n)$ *whose dimension is equal to* $\dim R(n)$, *then H is conjugate to* $R(n)$ *in* $M(n)$.

PROOF. It is easy to see that G is not compact. Let H_0 be a subgroup of G locally isomorphic to $R(n)$ and $\mathfrak{h}_0$ its Lie algebra. If we denote by K a maximal compact subgroup containing H, then the Lie algebra $\mathfrak{k}$ of K is invariant under $\mathrm{ad}(\mathfrak{h}_0)$ in $\mathfrak{g}$. But we have already shown that such an invariant subspace of $\mathfrak{g}$ must be either $\mathfrak{g}$ or $\mathfrak{h}_0$. From the fact $\mathfrak{k}\neq\mathfrak{g}$ it follows that $\mathfrak{k}=\mathfrak{h}_0$ and $K=H_0$.

Since maximal compact subgroups are connected and conjugate to each other, the last part of this theorem is clear.

2.4. Determination of the homogeneous space G/H. We first consider the invariant Riemannian connection on the homogeneous space G/H. For the terminology and notation concerning invariant affine connections we follow K. Nomizu [4]. G/H being a locally symmetric homogeneous space, we can define on it the canonical affine connection, which is clearly the unique invariant Riemannian connection. Then it has the curvature tensor $R(X, Y)\cdot Z = -[[X, Y], Z]$ for all $X, Y, Z \in \mathfrak{m}$. Therefore, in case (I) or (II), $[X_i^*, X_j^*] = \operatorname{sgn}(c) X_{ij}$, we have $-[[X_i^*, X_j^*], X_k^*] = \operatorname{sgn}(c)(\delta_{jk} X_i^* - \delta_{ik} X_j^*)$. Since $X_i^* = \sqrt{|c|}\, X_i$, we obtain the formula

$$R(X, Y)\cdot Z = \frac{1}{c}\,(B(Y, Z)\,X - B(X, Z)\,Y) \quad \text{for any} \quad X, Y, Z \in \mathfrak{m},$$

which shows that the Riemannian space G/H is of positive or negative constant curvature corresponding to $c>0$ or $c<0$. In case (III), $[\mathfrak{m}, \mathfrak{m}] = (0)$, we have $R(X, Y)\cdot Z = 0$ for any $X, Y, Z \in \mathfrak{m}$ and the Riemannian space G/H is locally flat.

We shall next consider the topological structure of the space. If case (I) in Lemma 6 occurs and the space G/H is simply connected, then, since G and H are locally isomorphic to $R(n+1)$ and $R(n)$ respectively, G/H can be considered as a homogeneous space $\tilde{R}(n+1)/\tilde{R}(n)$ which is an n-dimensional sphere, where $\tilde{R}(n+1)$ and $\tilde{R}(n)$ denote the simply connected covering groups of $R(n+1)$ and $R(n)$ respectively. Hence in case (I) the simply connected covering space of G/H is a sphere. If case (II) or (III) occurs, H is a maximal compact subgroup of G and therefore the space G/H is homeomorphic to an n-dimensional Euclidean space.

We have thus obtained the following results:

THEOREM 4. *Let G/H be an n-dimensional homogeneous space, where G is a connected Lie group of dimension $r \geqq n(n+1)/2$ and H a compact subgroup of G of dimension $r-n$. We assume that $n \geqq 2$ and G is almost effective on G/H as a transformation group. Then G/H is one of the following spaces:*

(I) *A Riemannian space of positive constant curvature whose simply connected covering space is an n-dimensional sphere,*

(II) *A Riemannian space of negative constant curvature homeomor-*

phic to an n-dimensional Euclidean space,

(III) *A locally flat Riemannian space homeomorphic to an n-dimensional Euclidean space.*

