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On birational invariance of classical groups.

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In the present paper, we define the notion of birational isomorphisms for algebraic groups by making use of the rational representations treated in Chevalley's book [1],[1] and formulate the following problem. Given a family $\mathfrak{F}$ of algebraic groups, is it birationally invariant (that is, does $\mathfrak{F}$ contain every algebraic group which is birationally isomorphic with a group in $\mathfrak{F}$)?

Here we take as $\mathfrak{F}$ the family composed of classical groups,[2] and shall show that the answer is affirmative if the field of definition of the groups is of characteristic zero and the dimension of the vector space on which groups operate is large enough. The proof depends on Weyl's representation theory and Dieudonné's structure theory of classical groups. We don't know whether the same assertion is true or not for algebraic groups defined over a field of characteristic p.

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§ 1. Birational isomorphism.

Let V be a finite dimensional vector space over an infinite field K,[3] and let E be the vector space of endomorphisms of V. We shall say that two algebraic groups G and G' on V are birationally isomorphic if there is an isomorphism ρ of G onto G' such that ρ and

1) [1] p. 101, Definition 4. In this paper, we shall freely use definitions and results in [1].

2) By classical groups we mean the following groups operating on a vector space V: $GL(V)$, $SL(V)$, $Sp(V,f)$, f being a non-degenerate skew symmetric bilinear form on V, $O(V,f)$, f being a non-degenerate symmetric bilinear form on V, and $SO(V,f)=SL(V)\cap O(V,f)$. we denote their Lie algebras by the corresponding small German letters.

3) Hereafter, by a field we shall always mean a field with infinitely many elements.

ρ^{-1} are rational representations of G and G' respectively. We shall denote the birational isomorphism by $G \underset{(r)}{\cong} G'$.

PROPOSITION 1. *Let G_1 and G_1' be the algebraic components of G and G' respectively. If ρ is a birational isomorshism of G with G', then it induces a birational isomorphism of G_1 with G_1'.*

PROOF. Let H be the smallest algebraic group containing $\rho(G_1)$. Then, H is irreducible and of finite index in G'. By the uniqueness of the subgroup in G' of this property, we have $H = G_1'$ and it follows that $\rho(G_1) \subset G_1'$. By the similar argument on ρ^{-1}, we have $\rho^{-1}(G_1') \subset G_1$. Therefore we see that $\rho(G_1) = G_1'$.

PROPOSITION 2. *Let $\mathfrak{g}$ and $\mathfrak{g}'$ be the Lie algebras of G and G' respectively. Then, the differential $d\rho$ of a birational isomorphism ρ gives an isomorphism $\mathfrak{g}$ with $\mathfrak{g}'$. Furthermore, if ρ is the restriction on G of a linear endomorphism λ of the vector space E, then $d\rho$ is also the restriction of λ on $\mathfrak{g}$.*

PROOF. By definition of the differential, we have $d\rho(\mathfrak{g}) \subset \mathfrak{g}'$ and $d\rho^{-1}(\mathfrak{g}') \subset \mathfrak{g}$. Since $d\rho^{-1} \circ d\rho = d(\rho^{-1} \circ \rho) = d\iota$, where ι is the identity mapping on G, $d\iota$ is the identity mapping on $\mathfrak{g}$. Hence $d\rho$ is a univalent mapping of $\mathfrak{g}$ into $\mathfrak{g}'$. In the same way, $d\rho^{-1}$ is a univalent mapping of $\mathfrak{g}'$ into $\mathfrak{g}$. This proves the first part. The second part comes from the equalities $d\rho(X) = d\rho(I, X) = d\lambda(I, X) = \lambda(X)$ for all $X \in \mathfrak{g}$, where I is the identity element of G.

PROPOSITION 3. *Suppose that K is of characteristic zero, and G, G' are irreducible. Let λ be a linear endomorphism of the vector space E such that the image $\lambda(G)$ is an algebraic group. Suppose that λ induces a birational isomorphism of G with $\lambda(G)$. If $\lambda(\mathfrak{g}) = \mathfrak{g}'$, then we have $\lambda(G) = G'$.*

PROOF. Let $\mathfrak{h}$ be the Lie algebra of $\lambda(G)$. By Proposition 2, we see that $\lambda(\mathfrak{g}) = \mathfrak{h}$. Therefore we have $\mathfrak{h} = \mathfrak{g}'$. By our assumption on K and G, G', we have $\lambda(G) = G'$.