3. The case $\dim G < n(n+1)/2$

Under the same situation as in § 1 we assume $n(n+1)/2 > r > n(n-1)/2$, $n \geqq 3$ and $n \neq 4$. Then $\mathrm{ad}(\mathfrak{h})$ in $\mathfrak{m}$ is a subalgebra of $\mathfrak{r}(n)$. Since the representation $\mathfrak{h} \to \mathrm{ad}(\mathfrak{h})$ in $\mathfrak{m}$ is faithful and $n(n-1)/2 > \dim H = r - n \geqq (n-1)(n-2)/2$, we see that $\mathrm{ad}(\mathfrak{h})$ in $\mathfrak{m}$ is isomorphic to $\mathfrak{r}(n-1)$ and of dimension $(n-1)(n-2)/2$. It follows from Theorem 1 that if $n \neq 8$ there exists one and only one 1-dimensional subspace $\mathfrak{m}_1$ of $\mathfrak{m}$ such that $\mathrm{ad}(\mathfrak{h})$ induces a trivial representation in it, $[\mathfrak{h}, \mathfrak{m}_1] = (0)$. Furthermore we can find a supplementary subspace $\mathfrak{m}_2$ of $\mathfrak{m}$ invariant under $\mathrm{ad}(\mathfrak{h})$ such that $\mathfrak{m}$ is the direct sum of $\mathfrak{m}_1$ and $\mathfrak{m}_2$. Then $\dim \mathfrak{m}_2 = n-1$ and $\mathrm{ad}(\mathfrak{h})$ in $\mathfrak{m}_2$ is irreducible. Since $\mathfrak{r}(n-1)$ is simple for $n \neq 5$ we have the irreducible decomposition $\mathfrak{g} = \mathfrak{m}_1 + \mathfrak{m}_2 + \mathfrak{h}$ of $\mathfrak{g}$ by $\mathrm{ad}(\mathfrak{h})$, where $[\mathfrak{h}, \mathfrak{m}_1] = (0)$, $[\mathfrak{h}, \mathfrak{m}_2] = \mathfrak{m}_2$. If $n = 5$, $\mathfrak{h} \cong \mathfrak{r}(4)$ is the direct sum of $\mathfrak{su}(2)$ by itself.

3.1. Determination of the spaces $[\mathfrak{m}_i, \mathfrak{m}_j]$. *In the following three sections we assume* $n \neq 8$. We consider the subspaces $[\mathfrak{m}_i, \mathfrak{m}_j]$ spanned by all elements of the form

$$[X, Y], \qquad X \in \mathfrak{m}_i, \qquad Y \in \mathfrak{m}_j, \qquad i, j = 1, 2.$$

LEMMA 8. *We have either* $[\mathfrak{m}_1, \mathfrak{m}_2] = (0)$ *or* $[\mathfrak{m}_1, \mathfrak{m}_2] = \mathfrak{m}_2$

PROOF. Let $X = X_1 + X_2 + U$, where $X_1 \in \mathfrak{m}_1$, $X_2 \in \mathfrak{m}_2$ and $U \in \mathfrak{h}$, be any element of $[\mathfrak{m}_1, \mathfrak{m}_2]$. Then for any $V \in \mathfrak{h}$ we have

$$[V, X] = [V, X_1] + [V, X_2] + [V, U] = [V, X_2] + [V, U],$$

where $[V, X_2] \in \mathfrak{m}_2$ and $[V, U] \in \mathfrak{h}$. This shows $[\mathfrak{h}, [\mathfrak{m}_1, \mathfrak{m}_2]] \subset \mathfrak{m}_2 + \mathfrak{h}$. On the other hand, the Jacobi identity shows that

$$[\mathfrak{h}, [\mathfrak{m}_1, \mathfrak{m}_2]] = [[\mathfrak{h}, \mathfrak{m}_1], \mathfrak{m}_2] + [\mathfrak{m}_1, [\mathfrak{h}, \mathfrak{m}_2]] = [\mathfrak{m}_1, \mathfrak{m}_2].$$

Therefore $[\mathfrak{m}_1, \mathfrak{m}_2]$ is contained in the subspace $\mathfrak{m}_2 + \mathfrak{h}$ of $\mathfrak{g}$ and is invariant under $\mathrm{ad}(\mathfrak{h})$. Since $\dim \mathfrak{m}_1 = 1$ and $\dim \mathfrak{m}_2 = n-1$, we must have $\dim[\mathfrak{m}_1, \mathfrak{m}_2] = n-1$ or 0. Hence we have either $[\mathfrak{m}_1, \mathfrak{m}_2] = (0)$ or $[\mathfrak{m}_1, \mathfrak{m}_2] = \mathfrak{m}_2$ by virtue of Lemma 3.

LEMMA 9. *There exists a linear function k on $\mathfrak{m}_1$ such that $[X, Y] = k(X)Y$ for any $X \in \mathfrak{m}_1$, $Y \in \mathfrak{m}_2$.*

PROOF. $\mathrm{ad}(\mathfrak{h})$ in $\mathfrak{m}_2$ coincides with $\mathfrak{r}(n-1)$ in $\mathfrak{m}_2$ and by Lemma 8, $\mathfrak{m}_1$ induces the adjoint representation $\mathrm{ad}(\mathfrak{m}_1)$ in $\mathfrak{m}_2$. Since $[\mathfrak{h}, \mathfrak{m}_1] = (0)$, the corresponding matrices of $\mathrm{ad}(\mathfrak{m}_1)$ commute with all matrices of $\mathfrak{r}(n-1)$. Therefore they are scalar multiples of the unit matrix, which proves the assertion of Lemma 9.

LEMMA 10. *We have either $[\mathfrak{m}_2, \mathfrak{m}_2] = (0)$ or $[\mathfrak{m}_2, \mathfrak{m}_2] = \mathfrak{h}$.*

PROOF. Since $\mathrm{ad}(\mathfrak{h})$ coincides with $\mathfrak{r}(n-1)$ in $\mathfrak{m}_2$, there are bases $\{X_i\}$, $1 \leqq i \leqq n-1$, of $\mathfrak{m}_2$ and $\{X_{ij}\}$, $1 \leqq i < j \leqq n-1$, of $\mathfrak{h}$ such that

$$[X_{ij}, X_k] = \delta_{ik}X_j - \delta_{jk}X_i, \qquad 1 \leqq i, j, k \leqq n-1.$$

Then the elements $[X_i, X_j]$, $1 \leqq i, j \leqq n-1$, span the subspace $[\mathfrak{m}_2, \mathfrak{m}_2]$ and the following relations hold:

$$\begin{aligned} &[X_{ik}, [X_i, X_j]] = [X_k, X_j], \\ &[X_{kj}, [X_i, X_j]] = -[X_i, X_k]. \end{aligned} \qquad (1 \leqq i, j, k \leqq n-1,\ i \neq j,\ j \neq k,\ k \neq i)$$

These relations show $[\mathfrak{h}, [\mathfrak{m}_2, \mathfrak{m}_2]] = [\mathfrak{m}_2, \mathfrak{m}_2]$.

Then in the same manner as in Lemma 8 we can easily see that $[\mathfrak{m}_2, \mathfrak{m}_2]$ is contained in $\mathfrak{m}_2 + \mathfrak{h}$. Therefore Lemma 2 and 2′ prove the statement of Lemma 10.

REMARK. By Lemma 8 and Lemma 10 the subspace $\mathfrak{g}' = \mathfrak{m}_2 + \mathfrak{h}$ is an ideal of $\mathfrak{g}$ and has the structure stated in Lemma 6.

LEMMA 11. *$[\mathfrak{m}_2, \mathfrak{m}_2] = \mathfrak{h}$ implies $[\mathfrak{m}_1, \mathfrak{m}_2] = (0)$.*

PROOF. Let ϕ be the fundamental bilinear form of $\mathfrak{g}$. Then $\mathfrak{g}' = \mathfrak{m}_2 + \mathfrak{h}$ being an ideal of $\mathfrak{g}$, the restriction of ϕ to $\mathfrak{g}' \times \mathfrak{g}'$ coincides with the fundamental bilinear form of the Lie algebra $\mathfrak{g}'$ itself. Since $\mathfrak{g}'$ is isomorphic to $\mathfrak{r}(n)$ or $\mathfrak{l}(n)$, for some $Y \in \mathfrak{m}_2$ we have $\phi(Y, Y) \neq 0$. As ϕ is invariant under $\mathrm{ad}(\mathfrak{g})$, in particular for any $X \in \mathfrak{m}_2$ we have $\phi([X, Y], Y) = 0$, i. e. $k(X)\phi(Y, Y) = 0$. Hence we have $k(X) = 0$ for any $X \in \mathfrak{m}_1$, which shows $[\mathfrak{m}_1, \mathfrak{m}_2] = (0)$.