If we apply the above two propositions to the linear endomorphism $\lambda(u) = tut^{-1}$, $u \in E$, $t \in GL(V)$, we have at once the following

PROPOSITION 4. *Under the same assumption as in Proposition 3, we have $G' = tGt^{-1}$ if and only if $\mathfrak{g}' = t\mathfrak{g}t^{-1}$, $t \in GL(V)$.*

§ 2. Birational invariance of irreducible classical groups.

First, we shall give an alternative proof of the irreducibility of classical groups by making use of their structure theory.

PROPOSITION 5. *Let K be an arbitrary field. Each of the groups $GL(V)$, $SL(V)$, $Sp(V,f)$ is irreducible. Suppose that the characteristic of K is not 2 and $\dim V \geqq 5$. Then, $SO(V,f)$ is irreducible.*[4]

PROOF. As to $GL(V)$ the proposition is trivial. As to $SL(V)$ and $Sp(V,f)$, we may assume that $\dim V \geqq 2$, since $SL(V)$ and $Sp(V,f)$ reduce to the identity for $\dim V=1$. Thus, these groups are of infinite order. Let G_1 be the algebraic component of $SL(V)$ or $Sp(V,f)$. Then, G_1 is an infinite group, and it can not be contained in the center which is of finite order. Thus, we have $G_1=SL(V)$ or $Sp(V,f)$ by the argument of Dieudonné.[5] As for the last group $SO(V,f)$ we may assume that K is algebraically closed, since the irreducibility of groups is independent of the extension of the scalar field. Thus, the bilinear form f is of index >0 and $SO(V,f)=\Omega(V,f)$: the commutator subgroup of $O(V,f)$. Thus, by the similar argument as above,[6] we conclude the irreducibility of $SO(V,f)$.

In the following propositions, we shall determine the algebraic groups which are birationally isomorphic with irreducible classical groups.

PROPOSITION 6. *Let K be an arbitrary field. If an algebraic group G is birationally isomorphic with $GL(V)$, then $G=GL(V)$.*

PROOF. We have an isomorphism of $\mathfrak{g}$ with $\mathfrak{gl}(V)$ and it follows at once that $\mathfrak{g}=\mathfrak{gl}(V)$. Therefore G is a subgroup in $GL(V)$ of finite index. Since both groups are irreducible, we have $G=GL(V)$.

PROPOSITION 7. *Let the characteristic of K be zero. If $G \underset{(r)}{\cong} SL(V)$, then $G=SL(V)$.*

PROOF. Let ρ be the birational isomorphism of G with $SL(V)$. Then $d\rho$ gives an isomorphism $\mathfrak{g} \cong \mathfrak{sl}(V)$. Since the statement is trivial for the case of $\dim V=1$, we may assume that $\dim \mathfrak{g} \geqq 3$. It is easy

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4) $SO(V,f)$ is irreducible without any assumption on $\dim V$. ([1], pp. 123–125) It is sufficient for our purpose to prove it under the above restriction.

5) [3], pp. 38–40. [2], p. 12 Theorem 1.

6) [2], p. 29 Theorem 2.

to see that the mapping $X \to t_r X$ of $\mathfrak{g}$ into K is a homomorphism of the Lie algebras. As is well known $\mathfrak{g}$ is a simple Lie algebra. Therefore the kernel of the trace mapping is the whole $\mathfrak{g}$ and it follows that $\mathfrak{g} \subset \mathfrak{sl}(V)$. Thus, we have $\mathfrak{g} = \mathfrak{sl}(V)$ and hence $G = SL(V)$.

To prove the Proposition 8, we need the following two lemmas.

LEMMA 1. *Let K be of characteristic zero. If $\mathfrak{sp}(V, f) = \mathfrak{sp}(V, g)$ or $\mathfrak{o}(V, f) = \mathfrak{o}(V, g)$, then $g = \lambda f$ with $\lambda \in K^*$.*[7]

PROOF. Let F and G be the skew symmetric matrices of f and g respectively with respect to some base of V. Then, we know that $\mathfrak{sp}(V, f) = \{X; X \in K_n, {}^tXF + FX = 0\}$ and $\mathfrak{sp}(V, g) = \{X; X \in K_n, {}^tXG + GX = 0\}$, $n = \dim V$. From the assumption, we have $-{}^tX = GXG^{-1} = FXF^{-1}$ for all $X \in \mathfrak{sp}(V, f)$. Therefore $F^{-1}G$ commutes with all $X \in \mathfrak{sp}(V, f)$. As is well known, $\mathfrak{sp}(V, f)$ is an absolutely irreducible matric system. Thus, from Schur's lemma it follows that $G = \lambda F$, $\lambda \in K^*$. In the same way, the statement on $\mathfrak{o}(V, f)$ is proved.