Summarizing the results obtained in this section we can state the following:

LEMMA 12. *Let G be a connected Lie group of dimension r and H*

a compact subgroup of dimension $r-n$. *If* $n(n+1)/2>r>n(n-1)/2$, $n\neq 3, 4, 8$, *then* G *is of dimension* $n(n-1)/2+1$. *Furthermore, let* $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{m}$, $\mathfrak{m}_1$, $\mathfrak{m}_2$ *and* $\mathfrak{g}'$ *be as before* ($\mathfrak{m}=\mathfrak{m}_1+\mathfrak{m}_2$, $\mathfrak{g}'=\mathfrak{m}_2+\mathfrak{h}$), *then the Lie algebra* $\mathfrak{g}$ *has one of the following structures:*

(I) $\qquad \mathfrak{g}=\mathfrak{m}_1+\mathfrak{g}'; \qquad [\mathfrak{m}_1, \mathfrak{g}']=(0), \qquad [\mathfrak{m}_2, \mathfrak{m}_2]=(0),$

i.e. $\mathfrak{g}$ *is the direct sum of the* 1*-dimensional ideal* $\mathfrak{m}_1$ *and the ideal* $\mathfrak{g}'$ *isomorphic to* $\mathfrak{m}(n-1)$, *where* $\mathfrak{m}(n-1)$ *is the Lie algebra of* $M(n-1)$.

(II) $\qquad \mathfrak{g}=\mathfrak{m}_1+\mathfrak{g}'; \qquad [\mathfrak{m}_1, \mathfrak{g}']=(0), \qquad [\mathfrak{m}_2, \mathfrak{m}_2]=\mathfrak{h},$

i.e. $\mathfrak{g}$ *is the direct sum of the* 1*-dimensional ideal* $\mathfrak{m}_1$ *and the ideal* $\mathfrak{g}'$ *isomorphic to* $\mathfrak{r}(n)$ *or* $\mathfrak{l}(n)$.

(III) $\qquad \mathfrak{g}=\mathfrak{m}_1+\mathfrak{m}_2+\mathfrak{h}: \qquad [\mathfrak{m}_1, \mathfrak{m}_2]=\mathfrak{m}_2, \qquad [\mathfrak{m}_2, \mathfrak{m}_2]=(0),$

i.e. $\mathfrak{g}$ *is the direct sum of the* 1*-dimensional subalgebra* $\mathfrak{m}_1$ *and the ideal* $\mathfrak{g}'$ *isomorphic to* $\mathfrak{m}(n-1)$.

It is to be noted that in the cases (I) and (II) of this lemma the homogeneous space G/H is a locally symmetric homogeneous space.

3.2. The homogeneous space G/H. We first consider the invariant Riemannian connection on the homogeneous space G/H. In the cases (I) and (II) in Lemma 12, as we have remarked above, G/H is a locally symmetric homogeneous space, so that the curvature tensor has the form $R(X, Y)\cdot Z=-[[X, Y], Z]$ for any $X, Y, Z\in\mathfrak{m}$ with respect to its canonical Riemannian connection. In the case (I), it is easily seen that the curvature tensor vanishes, $R(X, Y)\cdot Z=0$ for all $X, Y, Z\in\mathfrak{m}$, and G/H is a locally flat Riemannian space. In the case (II), we have the following formulas as in § 2:

$$R(X, Y)\cdot Z=\begin{cases}\dfrac{1}{c'}(B(Y, Z)X-B(X, Z)Y) & \text{for} \quad X, Y, Z\in\mathfrak{m}_2, \\ 0 & \text{otherwise,}\end{cases}$$

c' being a non-zero real number corresponding to the Lie algebra $\mathfrak{g}'=\mathfrak{m}_2+\mathfrak{h}$ as to $\mathfrak{g}$ in § 2. Therefore we see that the Riemannian space G/H is locally the product space of a straight line and an $(n-1)$-dimensional Riemannian space of non-zero constant curvature.

THEOREM 5. *The notation and assumptions being as in* Lemma 12, *assume further that the case* (*I*) *occurs; then the homogeneous space* G/H *is naturally a locally flat Riemannian space and is homeomorphic either to an n-dimensional Euclidean space or to the product space of a circle and an* $(n-1)$-*dimensional Euclidean space.*

PROOF. If we denote by M_1 and G' the Lie subgroups of G generated by $\mathfrak{m}_1$ and $\mathfrak{g}'$ respectively, then they are closed, since $\mathfrak{g}'\cong\mathfrak{m}(n-1)$ and $\mathfrak{m}_1$ is the centralizer of $\mathfrak{g}'$. We have then $G=M_1G'$ and $K=M_1\cap G'$ is contained in the centre of G' which is discrete. Let $V_1=M_1/H\cap M_1$ and $V_2=G'/H\cap G'$ be the orbits of the point $p_0=(H)$ in G/H by M_1 and G' respectively, then the invariant Riemannian metric of G/H is the Pythagorian product of those of V_1 and V_2. Since $V_1\cap V_2=K/K\cap H$ is discrete, by a theorem of Walker[3] [6] G/H is a fibre bundle with V_1 as the fibre and V_2' as the base space, where V_2' is represented as $(G'/K)/(H\cap G')/((K\cap H))$. On the other hand, since G' and G'/K are connected and locally isomorphic to $M(n-1)$ and $H\cap G'$, $H\cap G'/K\cap H$ is locally isomorphic to $R(n-1)$, it follows that V_2 and V_2' are homeomorphic to an $(n-1)$-dimensional Euclidean space and we have $V_2=V_2'$. It follows that V_2 is contractible to a point and therefore the bundle G/H over V_2 is the product bundle, i. e. G/H is homeomorphic to the product space $V_1\times V_2$, where V_1 is a straight line or a circle.

Taking account of Theorem 4, if we replace $M(n-1)$ in Theorem 5 by $L(n)$, we have easily the following

THEOREM 6. *The notation and assumptions being as in* Lemma 12, *assume further that the case* (*II*) *occurs and* $\mathfrak{g}'=\mathfrak{m}_2+\mathfrak{h}$ *is isomorphic to* $\mathfrak{l}(n)$; *then the homogeneous space* G/H *is naturally the product Riemannian space of a straight line and an* $(n-1)$-*dimensional Riemannian space of negative constant curvature. Furthermore* G/H *is homeomorphic either to an n-dimensional Euclidean space or to the product space of a straight line and an* $(n-1)$-*dimensional Euclidean space.*

In case $\mathfrak{g}'=\mathfrak{m}_2+\mathfrak{g}$ is isomorphic to $\mathfrak{r}(n)$, taking Theorem 4 into consideration we have easily the following

THEOREM 7. *The notation and assumptions being as in* Lemma 12, *assume further that the case* (*II*) *occurs and* $\mathfrak{g}'=\mathfrak{m}_2+\mathfrak{h}$ *is isomorphic to* $\mathfrak{r}(n)$; *then the homogeneous space* G/H *is naturally the product Rieman-*

3) Theorem 1 in [6].

nian space of a straight line and an $(n-1)$-dimensional Riemannian space of positive constant curvature. If moreover G/H is simply connected, it is homeomorphic to the product space of a straight line and an $(n-1)$-dimensional sphere.

3.3. Case (III) in Lemma 12. We shall next study the case (III) in Lemma 12. If the case (III) occurs, then the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ are isomorphic to the Lie algebra $\mathfrak{g}_0$ (or $\mathfrak{g}_0'$) and $\mathfrak{h}_0$ stated in Lemma 7 respectively. $\mathfrak{g}_0$ and $\mathfrak{h}_0$ being subalgebras of $\mathfrak{l}(n+1)$, there exist connected subgroups G_0 and H_0 of $L(n+1)$ having $\mathfrak{g}_0$ and $\mathfrak{h}_0$ as their Lie algebras respectively. H_0 is then isomorphic to $R(n-1)$ and by the relation between $\mathfrak{l}(n+1)$ and $\mathfrak{g}_0$ there exists the subgroup $R(n)$ of $L(n+1)$ containing H_0 naturally. Furthermore as is easily seen $G_0\cap R(n)=H_0$. Then the homogeneous space G_0/H_0 is connected and contained in the homogeneous space $L(n+1)/R(n)$. Since G_0 is a subgroup of $L(n+1)$, the canonical invariant Riemannian connection on $L(n+1)/R(n)$ is also that on G_0/H_0. The space $L(n+1)/R(n)$ being a Riemannian space of negative constant curvature with respect to this connection, so also is G_0/H_0. Furthermore it is homeomorphic to an n-dimensional Euclidean space. Since G and H are locally isomorphic to G_0 and H_0 respectively, the invariant Riemannian connection on the homogeneous space G/H is equivalent to that on G_0/H_0. Hence G/H is naturally of negative constant curvature.