LEMMA 2. *Let L be an over field of K, K being of characteristic zero. Let $\mathfrak{g}$ be a Lie algebra over K such that $\mathfrak{g}^L = \mathfrak{sp}(V^L, f)$ or $= \mathfrak{o}(V^L, f)$, where f is a form on V^L. Then, there is a form f_1 on V such that $f_1 = \lambda f$ with $\lambda \in L^*$, and $\mathfrak{g} = \mathfrak{sp}(V, f_1)$ or $= \mathfrak{o}(V, f_1)$.*

PROOF. Let F be the (skew) symmetric matrix of f with respect to some base of V. From our assumption, we have $\mathfrak{g}^L = \{X; X \in L_n, {}^tXF + FX = 0\}$. Let $\mathfrak{S}$ be the vector space composed of all (skew) symmetric matrices in K_n. Now, to each $X \in \mathfrak{g}$, we define an endomorphism A_X of $\mathfrak{S}$ by $A_X(S) = {}^tXS + SX$, $S \in \mathfrak{S}$. Let $\mathfrak{T}$ be the vector space composed of all $S \in \mathfrak{S}$ with $A_X(S) = 0$ for all $X \in \mathfrak{g}$. Then, it is easily verified that the scalar extension $\mathfrak{T}^L$ is the vector space composed of all $S' \in \mathfrak{S}^L$ with $A_X(S') = 0$ for all $X \in \mathfrak{g}^L$. Obviously we have $F \in \mathfrak{T}^L$. Conversely, take any $T \in \mathfrak{T}^L$. Since the determinant $D(T + \lambda F) = D(F)\lambda^n + \cdots$, $D(F) \neq 0$, we may take an $F' = T + \lambda F$ with $D(F') \neq 0$ for a suitable λ. We put $\mathfrak{g}' = \{X; X \in L_n, {}^tXF' + F'X = 0\}$. Since $F' \in \mathfrak{T}^L$, it follows that $\mathfrak{g}^L \subset \mathfrak{g}'$ by definition of $\mathfrak{T}$. By the above choice of F', we have $\mathfrak{g}^L = \mathfrak{g}'$. From Lemma 1, it follows that $F' = \lambda' F$, $\lambda' \in L^*$, and $T = F' - \lambda F = (\lambda' - \lambda)F$. Thus, the space $\mathfrak{T}^L$ is 1 dimensional: $\mathfrak{T}^L = \{F\}$. By taking a base F_1 of $\mathfrak{T}$, our lemma is proved.

PROPOSITION 8. *Let K be of characteristic zero. If $G \underset{(r)}{\cong} Sp(V, f)$,*

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7) K^* denotes the totality of non zero elements in K.

then $G=Sp(V,g)$. *Suppose that* $\dim V>6, \neq 8$. *If* $G \underset{(r)}{\cong} SO(V,f)$, *then* $G=SO(V,g)$.

PROOF. By taking the differential of the birational isomorphism, we have $\mathfrak{g} \cong \mathfrak{sp}(V,f)$ or $\cong \mathfrak{o}(V,f)$. Let L be the algebraic closure of K. Then, naturally we have $\mathfrak{g}^L \cong \mathfrak{sp}(V^L,f)$ or $\cong \mathfrak{o}(V^L,f)$. Thus, we have a faithful representation of $\mathfrak{sp}(V^L,f)$ or $\mathfrak{o}(V^L,f)$ of degree n. However, under the above restriction on the dimension of V, we know that the degree of the non trivial irreducible representation of $\mathfrak{sp}(V^L,f)$ or $\mathfrak{o}(V^L,f)$ is $\geqq n$ and there is one and only one representation class of degree n.[8] Therefore the above representation is equivalent to the identity representation of $\mathfrak{sp}(V^L,f)$ or $\mathfrak{o}(V^L,f)$. Thus, there exists a form g' on V^L such that $\mathfrak{g}^L=\mathfrak{sp}(V^L,g')$ or $=\mathfrak{o}(V^L,g')$. From Lemma 2 it follows that $\mathfrak{g}=\mathfrak{sp}(V,g)$ or $=\mathfrak{o}(V,g)$, where g is a form on V. By the irreducibility of the groups $Sp(V,g)$ and $SO(V,g)$, we get our proposition.

Now, let $\mathfrak{F}$ be a family of algebraic groups on V. We shall say that $\mathfrak{F}$ is birationally invariant if any algebraic groups G which is birationally isomorphic with some group F in $\mathfrak{F}$ is in $\mathfrak{F}$.

THEOREM 1. *Let K be a field of characteristic zero and let V be a vector space over K with* $\dim V>6, \neq 8$. *Let $\mathfrak{C}_1$ be the totality of irreducible classical groups on V. Then, $\mathfrak{C}_1$ is birationally invariant.*

PROOF. This follows immediately from Proposition 6, 7 and 8.

THEOREM 2. *Under the same assumption as in Theorem 1, let $\mathfrak{C}$ be the totality of the algebraic groups on V whose algebraic components are irreducible classical gronps on V. Then, $\mathfrak{C}$ is birationally invariant.*

PROOF. Let $G \underset{(r)}{\cong} C$, $C \in \mathfrak{C}$. Then, from Proposition 1, $G_1 \underset{(r)}{\cong} C_1$, where G_1 and C_1 are the algebraic components of G and C respectively. Since $C_1 \in \mathfrak{C}_1$, we have $G_1 \in \mathfrak{C}_1$ by Theorem 1. It implies that $G \in \mathfrak{C}$.

§ 3. Birational invariance of classical groups.

We begin with the explicit determination of certain groups which have $SO(V,f)$ as their algebraic components.

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8) This fact follows, for example, from Weyl's formula which gives the degrees of the irreducible representations of a semi-simple Lie algebra with given highest weights. [4], p. 342 Satz 4, p. 353 Satz 6.

LEMMA 3. *Let* $\mathfrak{g}$ *be the Lie algebra* $\mathfrak{o}(V, f)$ *over a field* K *of characteristic zero. If* $s \in GL(V)$ *satisfies the condition* $s\mathfrak{g}s^{-1} \subset \mathfrak{g}$, *then,* s *is a similarity transformation with respect to* f, *i. e.* $f(sx, sy) = \gamma f(x, y)$, $\gamma \in K^*$, $x, y \in V$.

PROOF. Let S, F be the matrices of s, f relative to some base of V. Then, we have $\mathfrak{g} = \{X ; {}^tXF + FX = 0\} = \{X ; FXF^{-1} = -{}^tX\}$. Since $SXS^{-1} \in \mathfrak{g}$ for all $X \in \mathfrak{g}$, we have $FSXS^{-1}F^{-1} = -{}^t(SXS^{-1}) = {}^tS^{-1}(-{}^tX){}^tS = {}^tS^{-1}FXF^{-1}\,{}^tS$. Therefore $F^{-1}\,{}^tSFS$ commutes with all $X \in \mathfrak{g}$. Since $\mathfrak{g}$ is an absolutely irreducible matric system, we have ${}^tSFS = \gamma F$ from Schur's lemma. This proves our lemma.

Now, let f be a symmetric form on V. We shall denote by $A(V, f)$ the group composed of all similarity transformations with respect to f, i. e. $A(V, f) = \{s : f(sx, sy) = N(s) f(x, y), N(s) \in K^*, x, y \in V\}$. It is easy to verify that the mapping $N : s \rightarrow N(s)$ is a homomorphism of $A(V, f)$ into K^*. We shall call $N(s)$ the norm of s. As is easily seen, we have

$$N(s)^n = D(s)^2, \qquad N(\alpha s) = \alpha^2 N(s) \qquad (\alpha \in K^*)$$

where $D(S)$ means the determinant of s.

PROPOSITION 9. *Suppose that* K *is of characteristic zero and algebraically closed. If* n *is odd, then* $O(V, f)$ *is the only extension of index* 2 *of* $SO(V, f)$. *If* n *is even, then there are three algebraic groups* $O(V, f)$, $P(V, f)$ *and* $Q(V, f)$ *which are extensions of index* 2 *of* $SO(V, f)$.