We shall now study the topological structure of G/H. In order to do this, we first prove the following lemma.

LEMMA 14. *If the case (III) occurs in Lemma 12, then the subgroup H is a maximal compact subgroup of the group G.*

PROOF. From the relation $\mathfrak{g}=\mathfrak{m}_1+\mathfrak{g}'$ and the fact that $\mathfrak{g}'=\mathfrak{m}_2+\mathfrak{h}$ is isomorphic to $\mathfrak{m}(n-1)$, it follows that G contains a closed subgroup G' which is locally isomorphic to $M(n-1)$. As we have seen in Theorem 3, G' is not compact, and therefore G is neither. If we denote by K a maximal compact subgroup of G containing H, then $\dim G>\dim K\geqq \dim H$. We shall show that $\dim K=\dim H$. Suppose $\dim K=\dim G-m>\dim H$, $m<n$, then K must contain a normal subgroup of G whose dimension is positive. In fact, if K does not contain such a subgroup, then Lemma 1 shows that $\dim G=n(n-1)/2+1\leqq m(m+1)/2$. This implies $m\geqq n$, contrary to the assumption

$m<n$. As we have seen just now, the Lie algebra $\mathfrak{k}$ of K must contain some non-trivial ideal. Since the decomposition of $\mathfrak{g}$, $\mathfrak{g}=\mathfrak{m}_1+\mathfrak{m}_2+\mathfrak{h}$, is irreducible under ad($\mathfrak{h}$) in $\mathfrak{g}$ in our case, we can easily see that the only non-trivial ideals of $\mathfrak{g}$ are $\mathfrak{m}_1+\mathfrak{m}_2$, $\mathfrak{g}'$ and $\mathfrak{m}_2$. Because of this fact and the inequality $\dim K<\dim G$, we have $\mathfrak{k}=\mathfrak{m}_2+\mathfrak{h}=\mathfrak{g}'$. On the other hand, the connected subgroup $K=G'$ of G having $\mathfrak{k}$ as its Lie algebra cannot be compact. We have thus proved that $\dim K=\dim H$ and therefore $K=H$. This shows that H is a maximal compact subgroup and connected.

From Lemma 13 it follows that the homogeneous space G/H is homeomorphic to an n-dimensional Euclidean space. The result established above becomes

THEOREM 8. *The notation and assumptions being as in* Lemma 12, *assume further that the case* (*III*) *occurs; then the homogeneous space* G/H *is naturally a Riemannian space of negative constant curvature and is homeomorphic to an* n*-dimensional Euclidean space.*

3.4. The case $n=8$. The notation and assumptions being as in the beginning of § 3, assume further $n=8$. Then ad($\mathfrak{h}$) in $\mathfrak{m}$ is a subalgebra of $\mathfrak{r}(8)$ in $\mathfrak{m}$ and isomorphic to $\mathfrak{r}(7)$.

If ad($\mathfrak{h}$) in $\mathfrak{m}$ is reducible, the same argument as in § 3.1–§ 3.3 applies and Theorem 5, 6 and 7 hold good.

Since $\mathfrak{r}(7)$ actually admits an irreducible representation in an 8-dimensional vector space, we have to study the case where ad($\mathfrak{h}$) in $\mathfrak{m}$ is irreducible. In the rest of this section we assume that this is the case.