PROOF. Let G be an algebraic group which contains $SO(V, f)$ as a subgroup of index 2. Thus, $SO(V, f)$ is normal in G and $\varphi_s(u) = sus^{-1}$, $u \in SO(V, f)$, gives a birational automorphism of $SO(V, f)$ for any $s \in G$. By Proposition 4, the differential $d\varphi_s$ induces an automorphism of the Lie algebra $\mathfrak{g} = \mathfrak{o}(V, f)$ such that $d\varphi_s(X) = sXs^{-1}$, $X \in \mathfrak{g}$. Therefore from Lemma 3, s belongs to $A(V, f)$. Now, put $R(V, f) = \{t ; t \in A(V, f), D(t)^2 = N(t)^2 = 1\}$. Since $s^2 \in SO(V, f)$, it is obvious that $s \in R(V, f)$ and it follows that $G \subset R(V, f)$. Suppose that n is odd. Then, we have $1 = D(t)^2 = N(t)^n = N(t)$ for any $t \in R(V, f)$. Thus, we have $R(V, f) = O(V, f)$. Therefore $G = O(V, f)$. This proves the statement for any odd n. Next, suppose that n is even. For any $t \in O(V, f)$, we have $N(\sqrt{-1}\,t) = -1$ and $D(\sqrt{-1}\,t) = (\sqrt{-1})^n D(t) = \pm 1$. Thus, $\sqrt{-1}\,t \in R(V, f)$ and $\notin O(V, f)$. This shows that $O(V, f)$ is the kernel of the norm homomorphism N of $R(V, f)$ onto $\{\pm 1\}$, and we see that the factor group $R(V,f)/SO(V,f)$ is an abelian group of order 4. Now let $P(V,f)$ be the kernel of the determinant homomorphism D of $R(V,f)$ into $\{\pm 1\}$. This is obviously an onto mapping. Thus, $(R(V,f):P(V,f))=2$. If $n\equiv 0 \bmod 4$, then thet element $\sqrt{-1}\,t$, where $t\in SO(V,f)$, belongs to $P(V,f)$ but not to $O(V,f)$, since $N(\sqrt{-1}\,t)=-1$ and $D(\sqrt{-1}\,t)=(\sqrt{-1})^n D(t)=1$. If $n\equiv 2 \bmod 4$, then the element $\sqrt{-1}\,t$, where $t\in O(V,f)$ and $\notin SO(V,f)$, belongs to $P(V,f)$ but not to $O(V,f)$, since $N(\sqrt{-1}\,t)=-1$ and $D(\sqrt{-1}\,t)=(\sqrt{-1})^n D(t)=1$. Therefore $P(V,f)\neq O(V,f)$ for any even n. Thus, we see that $R(V,f)/SO(V,f)$ is the abelian group of type $(2,2)$, and by the above argument G must be one of the three groups $O(V,f)$, $P(V,f)$ and $Q(V,f)$, where $Q(V,f)$ is the kernel of the homomorphism $t\to D(t)N(t)$ of $P(V,f)$ onto $\{\pm 1\}$. Thus, our proposition is proved.

To prove the next proposition we need a lemma.

LEMMA 4. *Suppose that n is even and $\geqq 6$. Let G be one of the three groups $O(V,f)$, $P(V,f)$ and $Q(V,f)$ defined in the proof of Proposition 9. Let $\sigma\in G$ be an involution and let V_i $(i=1,2)$ be the non isotropic subspaces associated with σ such that $\sigma(x)=-x$, $x\in V_1$, $\sigma(y)=y$, $y\in V_2$,[9] and let f_i be the restrictions of f on V_i $(i=1,2)$. Let Ω_σ be the commutator subgroup of the centralizer Z_σ of σ in G.*

i) If $G=O(V,f)$, then $\Omega_\sigma\cong SO(V_1,f_1)\times SO(V_2,f_2)$

ii) If $G=P(V,f)$ or $Q(V,f)$ and $n_i\neq 2$, $n_i=\dim V_i$, $i=1,2$, then $\Omega_\sigma\cong SO(V_1,f_1)\times SO(V_2,f_2)$

iii) If $G=P(V,f)$ or $Q(V,f)$ and if one of n_i $(i=1,2)$ is equal to 2, then the center of Ω_σ is at least of order 2.

PROOF. It follows easily that $\tau V_i=V_i$ $(i=1,2)$ for all $\tau\in Z_\sigma$ for each G. Let τ_i be the restrictions of τ on V_i. Then, the mapping $\varphi:\tau\to(\tau_1,\tau_2)$ gives an isomorphism of Z_σ into $A(V_1,f_1)\times A(V_2,f_2)$. Therefore φ maps Ω_σ into $SO(V_1,f_1)\times SO(V_2,f_2)$. If one of n_i $(i=1,2)$, say n_1, is equal to 0 or to 1, then $n_2\geqq 5$ and φ maps Ω_σ onto $\{1\}\times SO(V_2,f_2)$, since $SO(V_2,f_2)$ is identical with its commutator subgroup.[10] If both n_i $(i=1,2)$ are $\neq 2$, then for the same reason φ is an onto mapping. As for $G=O(V,f)$, this is also valid for $n_1=2$,