LEMMA 14. *If* $n=8$ *and* ad($\mathfrak{h}$) *in* $\mathfrak{m}$ *is irreducible, we have* $[\mathfrak{m},\mathfrak{m}]=(0)$.

PROOF. In the same manner as in Lemma 2 we can easily see that the space $[\mathfrak{m},\mathfrak{m}]$, which is invariant under ad($\mathfrak{h}$) in $\mathfrak{g}$, is either (0) or $\mathfrak{h}$. We shall prove that the case $[\mathfrak{m},\mathfrak{m}]=\mathfrak{h}$ cannot occur. If this were the case, $\mathfrak{g}$ must be simple by the same reason as in Lemma 4. On the other hand, $\dim\mathfrak{g}=29$ and there is not a 29-dimensional simple Lie algebra. Thus we have $[\mathfrak{m},\mathfrak{m}]=(0)$.

LEMMA 15. *H is a maximal compact subgroup of G.*

PROOF. Since $\mathfrak{m}$ is an abelian ideal of $\mathfrak{g}$ and H is almost effective on G/H, G is not compact. If we denote by K a maximal compact subgroup of G containing H and by $\mathfrak{k}$ its Lie algebra, then $\mathfrak{k}$ is invariant under $\mathrm{ad}(\mathfrak{h})$ in $\mathfrak{g}$. Since only invariant subspaces under $\mathrm{ad}(\mathfrak{h})$ are $\mathfrak{m}$, $\mathfrak{h}$ and $\mathfrak{g}$, and $\mathfrak{k} \neq \mathfrak{g}$, it follows that $\mathfrak{k} = \mathfrak{h}$ and therefore $K = H$, which proves Lemma 15.

Since $[\mathfrak{m}, \mathfrak{m}] = (0)$, the canonical invariant Riemannian connection on G/H, which is clearly unique, is locally flat. H being a maximal compact subgroup of G, G/H is homeomorphic to an n-dimensional Euclidean space. We have thus obtained the following result:

THEOREM 9. *The notation and assumptions being as in the beginning of* § 3, *assume further that* $n = 8$ *and the representation* $\mathfrak{h} \rightarrow \mathrm{ad}(\mathfrak{h})$ *in* $\mathfrak{m}$ *is irreducible; then the homogeneous space* G/H *is naturally a locally flat Riemannian space and is homeomorphic to an n-dimensional Euclidean space.*

Now we denote by C_+^n, C_-^n and C_0^n an n-dimensional Riemannian space of positive and negative constant curvature and a locally flat Riemannian space respectively, and denote by E^n and S^n an n-dimensional Euclidean space and sphere respectively. Using this notation we gather in the following statement the main results obtained in this § 3.

THEOREM 10. *Let G be a connected Lie group of dimension r and H a compact subgroup of dimension* $r-n$. *Assume that* $n(n-1)/2 < r < n(n+1)/2$, $n \geqq 3$, $n \neq 4$ *and G is almost effective on the homogeneous space G/H as a transformation group. Then G is of dimension* $n(n-1)/2+1$ *and the homogeneous space G/H is one of the followings:*

as a Riemannian space	*as a topological space*
$C_0^1 \times C_+^{n-1}$,	$E^1 \times S^{n-1}$ *if it is simply connected,*
$C_0^1 \times C_-^{n-1}$,	E^n *or* $S^1 \times E^{n-1}$,
C_0^n,	E^n *or* $S^1 \times E^{n-1}$,
C_-^n,	E^n.

The exceptional case $n = 4$ should be stated here, but, as was stated before, it has been studied by S. Ishihara [1], so that we shall only remark that $\mathfrak{h}$ is isomorphic either to $\mathfrak{r}(3)$ or to $\mathfrak{su}(2)$ and if $\mathfrak{h}$ is isomorphic to $\mathfrak{r}(3)$ then all the theorems and lemmas in § 3 hold good.

REMARK. If the case (III) in Lemma 12 occurs, then the homogeneous space G/H is a symmetric Riemannian space with respect to its (unique) invariant Riemannian connection, but not a symmetric homogeneous space because $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{m}$. In this case the largest connected group of isometries of the Riemannian space G/H is not G, but the Lorentz group $L(n+1)$.

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