9) The existence of such V_i for similarity involutions is proved in the same way as for orthogonal involutions. cf. [2], Prop. 3 (p. 9)

since $SO(V_1, f_1)$ is the commutator subgroup of $O(V_1, f_1)$.[10] On the other hand for the case iii), we have $\{1\}\times SO(V_2, f_2)\subset\varphi(\Omega_{\bar{\sigma}})\subset SO(V_1, f_1)\times SO(V_2, f_2)$, since $n_2\geqq 4$. Since n_2 is even, it follows from this relation that the center of $\varphi(\Omega_\sigma)$ is at least of order 2.

PROPOSITION 10. *Let K be the same as in Proposition 9 and suppose that n is even and $\geqq 6$. Then, any two of the three groups $O(V, f)$, $P(V, f)$ and $Q(V, f)$ are not isomorphic with each other.*

PROOF. By using the same notations as in Lemma 4, let $n_i=\dim V_i$ $(i=1, 2)$. As for $G=O(V, f)$, let σ be a symmetry with respect to a hyperplane V_2. Then, $n_1=1$, $n_2=n-1$ and $\Omega_\sigma\cong SO(V_2, f_2)$. Since $n-1$ is odd and $\geqq 5$, Ω_σ is a simple group.[11] On the other hand, as for $G=P(V, f)$ or $Q(V, f)$, any involution $(\neq 1)$ cannot be a symmetry, since $G\cap O(V, f)=SO(V, f)$. Thus, we have $n_i\geqq 2$, and if $n_i\neq 2$ $(i=1, 2)$, then $\Omega_\sigma\cong SO(V_1, f_1)\times SO(V_2, f_2)$ is not simple and if, say, $n_1=2$, then the center of Ω_σ is at least of order 2 and this is not simple too. Therefore $O(V, f)$ cannot be isomorphic neither with $P(V, f)$ nor with $Q(V, f)$. Next, let Z and Z' be the center of $P(V, f)$ and $Q(V, f)$ respectively. Since any $z\in Z$ or Z' must commute with every plane rotation, it leaves every non-isotropic plane invariant, and it follows that $z(x)=\lambda x$, $\lambda\in K^*$.[12] Therefore, we have $D(z)=\lambda^n=1$, $N(z)=\lambda^2=\pm 1$ for $P(V, f)$ and $D(z)N(z)=\lambda^{n+2}=1$ for $Q(V, f)$. From these relations, if follows that if $n\equiv 0 \bmod 4$, then $Z=\{\pm E, \pm\sqrt{-1}E\}$ and $Z'=\{\pm E\}$, and if $n\equiv 2 \bmod 4$, then $Z=\{\pm E\}$ and $Z'=\{\pm E, \sqrt{-1}E\}$.[13] Thus, $P(V, f)$ and $Q(V, f)$ cannot be isomorphic. Thus, our proposition is proved.

THEOREM 3. *Let K be a field of characteristic zero and let V be a vector space over K with $\dim V>6$, $\neq 8$. Let $\mathfrak{C}_2$ be the totality of the classical groups on V. Then, $\mathfrak{C}_2$ is birationally invariant.*

PROOF. From Theorem 1, it is sufficient to show that if $G\underset{(r)}{\cong}O(V, f)$, then $G=O(V, g)$ with some form g on V. From Proposition 1, we have $G_1\underset{(r)}{\cong}SO(V, f)$. Therefore $G_1=SO(V, g)$ from Proposition 8. Let L be the algebraic closure of K. It follows that

10) [3], p. 55.
11) [2], p. 29, Theorem 2.
12) See argument in [2], p. 24.
13) E denotes the identity matrix.

$G^L \underset{(r)}{\cong} O(V^L, f) \underset{(r)}{\cong} O(V^L, g)$, $G_1 = SO(V, g)$, since any two forms are equivalent. From Proposition 9 and 10, it follows that $G^L = O(V^L, g)$ and by taking the intersections with $GL(V)$ of both sides, we have $G = O(V, g)$.

REMARK. If the dimension of V is small, then the classical groups are not necessarily birationally invariant as the following simple example shows. Let $\dim V = 2$, and let $G = \{\lambda E; \lambda \in K^*\}$, $C = SO(V, f)$ with $\nu(f) = 1$, where $\nu(f)$ denotes the index of f. Then, $G \underset{(r)}{\cong} C$, but G is not classical.

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